Exactly solvable configuration mixing scheme in the vibrational limit of the interacting boson model

Feng Pan, Dongkang Li, Guoqing Cheng, Zheng Qiao, Jiachao Bai, and Jerry P. Draayer

Phys. Rev. C **97**, 034316 — Published 14 March 2018

DOI: 10.1103/PhysRevC.97.034316

Exactly solvable configuration mixing scheme in the vibrational limit of the interacting boson model

Feng Pan*,[1,2] Dongkang Li,[1] Guoqing Cheng,[1] Zheng Qiao,[1] Jiachao Bai,[1] and Jerry P. Draayer[2]

[1] *Department of Physics, Liaoning Normal University, Dalian 116029, China*

[2] *Department of Physics and Astronomy, Louisiana State University, Baton Rouge, LA 70803-4001, USA*

Abstract: An intruder configuration mixing scheme with $2n$-particle and $2n$-hole configurations from $n = 0$ up to $n \to \infty$ in the U(5) (vibrational) limit of the interacting boson model is proposed. A simple Hamiltonian suitable to describe the intruder and normal configuration mixing turns to be exactly solvable, of which the eigenstates can be expressed as the SU(1,1) coherent states built on the U(5) basis vectors of the interacting boson model. It is shown that the configuration mixing scheme keeps lower part of the vibrational spectrum unchanged and generates the intruder states due to the mixing. Some low-lying level energies and experimentally known B(E2) ratios of ^{108,110}Cd are fitted and compared with the experimental results.

PACS numbers: 21.60.Fw, 03.65.Fd, 02.20.Qs, 02.30.Ik

1. Introduction: As shown in the past a few decades, there are clear evidence for the presence of multi-particle-hole excitations in nuclei across the closed shells [2–4], in particular near closed-shell mass regions around proton number $Z \sim 50$ and $Z \sim 82$. Multi-particle-hole excitations are not easy to be incorporated in large-scale shell-model calculations due to the extremely large model space, which, however, can easily be handled within the Interacting Boson Model (IBM) [5, 6]. Generally, the IBM-2 in distinguishing neutron- from proton-pairs is often adopted in the configuration mixing schemes [5, 6], of which calculations including up to 6-particle and 6-hole excitations have been carried out [7, 8]. There have been a lot of investigations made in this direction. For example, a shell model interpretation of intruder states and the onset of deformation in even-even nuclei was provided in [9]; the microscopic study of the configuration mixing in the IBM was shown in [10]; and shell-model particle-hole intruder excitations and the collective bands built on them near closed shells were studied and classified in [11]. Generally, the configuration mixing due to the multi-particle-hole excitations was considered in understanding shape coexistence phenomena by taking different symmetry limits of the IBM for different configurations [12–18]. These IBM configuration mixing calculations have been proven to be successful in describing intruder states and related phenomena in near closed shell nuclei, in which, however, many parameters are involved.

Though most of the configuration mixing schemes were considered in the IBM-2 framework, the configuration mixing can also be considered equivalently in the IBM-1 with no distinction between neutron-type and proton-type bosons as shown in [13–16, 19, 20]. Furthermore, though configuration mixing schemes by taking different symmetry limits of the IBM for different configurations are certainly very much helpful in understanding shape coexistence phenomena, it will be shown in this paper that a reasonably simplified configuration mixing scheme based on the U(5) limit of the IBM-1 is not only analytically solvable, but also probable to describe intruder states in vibrational nuclei near the closed shells.

2. A solvable configuration mixing model: In the original IBM-1, a typical consistent-Q Hamiltonian may be schematically written as [1]

$$\hat{H}_{\rm CQ} = \epsilon_d \hat{n}_d - \kappa \hat{Q}(\chi) \cdot \hat{Q}(\chi), \tag{1}$$

where ϵ_d, κ, and $\chi \in [-\sqrt{7}/2, 0]$ are real parameters of the model, $\hat{n}_d = \sum_\mu d^\dagger_\mu d_\mu$ is the d-boson number operator, and $\hat{Q}_\mu(\chi) = d^\dagger_\mu s + s^\dagger \tilde{d}_\mu + \chi (d^\dagger \tilde{d})^{(2)}_\mu$ is the quadrupole operator, in which $(d^\dagger \tilde{d})^{(l)}_\mu$ stands for the tensor coupling of the d-boson creation and annihilation operators with $\tilde{d} = (-)^\mu d_{-\mu}$, and $\hat{Q}(\chi) \cdot \hat{Q}(\chi) = \sum_\mu (-)^\mu \hat{Q}_\mu(\chi) \hat{Q}_{-\mu}(\chi)$.

There are three standard IBM-1 dynamical symmetry limit cases: the U(5) limit for $\kappa = 0$, the SU(3) limit for $\epsilon_d = 0$ and $\chi = -\sqrt{7}/2$, and the O(6) limit for $\epsilon_d = 0$ and $\chi = 0$. In the U(5) limit, for example, the eigenstates of (1) denoted as $|N, n_d, \nu, \rho, L, M\rangle$ are nothing but the basis vectors of the group chain U(6)⊃U(5)⊃O(5)⊃O(3)⊃O(2), where N is the total number of bosons, n_d is the number of d-bosons, ν is the d-boson seniority number, ρ is an additional quantum

* The corresponding author's e-mail: daipan@dlut.edu.cn

number needed for the branching reduction of O(5)⊃O(3), and L and M are the quantum number of the angular momentum and that of its third projection, respectively. Actually, besides the terms included in (1), other $O(5)$ and $O(3)$ invariants, such as the Casimir (invariant) operator of $O(5)$, $C_2(O(5)) = 2(d^\dagger\tilde{d})^{(3)}\cdot(d^\dagger\tilde{d})^{(3)} + 2(d^\dagger\tilde{d})^{(1)}\cdot(d^\dagger\tilde{d})^{(1)}$, and $\hat{L}\cdot\hat{L}$, where $\hat{L}_\mu = \sqrt{10}(d^\dagger\tilde{d})^{(1)}_\mu$ $(\mu = +1, 0, -1)$ are the angular momentum operators, can also be considered in the U(5)-limit case.

Thus, the Hamiltonian suitable to describe $2n$-particle and $2n$-hole configuration mixing from $n = 0$ up to $n \to \infty$ in the U(5) limit of the interacting boson model may be written as

$$\hat{H} = \hat{P}\left(\Delta\,\hat{n}_s + \Delta\,\hat{n}_d + \hat{H}_0 + g(S^+ + S^-)\right)\hat{P}, \tag{2}$$

where $\hat{H}_0 = \epsilon_d\hat{n}_d + c\,C_2(O(5)) + f\,\hat{L}\cdot\hat{L}$ is an U(5)-limit Hamiltonian, $\hat{n}_s = s^\dagger s$ and $\hat{n}_d = d^\dagger\cdot\tilde{d}$ are s- and d-boson number operators, respectively, ϵ_d is the d-boson energy, c and f are real parameters needed to remove the degeneracy in level energies for fixed number of d-bosons n_d but with different ν and L, Δ is the energy needed to excite two more particles from the closed shell resulting in a configuration with two more particles and two more holes is taken to be a constant for simplicity when $2n$-particle and $2n$-hole excitations with $n = 0, 1, 2, \cdots$ are considered, g is the mixing parameter, of which the allowed range will be shown later on, $S^+ = S^+_d + S^+_s$ $(S^- = (S^+)^\dagger)$, in which $S^+_d = \frac{1}{2}d^\dagger\cdot d^\dagger$ and $S^+_s = \frac{1}{2}s^{\dagger\,2}$ are the d-boson and s-boson pairing operators. It should be noted that the Casimir operator of $O(5)$ and the $O(3)$ invarinat $\hat{L}\cdot\hat{L}$ in the $U(5)$ Hamiltonian $\hat{H}_0$ are commutative with $S^\pm_s$ and $S^\pm_d$. $\hat{P}$, satisfying $\hat{P}^2 = \hat{P}$ and $\hat{P}^\dagger = \hat{P}$, is the projection operator defined by

$$\hat{P}|N', n_d, \nu, \rho, L, M\rangle = \begin{cases} |N', n_d, \nu, \rho, L, M\rangle \text{ if } N' \geq N, \\ 0 \quad \text{otherwise,} \end{cases} \tag{3}$$

which keeps the Hamiltonian (2) to be effective only within the subspace spanned by $[N]\oplus[N+2]\oplus[N+4]\oplus\cdots$ mixed configurations, where N is the total boson number of the system without configuration mixing, $|N', n_d, \nu, \rho, L, M\rangle$ is a basis vector of U(6) ⊃ U(5) ⊃ O(5) ⊃ O(3) with total number of bosons $N' = N + 2n$ with $n = 0, 1, 2, \cdots$. (3) extends the projection introduced previously [5–8, 12–18]. With the projection (3), any operator $\hat{O}$ used in the model may be replaced by $\hat{P}\hat{O}\hat{P}$.

