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Bulk viscosity of strongly interacting matter in the relaxation time approximation

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This paper presents how thermal mean field effects are incorporated consistently in the hydrodynamical modelling of heavy-ion collisions. The nonequilibrium correction to the distribution function resulting from a temperature-dependent mass is obtained in a procedure which automatically satisfies the Landau matching condition and is thermodynamically consistent. The physics of the bulk viscosity is studied here for Boltzmann and Bose-Einstein gases within the Chapman-Enskog and 14-moment approaches in the relaxation time approximation. Constant and temperature-dependent masses are considered in turn. It is shown that, in the small mass limit, both methods lead to the same value of the ratio of the bulk viscosity over its relaxation time. The inclusion of a temperature-dependent mass leads to the emergence of the β_λ -function in that ratio, and it is of the expected parametric form for the Boltzmann gas, while for the Bose-Einstein case it is affected by the infrared cut-off. This suggests that the relaxation time approximation may be too crude to obtain a reliable form of ζ/τ_R for gases obeying Bose-Einstein statistics.

1. INTRODUCTION

The vibrant experiment program pursued at the Relativistic Heavy Ion Collider (RHIC) and at the Large Hadron Collider (LHC) have ushered in a new era of exploration of systems governed by the nuclear strong interaction. One of the remarkable features that emerged from investigating the physics of relativistic heavy-ion collisions is the fact that the created systems could be modelled theoretically by relativistic fluid dynamics [1, 2]. This realization led to developments in the formulation of relativistic viscous hydrodynamics in which observable consequences of the dissipative effects have been isolated [3–13]. Currently, second-order viscous hydrodynamics provides a description of the fluid behavior [11–14] which remedies the main failure of the Navier-Stokes – or first-order – formulation: acausal signal propagation and numerical instabilities plaguing relativistic systems.

While the hydrodynamic equations are universal and provide a macroscopic picture of a relativistic fluid behavior in terms of conservation laws, transport coefficients are governed by the underlying microscopic theory which must be used for their extraction. Although the first applications of viscous hydrodynamics focused on the shear viscosity, it has recently become clear that bulk viscosity also plays an important role in the evolution of the QGP system [15–17]. The calculation of bulk viscosity from first principles, however, remains a challenging project. It is on this aspect that we concentrate in this paper.

The equations of the second-order hydrodynamics describe very efficiently the expansion of the system produced in heavy-ion collisions. This is a strong indication that the system must thermalize very rapidly, which in turn indicates that the system is strongly interacting at presently achievable energies. Current estimates of the bulk viscosity of QCD are mainly based on the equation of state obtained from lattice QCD simulations [19, 20],

or rely on empirical extractions based on simulations of relativistic nuclear collisions [15–18]. Application of lattice QCD findings [21–23] and the hadron resonance gas results [24, 25] allowed to determine that the bulk viscosity is notably enhanced near the critical temperature of the QCD phase transition while the shear viscosity is substantially decreased in this region [26, 27]. Furthermore, the importance of bulk viscosity near the transition temperature region was shown to have a remarkable impact on the elliptic flow coefficient v_2 [25, 28, 29] and other heavy-ion observables [15–18, 30, 31]. Recently, the behavior of bulk viscosity was also obtained from hydrokinetic theory which incorporates thermal noise [32].

Despite the progress described above, there is still a need to develop methods which provide a better insight in the effects of bulk viscosity at different energy scales. In particular, one may be interested in having a consistent analytical approach to bulk viscosity physics in the regime of very high temperatures. At this energy scale the coupling constant is small and fundamental quantum field theoretical tools can be used to study bulk viscosity systematically. Having given a comprehensive fluid dynamic formulation of a weakly coupled gas may also provide an essential benchmark for different approaches and phenomenological applications.

In Refs. [33, 34] it was shown that quantum field theory is equivalent, at least at leading order of perturbative expansion, to kinetic theory. Later calculations then could use this efficient and intuitive kinetic theory framework to study transport phenomena, see [35–37]. It has also provided a natural language to formulate fluid dynamics concepts. Within the kinetic approaches, the Chapman-Enskog and the Grad's 14-moment methods are commonly employed to study the nonequilibrium processes of a fluid. They, however, rely on different treatments of the distribution function. While the Chapman-Enskog theory deals directly with solving the Boltzmann equation [38], the Grad's approach is based on an expansion

of the nonequilibrium function in terms of the powers of momenta [39]. To date, great progress has been made in extraction of different transport coefficients within different theories. It seems, however, that the comprehensive analysis of transport processes in a system exhibiting conformal anomaly is not yet complete, especially in cases involving a mean field interaction.

A violation of conformal symmetry has a different impact on different transport coefficients. It does not affect shear viscosity much, its leading order behavior is dominated by the kinetic energy scale in weakly interacting systems. On the other hand, the breaking of scale invariance dominates the physics of bulk viscosity. Consequently, the behavior of bulk viscosity is largely determined by the sources of conformal symmetry breaking, either the physical mass of plasma constituents or the Callan-Symanzyk β_λ -function, which fixes the coupling as a function of the energy scale [33]. The parametric form of bulk viscosity should then be dictated by the sources of scale invariance breaking squared, as shown in Ref. [40] for QCD. The bulk viscosity of systems exhibiting conformal anomaly due to presence of a constant mass only, was later studied within the Chapman-Enskog approach and 14-moment approximation, mostly in the relaxation time approximation [41–43], and also within other approaches [44]. Moreover, quasiparticle models were also examined for systems of various matter content in Refs. [45–53].

We observe, however, that there is still a need to revisit a formulation of nonequilibrium fluid dynamics with the mean field background. Such a formulation is essential when one needs to include variable thermal masses consistently in the equations of viscous hydrodynamics. Having the correct form of a nonequilibrium momentum distribution is also critical while studying some aspects of nuclear matter behavior phenomenologically, in particular, when implementing the Cooper-Frye prescription in hydrodynamic simulations or examining electromagnetic probes in heavy-ion collisions [17, 54–56]. Furthermore, such a consistent approach allows for an exhaustive calculation of transport coefficients.

The central part of this paper is devoted to derivation of the nonequilibrium correction to the distribution function where thermal effects are consistently included. Subsequently, it is shown how the correction influences the bulk viscosity behavior in the relaxation time approximation. The analysis is done systematically and it comprises different cases, namely, formulation of equilibrium and nonequilibrium fluid dynamics and then computation of the ratio of bulk viscosity over relaxation time. A computation is provided for gases of Boltzmann and Bose-Einstein statistics in both the Anderson-Witting model of the Chapman-Enskog method and the 14-moment approximation. The analysis performed in this paper is specific to single-component bosonic degrees of freedom. Consequently, when the explicit form of the thermal mass and the β_λ function is needed, we will use those of the scalar $\lambda\phi^4$ theory [33, 34]. The method developed here

is not appropriate for one-component systems of Fermi-Dirac distribution function. Such a system would be a system of non-interacting fermionic degrees of freedom where the thermal mass and bulk viscosity cannot be determined. To count fermions accurately one needs to consider a many-component system with the inclusion of bosons mediating the interaction. This is not done here and is left for future work.

The correction to the distribution function is found by noticing that there is a twofold source of departure from equilibrium. Firstly, there are hydrodynamic forces that generate a deviation in the distribution function δf , that is, they change the functional form of the distribution function. The other source is related directly to interparticle interactions, the effect of which is statistically averaged and emerges as the mean field. Therefore, the correction is expressed by two terms, which, for the Bose-Einstein gas, is

$$\Delta f = \delta f - T^2 \frac{dm_{\text{eq}}^2}{dT^2} \frac{f_0(1+f_0)}{E_k} \frac{\int dK \delta f}{\int dK E_k f_0(1+f_0)}. \quad (1.1)$$

For the description of quantities, see Table I. The obtained form of the correction allows one to formulate hydrodynamic equations in a coherent way, where the Landau matching condition and thermodynamic relations are guaranteed. Since the thermal mean field has a negligible impact on shear viscosity, we further concentrate on bulk viscosity dynamics, where the influence of the thermal background reveals itself through the Landau condition and the speed of sound.

We show that both the Chapman-Enskog and the 14-moment approaches lead to the same final expressions for the ζ/τ_R ratio in the small mass limit, where τ_R is the bulk relaxation time. In general, temperature dependent mass results in the emergence of the β_λ -function, which dictates the very high temperature form of the ratio. In the Boltzmann case the ratio is

$$\frac{\zeta_{\text{Boltz}}}{\tau_R} \approx T^4 \left(\frac{1}{3} - c_s^2 \right)^2 \left(\frac{60}{\pi^2} - \frac{36m_x}{\pi T} \right), \quad (1.2)$$

where $(1/3 - c_s^2)$ is directly related to M_c , the nonconformality parameter: see Table I. This shows the expected behavior of the source of scale invariance breaking. One may observe that one factor of the scale invariance breaking parameter is introduced directly by the Landau matching, which comes from the small departure from equilibrium. The other factor emerges as a correction to the pressure given by purely equilibrium quantities, but not provided by the equation of state, as argued in [40]. For a system with Bose-Einstein statistics, the result is

$$\frac{\zeta}{\tau_R} \approx T^4 \left(\frac{1}{3} - c_s^2 \right)^2 \left(\frac{2\pi^3 T}{25m_x} - \frac{4\pi^2}{75} \left(1 - \frac{9m_{\text{eq}}^2}{8m_x^2} \right) \right). \quad (1.3)$$

The leading order term is not of the expected dependence because of the factor T/m_x , which comes from an infrared cut-off. The same behavior is reflected either if we neglect the constant mass term or thermally affected quantities. Therefore, it rather indicates that the relaxation time approximation, which assumes that τ_R is energy independent, may not allow one to entirely capture microscopic physics, in particular, of soft momenta in quantum gases of Bose-Einstein distribution function. A similar conclusion was reached in Ref. [40].

The paper is organized as follows. In Sec. 2 the ingredients of the effective kinetic theory are briefly summarized and the derivation of the nonequilibrium ther-

mal correction is provided. Sec. 3 is devoted to the formulation of fluid dynamic basic equations with the mean field background. In Sec. 4, the analysis of the bulk viscosity over relaxation time ratio is presented in the Chapman-Enskog theory within which we solve the Anderson-Witting model. In Sec. 5 we use the 14-moment approximation to derive the evolution equation for the bulk pressure and then to calculate the bulk viscosity over the relaxation time ratio and other transport coefficients in the bulk channel in the relaxation time approximation. Sec. 6 summarizes and concludes the work. Appendices contain some technical details.

description	equilibrium quantity	nonequilibrium quantity
physical, zero-temperature mass of a particle	m_0	m_0
quasiparticle thermal mass	m_{eq}	m_{th}
quasiparticle mass	$m_x = \sqrt{m_0^2 + m_{\text{eq}}^2}$	$\tilde{m}_x = \sqrt{m_0^2 + m_{\text{th}}^2}$
quasiparticle energy	$E_k = \sqrt{\mathbf{k}^2 + m_x^2}$	$\mathcal{E}_k = \sqrt{\mathbf{k}^2 + \tilde{m}_x^2}$
quasiparticle four-momentum	$k^\mu \equiv (k_0, \mathbf{k}) = (E_k, \mathbf{k})$	$\tilde{k}^\mu \equiv (\tilde{k}_0, \mathbf{k}) = (\mathcal{E}_k, \mathbf{k})$
Lorentz invariant measure	$dK = d^3\mathbf{k}/[(2\pi)^3 E_k]$	$d\mathcal{K} = d^3\mathbf{k}/[(2\pi)^3 \mathcal{E}_k]$
distribution function (in the local rest frame)	$f_0 = 1/[e^{\beta E_k} - 1]$, with $\beta = 1/T$	$f = f_0 + \Delta f$
beta function for a coupling constant λ	$\beta_\lambda = T d\lambda/dT = 3\lambda^2/(16\pi^2)$	
temperature dependence of the thermal mass	$T^2 dm_{\text{eq}}^2/dT^2 = m_{\text{eq}}^2 + aT^2\beta_\lambda$, with $a = 1/48$	
nonconformality parameter	$M = (-m_0^2 + aT^2\beta_\lambda)/3$	

TABLE I. The quantities characterizing the equilibrium and nonequilibrium dynamics of a gas with the Bose-Einstein statistics. For the classical gas with the Boltzmann statistics some of these quantities have different values or forms and whenever there is a need to distinguish them we add the subscript ‘c’: $m_{\text{eq},c}$, $f_{0,c} = e^{-\beta E_k}$, f_c , $m_{\text{th},c}$, $a_c = 1/(8\pi^2)$, and M_c .