The configuration mixing scheme of the IBM [5–8, 12–18], such as the U(5) limit one shown in (2) is aimed at accommodating core-excited intruder configurations. The "normal" configuration is described in a basis of N bosons with angular momenta 0^+ or 2^+ and interacting via the usual IBM Hamiltonian. Here and in Eq. (3), N is half the number of valence nucleons or half the number of valence holes, which is always fixed for a given nucleus though the total boson number in the scheme is not a conserved quantity due to the configuration mixing. The "intruder" configurations, which we will assume throughout, involves $2n$-particle and $2n$-hole excitation, is described by states with $N + 2n$ bosons with $n = 0, 1, \cdots$ again restricted to angular momenta 0^+ or 2^+ and interacting via a specific IBM Hamiltonian. Basis vectors of the $[N + 2n]$ and those of the $[N + 2n + 2]$ configurations are mixed by the mixing term, such as $g\hat{P}\,(S^+ + S^-)\,\hat{P}$ used in (2). These essential assumptions of the model were outlined, for example, in [10, 17]. Furthermore, in previous configuration mixing schemes, the gap $\Delta > 0$ was taken to be configuration dependent, while, for simplicity, it is taken to be the same for any n in (2), with which the configuration of the core with N bosons is lowest in energy, while the configuration with $N + 2$ bosons is the next lowest in energy, and so on, where there is a constant energy gap $\sim 2\Delta$ between $[N+2n+2]$ and $[N+2n]$ configurations for any n. It is clear that the configuration mixing scheme described above is similar to the shell model calculations in taking $2n\hbar\omega$ excitations with $n = 0, 1, \cdots$ into account [21, 22]. However, other possible excitations, for example, broken pair excitations with explicit valence nucleon degrees of freedom, are not considered in this scheme.

As shown in the following, the first set of eigenstates (called normal states in the following) labeled with an additional quantum number $\zeta = 1$, which is similar to those provided by the $U(5)$-limit of the IBM without configuration mixing, up to a normalization factor, are given by

$$|\zeta = 1, \omega\rangle = e^{\alpha\tilde{S}^+_s + \beta\tilde{S}^+_d}|\omega\rangle, \tag{4}$$

where $\omega \equiv \{N, n_d, \nu, \rho, L, M\}$, in which $N = n_s + n_d$, $\tilde{S}^+_d = \hat{P}S^+_d\hat{P}$, $\tilde{S}^+_s = \hat{P}S^+_s\hat{P}$, and α and β are parameters to be determined by the corresponding eigen-equation. It should be understood that n_d, n_s, and N are used to label the basis vectors $\{|\omega\rangle\}$ of $U(6) \supset U(5) \supset O(5) \supset O(3)$ in the original U(5) limit of the IBM shown in (4), in which ω is a set of good quantum numbers, while neither the number of s-bosons nor that of d-bosons is good quantum number of $|\zeta = 1, \omega\rangle$ given in (4) for given ω when $g \neq 0$, which applies to other excited states as well.

Actually, $\{S_d^+,\ S_d^-,\ S_d^0=\frac{1}{2}\hat{n}_d+\frac{5}{4}\}$ and $\{S_s^+,\ S_s^-,\ S_s^0=\frac{1}{2}\hat{n}_s+\frac{1}{4}\}$ are two sets of $SU(1,1)$ generators, satisfying

$$[S_\xi^0, S_{\xi'}^\pm]=\pm\delta_{\xi\xi'}S_\xi^\pm,\quad [S_\xi^+, S_{\xi'}^-]=-2\delta_{\xi\xi'}S_\xi^0 \tag{5}$$

for $\xi,\ \xi'=s$ or d, from which one also has

$$[S_{\xi'}^-, S_\xi^{+k}]=\delta_{\xi'\xi}\,k\,S_\xi^{+k-1}(2S_\xi^0+k-1). \tag{6}$$

Since the s-boson part and the d-boson part of the Hamiltonian (2) are separated, only the s-boson part of the solution will be presented in the following. The same procedure obviously applies to the d-boson part as well. The s-boson part of (2) is

$$\hat{H}_s=\hat{P}\left(2\Delta S_s^0+g(S_s^++S_s^-)-\frac{1}{2}\Delta\right)\hat{P}. \tag{7}$$

For a given eigenstate $|N,n_d,\nu,\rho,L,M\rangle$ of the original $U(5)$ limit, the eigenstate of (7) can be written as

$$|\zeta=1,\omega\rangle\equiv|\zeta=1,N,n_d,\nu,\rho,L,M\rangle=e^{\alpha\tilde{S}_s^+}|N,n_d,\nu,\rho,L,M\rangle, \tag{8}$$

where the number of s-bosons in $|N,n_d,\nu,\rho,L,M\rangle$ is fixed with $n_s=N-n_d$ representing the number of s-bosons in $|N,n_d,\nu,\rho,L,M\rangle$ of the original U(5) limit. It is obvious that the number of s-bosons and that of d-bosons are not good quantum numbers of $|\zeta=1,\omega\rangle\equiv|\zeta=1,N,n_d,\nu,\rho,L,M\rangle$ shown in (8).

Using the Hausdorff-Campbell relation

$$\hat{A}e^{\hat{B}}=e^{\hat{B}}\left(\hat{A}+\frac{1}{1!}[\hat{A},\hat{B}]+\frac{1}{2!}[[\hat{A},\hat{B}],\hat{B}]+\cdots\right), \tag{9}$$

we have

$$\hat{H}_s|\zeta=1,\omega\rangle=e^{\alpha\tilde{S}_s^+}\left(2\Delta\Lambda_s-\frac{1}{2}\Delta+2\alpha\Delta\tilde{S}_s^++g\tilde{S}_s^++2g\alpha\Lambda_s+g\alpha^2\tilde{S}_s^+\right)|N,n_d,\nu,\rho,L,M\rangle, \tag{10}$$

where $\Lambda_s=\frac{1}{2}(N-n_d+\frac{1}{2})$. Thus, we get the corresponding eigen-energy

$$E_{N,n_d}^{(\zeta=1)}=2\Delta\Lambda_s-\frac{1}{2}\Delta+2g\alpha\Lambda_s, \tag{11}$$

while the term proportional to $\tilde{S}_s^+e^{\alpha\tilde{S}_s^+}$ in (10) must be zero, which leads to

$$g+2\alpha\Delta+g\alpha^2=0 \tag{12}$$

with two possible solutions

$$\alpha^\pm=\frac{1}{g}(-\Delta\pm\sqrt{\Delta^2-g^2}), \tag{13}$$

where $g\neq 0$ should be assumed. As shown in (12), when $g=0$, $\alpha=0$ is the only solution corresponding to the original U(5) limit Hamiltonian without configuration mixing. Since $\hat{H}_s$ is Hermitian, $\alpha^\pm$ must be real, which requires $|g|\leq\Delta$. It is obvious that the sign of g does not affect the eigen-energy (11).

Using a similar procedure, one can also get excited states of the first set of (7):

$$|\zeta=2,k_s=1;N,n_d,\nu,\rho,L,M\rangle=(1+a_{n_d}\tilde{S}_s^+)e^{\alpha\tilde{S}_s^+}|N,n_d,\nu,\rho,L,M\rangle \tag{14}$$

with fixed quantum numbers N, n_d, ν, ρ, L, M, which implies $g \neq 0$. As shown in (2) and (7), when $g = 0$ there is no excited state similar to that shown in (14) built on $|N, n_d, \nu, \rho, L, M\rangle$. Namely, excited states with $\zeta > 1$ become null when $g = 0$. The corresponding eigen-energy of (7) under (14) is

$$E^{(k_s=1)}_{N,n_d} = E^{(\zeta=1)}_{N,n_d} + 2g\, a_{n_d}\, \Lambda_s \tag{15}$$

with

$$a_{n_d} = \frac{2\Delta + 2g\alpha^{\pm}}{2g\Lambda_s} = \pm\frac{\sqrt{\Delta^2 - g^2}}{g\Lambda_s} \tag{16}$$

when $g \neq 0$. Substituting (16) into (15), we get

$$E^{(k_s=1)}_{N,n_d} = E^{(\zeta=1)}_{N,n_d} \pm 2\sqrt{\Delta^2 - g^2} \tag{17}$$

for $g \neq 0$. Since physical spectrum should be lower-bound, only α^+ shown in (13) is possible, which corresponds to a_{n_d} given in (16) and the eigen-energy provided in (17) with positive sign. Otherwise, the spectrum will become upper-bound. In (17), $2\sqrt{\Delta^2 - g^2}$ is the energy of the collective 2p-2h excitation. In the following, $g \neq 0$ is always assumed.

Moreover, one can prove that the eigenstate with $\zeta = 1$ is orthogonal to that with $\zeta = 2$. Since α is real,

$$\langle e^{\alpha\tilde{S}^-_s}\, \tilde{S}^+_s e^{\alpha\tilde{S}^+_s}\rangle = \langle e^{\alpha\tilde{S}^-_s}\, \tilde{S}^-_s e^{\alpha\tilde{S}^+_s}\rangle = \langle e^{\alpha\tilde{S}^-_s} e^{\alpha\tilde{S}^+_s}(2\alpha\Lambda_s + \alpha^2\tilde{S}^+_s)\rangle, \tag{18}$$

where $\langle\hat{O}\rangle \equiv \langle\omega|\hat{O}|\omega\rangle$ stands for the expectation value of $\hat{O}$ under an eigenstate in the original U(5) limit, and the Hausdorff-Campbell relation is used to get the final result. (18) leads to

$$\langle e^{\alpha\tilde{S}^-_s}\, \tilde{S}^+_s e^{\alpha\tilde{S}^+_s}\rangle = \frac{2\alpha\Lambda_s}{1-\alpha^2}\langle e^{\alpha\tilde{S}^-_s} e^{\alpha\tilde{S}^+_s}\rangle. \tag{19}$$

Thus, we have

$$\langle e^{\alpha\tilde{S}^-_s}(1 + a_{n_d}\tilde{S}^+_s)e^{\alpha\tilde{S}^+_s}\rangle = \langle e^{\alpha\tilde{S}^-_s} e^{\alpha\tilde{S}^+_s}\rangle(1 + \frac{2a_{n_d}\,\alpha\,\Lambda_s}{1-\alpha^2}). \tag{20}$$

Since $\langle e^{\alpha\tilde{S}^-_s} e^{\alpha\tilde{S}^+_s}\rangle$ is always nonzero, substituting (16) into (20) and using (12), one can easily verify that $1+\frac{2a\,\alpha_{n_d}\Lambda_s}{1-\alpha^2}$ in (20) is obviously zero with

$$\langle e^{\alpha\tilde{S}^-_s}(1 + a_{n_d}\tilde{S}^+_s)e^{\alpha\tilde{S}^+_s}\rangle = 0, \tag{21}$$

which leads to the orthogonality relation:

$$\langle\zeta = 1, N', n'_d, \nu', \rho', L', M'|\zeta = 2, k_s = 1, N, n_d, \nu, \rho, L, M\rangle = 0. \tag{22}$$

According to the procedure for the s-boson part of (2) shown above, the unknowns α and β shown in (4) are given by

$$\begin{aligned}\alpha &= \frac{1}{g}\left(-\Delta + \sqrt{\Delta^2 - g^2}\right),\\ \beta &= \frac{1}{g}\left(-\Delta - \epsilon_d + \sqrt{(\Delta + \epsilon_d)^2 - g^2}\right).\end{aligned} \tag{23}$$

The eigen-energy corresponding to (4) is given by

$$\begin{aligned}E^{(\zeta=1)}_{N,n_d,\nu,L} &= \Delta N + \left(\sqrt{\Delta^2 - g^2} - \Delta\right)(N - n_d) +\\ &\left(\sqrt{(\Delta + \epsilon_d)^2 - g^2} - \Delta\right) n_d + c\,\nu(\nu + 3) + f\,L(L + 1).\end{aligned} \tag{24}$$

It is clearly shown that the modified single d-boson energy after the configuration mixing becomes

$$\tilde{\epsilon}_d = \sqrt{(\Delta+\epsilon_d)^2 - g^2} - \sqrt{\Delta^2 - g^2}, \tag{25}$$

which requires that $\Delta > |g|$ should be satisfied. In the following, we assume $\Delta > g > 0$. Moreover, the solution for both α and β with negative sign in front of the square root corresponding to a system with upper-bound in energy, which is non-physical and discarded. Only the solution shown in (23) corresponding to a lower-bound system is acceptable.

Other sets of excited eigenstates (called intruder states) labeled with $\zeta \geq 2$, which is not possible in the U(5)-limit of the IBM without configuration mixing ($g = 0$), also emerge due to the configuration mixing ($g \neq 0$). For example, by using a similar procedure shown above, one can prove that the first a few sets of excited eigenstates labeled with $\zeta = 2$ ($k_s = 1$ and $k_d = 0$), $\zeta = 3$ ($k_s = 0$ and $k_d = 1$), or $\zeta = 4$ ($k_s = 1$ and $k_d = 1$), up to a normalization factor, are given by

$$|k_s, k_d, \omega\rangle = (1 + a_{n_d}\tilde{S}_s^+)^{k_s}(1 + b_{n_d}\tilde{S}_d^+)^{k_d} e^{\alpha\tilde{S}_s^+ + \beta\tilde{S}_d^+}|\omega\rangle, \tag{26}$$

where

$$a_{n_d} = \frac{4\sqrt{\Delta^2 - g^2}}{g(2N - 2n_d + 1)}, \quad b_{n_d} = \frac{4\sqrt{(\Delta+\epsilon_d)^2 - g^2}}{g(2n_d + 5)}. \tag{27}$$

The corresponding eigen-energy is

$$E^{(k_s,k_d)}_{N,n_d,\nu,L} = E^{(\zeta=1)}_{N,n_d,\nu,L} + 2k_d\sqrt{(\Delta+\epsilon_d)^2 - g^2} + 2k_s\sqrt{\Delta^2 - g^2}. \tag{28}$$

It is obvious that there are additional collective 2-particle and 2-hole excitations of both s- and d-type with energy $\Delta_s = 2\sqrt{\Delta^2 - g^2}$ and $\Delta_d = 2\sqrt{(\Delta+\epsilon_d)^2 - g^2}$, respectively, which are absent in the original U(5) limit ($g = 0$) description. Since the energy needed to excite two more particles from below the closed shell resulting in a configuration with two more particles and two more holes is relatively large, the excited levels with $\zeta > 3$ lie comparatively higher in energy, which are not considered in the following. It is clear that the spectrum generated from (2) with the configuration mixing is still vibrational, of which some low-lying levels of the model with $c = f = 0$ are shown in Fig. 1. In Fig. 1, the first set of levels with $\zeta = 1$ are the same as those generated from the model without configuration mixing, the second set of levels with $\zeta = 2$ are built on the $0_1^+(i)$ level with the gap Δ_s to the ground level, and the third set of levels with $\zeta = 3$ are built on the $0_3^+(i)$ level with the gap Δ_d to the ground level, and so on. Each set of the levels with $\zeta \geq 2$ is a replica of those with $\zeta = 1$ generated from the original U(5) limit without configuration mixing, but there are energy gaps among different sets.

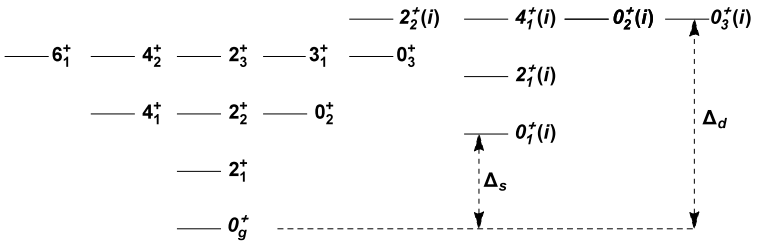

FIG. 1: (Color online) Low-lying level pattern of the solvable configuration mixing U(5) model with the Hamiltonian given by (2) with $c = f = 0$, where the lower left 10 levels belong to $\zeta = 1$ family, which are the same as those generated from the U(5) limit Hamiltonian of the IBM $\hat{H}_0$ only, but with the modified single d-boson energy $\tilde{\epsilon}_d$, the levels labeled with (i) are intruder levels of the model with $\Delta_s = 2\sqrt{\Delta^2 - g^2}$ or $\Delta_d = 2\sqrt{(\Delta+\epsilon_d)^2 - g^2}$, $0_1^+(i)$, $2_1^+(i)$, $0_2^+(i)$, $2_2^+(i)$, $4_2^+(i)$ belong to $\zeta = 2$ family, and $0_3^+(i)$ belongs to $\zeta = 3$ family.