2. NONEQUILIBRIUM DEVIATION FROM THE EQUILIBRIUM DISTRIBUTION FUNCTION

A. Boltzmann equation with the mean field effect

Kinetic theory provides an efficient classical description of complex microscopic dynamics of an interacting many-body system. It is a good alternative to quantum field theory to study transport phenomena in the weakly coupled limit dominated by quasi-particle dynamics. By quasiparticles one shall understand particles which, apart from zero temperature mass, gain additional thermal mass due to interactions with the medium: the effect of the mean field. They are characterized by a mean free path which is much larger than the Compton wavelength of system’s constituents, and by a mean free time, which is much larger than the time between collisions [36]. The dynamics of quasiparticles is encoded in the phase-space distribution function which evolves according to the Boltzmann equation.

We consider a system of uncharged thermally influenced particles of a single species for which the Boltzmann equation reads

$$(\tilde{k}^\mu \partial_\mu - \mathcal{E}_k \nabla \mathcal{E}_k \cdot \nabla_k) f = C[f], \quad (2.1)$$

where $C[f]$ is the collision term, $f = f(x, k)$ is a distribution function of quasiparticles¹ and the second term of the left hand side involves the force $\mathbf{F} = d\mathbf{k}/dt \equiv -\nabla \mathcal{E}_k$. The quasiparticle four-momentum is defined as $\tilde{k}^\mu = (\tilde{k}^0, \mathbf{k})$, where $\tilde{k}_0 \equiv \mathcal{E}_k$ is the nonequilibrium energy given by

$$\mathcal{E}_k = \sqrt{\mathbf{k}^2 + \tilde{m}_x^2}, \quad (2.2)$$

which is a time and space dependent variable since $\tilde{m}_x^2 \equiv \tilde{m}^2(x) = m_0^2 + m_{\text{th}}^2(x)$, where m_0 is the physical mass

¹ We use here such a notation that whenever x and k appear as arguments of a function, we mean x^μ and \tilde{k}^μ (or k^μ in case of f_0), respectively.

and $m_{\text{th}}(x)$ is the nonequilibrium thermal mass, which varies in time and space. Knowing the x -dependence of the energy, one may rewrite Eq. (2.1) as

$$(\tilde{k}^\mu \partial_\mu - \frac{1}{2} \nabla \tilde{m}_x^2 \cdot \nabla_k) f = C[f]. \quad (2.3)$$

The central object of the kinetic theory is the phase-space density function $f(x, k)$. What we assume about the system is that its departure from the equilibrium state is small, which, in turn, means that the process of the system equilibration is controlled by the small deviation in the distribution function, which we denote as

$$\Delta f(x, k) = f(x, k) - f_0(x, k), \quad (2.4)$$

where $f_0(x, k)$ is the equilibrium Bose-Einstein distribution function and, in a general frame, it has the form

$$f_0(x, k) = \frac{1}{\exp[u_\mu(x) k^\mu(x) \beta(x)] - 1}, \quad (2.5)$$

where $\beta \equiv \beta(x) = 1/T(x)$ with $T(x)$ being the local temperature, and $u_\mu \equiv u_\mu(x)$ is the fluid four-velocity. The four-velocity in the local rest frame is $u^\mu = (1, 0, 0, 0)$. The quasiparticle four-momentum is $k^\mu = (k^0, \mathbf{k})$, where k^0 component is the equilibrium x -dependent energy

$$E_k = \sqrt{\mathbf{k}^2 + m_x^2}, \quad (2.6)$$

where the dependence of x enters through the mass $m_x^2 \equiv m^2(x) = m_0^2 + m_{\text{eq}}^2(x)$ with $m_{\text{eq}}^2(x)$ being the equilibrium thermal mass, which is not the same as $m_{\text{th}}^2(x)$, the nonequilibrium thermal mass. The Bose-Einstein density function in the fluid rest frame takes the form

$$f_0(x, k) = \frac{1}{\exp(E_k(x) \beta(x)) - 1}. \quad (2.7)$$

Let us add that in the forthcoming parts we will be deriving all equations for the Bose-Einstein gas, but these equations may be analogously found for the classical Boltzmann gas with the distribution function

$$f_{0,c}(x, k) = \exp(-\beta(x) u_\mu(x) k^\mu(x)) \quad (2.8)$$

and these will be briefly presented as well. Our aim is to reformulate the equations of the viscous hydrodynamics when the effect of fluctuating thermal mass is incorporated. Therefore, we assume that thermal influence on the process of the system equilibration is controlled by the nonequilibrium correction to the thermal mass, $\Delta m_{\text{th}}^2 = m_{\text{th}}^2 - m_{\text{eq}}^2$, which will be specified further.

B. Form of Δf

As stated earlier, in this work we study systems with distribution functions that are perturbed from their equilibrium value. More specifically, the nonequilibrium phase space density can be written as

$$f(x, k) = f_{\text{th}}(x, k) + \delta f(x, k) \quad (2.9)$$

The first part, $f_{\text{th}}(x, k)$, still retains the local-equilibrium form of the distribution function, but the thermal mass contains the nonequilibrium corrections

$$f_{\text{th}}(x, k) \equiv f_0(x, k) \Big|_{m_0^2 + m_{\text{eq}}^2(x) \rightarrow m_0^2 + m_{\text{eq}}^2(x) + \Delta m_{\text{th}}^2(x)} \quad (2.10)$$

$$= \left[\exp \left(\sqrt{\mathbf{k}^2 + m_0^2 + m_{\text{eq}}^2(x) + \Delta m_{\text{th}}^2(x)} \beta(x) \right) - 1 \right]^{-1}.$$

The second part, $\delta f(x, k)$, is a change in the functional form of $f_0(x, k)$ caused by hydrodynamic forces, or equivalently, non-vanishing gradients of energy and momentum densities. The nonequilibrium correction Δf then has two parts

$$\Delta f(x, k) = f(x, k) - f_0(x, k) = \delta f(x, k) + \delta f_{\text{th}}(x, k), \quad (2.11)$$

where to the leading order in small change, $\delta f_{\text{th}}(x, k) = f_{\text{th}}(x, k) - f_0(x, k)$ is

$$\delta f_{\text{th}}(x, k) = -f_0(x, k) (1 + f_0(x, k)) \frac{\Delta m_{\text{th}}^2(x)}{2E_k(x)} \beta(x), \quad (2.12)$$

which is obtained by expanding f_{th} . Since Δm_{th}^2 is the nonequilibrium deviation, it itself is going to be a functional of Δf . Hence, the equation

$$\Delta f = \delta f - \beta f_0 (1 + f_0) \frac{\Delta m_{\text{th}}^2}{2E_k} \quad (2.13)$$

must be solved self-consistently for Δf .

C. Form of Δm_{th}^2

Recalling the basic foundations of effective kinetic theory, the analysis here relies heavily on findings within the scalar $\lambda\phi^4$ theory, as provided in Refs. [33, 34], which makes the introduction of thermal corrections analytically feasible. But the analysis presented here works equally well whenever the equilibrium thermal mass has the form $\sim g^n T^2$ where g is the dimensionless coupling constant and n is a positive integer. We intend to provide an effective macroscopic framework to study weakly interacting systems, where the strength of interaction is determined by the coupling constant $\lambda \ll 1$. The coupling constant is scale (temperature) dependent and the analysis performed here pertains only to the perturbative regime. Within this approach the equilibrium thermal mass is found to be

$$m_{\text{eq}}^2 = \frac{\lambda(q_0)}{2} q_0, \quad (2.14)$$

where we have introduced the equilibrium scalar quantity q_0 . The function q_0 and its nonequilibrium counterpart q are defined through the corresponding distribution functions as

$$q_0 = \int dK f_0, \quad (2.15)$$

$$q = \int dK f. \quad (2.16)$$

For the definitions of the symbols, see Table I. Therefore, one can observe that Eq. (2.14) contains the coupling constant $\lambda(q_0)$, which is temperature dependent since q_0 is temperature dependent.

Throughout the analysis we always keep the assumption that all nonequilibrium quantities are slowly varying functions of space points, which justifies that the nonequilibrium dynamics is governed by small deviations of the quantities from their equilibrium values. Therefore, we further assume that the nonequilibrium thermal mass is a function of the scalar quantity q only. The same assumption is applied to the running coupling $\lambda(q)$. Thus, the nonequilibrium thermal mass can be expanded as

$$m_{\text{th}}^2(q) = m_{\text{th}}^2(q_0 + \Delta q) = m_{\text{eq}}^2(q_0) + \Delta m_{\text{th}}^2 \quad (2.17)$$

with

$$\Delta m_{\text{th}}^2 = \frac{dm_{\text{eq}}^2}{dq_0} \Delta q. \quad (2.18)$$

The function q is uniquely defined by Eq. (2.16) and should be obtained self-consistently from this equation. Hence to evaluate Δm_{th}^2 , we need to find Δq which is itself a function of Δm_{th}^2 . The deviation of the scalar quantity q can be written as

$$\Delta q = \int dK \delta f + \frac{\partial q_0}{\partial m_{\text{eq}}^2} \Delta m_{\text{th}}^2. \quad (2.19)$$

Eq. (2.18) then takes the form

$$\Delta m_{\text{th}}^2 = \frac{1}{1 - \frac{dm_{\text{eq}}^2}{dq_0} \frac{\partial q_0}{\partial m_{\text{eq}}^2}} \frac{dm_{\text{eq}}^2}{dq_0} \int dK \delta f. \quad (2.20)$$

On the other hand both m_{eq}^2 and q_0 are related by temperature, so that one can find

$$\begin{aligned} \frac{dm_{\text{eq}}^2}{dT} &= \frac{dm_{\text{eq}}^2}{dq_0} \frac{dq_0}{dT} \\ &= \frac{dm_{\text{eq}}^2}{dq_0} \left(\beta^2 \int dK E_k f_0 (1 + f_0) + \frac{dm_{\text{eq}}^2}{dT} \frac{\partial q_0}{\partial m_{\text{eq}}^2} \right). \end{aligned} \quad (2.21)$$

Extracting further $\frac{dm_{\text{eq}}^2}{dq_0} \frac{\partial q_0}{\partial m_{\text{eq}}^2}$ and inserting it to Eq. (2.20) leads to

$$\Delta m_{\text{th}}^2 = 2T^2 \frac{dm_{\text{eq}}^2}{dT^2} \frac{\int dK \delta f}{\beta \int dK E_k f_0 (1 + f_0)}, \quad (2.22)$$

where we used $dm_{\text{eq}}^2/dT = 2T dm_{\text{eq}}^2/dT^2$.