The main feature of the model lies in the fact that the eigenstates (4), (26), *etc* are SU(1,1) coherent states built on the U(5) limit state of the IBM due to the projection (3). A similar bosonic Hamiltonian with $2n\hbar\omega$ excitations built on $0\hbar\omega$ states of a unitary group was discussed in [23]. A comprehensive review on the theory of coherent states

is provided, for example, in [24]. As a consequence, matrix elements of any operator in the model can be derived analytically. For example, one can prove that the norm

$$\mathcal{F}_{\zeta=1,\omega} = \langle\omega|e^{\alpha_1\tilde{S}_s^- + \beta_1\tilde{S}_d^-}\, e^{\alpha\tilde{S}_s^+ + \beta\tilde{S}_d^+}|\omega\rangle = (1-\alpha_1\alpha)^{-N+n_d-\frac{1}{2}}(1-\beta_1\beta)^{-n_d-\frac{5}{2}}, \tag{29}$$

which only depends on n_d for fixed N. It is obvious that the normalization factor $\mathcal{N}_{\zeta=1,\omega} \equiv \mathcal{N}_{n_d} = (\mathcal{F}_{\zeta=1,\omega})^{-1/2}$ with $\alpha_1 = \alpha$ and $\beta_1 = \beta$ of the normal states provided by (29) satisfies $\mathcal{N}_{k_1}\mathcal{N}_{k_2} = \mathcal{N}_{k_3}\mathcal{N}_{k_4}$ if $k_1 + k_2 = k_3 + k_4$. One can also derive

$$\begin{aligned}\langle\zeta=1,\omega|S_s^\pm|\zeta=1,\omega\rangle &= \tfrac{\partial}{\partial\alpha}\mathcal{F}_{\zeta=1,\omega}\big|_{\alpha_1=\alpha,\beta_1=\beta}\,,\\ \langle\zeta=1,\omega|S_d^\pm|\zeta=1,\omega\rangle &= \tfrac{\partial}{\partial\beta}\mathcal{F}_{\zeta=1,\omega}\Big|_{\alpha_1=\alpha,\beta_1=\beta}\,.\end{aligned} \tag{30}$$

Thus, the normalization factor of (26) with $k_d = 0$ can be expressed as

$$\mathcal{N}_{\zeta=2,n_d} = \sqrt{\frac{\alpha(\alpha^2-1)}{a_{n_d}}}\mathcal{N}_{n_d}. \tag{31}$$

Similarly, one can prove

$$\langle\zeta',\omega'|\zeta,\omega\rangle = \delta_{\zeta',\zeta}\,\delta_{\omega',\omega}\,(\mathcal{N}_{\zeta,\omega})^{-2}. \tag{32}$$

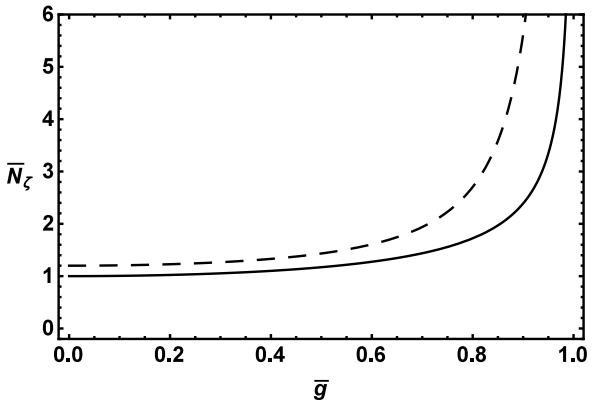

FIG. 2: (Color online) The ground-state expectation value of total number of bosons $\bar{N}_{\zeta=1}$ (solid curve) and that of the first excited intruder 0_1^+(i) state $\bar{N}_{\zeta=2}$ (dashed curve) defined by (34) and (35), respectively, as functions of the mixing parameter $\bar{g} = g/\epsilon_d$, where $N = 10$ and $\bar{\Delta} = \Delta/\epsilon_d = 1$ is taken in the plot.

Due to the configuration mixing, the number operator of total bosons $\hat{N} = \hat{n}_s + \hat{n}_d$ is not a conserved quantity. Actually, neither $\hat{n}_s$ nor $\hat{n}_d$ is conserved when $g \neq 0$. The degree of the configuration mixing may be measured by the expectation value and fluctuation of $\hat{N}$ at the ground state defined as

$$\begin{aligned}\bar{N}_{\zeta=1} &= \tfrac{1}{N}(\zeta=1|\hat{N}|\zeta=1),\\ \delta N_{\zeta=1} &= \tfrac{1}{N}\left((\zeta=1|\hat{N}^2|\zeta=1) - \bar{N}^2\right)^{\frac{1}{2}}\end{aligned} \tag{33}$$

with $n_d = \nu = L = 0$ and $N \neq 0$, where $|\zeta=1) \equiv |\zeta=1,\omega)$ is the normalized ground state. By using (29) and (30), (33) can be expressed as

$$\begin{aligned}\bar{N}_{\zeta=1} &= 1 - \tfrac{2\alpha}{a_0 N} - \tfrac{2\beta}{b_0 N} = 1 - \tfrac{5(\sqrt{(1+\bar{\Delta})^2-\bar{g}^2}-\bar{\Delta}-1)}{2N\sqrt{(1+\bar{\Delta})^2-\bar{g}^2}} - \tfrac{(2N+1)(\sqrt{\bar{\Delta}^2-\bar{g}^2}-\bar{\Delta})}{2N\sqrt{\bar{\Delta}^2-\bar{g}^2}},\\ \delta N_{\zeta=1} &= \tfrac{2}{N}\sqrt{\tfrac{\alpha}{a_0(\alpha^2-1)} + \tfrac{\beta}{b_0(\beta^2-1)}}.\end{aligned} \tag{34}$$

It can be expected that the mixing is mainly driven by the mixing term in the Hamiltonian (2). When $g = 0$, $\bar{N}_{\zeta=1} = 1$ and $\delta N_{\zeta=1} = 0$ indicating that the ground state of the system is in the U(5) limit without configuration mixing. With the increasing of $g > 0$, the mixing occurs with $\bar{N}_{\zeta=1} > 1$ and $\delta N_{\zeta=1} > 0$. We observe that both $\bar{N}_{\zeta=1}$ and $\delta N_{\zeta=1}$ increase with the increasing of g, but the increasing in $\bar{N}_{\zeta=1}$ with the increasing of g is more noticeable. The expectation value of the total number of bosons of the first excited intruder state with $n_d = \nu = L = 0$ corresponding to 0_1^+(i) state with $\zeta = 2$ shown in the caption of Fig. 1 can be derived similarly, which is given by

$$\bar{N}_{\zeta=2} = \tfrac{1}{N}(\zeta = 2|\hat{N}|\zeta = 2) = \\ \tfrac{N+2}{N} - \tfrac{2\beta}{Nb_0} + \tfrac{2\alpha^2(\alpha^2-1)}{Na_0}\left(\tfrac{1}{\alpha(\alpha^2-1)} + \tfrac{1}{a_0} + \tfrac{\alpha}{\alpha^2-1} - (\alpha + \tfrac{2a_0}{\alpha^2-1})(\tfrac{1}{\alpha a_0} + \tfrac{1}{\alpha^2-1})\right). \quad (35)$$

In this case $g \neq 0$ should be assumed. When $g \to 0$, $\bar{N}_{\zeta=2} \to 1 + 2/N$ indicating that the collective 2p-2h excitation component is dominating in the 0_1^+(i) state. With the increasing of $g > 0$, the mixing of $[N + 2n]$ configurations with $n \geq 2$ in this intruder state occurs with $\bar{N}_{\zeta=2} > 1 + 2/N$. Since the increasing in $\bar{N}_\zeta$ with the increasing of g is more noticeable, only $\bar{N}_\zeta$ with $\zeta = 1$ and $\zeta = 2$ are shown in Fig. 2. As shown in Fig. 2, the ground state with $\zeta = 1$ is dominated by the $[N]$ configuration, while 0_1^+(i) state with $\zeta = 2$ is dominated by the $[N + 2]$ configuration, as long as the mixing parameter is small with $g \sim 0$. In addition, the $O(5)$ and $O(3)$ invariants $C_2(O(5))$ and $\hat{L} \cdot \hat{L}$ added in $\hat{H}_0$ only affect the level energies, but do not alter the eigenstates of the model. Thus, these quantities do not affect expectation values of $\hat{N}$ under any eigenstates and matrix elements of other operators in the model.

As shown in [5–8, 12–18], effective boson charge needs to be taken with different values for different multi-particle-hole configurations. The E2 operator used in this work is simply chosen as

$$T_\mu(\text{E2}) = q_2\hat{P}_N\left(d_\mu^\dagger s + s^\dagger\tilde{d}_\mu\right)\hat{P}_N + q_2'\hat{P}\left(d_\mu^\dagger s + s^\dagger\tilde{d}_\mu\right)\hat{P}, \quad (36)$$

where $\hat{P}_N$ is a projection onto the configuration without multi-particle-hole excitations, with which the B(E2) values are given by

$$\text{B(E2};\omega_i, L_i \to \omega_f, L_f) = \frac{2L_f+1}{2L_i+1}(\mathcal{N}_{\omega_f}\mathcal{N}_{\omega_i})^2|\langle\omega_f, L_f\|T(\text{E2})\|\omega_i, L_i\rangle|^2, \quad (37)$$

where the reduced matrix element is defined as $\langle\omega_f, L_f||\hat{I}||\omega_i, L_i\rangle = \delta_{\omega_f,\omega_i}\delta_{L_f,L_i}$ with unit identity operator $\hat{I}$. The selection rules of the E2 transition are still those in the U(5) limit of the IBM with $\delta n_d = n_{d,f} - n_{d,i} = \pm 1$, $\delta\nu = \nu_f - \nu_i = \pm 1$.

The SU(1,1) coherent state formulism shown above is also helpful to evaluate reduced matrix elements of T(E2). For example, the reduced matrix element of the second part of T(E2) between normal states can be expressed as

$$\langle\zeta = 1, \omega'||\hat{P}(s^\dagger\tilde{d} + d^\dagger s)\hat{P}||\zeta = 1, \omega\rangle = \\ \langle N, n_d', \nu', \rho', L'||e^{\alpha\tilde{S}_s^- + \beta\tilde{S}_d^-}(s^\dagger\tilde{d} + d^\dagger s)e^{\alpha\tilde{S}_s^+ + \beta\tilde{S}_d^+}||N, n_d, \nu, \rho, L\rangle = \\ \langle N, n_d', \nu', \rho', L'||e^{\alpha\tilde{S}_s^- + \beta\tilde{S}_d^-}e^{\alpha\tilde{S}_s^+ + \beta\tilde{S}_d^+}(s^\dagger\tilde{d} + d^\dagger s + (\alpha+\beta)d^\dagger s^\dagger)||N, n_d, \nu, \rho, L\rangle, \quad (38)$$

where the Hausdorff-Campbell relation is applied. In the following, we only provide an example of (38) with $n_d' = n_d - 1$. In this case,

$$\langle N, n_d - 1, \nu', \rho', L'||e^{\alpha\tilde{S}_s^- + \beta\tilde{S}_d^-}(s^\dagger\tilde{d} + d^\dagger s)e^{\alpha\tilde{S}_s^+ + \beta\tilde{S}_d^+}||N, n_d, \nu, \rho, L\rangle = \\ \langle N, n_d - 1, \nu', \rho', L'||e^{\alpha\tilde{S}_s^- + \beta\tilde{S}_d^-}e^{\alpha\tilde{S}_s^+ + \beta\tilde{S}_d^+}||N, n_d - 1, \nu', \rho', L'\rangle\langle N, n_d - 1, \nu', L'||s^\dagger\tilde{d}||N, n_d, \nu, \rho, L\rangle + \\ (\alpha+\beta)\langle N, n_d - 1, \nu', \rho', L'||e^{\alpha\tilde{S}_s^- + \beta\tilde{S}_d^-}e^{\alpha\tilde{S}_s^+ + \beta\tilde{S}_d^+}||N + 2, n_d + 1, \nu', \rho', L'\rangle \times \\ \langle N + 2, n_d + 1, \nu', L'||d^\dagger s^\dagger||N, n_d, \nu, \rho, L\rangle. \quad (39)$$

Using the identity

$$|N + 2, n_d + 1, \nu, \rho, L, M\rangle = \sqrt{\frac{4}{(n_d - \nu + 1)(n_d + \nu + 4)}}S_d^+|N, n_d - 1, \nu, \rho, L, M\rangle, \quad (40)$$

we have

$$\langle N, n_d-1, \nu', \rho', L'||e^{\alpha\tilde{S}_s^- +\beta\tilde{S}_d^-}\, e^{\alpha\tilde{S}_s^+ +\beta\tilde{S}_d^+}||N+2, n_d+1, \nu', \rho', L'\rangle = \sqrt{\frac{4}{(n_d-\nu'+1)(n_d+\nu'+4)}}\langle N, n_d-1, \nu', \rho', L'||e^{\alpha\tilde{S}_s^- +\beta\tilde{S}_d^-}\, e^{\alpha\tilde{S}_s^+ +\beta\tilde{S}_d^+}\, S_d^+||N, n_d-1, \nu', \rho', L'\rangle. \quad (41)$$

By using (30), (41) can be simplified as

$$\langle N, n_d-1, \nu', \rho', L'||e^{\alpha\tilde{S}_s^- +\beta\tilde{S}_d^-}\, e^{\alpha\tilde{S}_s^+ +\beta\tilde{S}_d^+}||N+2, n_d+1, \nu', \rho', L'\rangle = -\frac{1}{b_{n_d-1}}\sqrt{\frac{4}{(n_d-\nu'+1)(n_d+\nu'+4)}}\langle N, n_d-1, \nu', \rho', L'||e^{\alpha\tilde{S}_s^- +\beta\tilde{S}_d^-}\, e^{\alpha\tilde{S}_s^+ +\beta\tilde{S}_d^+}||N, n_d-1, \nu', \rho', L'\rangle. \quad (42)$$

Hence, (39) can be rewritten as

$$\langle \zeta=1, N, n_d-1, \nu', \rho', L'||\hat{P}(s^\dagger\tilde{d}+d^\dagger s)\hat{P}||\zeta=1, N, n_d, \nu, \rho, L\rangle = \frac{1}{(\mathcal{N}^{\zeta=1}_{N,n_d-1})^2}\times \left(\langle N, n_d-1, \nu', \rho', L'||s^\dagger\tilde{d}||N, n_d, \nu, \rho, L\rangle - \frac{2(\alpha+\beta)/b_{n_d-1}}{\sqrt{(n_d-\nu'+1)(n_d+\nu'+4)}}\langle N+2, n_d+1, \nu', \rho', L'||d^\dagger s^\dagger||N, n_d, \nu, \rho, L\rangle\right). \quad (43)$$

On the right-hand-side of (43), the first term within the parentheses is the reduced matrix element of $T(E2)$ in the original U(5) limit of the IBM, while the second term is due to the intruder configuration mixing, which is always on even only $[N]\oplus[N+2]$ configuration mixing is considered. Therefore, the only difference of a configuration mixing scheme with finite number of configurations from that with different or infinite number of configurations is in the normalization factors shown in (37) and those in matrix elements similar to (43).

By using the results of Winger-Racah calculus for the $U(5)$ limit of the IBM, the reduced matrix elements shown in (43) can further be expressed as

$$\langle \zeta, k_s, k_d=0, N, n_d-1, \nu-1, \rho', L'||\hat{P}(s^\dagger\tilde{d}+d^\dagger s)\hat{P}||\zeta, k_s, k_d=0, N, n_d, \nu, \rho, L\rangle = \frac{(-1)^{L'-L}}{(\mathcal{N}^{\zeta,k_s}_{N,n_d-1,\nu-1})^2}\left(\frac{(N-n_d+1)\nu(2L+1)}{(n_d+\nu+3)(2\nu+3)(2L'+1)}\right)^{\frac{1}{2}}\left(n_d+\nu+3-\frac{2(\alpha+\beta)}{b_{n_d-1}}\right)\langle \nu-1, \rho', L';\ 1,2||\nu, \rho, L\rangle, \quad (44)$$

$$\langle \zeta, k_s, k_d=0, N, n_d-1, \nu+1, \rho', L'||\hat{P}(s^\dagger\tilde{d}+d^\dagger s)\hat{P}||\zeta, k_s, k_d=0, N, n_d, \nu, \rho, L\rangle = \frac{1}{(\mathcal{N}^{\zeta,k_s}_{N,n_d-1,\nu+1})^2}\left(\frac{(N-n_d+1)(\nu+1)}{(n_d-\nu)(2\nu+5)}\right)^{\frac{1}{2}}\left(n_d-\nu-\frac{2(\alpha+\beta)}{b_{n_d-1}}\right)\langle \nu, \rho, L;\ 1,2||\nu+1, \rho', L'\rangle, \quad (45)$$

for $\zeta=1$, $k_s=0$ or $\zeta=2$, $k_s=1$, where $\langle \nu, \rho, L;\ 1,2||\nu+1, \rho', L'\rangle$ is the elementary Wigner coefficient (isoscalar factor) of $O(5)\supset O(3)$, which was shown, for example, in [25].

However, due to the projection $\hat{P}_N$, the reduced matrix elements of the first part of T(E2) is simply given by

$$\langle \zeta', N, n_d', \nu', \rho', L'||\hat{P}_N(s^\dagger\tilde{d}+d^\dagger s)\hat{P}_N||\zeta, N, n_d, \nu, \rho, L\rangle = \langle N, n_d', \nu', \rho', L'||(s^\dagger\tilde{d}+d^\dagger s)||N, n_d, \nu, \rho, L\rangle, \quad (46)$$

which is independent of ζ' and ζ, and equal to the reduced matrix element of T(E2) in the original U(5) limit.