Inserting Eq. (2.22) to Eq. (2.13), one gets

$$\Delta f = \delta f - T^2 \frac{dm_{\text{eq}}^2}{dT^2} \frac{f_0 (1 + f_0)}{E_k} \frac{\int dK \delta f}{\int dK E_k f_0 (1 + f_0)}. \quad (2.23)$$

Analogously, the correction for the Boltzmann gas is

$$\Delta f_c = \delta f_c - T^2 \frac{dm_{\text{eq},c}^2}{dT^2} \frac{f_{0,c}}{E_k} \frac{\int dK \delta f_c}{\int dK E_k f_{0,c}}, \quad (2.24)$$

where the subscript 'c' has been used to emphasize that the formula holds for the classical gas. Eqs. (2.23) and (2.24) are one of the main results of this paper. In previous analyses [45–47, 49–53], the second term in Eq. (2.23) was missing or was incomplete. When applying Cooper-Frye formula in viscous hydrodynamics, it is Δf , not δf that should be used.

D. Temperature dependence of the thermal mass

The thermal mass is a function of the scalar quantity q_0 and is defined by Eq. (2.14). Its temperature dependence is dictated by

$$\frac{dm_{\text{eq}}^2}{dT} = \frac{\lambda(q_0)}{2} \frac{dq_0}{dT} + \frac{q_0}{2} \frac{d\lambda(q_0)}{dT}. \quad (2.25)$$

q_0 is one of the thermodynamic functions discussed in detail in Appendix B and its leading order value is found to be $T^2/12$. Additionally, the second term in Eq. (2.25) encodes the running of the coupling constant as a function of the energy scale, which is the essence of the renormalization group β_λ -function, defined by

$$\beta_\lambda \equiv \beta(\lambda) = T \frac{d\lambda(q_0)}{dT} \quad (2.26)$$

It should be obtained using diagrammatic methods. In case of the scalar theory, β_λ is positive and proportional to λ^2 . Collecting these contributions, one finds

$$T^2 \frac{dm_{\text{eq}}^2}{dT^2} = m_{\text{eq}}^2 + aT^2 \beta_\lambda. \quad (2.27)$$

where $m_{\text{eq}}^2 = \lambda T^2/24$ and $a = 1/48$.

One can analogously consider a temperature-dependent scaling for the classical Boltzmann gas. In this case, the thermal effective mass may be assumed to have the same form as (2.14). The only difference is that one uses the Boltzmann distribution function $f_{0,c}$ instead of f_0 . This gives $q_{0c} = T^2/(2\pi^2) + \mathcal{O}(m_x^2)$, as given by Eq. (B.16) and it leads to

$$T^2 \frac{dm_{\text{eq},c}^2}{dT^2} = m_{\text{eq},c}^2 + a_c T^2 \beta_\lambda, \quad (2.28)$$

where $m_{\text{eq},c}^2 = \lambda T^2/(4\pi^2)$ and $a_c = 1/(8\pi^2)$.

3. EQUATIONS OF HYDRODYNAMICS WITH THERMAL CORRECTIONS

A. Local equilibrium hydrodynamics

First consider the system under strict local equilibrium. By that we mean that the functional form of the distribution function is still f_0 given in Eq. (2.5) or in Eq. (2.8), but the temperature as well as the thermal

mass is x -dependent. Such a system possesses a conserved stress-energy tensor of the form

$$T_0^{\mu\nu} = \int dK k^\mu k^\nu f_0 - g^{\mu\nu} U_0, \quad (3.1)$$

where the metric tensor we use is $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$. The extra term $U_0 \equiv U_0(x)$ is the mean-field contribution that guarantees the thermodynamic consistency of hydrodynamic equations and the conservation of energy and momentum, via the following condition

$$dU_0 = \frac{q_0}{2} dm_{\text{eq}}^2, \quad (3.2)$$

where q_0 is the Lorentz scalar defined by Eq. (2.15).

Since we study here a system with no conserved charges, the Landau frame is a natural kinetic framework to define the four-velocity u^μ via

$$u_\mu T_0^{\mu\nu} = \epsilon_0 u^\nu, \quad (3.3)$$

where the eigenvalue ϵ_0 can be identified as the local energy density. With this definition the energy-momentum tensor may be decomposed using two orthogonal projections $u^\mu u^\nu$ and $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$. The equilibrium energy-momentum tensor becomes

$$T_0^{\mu\nu} = \epsilon_0 u^\mu u^\nu - P_0 \Delta^{\mu\nu}, \quad (3.4)$$

where P_0 is the local thermodynamic pressure. The energy density and the pressure are in turn given by

$$\epsilon_0 = \bar{\epsilon}_0 - U_0, \quad (3.5)$$

$$P_0 = \bar{P}_0 + U_0, \quad (3.6)$$

where

$$\bar{\epsilon}_0 = \langle (u_\mu k^\mu)^2 \rangle_0, \quad (3.7)$$

$$\bar{P}_0 = -\frac{1}{3} \langle \Delta^{\mu\nu} k_\mu k_\nu \rangle_0 \quad (3.8)$$

with the notation $\langle \dots \rangle_0 = \int dK \dots f_0$. Let us point out that the enthalpy is not changed by the mean field $\bar{\epsilon}_0 + \bar{P}_0 = \epsilon_0 + P_0$. One may also check that the definition of energy density (3.5) and pressure (3.6), together with the condition (3.2), guarantee that the thermodynamic relation

$$T s_0 = T \frac{dP_0}{dT} = \epsilon_0 + P_0, \quad (3.9)$$

where s_0 is the entropy density, is satisfied.

B. Nonequilibrium hydrodynamics

The stress-energy tensor of fluid dynamics out of equilibrium takes the following form

$$T^{\mu\nu} = \int dK \tilde{k}^\mu \tilde{k}^\nu f - g^{\mu\nu} U, \quad (3.10)$$

which is formally the same as Eq. (3.1). The mean-field correction U must be now a function of $q = \int dK f$ only [34]. We emphasize that the formulation of fluid hydrodynamic framework with the thermal correction still has to conform with all assumptions that were made to provide the effective kinetic theory, discussed in Sec. 2. In particular, such a description requires the system to be sufficiently dilute and quasiparticles' mean free paths to be much longer than the thermal width of its constituents, which is maintained when the strength of interaction is weak. Furthermore, to allow for validity of hydrodynamics, the system has to be characterized by some macroscopic length scale at which macroscopic variables, such as pressure and energy density, vary. Under these assumptions, nonequilibrium hydrodynamic description applies to systems where departures of all quantities from its equilibrium values are characterized by small corrections. Therefore, the nonequilibrium function U , in particular, may be expanded as

$$U = U_0 + \Delta U, \quad (3.11)$$

where

$$\Delta U = \frac{dU_0}{dq_0} \Delta q. \quad (3.12)$$

However, as discussed before and explicitly shown by Eqs. (2.17) and (2.18), the thermal mass is also a function of q only. Therefore, applying the relation (2.18) to (3.12), one finds

$$\Delta U = \frac{q_0}{2} \Delta m_{\text{th}}^2, \quad (3.13)$$

As before, this is also the condition that U must satisfy to maintain the energy-momentum conservation law $\partial_\mu T^{\mu\nu} = 0$.

The stress-energy tensor of the viscous hydrodynamics (3.10) may be next decomposed into the local equilibrium part and the nonequilibrium deviation

$$T^{\mu\nu} = T_0^{\mu\nu} + \Delta T^{\mu\nu}, \quad (3.14)$$

where $T_0^{\mu\nu}$ is given by (3.4) and $\Delta T^{\mu\nu}$ carries all dynamical information needed in order to determine how the nonequilibrium system evolves into equilibrium. Let us notice that a separation of viscous correction from the equilibrium part in Eq. (3.14) has been done not as a rearrangement of Eq. (3.10) but rather as an expansion of the stress-energy tensor around its local equilibrium value. As shown in Appendix A, we have

$$\Delta T^{00} = \int dK E_k^2 \Delta f, \quad (3.15)$$

$$\Delta T^{0i} = \int dK E_k k^i \Delta f, \quad (3.16)$$

$$\begin{aligned} \Delta T^{ij} = & \int dK k^i k^j \Delta f - \frac{\Delta m_{\text{th}}^2}{2} \int dK \frac{k^i k^j}{E_k^2} f_0 \\ & + \delta^{ij} \frac{\Delta m_{\text{th}}^2}{2} \int dK f_0, \end{aligned} \quad (3.17)$$

where Δm_{th}^2 and Δf are given by (2.22) and (2.23), respectively. Eqs. (3.15) and (3.16) shall dictate the form of the Landau matching condition, and Eq. (3.17) contains the definitions of the viscous corrections.

C. Landau matching condition in the rest frame

The Landau matching is defined by the eigenvalue problem

$$u_\mu T^{\mu\nu} = \epsilon u^\nu, \quad (3.18)$$

where ϵ is the energy density of the nonequilibrium state including the thermal correction U . In the fluid rest frame it comes down to two equations, corresponding to the conditions on the energy density and the momentum density:

$$T^{00} = \epsilon, \quad T^{0i} = 0. \quad (3.19)$$

Under the Landau matching condition, the local equilibrium is defined to have the same local energy and the momentum density

$$\Delta T^{00} = 0, \quad \Delta T^{0i} = 0. \quad (3.20)$$

Using Eqs. (3.15) and (3.16) with the correction to the distribution function Δf given by Eq. (2.23), we obtain

$$\Delta\epsilon = \int dK \left[E_k^2 - T^2 \frac{dm_{\text{eq}}^2}{dT^2} \right] \delta f, \quad (3.21)$$

$$0 = \int dK \left[E_k k^i - T^2 \frac{dm_{\text{eq}}^2}{dT^2} \frac{\int dK' k'^i f_0(f_0 + 1)}{\int dK' E'_k f_0(f_0 + 1)} \right] \delta f. \quad (3.22)$$

However, the second term in Eq. (3.22) vanishes because of rotational symmetry in equilibrium. Hence the Landau matching conditions are

$$\int dK \left[E_k^2 - T^2 \frac{dm_{\text{eq}}^2}{dT^2} \right] \delta f = 0, \quad (3.23)$$

$$\int dK E_k k^i \delta f = 0. \quad (3.24)$$

The second condition indicates that δf cannot have a vector component: it can only contain a spin 0 part and a spin 2 part.

D. Shear-stress tensor and bulk pressure in the local rest frame

The shear tensor π^{ij} and the bulk pressure Π are found from Eq. (3.17) in the local rest frame, where the Eqs. (2.22) and (2.23) are inserted. Then, as shown in Appendix A, one obtains

$$\Delta T^{ij} = \int dK k^i k^j \delta f. \quad (3.25)$$

We can reorganize (3.25) to separate the spin 0 part and the spin 2 part as follows

$$\Delta T^{ij} = \pi^{ij} + \delta^{ij} \Pi, \quad (3.26)$$

where

$$\pi^{ij} = \int dK k^{\langle i} k^{j \rangle} \delta f, \quad (3.27)$$

$$\Pi = \frac{1}{3} \int dK \mathbf{k}^2 \delta f, \quad (3.28)$$

where $k^{\langle i} k^{j \rangle} = k^i k^j - \mathbf{k}^2 \delta^{ij}/3$. These coincide with the commonly known forms of the shear-stress tensor and bulk pressure in the local rest frame.

E. General frame

In a general frame where the flow velocity u^μ may be arbitrary, the energy-momentum tensor is²

$$T^{\mu\nu} = \int dK k^\mu k^\nu f_0 - g^{\mu\nu} U_0 + \int dK \left[k^\mu k^\nu - u^\mu u^\nu T^2 \frac{dm_{\text{eq}}^2}{dT^2} \right] \delta f. \quad (3.29)$$

The Landau condition then becomes

$$\int dK \left[(u_\mu k^\mu) k^\nu - u^\nu T^2 \frac{dm_{\text{eq}}^2}{dT^2} \right] \delta f = 0 \quad (3.30)$$

and the viscous corrections are given by

$$\pi^{\mu\nu} = \langle k^{\langle \mu} k^{\nu \rangle} \rangle_\delta, \quad \Pi = -\frac{1}{3} \langle \Delta_{\mu\nu} k^\mu k^\nu \rangle_\delta, \quad (3.31)$$

where $\langle \dots \rangle_\delta \equiv \int dK (\dots) \delta f$. We have also used the notation $A^{\langle \mu\nu \rangle} \equiv \Delta_{\alpha\beta}^{\mu\nu} A^{\alpha\beta}$, where $\Delta_{\alpha\beta}^{\mu\nu} \equiv (\Delta_\alpha^\mu \Delta_\beta^\nu + \Delta_\beta^\mu \Delta_\alpha^\nu - 2/3 \Delta^{\mu\nu} \Delta_{\alpha\beta})/2$. The definitions (3.31) have the well-known structures, but the thermal mass that enters them is now x -dependent and the Landau matching contains a correction due to the temperature dependent mass. These arguments are essential when one aims at examining transport properties of the medium.

4. NONEQUILIBRIUM CORRECTION IN THE CHAPMAN-ENSKOG APPROACH

Chapman-Enskog theory provides a way to directly find the solution to the Boltzmann equation for near-equilibrium systems. Solving the full Boltzmann equation, however, is formidable task. In this paper, we

² In Ref. [33], the energy-momentum tensor correction was written down incorrectly, but the mistake vanished with the imposition of the Landau matching condition, ensuring the validity of the subsequent derivations.

use the Anderson-Witting model [57] to find the explicit leading order solution. In this section, we focus on the bosonic quantum gas case. Treatment for the Boltzmann gas case is identical if one replaces $f_0(1 + f_0)$ with the Boltzmann factor $f_{0,c}$.

A. Solution of the Anderson-Witting equation in the rest frame

With the medium dependent thermal mass, the Anderson-Witting model is given by

$$\left(\tilde{k}^\mu \partial_\mu - \mathcal{E}_k \nabla \mathcal{E}_k \cdot \nabla_k\right) f = -\frac{(u \cdot \tilde{k})}{\tau_R} \Delta f, \quad (4.1)$$

where $\tilde{k}^\mu = (\mathcal{E}_k, \mathbf{k})$. In the fluid cell rest frame $u^\mu = (1, 0, 0, 0)$ and $u \cdot \tilde{k} = \mathcal{E}_k$.

To use the Chapman-Enskog method, we let

$$f = f_0 + f_1 + f_2 + \dots \quad (4.2)$$

where each f_n contains only the n -th derivatives of the thermodynamic quantities and the flow velocity. The first order equation is obtained by identifying $\Delta f = f_1$ in the right hand side and using all other quantities in their equilibrium forms

$$\left(k^\mu \partial_\mu - \frac{1}{2} \partial_i m_{\text{eq}}^2 \frac{\partial}{\partial k_i}\right) f_0(x, k) = -\frac{E_k}{\tau_R} \Delta f(x, k), \quad (4.3)$$

where now $k^\mu = (E_k, \mathbf{k})$.

Evaluating the left hand side yields

$$\begin{aligned} &\left(k^\mu \partial_\mu - \frac{1}{2} \partial_i m_{\text{eq}}^2 \frac{\partial}{\partial k_i}\right) f_0(x, k) = -\beta f_0(x, k) (1 + f_0(x, k)) \\ &\times \left[\left(c_s^2 \left(E_k^2 - T^2 \frac{dm_{\text{eq}}^2}{dT^2} \right) - \frac{\mathbf{k}^2}{3} \right) (\partial_i u^i) - k^{(j} k^{i)} \partial_j u_i \right], \end{aligned} \quad (4.4)$$

where the equations of motion from the ideal hydrodynamics

$$\partial_0 u^i = \frac{\partial^i T}{T}, \quad (4.5)$$

$$\partial_0 T = -T c_s^2 \partial_i u^i \quad (4.6)$$

are used to remove time derivatives.

The Δf in the right hand side of the Anderson-Witting model is just Eq. (2.23). Letting $\delta f = f_0(1 + f_0)\phi$, we get

$$\begin{aligned} \Delta f(k) &= f_0(k)(1 + f_0(k)) \\ &\times \left(\phi(k) - \frac{T^2}{E_k} \frac{dm_{\text{eq}}^2}{dT^2} \frac{\int dK \phi(k) f_0(k)(1 + f_0(k))}{\int dK E_k f_0(k)(1 + f_0(k))} \right), \end{aligned} \quad (4.7)$$

where the x dependence of all quantities is suppressed for the sake of brevity. In previous derivations, the last term was missing [47, 52, 53]. Dividing ϕ into the shear

and the bulk parts $\phi = \phi_s + \phi_b$, and comparing Eqs. (4.4) and (4.7), the shear part of ϕ is trivially obtained as

$$\phi_s(k) = -\frac{\tau_R}{TE_k} k^{(j} k^{i)} \partial_j u_i, \quad (4.8)$$

since the angle integration over the spin-2 tensor $k^{(j} k^{i)}$ vanishes. For the bulk part, letting

$$\phi_b(k) = \left(a E_k + \frac{b}{E_k} \right) \partial_i u^i \quad (4.9)$$

and comparing Eqs. (4.7) and (4.4), we get

$$a = \tau_R \beta \left(c_s^2 - \frac{1}{3} \right) \quad (4.10)$$

and

$$b = \frac{-M \beta \tau_R J_{1,0}}{J_{1,0} - T^2 (dm_{\text{eq}}^2/dT^2) J_{-1,0}}, \quad (4.11)$$

where we defined

$$M = -\frac{1}{3} \left(m_x^2 - T^2 \frac{dm_{\text{eq}}^2}{dT^2} \right). \quad (4.12)$$

With $m_{\text{eq}}^2 \propto \lambda T^2$, we have

$$M = -\frac{1}{3} (m_0^2 - a \beta_\lambda T^2), \quad (4.13)$$

where β_λ is the coefficient function of the coupling constant renormalization group and $a = O(1)$ depends on the theory. The parameter M can be identified as the parameter of nonconformality of the system (or the source of the conformal invariance violation). We have also introduced a notation for thermodynamic integrals

$$J_{n,q} = a_q \int dK (u \cdot k)^{n-2q} (-\Delta_{\mu\nu} k^\mu k^\nu)^q f_0(k) (1 + f_0(k)), \quad (4.14)$$

where $a_q = 1/(2q+1)!!$, which can be evaluated in the fluid cell rest frame. The bulk part of the leading order Chapman-Enskog solution of the Anderson-Witting equation is then

$$\begin{aligned} \phi_b(k) &= \tau_R \beta (\partial_i u^i) \\ &\times \left((c_s^2 - 1/3) E_k - \frac{1}{E_k} \frac{M J_{1,0}}{J_{1,0} - T^2 (dm_{\text{eq}}^2/dT^2) J_{-1,0}} \right). \end{aligned} \quad (4.15)$$

To show that $\phi_b(k)$ is in fact proportional to $(c_s^2 - 1/3)$, we can use

$$c_s^2 = \frac{dP_0/dT}{d\epsilon_0/dT} = \frac{J_{3,1}}{J_{3,0} - (T^2 dm_{\text{eq}}^2/dT^2) J_{1,0}}, \quad (4.16)$$

where P_0 and ϵ_0 are the pressure and the energy density given in Eqs. (3.6) and (3.5). Using the identities from Appendix B 2, one can also show that

$$\frac{1}{3} - c_s^2 = -\frac{M J_{1,0}}{J_{3,0} - T^2 (dm_{\text{eq}}^2/dT^2) J_{1,0}}. \quad (4.17)$$

Hence finally

$$\begin{aligned} \phi_b(k) &= \tau_R \beta (\partial_i u^i) (c_s^2 - 1/3) \\ &\times \left(E_k - \frac{1}{E_k} \frac{J_{3,0} - T^2 (dm_{\text{eq}}^2/dT^2) J_{1,0}}{J_{1,0} - T^2 (dm_{\text{eq}}^2/dT^2) J_{-1,0}} \right). \end{aligned} \quad (4.18)$$

Eq. (4.18) is another main result in this work. This equation slightly differs from the analogous one for the Boltzmann statistics shown in Ref. [17, 46, 49].

In hydrodynamic simulations, it is practical to replace the system expansion rate by the bulk viscous pressure using the Navier-Stokes relation $\Pi = -\zeta\theta$, which gives

$$\begin{aligned} \phi_b(k) &= \beta \left(-\frac{\Pi}{\zeta/\tau_R} \right) (c_s^2 - 1/3) \\ &\times \left(E_k - \frac{1}{E_k} \frac{J_{3,0} - T^2 (dm_{\text{eq}}^2/dT^2) J_{1,0}}{J_{1,0} - T^2 (dm_{\text{eq}}^2/dT^2) J_{-1,0}} \right). \end{aligned} \quad (4.19)$$

Having given the solution of the Anderson-Witting equation, one can also find Δf explicitly. Inserting Eqs. (4.18) and (4.8) into (4.7) one finds

$$\begin{aligned} \Delta f(k) &= f_0(k) (1 + f_0(k)) \tau_R \beta \left[-(\partial_j u_i) \frac{k^{(j} k^{i)}}{E_k} \right. \\ &\quad \left. + (\partial_i u^i) (c_s^2 - 1/3) \left(E_k - \frac{1}{E_k} \frac{J_{3,0}}{J_{1,0}} \right) \right]. \end{aligned} \quad (4.20)$$

The phase space density correction Δf has a much simpler form than ϕ . However, for transport coefficient calculations, it is ϕ (equivalently δf), rather than Δf , that is needed.

B. Energy conservation and Landau matching in the Anderson-Witting case

By multiplying $\tilde{k}^\nu = (\mathcal{E}_k, \mathbf{k})$ and integrating over dK , the left hand side of Anderson-Witting equation Eq. (4.1) turns into $\partial_\mu T^{\mu\nu}$, where the stress-energy tensor $T^{\mu\nu}$ is defined in Eq. (3.10). Assuming that the mean-field contribution U satisfies

$$\partial_\mu U(x) = \frac{\partial_\mu \tilde{m}_x^2(x)}{2} \int dK f(x, k), \quad (4.21)$$

we get $\partial_\mu T^{\mu\nu} = 0$.