According to (37) and the evaluation procedure of the reduced matrix elements of T(E2) shown above, B(E2) values can be expressed as

$$\mathrm{B(E2};\zeta=1, n_d, \nu, \rho, L_i\to\zeta=1, n_d-1, \nu-1, \rho', L_f) = (q_2')^2\frac{(N-n_d+1)(n_d+\nu+3)\nu}{2\nu+3}\left(\frac{\mathcal{N}_{n_d}}{\mathcal{N}_{n_d-1}}\right)^2\left(1-\frac{2(\alpha+\beta)}{(n_d+\nu+3)\,b_{n_d-1}}+\lambda\mathcal{N}^2_{n_d-1}\right)^2\langle \nu-1, \rho', L_f;\ 1,2||\nu, \rho, L_i\rangle^2, \quad (47)$$

$$\mathrm{B(E2};\zeta=1, n_d, \nu, \rho, L_i\to\zeta=1, n_d-1, \nu+1, \rho', L_f) = (q_2')^2\frac{(N-n_d+1)(n_d-\nu)(\nu+1)(2L_f+1)}{(2\nu+5)(2L_i+1)}\left(\frac{\mathcal{N}_{n_d}}{\mathcal{N}_{n_d-1}}\right)^2\left(1-\frac{2(\alpha+\beta)}{(n_d-\nu)\,b_{n_d-1}}+\lambda\mathcal{N}^2_{n_d-1}\right)^2\langle \nu, \rho, L_i;\ 1,2||\nu+1, \rho', L_f\rangle^2, \quad (48)$$

$$\mathrm{B(E2};\zeta=2,n_d,\nu,\rho,L_i\to\zeta=2,n_d-1,\nu-1,\rho',L_f)$$
$$=(q_2')^2\tfrac{(N-n_d+1)(n_d+\nu+3)\nu}{2\nu+3}(\tfrac{\mathcal{N}_{n_d}}{\mathcal{N}_{n_d-1}})^2\tfrac{a_{n_d}}{a_{n_d-1}}\times$$
$$(1-\tfrac{2(\alpha+\beta)}{(n_d+\nu+3)\,b_{n_d-1}}+\lambda\mathcal{N}^2_{n_d-1}\tfrac{\alpha(\alpha^2-1)}{a_{n_d}})^2\langle\nu-1,\rho',L_f;\,1,2\|\nu,\rho,L_i\rangle^2, \tag{49}$$

$$\mathrm{B(E2};\zeta=2,n_d,\nu,\rho,L_i\to\zeta=2,n_d-1,\nu+1,\rho',L_f)=$$
$$(q_2')^2\tfrac{(N-n_d+1)(n_d-\nu)(\nu+1)(2L_f+1)}{(2\nu+5)(2L_i+1)}(\tfrac{\mathcal{N}_{n_d}}{\mathcal{N}_{n_d-1}})^2\times$$
$$\tfrac{a_{n_d}}{a_{n_d-1}}(1-\tfrac{2(\alpha+\beta)}{(n_d-\nu)\,b_{n_d-1}}+\lambda\mathcal{N}^2_{n_d-1}\tfrac{\alpha(\alpha^2-1)}{a_{n_d}})^2\langle\nu,\rho,L_i;\,1,2\|\nu+1,\rho',L_f\rangle^2, \tag{50}$$

$$\mathrm{B(E2};\zeta=2,n_d+1,\nu+1,\rho,L_i\to\zeta=1,n_d,\nu,\rho',L_f)=$$
$$(q_2')^2\tfrac{\alpha(\alpha^2-1)}{a_{n_d+1}}\tfrac{(N-n_d)(n_d+\nu+5)(\nu+1)}{2\nu+5}(\tfrac{\mathcal{N}_{n_d}}{\mathcal{N}_{n_d+1}})^2\times$$
$$(1-\tfrac{a_{n_d}}{a_{n_d+1}}-\tfrac{2(\alpha+\beta)}{(N-n_d)\alpha\,a_{n_d+1}}(1-\tfrac{\alpha}{2a_{n_d}})+\lambda\mathcal{N}^2_{n_d+1})^2\langle\nu,\rho',L_f;\,1,2\|\nu+1,\rho,L_i\rangle^2, \tag{51}$$

where $\lambda=q_2/q_2'$. Some nonzero B(E2) values for the transitions among low-lying states of the model are shown in Table I. It can be verified that the transitions among both normal and intruder states are similar to those in the U(5) limit without configuration mixing, of which the relative strength depends on the mixing parameter g and the effective charge ratio $\lambda=q_2/q_2'$.

TABLE I: Some nonzero B(E2; $L_i\to L_f$) values for the transitions among low-lying states of the model, where $\lambda=q_2/q_2'$.

Transition	B(E2; $L_i\to L_f$)/$(q_2')^2$
$2_1^+\to 0_g^+$	$N(\frac{\mathcal{N}_1}{\mathcal{N}_0})^2(1-\frac{2(\alpha+\beta)}{5b_0}+\lambda\mathcal{N}_0^2)^2$
$4_1^+\to 2_1^+$	$2(N-1)(\frac{\mathcal{N}_2}{\mathcal{N}_1})^2(1-\frac{2(\alpha+\beta)}{7b_1}+\lambda\mathcal{N}_1^2)^2$
$2_1^+(i)\to 0_1^+(i)$	$N(\frac{\mathcal{N}_1}{\mathcal{N}_0})^2\frac{a_1}{a_0}(1-\frac{2(\alpha+\beta)}{5b_0}+\mathcal{N}_0^2\alpha(\alpha^2-1)\frac{\lambda}{a_1})^2$
$4_1^+(i)\to 2_1^+(i)$	$2(N-1)(\frac{\mathcal{N}_2}{\mathcal{N}_1})^2\frac{a_2}{a_1}(1-\frac{2(\alpha+\beta)}{7b_1}+\mathcal{N}_1^2\alpha(\alpha^2-1)\frac{\lambda}{a_2})^2$
$0_1^+(i)\to 2_1^+$	$5N\frac{\alpha(\alpha^2-1)}{a_0}(\frac{\mathcal{N}_1}{\mathcal{N}_0}(1-\frac{2(\alpha+\beta)}{5b_0})(1-\frac{a_1}{a_0})+\lambda\mathcal{N}_0\mathcal{N}_1)^2$
$2_1^+(i)\to 0_g^+$	$N\frac{\alpha(\alpha^2-1)}{a_1}(\frac{\mathcal{N}_0}{\mathcal{N}_1}(1-\frac{a_0}{a_1}-\frac{1+\beta/\alpha}{a_1N}(1-\frac{\alpha}{2a_0}))+\lambda\mathcal{N}_0\mathcal{N}_1)^2$
$4_1^+(i)\to 2_1^+$	$2(N-1)\frac{\alpha(\alpha^2-1)}{a_2}(\frac{\mathcal{N}_1}{\mathcal{N}_2}(1-\frac{a_1}{a_2}-\frac{1+\beta/\alpha}{a_2(N-1)}(1-\frac{\alpha}{2a_1}))+\lambda\mathcal{N}_1\mathcal{N}_2)^2$

It can now be understood that the differences of the mixing scheme with infinite number of configurations considered in this work from those with finite number of configurations studied previously are in two aspects. Firstly, the eigen-energies and eigen-states of the two schemes are quite the same, but the quantum numbers k_s and k_d shown in (26) should be terminated in the latter. For example, when $m+1$ configurations $[N]\oplus[N+2]\oplus[N+4]\oplus\cdots\oplus[N+2m]$ are adopted, the projection $\hat{P}$ should be restricted by another additional condition with $\hat{P}|N',\omega\rangle=0$ for $N'>N+2m$. Hence, the term $\exp[\alpha S_s^+ + \beta S_d^+]$ in (4) and (26) in this case is equivalent to its Taylor expansion in terms of α and β with $m+1$ terms due to the projection, resulting in a system with finite number of s- and d-type of collective 2-particle and 2-hole excitations. Secondly, the Taylor expansion in terms of α and β for the norm $\mathcal{N}$ with $m+1$ terms in the current scheme is equivalent to that in the mixing scheme with finite number of configurations if the same form of the eigenstates is adopted. Since physical quantities, such as B(E2) values, are related to the norms involved, the results of the Taylor expansion for the norms involved with $m+1$ terms are equivalent to those of the mixing scheme with $m+1$ configurations, though the results may be a little different due to the finite-term cutoff. Since $g/\Delta<1$ is always satisfied, there is no much difference of the exact results provided in this work from those with the finite-term Taylor expansions because $\alpha\sim -g/(2\Delta)$ and $\beta\sim -g/(2(\Delta+\epsilon_d))$ are small quantities, which justifies that the configuration mixing scheme with infinite number of configurations is an acceptable approximation.

Though the above SU(1,1) coherent state technique is applied to the U(5) limit case in this paper, it can also be applied to more general cases, for example, by replacing $\hat{H}_0$ used in (2) with configuration dependent consistent-Q IBM

Hamiltonian (1) in the same configuration mixing scheme because, for fixed N, the basis vectors $\{|N, n_d, \nu, \rho, L, M\rangle\}$ of the U(5) limit form a complete set. Then, the Hamiltonian may be diagonalized under the SU(1,1) coherent states $\{\exp[\alpha\tilde{S}_s^+ + \beta\tilde{S}_d^+]|N, n_d, \nu, \rho, L, M\rangle\}$, where the parameters α and β may be determined according to the first and second intruder band heads of the model similarly. With such extension, the model, however, can only be solved numerically.