Under the same condition, the right hand side of the Anderson-Witting model within the Chapman-Enskog approach must also vanish

$$-\frac{1}{\tau_R} \int dK E_k k^\mu \Delta f = 0 \quad (4.22)$$

to ensure energy-momentum conservation. This condition for energy-momentum conservation is actually exactly the same as the Landau conditions we derived in Section 3C. Upon using Δf in Eq. (2.23) in the fluid rest frame, these become

$$0 = \int dK \left(E_k^2 - T^2 \frac{dm_{\text{eq}}^2}{dT^2} \right) \delta f \quad (4.23)$$

and

$$0 = \int dK E_k k^i \delta f. \quad (4.24)$$

Eq. (4.24) is automatically satisfied by the $\delta f = f_0(1 + f_0)(\phi_s + \phi_b)$ obtained in the previous subsection since it does not contain a vector part. In the condition (4.23), the shear part ϕ_s also vanishes because it contains a spin-2 tensor. Using Eqs. (4.18) and (4.14), it is easy to show that the energy conservation and the Landau condition are indeed fulfilled. This automatic fulfillment of the Landau condition for the quasi-particle case would not have been possible if one missed the Δm_{th}^2 correction in Δf .

C. The shear and the bulk viscosities in the Anderson-Witting model

The full leading order Chapman-Enskog solution to the Anderson-Witting model is given by Eq. (4.7) with ϕ_s and ϕ_b obtained above. The shear viscosity can be evaluated by using Eq. (3.27) for π^{ij} and Eq. (4.8) for ϕ_s as

$$\pi^{ij} = \frac{2\beta}{15} \tau_R \int dK f_0(1 + f_0) \frac{\mathbf{k}^4}{E_k} \sigma^{ij}, \quad (4.25)$$

where $\sigma^{ij} = -1/2(\partial^i u^j + \partial^j u^i - 2/3 g^{ij} \partial_k u^k)$. Identifying $\pi^{ij} = 2\eta \sigma^{ij}$, we get

$$\frac{\eta}{\tau_R} = \beta J_{3,2} \quad (4.26)$$

and subsequently find the shear viscosity in the relaxation time approximation, which was examined in few papers, see, for example, [41, 43, 49], and has the form

$$\frac{\eta}{\tau_R} = \frac{\epsilon_0 + P_0}{5}. \quad (4.27)$$

For the bulk viscosity, we start with Eq. (3.28)

$$\Pi = \int dK \frac{\mathbf{k}^2}{3} \delta f. \quad (4.28)$$

Using the Landau condition, Eq. (3.23), one gets

$$\Pi = M \int dK \delta f, \quad (4.29)$$

in which only the bulk part is relevant

$$\Pi = M \int dK f_0(k) (1 + f_0(k)) \phi_b(k) \quad (4.30)$$

with $\phi_b(k)$ given by Eq. (4.18). Since $\Pi = -\zeta \partial_i u^i$, one can read off the bulk viscosity to the relaxation time ratio from Eq. (4.30) as

$$\begin{aligned} \frac{\zeta}{\tau_R} &= \beta M^2 \left(\frac{J_{-1,0} J_{1,0}}{J_{1,0} - T^2 (dm_{\text{eq}}^2/dT^2) J_{-1,0}} \right. \\ &\quad \left. - \frac{J_{1,0} J_{1,0}}{J_{3,0} - T^2 (dm_{\text{eq}}^2/dT^2) J_{1,0}} \right) \end{aligned} \quad (4.31)$$

The integrals present in Eq. (4.31) have been computed in Appendix B2. Using them one gets the value of the ratio as

$$\frac{\zeta}{\tau_R} \approx \frac{M^2}{2\pi^2} \left(\frac{\pi T}{4m_x} - \frac{11}{12} \left(1 - \frac{9m_{\text{eq}}^2}{44m_x^2} \right) \right). \quad (4.32)$$

For the application in relativistic viscous hydrodynamics, it is more useful to use the speed of sound. Applying Eq. (4.17) and (4.31) one can explicitly show that the ratio is proportional to $(1/3 - c_s^2)^2$, namely

$$\frac{\zeta}{\tau_R} \approx T^4 \left(\frac{1}{3} - c_s^2 \right)^2 \left(\frac{2\pi^3 T}{25m_x} - \frac{4\pi^2}{75} \left(1 - \frac{9m_{\text{eq}}^2}{8m_x^2} \right) \right). \quad (4.33)$$

Note the appearance of T/m_x in the expression (4.33). This is in clear contrast to the Boltzmann statistics case which does not show such a behavior. The analysis for the Boltzmann statistics case is identical to the analysis above except that in place of $J_{n,q}$, we have

$$I_{n,q} = a_q \int dK (u \cdot k)^{n-2q} (-\Delta_{\mu\nu} k^\mu k^\nu)^q f_{0,c}(k), \quad (4.34)$$

where $f_{0,c}(k) = e^{-\beta k^\mu u_\mu}$. In this case, one gets

$$\frac{\zeta_{\text{Boltz}}}{\tau_R} \approx T^4 \left(\frac{1}{3} - c_s^2 \right)^2 \left(\frac{60}{\pi^2} - \frac{36m_x}{\pi T} \right). \quad (4.35)$$

The origin of this discrepancy is the fact that the Bose-Einstein factor behaves like $f(k) \sim T/E_k$ in the infrared limit which makes the thermodynamic integral $J_{-1,0}$ in Eq. (4.31) diverge in the $m_x \rightarrow 0$ limit while $I_{-1,0}$ does not. As a result, soft momenta govern the structure of ζ/τ_R . However, since the calculation was performed in the relaxation time approximation, which assumes that τ_R is independent of energy, it may not capture the right soft physics. A similar behavior was seen in Ref. [40], where QCD bulk viscosity is studied. The authors claim that the correct behavior of bulk viscosity is obtained in the relaxation time approximation by neglecting the infrared divergent term. But in principle there is no reason why this term should be ignored within the present framework.

Further, notice that starting from Eq. (4.4), the spin 0 part (the bulk part) and the spin 2 part (the shear part) of the analysis are totally independent. Hence, it is possible to generalize the leading order Anderson-Witting equation as

$$\begin{aligned} & \left(k^\mu \partial_\mu - \frac{1}{2} \partial_i m_{\text{eq}}^2 \frac{\partial}{\partial k_i} \right) f_0(x, k) \\ & = -\frac{E_k}{\tau_\pi} \Delta f_s(x, k) - \frac{E_k}{\tau_\Pi} \Delta f_b(x, k), \end{aligned} \quad (4.36)$$

where Δf_s and Δf_b is the shear and the bulk part of Δf . In fact, when the dominant physical processes for the shear relaxation and the bulk relaxation are different, this is the most natural form of the Anderson-Witting model.

The analysis of this generalized Anderson-Witting model follows exactly the same route as for the single τ_R except that the shear viscosity and the bulk viscosity have different relaxation times.

As discussed in Ref. [33, 34], the dominant physical process for the shear relaxation and the bulk relaxation can be indeed very different and the bulk relaxation can be dominated by the soft sector. Hence, the appearance of T/m_x is not entirely unnatural given that τ_Π can have very different m_x dependence from τ_π and the bulk relaxation is dominated by the soft number-changing process.

D. Comparison of Δf to previous works

The phase space correction Δf in Eq. (4.20) ultimately comes from solving the first order Chapman-Enskog approximation. Hence, it should come as no surprise that Eq. (4.20) is consistent to similar results found in other similar works provided that the right expression for the speed of sound is used. For instance, in Ref. [49] one finds that the bulk part of the phase space correction in the Boltzmann case is derived to be

$$\Delta f_R(k) = f_{0,c}(k) \phi_R(k) \quad (4.37)$$

with

$$\begin{aligned} \phi_R(k) = \tau_R \beta (\partial_i u^i) & \left((c_{sR}^2 - 1/3) E_k \right. \\ & \left. - \frac{1}{E_k} \left(c_{sR}^2 m_x T \frac{dm_x}{dT} - \frac{m_x^2}{3} \right) \right), \end{aligned} \quad (4.38)$$

where the speed of sound is $c_{sR}^2 = (3 + z K_2(z)/K_3(z))^{-1}$, with $z = m_x/T$ and $K_n(z)$ being the modified Bessel functions of the second kind. This ϕ_R is different than ϕ_b in Eq. (4.19) since ϕ_R is a part of Δf while ϕ_b is a part of δf . The phase space correction Δf_R is, however, equivalent to the bulk part of Δf in Eq. (4.20) if one uses the speed of sound expression (4.16) with $J_{n,q} \rightarrow I_{n,q}$. As mentioned above, this is as it should be since both are solutions of the first order Chapman-Enskog approximation.

The big difference between the previous treatments and ours is in computing the bulk viscosity. The bulk viscosity must be calculated using δf and *not* Δf as explained in the previous section. If one uses Δf (or Δf_R) instead of δf , the ratio ζ/τ_R would be incorrectly calculated.

5. TRANSPORT COEFFICIENTS IN THE 14-MOMENT APPROXIMATION

When a system features a conformal anomaly, first order transport coefficients reveal different sensitivity to the source of the conformal symmetry violation, as explicitly shown in the previous section. In particular,

shear viscosity is fully determined by the dominant energy scale, which is the temperature T , and thus the shear viscosity over its relaxation time ratio behaves as T^4 at leading order in the conformal symmetry breaking making the effects of scale anomaly negligible. On the other hand, bulk viscosity over the relaxation time is fully determined by breaking of conformal symmetry. Such a difference makes it justified to omit the analysis of shear viscous effects and to evaluate first and second order transport coefficients related to bulk pressure because the additional term in Eq. (2.23) indeed concerns only the scalar part. The analysis is performed at leading order in the conformal breaking parameter while including the thermal mass consistently.

The bulk pressure is given by Eq. (4.29). Noting that Eq. (2.23) can be expressed as

$$M\Delta f = M\delta f - T^2 \frac{dm_{\text{eq}}^2}{dT^2} \frac{f_0(1+f_0)}{E_k} \frac{\Pi}{\int dK E_k f_0(1+f_0)}, \quad (5.1)$$

one can rewrite Eq. (4.29) as

$$\Pi = \tilde{M} \int dK \Delta f, \quad (5.2)$$

where

$$\tilde{M} = \frac{MJ_{1,0}}{J_{1,0} - T^2(dm_{\text{eq}}^2/dT^2)J_{-1,0}}. \quad (5.3)$$

To obtain the equation of motion for the bulk pressure, we first take the time derivative of Π

$$\begin{aligned} \dot{\Pi} &= \dot{\tilde{M}} \int dK \Delta f \\ &+ \tilde{M} \left[\int dK \Delta \dot{f} - \frac{\dot{m}_{\text{eq}}^2}{2} \int dK \frac{1}{E_k^2} \Delta f \right], \end{aligned} \quad (5.4)$$

where we adopted the notation $\dot{A} = u^\mu \partial_\mu A$ for an arbitrary quantity A , which reduces to the time derivative in the rest frame of the fluid. From the Boltzmann equation

$$\left(\tilde{k}^\mu \partial_\mu - \mathcal{E}_k \nabla \mathcal{E}_k \cdot \nabla_k \right) f = C[f], \quad (5.5)$$

where $C[f]$ is the collision integral, one finds

$$\begin{aligned} u^\mu \partial_\mu (\Delta f) &= \frac{1}{(u \cdot \tilde{k})} \left[C[f] - \tilde{k}^\mu \partial_\mu f_0 - \tilde{k}^\mu \nabla_\mu \Delta f \right. \\ &\quad \left. + \frac{1}{2} \nabla \tilde{m}_x^2 \nabla_k f_0 + \frac{1}{2} \nabla \tilde{m}_x^2 \nabla_k \Delta f \right]. \end{aligned} \quad (5.6)$$