3. Model applications to ^{108,110}Cd: Even-even Cd isotopes are generally regarded as good examples of U(5)-limit nuclei. However, there are additional 0^+ and 2^+ levels near the two-phonon triplet, which can not be described by the IBM without configuration mixing. These intruder states have been interpreted as arising from proton excitations across the $Z = 50$ shell leading to 2p-4h configurations [12, 26], in which the configuration mixing based on the IBM was adopted to describe ^{112,114,118}Cd. Similar configuration mixing schemes for even-even Cd isotopes, especially for ^{112,114}Cd, were also considered in [13, 27–29]. In fact, in order to describe the shape evolution in even-even Cd isotopes, a consistent-Q Hamiltonian mainly along the U(5)-O(6) transitional leg of the Casten triangle should be adopted [13, 27–29], because the γ-soft configurations are inevitable in most vibrational nuclei, especially with the increasing of the boson number N. Since the present model can only be used to describe U(5) limit nuclei, ^{108,110}Cd seem to be the suitable candidates. As shown in [30], ^{108}Cd was considered as a good U(5) limit candidate though E2 decay rates from its normal three-phonon states have not been determined experimentally.

The level energies of the intrude states $0_1^+(i)$, $2_1^+(i)$, and $4_1^+(i)$ of ^{108}Cd deduced in [30] are used, while the other level energies of the intrude states are predicted in this model. The model parameters are determined from a best global fit to the experimental level energies and B(E2) values concerned, from which we get $\Delta = 0.8635$MeV, $g = 0.0782$MeV, $\epsilon_d = 0.6305$MeV, $c = 0.0200$MeV, $f = 0.0010$MeV, and $\lambda = q_2/q_2' = 0.89$. Some low-lying level energies and B(E2) ratios of ^{108}Cd are shown in Table II, where level energies of the normal four-phonon states and above are not shown, and only experimentally determined B(E2) values are provided except $B(E2; 0_1^+(i) \to 2_1^+)$, which demonstrates that both low-lying level energies and B(E2) ratios of ^{108}Cd are fitted by this simple model approximately. Particularly, the E2 decays out of the intruder band are indeed very week [30] as predicted by the model, and the transitions among intruder states are weaker than those among the normal states. In addition, the B(E2) ratio $R_{4,2} = B(E2, 4_1^+ \to 2_1^+)/B(E2, 2_1^+ \to 0_g^+)$ of the normal ($\zeta = 1$) band keeps almost the same as that in the U(5) limit without configuration mixing. As shown In Table II, $R_{4,2} = 1.6688$ in this work, while $R_{4,2} = 1.6667$ in the U(5) limit without configuration mixing. It is also shown that the transitions with $\delta\nu = \pm 2$ or 0 are also experimentally allowed, while they are forbidden in this simple model, which indicates that the O(5)-symmetry is a little broken in ^{108}Cd. Moreover, the assignments of relatively higher-lying intruder states $0_2^+(i)$, $0_3^+(i)$, and $2_2^+(i)$ need to be confirmed because their E2 decay rates were not determined experimentally. It should be noted that B(E2) values of the transitions among intruder states and those from intruder states to the normal states are taken from [30], while those among normal states are taken from [31].

Though there are inevitable γ-soft configurations, this simple model is also applied to describe low-lying level energies up to the three phonon states and related known B(E2) ratios of ^{110}Cd. From the best global fit to the experimental level energies and B(E2) values for ^{110}Cd concerned, we get $\Delta = 0.7850$MeV, $g = 0.5493$MeV, $\epsilon_d = 0.5800$MeV, $c = -0.0080$MeV, $f = 0.0096$MeV, and $\lambda = q_2/q_2' = 1.99$ for ^{110}Cd. The fitting results are shown in Table III, in which the related B(E2) ratios calculated from the IBM-2 with configuration mixing [33] are also provided. Actually, there are two schemes with or without strong mixing between the phonon and intruder configurations adopted in the IBM-2 calculations [33]. Since there is no much difference in the fitting quality of the mixed scheme form that of the unmixed one as concluded in [33], only the results of the mixed scheme are provided in Table III. Relative B(E2) ratios of transitions from the three-phonon 0_3^+ state shown in [33] are not included. It can be observed from Table III that the fitting quality of the IBM-2 with configuration mixing [33] for the B(E2) ratios of ^{110}Cd is a little better than that of this simple model. It can be seen that there are obvious discrepancies in the B(E2) ratios calculated from this model. Firstly, the increasing in the B(E2) value along the normal yrast states is a little larger than the corresponding experimental data. Secondly, the B(E2) value for the transition $0_2^+ \to 2_1^+$ is at least 9.25 times too large, while that for 4_1^+(i) $\to 2_1^+$(i) is about 1.93 times smaller than the corresponding experimental value. Discrepancies in some other B(E2) values are also noticeable. It can be expected that the O(6) content in both normal and intruder bands are not negligible in ^{110}Cd as noted in [33]. Moreover, the value of the mixing parameter adopted indicates that the configuration mixing in ^{110}Cd is stronger. Taking the parameter Δ and the T(E2) operator to be configuration-dependent in the model as previously did [5–8, 12–18] is also possible. To implement an additional O(6) term in (2) would certainly improve the quality of fits, of which the solution, however, will be complicated.

TABLE II: Some low-lying level energies (the upper part) and B(E2) ratios $B(E2;L_i \to L_f)/B(E2;2_1^+ \to 0_g^+)$ (the lower part) of ^{108}Cd, where $*$ indicates that the corresponding spin assignment is not fully confirmed, and – denotes undetermined experimental value. The spin of both $0_2^+(i)$ and $0_3^+(i)$ states was assigned with $(0^+, 1^+, 2^+)$ as shown in [31]. The model parameters are taken as $\Delta = 0.8635$MeV, $g = 0.0782$MeV, $\epsilon_d = 0.6305$MeV, $c = 0.0200$MeV, $f = 0.0010$MeV, and $\lambda = q_2/q_2' = 0.89$.

Level energy (MeV)	This work	Exp. [30, 31]
$E(2_1^+)$	0.718	0.632
$E(4_1^+)$	1.484	1.508
$E(2_2^+)$	1.470	1.601
$E(0_2^+)$	1.264	1.913
$E(3_1^+)$	2.268	2.146
$E(4_2^+)$	2.276	2.239
$E(0_3^+)$	1.896	2.375
$E(2_3^+)$	1.982	2.486
$E(6_1^+)$	2.298	2.541
$E(0_1^+(i))$	1.720	1.720
$E(2_1^+(i))$	2.438	2.163
$E(4_1^+(i))$	3.084	2.739
$E(2_2^+(i))$	3.070	2.366
$E(0_2^+(i))$	2.984	2.740$*$
$E(0_3^+(i))$	2.984	2.936$*$
Transition	This work	Exp. [30, 31]
$4_1^+ \to 2_1^+$	1.6688	1.5639
$2_2^+ \to 2_1^+$	1.6688	0.6579
$2_2^+ \to 0_g^+$	0	0.0676
$2_1^+(i) \to 0_1^+(i)$	0.3428	≥ 0.338
$4_1^+(i) \to 2_1^+(i)$	0.5901	≥ 0.226
$0_1^+(i) \to 2_1^+$	0.0091	–
$2_1^+(i) \to 0_g^+$	0.0029	≥ 0.002
$4_1^+(i) \to 2_1^+$	0.0043	≥ 0.005
$2_1^+(i) \to 2_1^+$	0	≥ 0.015
$4_1^+(i) \to 4_1^+$	0	≥ 0.015

TABLE III: Some low-lying level energies (the upper part) and B(E2) ratios $B(E2;L_i \to L_f)/B(E2;2_1^+ \to 0_g^+)$ (the lower part) of ^{110}Cd, where $^\sharp$ indicates the corresponding level energy being chosen to be fitted, and $*$ indicates that the corresponding spin assignment is not fully confirmed, and – indicates that the quantity was not provided in [33]. The spin of the $0_3^+(i)$ state was assigned with $(0^+, 2^-)$ as shown in [32]. The model parameters are taken as $\Delta = 0.7850$MeV, $g = 0.5493$MeV, $\epsilon_d = 0.5800$MeV, $c = -0.0080$MeV, $f = 0.0096$MeV, and $\lambda = q_2/q_2' = 1.99$.