Inserting the expression (5.6) to Eq. (5.4) and keeping only leading order terms, that is, terms which are evaluated with $\tilde{k} \rightarrow k$, we have

$$\begin{aligned} \dot{\Pi} - C &= -\tilde{M} \left[-\dot{\beta} \left(J_{1,0} - T^2 (dm_{\text{eq}}^2/dT^2) J_{-1,0} \right) \right. \\ &\quad \left. + \frac{\beta}{3} \theta \left(J_{1,0} - m_x^2 J_{-1,0} \right) \right] + \left(\frac{\dot{\tilde{M}}}{\tilde{M}} - \frac{2}{3} \theta \right) \Pi \\ &- M \left(\frac{\dot{m}_{\text{eq}}^2}{2} + \frac{m_x^2}{3} \theta \right) \rho_{-2} - M \rho_{-2}^{\mu\nu} \sigma_{\mu\nu}, \end{aligned} \quad (5.7)$$

where $\theta \equiv \nabla_\mu u^\mu$ and $\sigma_{\mu\nu} = \partial_{\langle\mu} u_{\nu\rangle}$ is the Navier-Stokes shear tensor. In Eq. (5.7) we adopted the following notation for the collision term

$$C = \tilde{M} \int dK (u \cdot k)^{-1} C[f] \quad (5.8)$$

and for the irreducible moments

$$\rho_n = \langle (u^\alpha k_\alpha)^n \rangle_\delta, \quad \rho_n^{\mu\nu} = \langle (u^\alpha k_\alpha)^n k^{(\mu} k^{\nu)} \rangle_\delta. \quad (5.9)$$

Evaluating $u_\nu \partial_\mu T^{\mu\nu} = 0$ and implementing the formula (4.16) for the speed of sound squared, one obtains

$$\dot{\beta} = \frac{\Pi\theta - \pi^{\mu\nu} \sigma_{\mu\nu}}{J_{3,0} - T^2 (dm_{\text{eq}}^2/dT^2) J_{1,0}} + c_s^2 \beta \theta. \quad (5.10)$$

Next, calculating time derivatives $\dot{\tilde{M}}$ and \dot{m}_{eq}^2 , Eq. (5.7) simplifies to

$$\begin{aligned} \dot{\Pi} - C &= -\beta \tilde{M} \left[\left(\frac{1}{3} - c_s^2 \right) \left(J_{1,0} - T^2 \frac{dm_{\text{eq}}^2}{dT^2} J_{-1,0} \right) \right. \\ &\quad \left. + M J_{-1,0} \right] \theta - \left(\frac{2}{3} + \frac{2c_s^2 a T^2 \beta_\lambda}{3\tilde{M}} - A \right) \theta \Pi \\ &\quad - \pi^{\mu\nu} \sigma_{\mu\nu} A + M^2 \rho_{-2} \theta - M \rho_{-2}^{\mu\nu} \sigma_{\mu\nu}, \end{aligned} \quad (5.11)$$

where

$$A = \tilde{M} \frac{J_{1,0} - T^2 (dm_{\text{eq}}^2/dT^2) J_{-1,0}}{J_{3,0} - T^2 (dm_{\text{eq}}^2/dT^2) J_{1,0}} = c_s^2 - \frac{1}{3} \quad (5.12)$$

with the quantity $(c_s^2 - 1/3)$ given by Eq. (4.17).

To close Eq. (5.11) in terms of Π and $\pi^{\mu\nu}$, one can apply the 14-moment approximation, which allows to express the irreducible moments by Π and $\pi^{\mu\nu}$ as follows

$$\rho_{-2} = \gamma_2^{(0)} \Pi, \quad (5.13)$$

$$\rho_{-2}^{\mu\nu} = \gamma_2^{(2)} \pi^{\mu\nu}, \quad (5.14)$$

where the coefficients $\gamma_2^{(0)}$ and $\gamma_2^{(2)}$ are combinations of different thermal functions $J_{n,q}$. Their particular forms are presented in Appendix C. Also, using the Anderson-Witting model for the collision term

$$C[f] = -(u \cdot k) \frac{\Delta f}{\tau_R}, \quad (5.15)$$

where Δf is given by Eq. (2.23), the collision integral becomes

$$C = -\frac{\Pi}{\tau_R}. \quad (5.16)$$

Applying the collision term in the relaxation time approximation (5.16), the irreducible moments, Eqs. (5.13) and (5.14), and the relation for the speed of sound (4.17) to the evolution equation (5.11), one obtains

$$\dot{\Pi} + \frac{\Pi}{\tau_R} = -\frac{\zeta\theta}{\tau_R} - \frac{\delta_{\Pi\Pi}}{\tau_R} \theta \Pi + \frac{\lambda_{\Pi\pi}}{\tau_R} \pi^{\mu\nu} \sigma_{\mu\nu}, \quad (5.17)$$

where

$$\frac{\zeta}{\tau_R} = \beta M^2 \left[\frac{J_{1,0} J_{-1,0}}{J_{1,0} - T^2 (dm_{\text{eq}}^2/dT^2) J_{-1,0}} - \frac{J_{1,0} J_{1,0}}{J_{3,0} - T^2 (dm_{\text{eq}}^2/dT^2) J_{1,0}} \right] \quad (5.18)$$

is identical to the expression obtained in the Chapman-Enskog approach found in the previous section, Eq. (4.31). The remaining transport coefficients are

$$\frac{\delta_{\text{III}}}{\tau_R} = 1 - c_s^2 + M^2 \gamma_2^{(0)} + \frac{2aT^2 \beta \lambda}{9M}, \quad (5.19)$$

$$\frac{\lambda_{\text{II}\pi}}{\tau_R} = \frac{1}{3} - c_s^2 - M \gamma_2^{(2)}. \quad (5.20)$$

Converting M to the speed of sound and taking $m_0 \rightarrow 0$ limit, one gets

$$\frac{\delta_{\text{III}}}{\tau_R} \approx \frac{4}{3} \left(1 + \frac{T^2}{2} \frac{dm_{\text{eq}}^2}{dT^2} \frac{J_{-1,0}}{J_{1,0}} \right) + \left(\frac{1}{3} - c_s^2 \right) + \gamma_2^{(0)} \left(\frac{J_{3,0}}{J_{1,0}} - T^2 \frac{dm_{\text{eq}}^2}{dT^2} \right)^2 \left(\frac{1}{3} - c_s^2 \right)^2, \quad (5.21)$$

$$\frac{\lambda_{\text{II}\pi}}{\tau_R} \approx \left(1 + \gamma_2^{(2)} \left(\frac{J_{3,0}}{J_{1,0}} - T^2 \frac{dm_{\text{eq}}^2}{dT^2} \right) \right) \left(\frac{1}{3} - c_s^2 \right), \quad (5.22)$$

where $\gamma_2^{(0)}$ and $\gamma_2^{(2)}$ are calculated in Appendix C and are given by Eqs. (C.14) and (C.15), respectively. When inserted, one gets the leading orders of the coefficients

$$\frac{\delta_{\text{III}}}{\tau_R} \approx \frac{4}{3} \left(1 + \frac{3}{8\pi} \frac{m_{\text{eq}}}{T} - \frac{3}{16\pi^2} \frac{m_{\text{eq}}^2}{T^2} \right) + \left(\frac{1}{3} - c_s^2 \right) \left(\frac{6}{15\pi} \frac{T}{m_{\text{eq}}} + 1 \right) + 0.97 \left(\frac{1}{3} - c_s^2 \right)^2 \frac{T^4}{m_{\text{eq}}^4}, \quad (5.23)$$

$$\frac{\lambda_{\text{II}\pi}}{\tau_R} \approx 1.05 \left(\frac{1}{3} - c_s^2 \right), \quad (5.24)$$

where the numerical factors come from evaluating g_0 ($12/15$)² ≈ 0.97 and $(1 + 12g_2/15) \approx 1.05$ with g_0 and g_2 given by Eqs. (C.16) and (C.17). As seen, the coefficient $\delta_{\text{III}}/\tau_R$ is affected by the soft physics even more strongly than the bulk viscosity which is manifested by the factors $1/m_{\text{eq}}$ and $1/m_{\text{eq}}^4$.

Repeating the same analysis for the Boltzmann gas, which leads simply to replacement of the thermodynamic functions $J_{n,q} \rightarrow I_{n,q}$, one obtains the same value of $\zeta_{\text{Boltz}}/\tau_R$ as within the Chapman-Enskog approach, Eq. (4.35). The other two coefficients have the forms (5.21) and (5.22) with $\gamma_2^{(0)}$ and $\gamma_2^{(2)}$ given by Eqs. (C.19) and (C.20). The explicit expressions in the $m_0 \rightarrow 0$ limit

are then found to be

$$\frac{\delta_{\text{III,Boltz}}}{\tau_R} \approx \frac{4}{3} \left(1 + \frac{1}{4} \frac{m_{\text{eq},c}^2}{T^2} \right) + 5 \left(\frac{1}{3} - c_s^2 \right) - 10.8 \left(\frac{1}{3} - c_s^2 \right)^2, \quad (5.25)$$

$$\frac{\lambda_{\text{II}\pi,\text{Boltz}}}{\tau_R} \approx 1.6 \left(\frac{1}{3} - c_s^2 \right), \quad (5.26)$$

where the numerical factors were found from $144g_{0c} \approx -10.8$ and $(1 + 12g_{2c}) \approx 1.6$ with g_{0c} and g_{2c} written up below Eq. (C.20). One can also see from Eqs. (5.19) and (5.20) that when thermal quantities are neglected and the constant mass is kept, we reproduce $\zeta_{\text{Boltz}}/\tau_R$, $\lambda_{\text{II}\pi,\text{Boltz}}/\tau_R$ and the two first terms of $\delta_{\text{III,Boltz}}/\tau_R$ from Ref. [41].

6. SUMMARY AND CONCLUSIONS

In this paper we analyzed the influence of the mean field on fluid dynamics in weakly interacting systems of a single species, where all occurring masses are much smaller than the system's temperature. The main attention was paid to proper determination of the form of the nonequilibrium correction to the distribution function which depends on the mass varying as the temperature varies. The correction guarantees a consistent hydrodynamic description which satisfies thermodynamic relations and the conservation of energy and momentum and furthermore gives an accurate fixing of the temperature through Landau matching. The correction plays a central role in studying thermal dependence of bulk viscous dynamics. Therefore, we further considered the Anderson-Witting model of the Chapman-Enskog approach and computed ζ/τ_R of single-component Bose-Einstein and Boltzmann gases. We also derived the evolution equation for the bulk pressure in the 14-moment approximation and obtained relevant transport coefficients. Both methods provide the same result for ζ/τ_R .