Level energy (MeV)	This work	Exp. [32, 33]	
$E(2_1^+)$	0.726	0.658	
$E(2_2^+)$	1.378	1.476	
$E(4_1^+)$	1.512	1.542	
$E(0_2^+)$	1.400	1.731	
$E(0_3^+)$	1.956	2.079	
$E(3_1^+)$	2.071	2.163	
$E(4_2^+)$	2.148	2.220	
$E(2_3^+)$	2.126	2.355	
$E(6_1^+)$	2.359	2.479	
$E(0_1^+(i))$	1.081	1.473	
$E(2_1^+(i))$	1.807	1.783	
$E(4_1^+(i))$	2.641	2.251	
$E(2_2^+(i))$	2.507	2.288	
$E(0_2^+(i))$	2.481	2.332	
$E(0_3^+(i))$	2.481	2.405$*$	
Transition	This work	IBM-2 [33]	Exp. [33]
$4_1^+ \to 2_1^+$	1.861	1.593	1.549
$2_2^+ \to 2_1^+$	1.861	1.333	0.701
$2_2^+ \to 0_g^+$	0	0.007	0.025
$0_2^+ \to 2_2^+$	0	2.037	< 62
$0_2^+ \to 2_1^+$	2.703	0.041	< 0.292
$3_1^+ \to 2_1^+(i)$	0	0.016	0.185
$3_1^+ \to 4_1^+$	0.727	0.481	0.089, 1.440
$3_1^+ \to 2_2^+$	1.818	1.370	0.838
$3_1^+ \to 2_1^+$	0	0.008	0.031
$4_2^+ \to 4_1^+$	1.212	0.815	0.395
$4_2^+ \to 2_2^+$	1.333	1.296	0.812
$4_2^+ \to 2_1^+$	0	0.007	0.005
$4_2^+ \to 2_1^+(i)$	0	1.630	0.018
$6_1^+ \to 4_1^+(i)$	1.061	0.244	1.328
$6_1^+ \to 4_2^+$	0	0.096	0.185
$6_1^+ \to 4_1^+$	2.546	2.000	2.288
$2_3^+ \to 0_2^+$	1.242	0.593	0.893
$2_3^+ \to 4_1^+$	1.418	0.167	< 0.185
$2_3^+ \to 2_2^+$	0.788	0.031	0.026
$2_3^+ \to 2_1^+$	0	0.001	0.118, 0.0003
$2_3^+ \to 0_1^+(i)$	0	0.032	< 0.070
$2_3^+ \to 2_1^+(i)$	0	0.852	< 0.185
$2_1^+(i) \to 0_1^+(i)$	1.209	2.185	1.070
$0_1^+(i) \to 2_1^+$	0.192	1.148	$<$1.476
$2_1^+(i) \to 0_g^+$	0.099	–	0.010
$2_1^+(i) \to 2_1^+$	0	0.0002	0.247, 0.012
$2_1^+(i) \to 2_2^+$	1.388	0.370	< 0.295
$4_1^+(i) \to 2_1^+(i)$	2.193	3.963	4.244
$4_1^+(i) \to 2_1^+$	0.065	0.008	0.005
$4_1^+(i) \to 4_1^+$	0	0.044	0.066
$4_1^+(i) \to 2_2^+$	0	0.002	0.044

4. Summary: An intruder configuration mixing scheme with $2n$-particle and $2n$-hole configurations from $n = 0$ up to $n \to \infty$ in the U(5) (vibrational) limit of the interacting boson model is proposed. A simple Hamiltonian suitable to describe the configuration mixing turns to be exactly solvable, which may be helpful in the analysis of experimental data of vibrational nuclei near the closed shells, because physical quantities in the model can be expressed analytically. As examples of the model application, low-lying level energies and experimentally known B(E2) ratios of ^{108,110}Cd are fitted. As far as the lower energy region is concerned, the fitting results show that the U(5) based SU(1,1) symmetry seems dominating in ^{108,110}Cd, especially in the low-lying intruder states. Since this simple model is only suitable to describe U(5) limit nuclei, the fitting quality for ^{108}Cd is better than that for ^{110}Cd, because O(6) configuration mixing, especially in the intruder bands, becomes stronger with the increasing of the total boson number N. Therefore, the model may also be applicable to other vibrational nuclei near closed shells, such as those among Ru and Pd isotopes. Finally, the SU(1,1) coherent state technique proposed in this work may also be useful in diagonalizing a more general configuration dependent consistent-Q IBM Hamiltonian in the same configuration mixing scheme though only numerical solution is possible in such a case, which will be considered in our future work.

Acknowledgments

Support from the National Natural Science Foundation of China (11675071 and 11375080), the U. S. National Science Foundation (ACI-1516338), U. S. Department of Energy (DE-SC0005248), the Southeastern Universities Research Association, the China-U. S. Theory Institute for Physics with Exotic Nuclei (CUSTIPEN) (DE-SC0009971), and the LSU–LNNU joint research program (9961) is acknowledged.

[1] F. Iachello and A. Arima, The Interacting Boson Model (Cambridge University Press, Cambridge, 1987).
[2] K. Heyde, P. Van Isacker, M. Waroquier, J. L. Wood, and R. A. Meyer, Phys. Rep. **102** (1983) 291.
[3] J. L. Wood, K. Heyde, W. Nazarewicz, M. Huyse, and P. Van Duppen, Phys. Rep. **215** (1992) 101.
[4] K. Heyde and J. L. Wood, Rev. Mod. Phys. **83** (2011) 1467; 1655.
[5] P. D. Duval and B. R. Barrett, Phys. Lett. B **100** (1981) 223.
[6] P. D. Duval and B. R. Barrett, Nucl. Phys. A **376** (1982) 213.
[7] K. Heyde, P. Van Isacker, J. Jolie, J. Moreau and M. Waroquier, Phys. Lett. B **132** (1983) 15.
[8] K. Heyde, J. Jolie, P. Van Isacker, J. Moreau and M. Waroquier, Phys. Rev. C **29** (1984) 1428.
[9] K. Heyde, P. Van Isacker, R. F. Casten, and J. L. Wood, Phys. Lett. B **155** (1985) 303.
[10] P. Van Isacker, S. Pittel, A. Frank, and P. D. Duval, Nucl. Phys. A **451** (1986) 202.
[11] K. Heyde, C. De Coster, J. Jolie, and J. L. Wood, Phys. Rev. C **46** (1992) 541.
[12] K. Heyde, P. Van Isacker, M. Waroquier, G. Wenes, and M. Sambataro, Phys. Rev. C **25** (1982) 3160.
[13] H. Lehmann, J. Jolie, C. De Coster, B. Decroix, K. Heyde, J. L. Wood, Nucl. Phys. A **621** (1997) 767.
[14] R. Fossion, K. Heyde, G. Thiamova, and P. Van Isacker, Phys. Rev. C **67** (2003) 024306.
[15] V. Hellemans, R. Fossion, S. D. Baerdemacker, and K. Heyde, Phys. Rev. C **71** (2005) 034308.
[16] J. E. García-Ramos, V. Hellemans, and K. Heyde, Phys. Rev. C **84** (2011) 014331.
[17] K. Nomura, T. Otsuka, and P. Van Isacker, J. Phys. G: Nucl. Part. **43** (2016) 024008.
[18] T. Thomas, V. Werner, J. Jolie, K. Nomura, T. Ahn, N. Cooper, H. Duckwitz, A. Fitzler, C. Fransen, A. Gade, M. Hinton, G. Ilie, K. Jessen, A. Linnemann, P. Petkov, N. Pietralla, D. Radeck, Nucl. Phys. A **947** (2016) 203.
[19] M. K. Harder, K. T. Tanga, P. Van Isacker, Phys. Lett. B **405**(1997) 25.
[20] J. Jolie, F. Corminboeuf, N. Warr, and S. W. Yates, Nucl. Phys. A **690** (2001) 259c.
[21] G. Rosensteel and D. J. Rowe Phys. Rev. Lett. **38** (1977) 10.
[22] T. Dytrych, K. D. Sviratcheva, C. Bahri, J. P. Draayer, and J. P. Vary, Phys. Rev. Lett. **98** (2007) 162503.
[23] F. Pan and J. P. Draayer, J. Phys. A: Math. Gen. **34** (2001) 2637.
[24] J. R. Klauder and B. S. Skagerstam, Coherent States (World Scientific, Singapore, 1995)
[25] F. Pan, L. Bao, Y.-Z. Zhang, and J. P. Draayer, Eur. Phys. Plus **129** (2014) 169.
[26] A. Aprahamian, D. S. Brenner, R. F. Casten, R. L. Gill, and A. Piotrowski, Phys. Rev. Lett. 59, (1987) 535.
[27] J. Kumpulainen, R. Julin, J. Kantele, A. Passoja, W. H. Trzaska, E. Verho, J. Väärämäki, D. Cutoiu, and M. Ivascu, Phys. Rev. C **45** (1992) 640.
[28] H. Lehmann and J. Jolie, Nucl. Phys. A **588** (1995) 623.
[29] K. Heyde, J. Jolie, H. Lehmann, C. De Coster, and J. L. Wood, Nucl. Phys. A **586** (1995) 1.
[30] A. Gade, J. Jolie, P. von Brentano, Phys. Rev. C **65** (2002) 041305(R).
[31] J. Blachot, Nucl. Data Sheets **91** (2000) 135.
[32] G. Gürdal and F. G. Kondev, Nucl. Data Sheets **113** (2012) 1315.
[33] P. E. Garrett, J. Bangay, A. Diaz Varela, G. C. Ball, D. S. Cross, G. A. Demand, P. Finlay, A. B. Garnsworthy, K. L. Green, G. Hackman, C. D. Hannant, B. Jigmeddorj, J. Jolie, W. D. Kulp, K. G. Leach, J. N. Orce, A. A. Phillips, A. J. Radich, E. T. Rand, M. A. Schumaker, C. E. Svensson, C. Sumithrarachchi, S. Triambak, N. Warr, J. Wong, J. L. Wood, and S. W. Yates, Phys. Rev. C **86** (2012) 044304.