The ratio ζ/τ_R obtained for the Boltzmann statistics behaves as expected, that is, it is given by the nonconformality parameter squared. When thermal effects are omitted, we reproduce the result from Refs. [41, 43]. On the other hand, for very high temperatures the ratio gets dominated by the β_λ -function. We also see that in spite of breaking conformal invariance, bulk viscosity vanishes at some critical temperature where $c_s^2 = 1/3$. In case of the Bose-Einstein gas, we have shown that the leading order term of ζ/τ_R is different than expected either if we neglect the physical mass or thermal effects. The ratio in this case is strongly redounded by the infrared physics, which introduces an additional energy scale dependent factor T/m_x . We suspect that the relaxation time approximation used here does not include the entire microscopic physics of a quantum gas, in particular, it is insensitive to phenomena at the soft scale. Therefore, we conclude that to compute the bulk viscosity over

its relaxation time for quantum gases of Bose-Einstein statistics, one needs to use more advanced methods and solve an integral equation. It can be done starting from either the linearized Boltzmann equation or Kubo formulas, in which case the formula for the bulk relaxation time has been recently found [59].

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Appendix A: Components of the energy-momentum tensor correction

The correction to the energy-momentum tensor is

$$\Delta T^{\mu\nu} = \int dK k^\mu k^\nu \delta f + \frac{\partial T_0^{\mu\nu}}{\partial m_{\text{eq}}^2} \Delta m_{\text{th}}^2 \quad (\text{A.1})$$

and its particular components are derived as follows. For $\mu = \nu = 0$, one gets

$$\begin{aligned} \Delta T^{00} &= \int dK E_k^2 \delta f + \frac{\Delta m_{\text{th}}^2}{2} \int dK f_0 \\ &\quad - \frac{\Delta m_{\text{th}}^2}{2} \beta \int dK E_k f_0 (1 + f_0) - \Delta U \\ &= \int dK E_k^2 \Delta f, \end{aligned} \quad (\text{A.2})$$

where the condition on ΔU , given by (3.13), and Eq. (2.13) have been used. Using Eq. (2.23) for Δf , one has

$$\begin{aligned} \Delta T^{00} &= \int dK E_k^2 \Delta f, \\ &= \int dK E_k^2 \left[\delta f \right. \\ &\quad \left. - T^2 \frac{dm_{\text{eq}}^2}{dT^2} \frac{f_0(1+f_0)}{E_k} \frac{\int dK \delta f}{\int dK E_k f_0(1+f_0)} \right] \\ &= \int dK \left(E_k^2 - T^2 \frac{dm_{\text{eq}}^2}{dT^2} \right) \delta f \end{aligned} \quad (\text{A.3})$$

Analogously, one gets the momentum density variation

$$\begin{aligned} \Delta T^{0i} &= \int dK E_k k^i \delta f - \frac{\Delta m_{\text{th}}^2}{2} \beta \int dK k^i f_0 (1 + f_0) \\ &= \int dK E_k k^i \Delta f. \end{aligned} \quad (\text{A.4})$$

The stress tensor variation is

$$\begin{aligned} \Delta T^{ij} &= \int dK k^i k^j \delta f - \frac{\Delta m_{\text{th}}^2}{2} \beta \int dK \frac{k^i k^j}{E_k} f_0 (1 + f_0) \\ &\quad - \frac{\Delta m_{\text{th}}^2}{2} \int dK \frac{k^i k^j}{E_k^2} f_0 + \delta^{ij} \Delta U \\ &= \int dK k^i k^j \Delta f - \frac{\Delta m_{\text{th}}^2}{2} \int dK \frac{k^i k^j}{E_k^2} f_0 \\ &\quad + \delta^{ij} \frac{\Delta m_{\text{th}}^2}{2} \int dK f_0, \end{aligned} \quad (\text{A.5})$$

where Eq. (2.23) has been applied. Eqs. (A.3) - (A.5) reproduce these provided by Eqs. (3.15) - (3.17). Among all these expressions, ΔT^{ij} needs further simplifications to show how one can obtain Eq. (3.25). The second and the third term of the first line in Eq. (A.5) may be combined to get

$$\begin{aligned} \Delta T^{ij} &= \int dK k^i k^j \delta f - \frac{\Delta m_{\text{th}}^2}{2} \int dK k^i k^j \partial_{E_k} \left(\frac{f_0}{E_k} \right) \\ &\quad + \delta^{ij} \Delta U. \end{aligned} \quad (\text{A.6})$$

Next, using $\partial_{E_k}(\dots) = \frac{E_k}{k} \partial_k(\dots)$ and then integrating by parts leads to

$$\begin{aligned} \Delta T^{ij} &= \int dK k^i k^j \delta f - \delta^{ij} \frac{\Delta m_{\text{th}}^2}{2} \int dK f_0 + \delta^{ij} \Delta U \\ &= \int dK k^i k^j \delta f, \end{aligned} \quad (\text{A.7})$$

where the condition (3.13) has been used.

Appendix B: Details of the thermodynamic integrals

1. Boltzmann statistics

Our strategy to evaluate the integrals with the Boltzmann statistics is to use the integral representation of the modified Bessel functions of the second kind

$$K_n(z) = \int_0^\infty d\theta \cosh(n\theta) \exp(-z \cosh \theta), \quad (\text{B.1})$$

We will also need the Bickley functions defined by

$$\text{Ki}_r(z) = \int_0^\infty d\theta \frac{\exp(-z \cosh \theta)}{(\cosh \theta)^r}. \quad (\text{B.2})$$

We will need the following series in the small z limit

$$K_1(z) \approx \frac{1}{z} - \frac{z}{4} (1 - 2\gamma_E + \ln 4 - 2 \ln z), \quad (\text{B.3})$$

$$K_2(z) \approx \frac{2}{z^2} - \frac{1}{2} + \frac{z^2}{32} (3 - 4\gamma_E + 2 \ln 4 - 4 \ln z), \quad (\text{B.4})$$

$$K_3(z) \approx \frac{8}{z^3} - \frac{1}{z} + \frac{z}{8}, \quad (\text{B.5})$$

$$K_4(z) \approx \frac{48}{z^4} - \frac{4}{z^2} + \frac{1}{4}, \quad (\text{B.6})$$

$$K_5(z) \approx \frac{384}{z^5} - \frac{24}{z^3} + \frac{1}{z}, \quad (\text{B.7})$$

where $\gamma_E = 0.577$ is the Euler constant and $\ln 4 = 1.386$ and the higher order terms in z were neglected. For the Bickley function [58], we need

$$\text{Ki}_1(z) \approx \frac{\pi}{2} - z(1 - \gamma_E - \ln(z/2)) \quad (\text{B.8})$$

Using $|\mathbf{k}| \equiv k = m_x \sinh \theta$, the thermodynamic functions $I_{n,q}$, defined by (4.34) and evaluated in the fluid rest frame, can be expressed as

$$I_{n,q}(T, z) = a_q \frac{T^{n+2} z^{n+2}}{2\pi^2} \times \int_0^\infty d\theta (\cosh \theta)^{n-2q} (\sinh \theta)^{2q+2} \exp(-z \cosh \theta), \quad (\text{B.9})$$

where $z = \frac{m_x}{T}$ and $a_q = 1/((1+2q)!!)$. Using $\cosh x = (e^x + e^{-x})/2$ and $\sinh x = (e^x - e^{-x})/2$, and the definition Eq. (B.1), these integrals can be expressed in terms of the modified Bessel functions of the second kind.

Let us consider $I_{3,0}$ first. After the angle integral, we have

$$I_{3,0} = \frac{1}{2\pi^2} \int_0^\infty dk k^2 E_k^2 e^{-E_k/T}. \quad (\text{B.10})$$

Using $k = m_x \sinh \theta$, this becomes

$$I_{3,0} = \frac{m_x^5}{2\pi^2} \int_0^\infty d\theta \sinh^2 \theta \cosh^3 \theta e^{-z \cosh \theta}, \quad (\text{B.11})$$

where $z = m_x/T$. By using $\cosh x = (e^x + e^{-x})/2$ and $\sinh x = (e^x - e^{-x})/2$ and the definition Eq. (B.1), one gets

$$I_{3,0} = -\frac{m_x^5}{32\pi^2} (2K_1(z) - K_3(z) - K_5(z)) \approx \frac{m_x^5}{32\pi^2} \left(\frac{384}{z^5} - \frac{16}{z^3} \right) = \frac{12T^5}{\pi^2} \left(1 - \frac{z^2}{24} \right). \quad (\text{B.12})$$

The other useful integrals are found in a similar way

$$I_{1,0} \approx \frac{T^3}{\pi^2} \left(1 - \frac{z^2}{4} \right), \quad (\text{B.13})$$

$$I_{-1,0} \approx \frac{T}{2\pi^2} \left(1 - \frac{z\pi}{2} \right), \quad (\text{B.14})$$

$$I_{3,1} \approx \frac{4T^5}{\pi^2} \left(1 - \frac{z^2}{8} \right), \quad (\text{B.15})$$

$$I_{0,0} \approx \frac{T^2}{2\pi^2} \left(1 - \frac{z^2}{4} (1 - 2\gamma + \ln 4 - 2 \ln z) \right), \quad (\text{B.16})$$

where $I_{0,0} \equiv q_0$ is needed for the thermal mass evaluation. For $\epsilon + P$, we have

$$\epsilon + P \approx \frac{T^4}{\pi^2} \left(4 - \frac{z^2}{2} + \frac{z^4}{16} \right). \quad (\text{B.17})$$

2. Bose-Einstein statistics

The thermodynamic integrals for the Bose-Einstein gas are defined by Eqs. (4.14)

$$J_{n,q} = a_q \int dK (u^\mu k_\mu)^{n-2q} (-\Delta_{\mu\nu} k^\mu k^\nu)^q [f_0(1 + f_0)]. \quad (\text{B.18})$$

In the fluid rest frame and after the angle integrals $J_{n,q}$ becomes

$$J_{n,q} = \frac{a_q}{2\pi^2} \int_0^\infty dk \frac{k^2}{E_k} F_{n,q}(E_k) f_0(E_k) (1 + f_0(E_k)) \quad (\text{B.19})$$

and

$$F_{n,q}(E_k) = E_k^{n-2q} k^{2q} = E_k^{n-2q} (E_k^2 - m_x^2)^q. \quad (\text{B.20})$$

Using $\partial_k f_0(E_k) = -\frac{k}{TE_k} f_0(E_k) (1 + f_0(E_k))$ and integrating by part, we can rewrite the above as

$$J_{n,q} = \frac{a_q T}{2\pi^2} \int_0^\infty dk f_0(E_k) \partial_k (k F_{n,q}(E_k)). \quad (\text{B.21})$$

Changing the integration variable to E_k , we further get

$$J_{n,q} = \frac{a_q T}{2\pi^2} \int_{m_x}^\infty dE_k G_{n,q}(E_k) f_0(E_k), \quad (\text{B.22})$$

where

$$G_{n,q}(E_k) = (E_k/k) \partial_k (k F_{n,q}(E_k)) = \frac{(E_k^2 - m_x^2)^q E_k^{n-2q-1} ((n+1)E_k^2 + m_x^2(2q-n))}{\sqrt{E_k^2 - m_x^2}} \quad (\text{B.23})$$

using $k = \sqrt{E_k^2 - m_x^2}$.

Our strategy to evaluate this integral is to separate the high momentum contribution and the low momentum contribution. We know how to evaluate

$$\int_{m_x}^\infty dE_k E_k^l f_0(E_k) = T^{l+1} \int_z^\infty dx x^l \sum_{n=1}^\infty e^{-nx} \quad (\text{B.24})$$

where $x = E_k/T$ and $z = m_x/T$ in terms of the polylogarithmic functions $\text{Li}_n(z)$. Hence, we first expand the square-root in m_x^2/E_k^2 and identify the non-negative power terms in E_k . Denoting the collection of such terms as $H_{n,q}(E_k)$, we then separate the integral as

$$J_{n,q} = \frac{a_q T}{2\pi^2} \int_{m_x}^\infty dE_k H_{n,q}(E_k) f_0(E_k) + \frac{a_q T}{2\pi^2} \int_{m_x}^\infty dE_k (G_{n,q}(E_k) - H_{n,q}(E_k)) f_0(E_k) \quad (\text{B.25})$$

One can show that the reminder $G_{n,q}(E_k) - H_{n,q}(E_k) = O(1/E_k^3)$ for all n and q . Then expanding f_0 in the small E_k/T limit,

$$f_0(E_k) = \frac{T}{E_k} - \frac{1}{2} + \frac{E_k}{12T} + O((E_k/T)^3) \quad (\text{B.26})$$

we can keep the first three terms in the integrand to calculate the soft contribution. This integral can usually be exactly evaluated.

Let us consider $J_{3,0}$. From Eq. (B.22), we have

$$J_{3,0} = \frac{T}{2\pi^2} \int_{m_x}^{\infty} dE_k \frac{4E_k^4 - 3m_x^2 E_k^2}{\sqrt{E_k^2 - m_x^2}} f_0(E_k). \quad (\text{B.27})$$

Expanding the square-root in powers of m_x^2/E_k^2 , we get

$$G_{3,0}(E_k) = 4E_k^3 - m_x^2 E_k + O(1/E_k^3). \quad (\text{B.28})$$

We can then separate the hard and the soft parts

$$\begin{aligned} J_{3,0} &= \frac{T}{2\pi^2} \int_{m_x}^{\infty} dE_k (4E_k^3 - m_x^2 E_k) f_0(E_k) \\ &+ \frac{T}{2\pi^2} \int_m^{\infty} dE_k \left[\frac{4E_k^4 - 3m_x^2 E_k^2}{\sqrt{E_k^2 - m_x^2}} - 4E_k^3 + m_x^2 E_k \right] f_0(E_k). \end{aligned} \quad (\text{B.29})$$

Since the square bracket behaves like $1/E_k^3$, we can use $f_0(E_k) \approx T/E_k - 1/2 + E_k/12T$ to evaluate the second integral. It is

$$J_{n,q}^{\text{soft}} \approx \frac{T^5}{2\pi^2} \left(\frac{z^3}{3} - \frac{3z^4}{16} + \frac{7z^5}{180} \right) \quad (\text{B.30})$$

with $z = m_x/T$. The hard part is

$$\begin{aligned} J_{3,0}^{\text{hard}} &= \frac{T}{2\pi^2} \int_{m_x}^{\infty} dE_k (4E_k^3 - m_x^2 E_k) f_0(E_k) \\ &= \frac{T^5}{2\pi^2} \int_z^{\infty} dx (4x^3 - z^2 x) \frac{1}{e^x - 1} \\ &= \frac{T^5}{2\pi^2} \left(24 \text{Li}_4(e^{-z}) + 24z \text{Li}_3(e^{-z}) + 11z^2 \text{Li}_2(e^{-z}) \right. \\ &\quad \left. + 3z^3 \text{Li}_1(e^{-z}) \right) \\ &= \frac{T^5}{2\pi^2} \left(\frac{4\pi^4}{15} - \frac{\pi^2 z^2}{6} - \frac{z^3}{3} + \frac{z^4}{4} - \frac{7z^5}{180} O(z^6) \right). \end{aligned} \quad (\text{B.31})$$

Adding the two yields

$$J_{3,0} = J_{3,0}^{\text{hard}} + J_{3,0}^{\text{soft}} \approx \frac{T^5}{2\pi^2} \left(\frac{4\pi^4}{15} - \frac{\pi^2 z^2}{6} + \frac{z^4}{16} \right). \quad (\text{B.32})$$

This formula works better than 1 part in 10^4 up to $z = m_x/T = 1$.

The usual way of evaluating Bose-Einstein integrals is to use the modified Bessel functions of the second kind

$$\begin{aligned} J_{3,0} &= \frac{T}{2\pi^2} \int_0^{\infty} dk \frac{k}{E_k} \frac{4E_k^4 - 3m_x^2 E_k^2}{k} \sum_{n=1}^{\infty} e^{-nE_k/T} \\ &= \frac{Tm_x^4}{2\pi^2} \sum_{n=1}^{\infty} \left(\frac{1}{2} K_4(nz) + \frac{1}{2} K_2(nz) \right). \end{aligned} \quad (\text{B.33})$$

Using the small x expansion of $K_n(x)$ and collecting only the terms converging under the infinite sum, we get

$$\begin{aligned} J_{3,0}^{\text{Bessel}} &\approx \frac{Tm_x^4}{2\pi^2} \sum_{n=1}^{\infty} \left(\frac{24}{n^4 z^4} - \frac{1}{n^2 z^2} \right) \\ &= \frac{T^5}{2\pi^2} \left(\frac{4\pi^4}{15} - \frac{\pi^2 z^2}{6} \right). \end{aligned} \quad (\text{B.34})$$

which gets only the first two terms.

The useful integrals are then found using the former method

$$J_{1,0} \approx \frac{T^3}{6} \left(1 - \frac{3z}{2\pi} + \frac{3z^2}{4\pi^2} \right), \quad (\text{B.35})$$

$$J_{3,1} \approx \frac{2T^5 \pi^2}{45} \left(1 - \frac{15z^2}{8\pi^2} \right), \quad (\text{B.36})$$

$$J_{-1,0} \approx \frac{1}{8\pi} \frac{T}{z} \left(1 - \frac{2z}{\pi} + \frac{z^2}{6} \right). \quad (\text{B.37})$$

These formulas provide very good approximation up to $z = 1$. In evaluating $J_{-1,0}$ one would expect to use the Bickley functions equivalently but this method does not work because the sum cannot be easily evaluated, even for the leading behavior.

For the enthalpy, one gets

$$\epsilon + P \approx \frac{2\pi^2 T^4}{45} \left(1 - \frac{15z^2}{8\pi^2} \right) \quad (\text{B.38})$$

and for the thermal mass

$$q_0 \approx \frac{T^2}{12} \left(1 - \frac{3z}{\pi} \right). \quad (\text{B.39})$$

Appendix C: Irreducible moments

To express the irreducible moments of the distribution function one can apply the Grad's 14-moment approximation, where the correction to the distribution function of the Bose-Einstein gas is a generalization of the Boltzmann one, shown in [14, 41, 43], and takes the form

$$\begin{aligned} \delta f &= f_0(1 + f_0) \left[E_0 + B_0 m_x^2 + D_0(u \cdot k) - 4B_0(u \cdot k)^2 \right] \Pi \\ &+ f_0(1 + f_0) B_2 p^\alpha p^\beta \pi_{\alpha\beta}. \end{aligned} \quad (\text{C.1})$$

The coefficients E_0 , B_0 , D_0 , and B_2 are functions of m_x , T , and $u^\mu k_\mu$ and they read

$$B_2 = \frac{1}{2J_{4,2}}, \quad (\text{C.2})$$

$$\frac{D_0}{3B_0} = -4 \frac{J_{3,1} J_{2,0} - J_{4,1} J_{1,0}}{J_{3,0} J_{1,0} - J_{2,0} J_{2,0}} \equiv -C_2, \quad (\text{C.3})$$

$$\frac{E_0}{3B_0} = m_x^2 + 4 \frac{J_{3,1} J_{3,0} - J_{4,1} J_{2,0}}{J_{3,0} J_{1,0} - J_{2,0} J_{2,0}} \equiv -C_1, \quad (\text{C.4})$$

$$B_0 = -\frac{1}{3C_1 J_{2,1} + 3C_2 J_{3,1} + 3J_{4,1} + 5J_{4,2}}, \quad (\text{C.5})$$

where terms related to the particle diffusion have been dropped. Therefore, the irreducible moments ρ_{-n} and $\rho_{-n}^{\mu\nu}$ can be expressed by Π and $\pi^{\mu\nu}$ as follows

$$\rho_{-n} = \gamma_n^{(0)} \Pi, \quad (\text{C.6})$$

$$\rho_{-n}^{\mu\nu} = \gamma_n^{(2)} \pi^{\mu\nu}, \quad (\text{C.7})$$

where the coefficients $\gamma_n^{(0)}$ and $\gamma_n^{(2)}$ are

$$\gamma_n^{(0)} = (E_0 + B_0 m_x^2) J_{-n,0} + D_0 J_{1-n,0} - 4B_0 J_{2-n,0}, \quad (\text{C.8})$$

$$\gamma_n^{(2)} = \frac{J_{4-n,2}}{J_{4,2}}. \quad (\text{C.9})$$

Only $\gamma_2^{(0)}$ and $\gamma_2^{(2)}$ are needed here.

Using the prescription shown in Appendix B for evaluating relevant thermodynamics functions one finds that the leading order terms of coefficients E_0 , B_0 , and D_0 , which are

$$E_0 \approx \frac{e_0}{z^2 T^4}, \quad D_0 \approx \frac{d_0}{z^2 T^5}, \quad B_0 \approx \frac{b_0}{z^2 T^6} \quad (\text{C.10})$$

where

$$e_0 = \frac{48\pi^2(\pi^8 - 10125\zeta(3)\zeta(5))}{5(19\pi^6\zeta(3) - 2835\zeta^3(3) - 300\pi^4\zeta(5))} \approx 22.36, \quad (\text{C.11})$$

$$d_0 = \frac{-216\pi^4(\pi^2\zeta(3) - 25\zeta(5))}{19\pi^6\zeta(3) - 2835\zeta^3(3) - 300\pi^4\zeta(5)} \approx -22.29, \quad (\text{C.12})$$

$$b_0 = \frac{3\pi^2(\pi^6 - 405\zeta^2(3))}{19\pi^6\zeta(3) - 2835\zeta^3(3) - 300\pi^4\zeta(5)} \approx -0.84. \quad (\text{C.13})$$

Therefore, the leading order of $\gamma_2^{(0)}$ and $\gamma_2^{(2)}$ is

$$\gamma_2^{(0)} \approx \frac{g_0}{z^4 T^4}, \quad (\text{C.14})$$

$$\gamma_2^{(2)} \approx \frac{g_2}{T^2} \quad (\text{C.15})$$

where

$$g_0 = \frac{32(\pi^8 - 10125\zeta(3)\zeta(5))}{5(19\pi^6\zeta(3) - 2835\zeta^3(3) - 300\pi^4\zeta(5))} \approx 1.51, \quad (\text{C.16})$$

$$g_2 = \frac{\zeta(3)}{20\zeta(5)} \approx 0.06 \quad (\text{C.17})$$

For the Boltzmann statistics, one needs to change all integrals $J_{n,q} \rightarrow I_{nq}$ and $M \rightarrow M_c$. The leading order results for the classical gas are

$$E_0 \approx \frac{\pi^2}{2T^4}, \quad D_0 \approx -\frac{\pi^2}{3T^5}, \quad B_0 \approx -\frac{\pi^2}{96T^6}, \quad (\text{C.18})$$

which lead to

$$\gamma_2^{(0)} \approx \frac{g_{0c}}{T^4}, \quad (\text{C.19})$$

$$\gamma_2^{(2)} \approx \frac{g_{2c}}{T^2}, \quad (\text{C.20})$$

where $g_{0c} = -(5 + 12\gamma_E - 12 \ln 2) \approx -0.075$ and $g_{2c} = 1/20$.

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