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Post-prior equivalence for transfer reactions with complex potentials

Jin Lei[1,*] and Antonio M. Moro[2,†]

[1] *Institute of Nuclear and Particle Physics, and Department of Physics and Astronomy, Ohio University, Athens, Ohio 45701, USA*
[2] *Departamento de FAMN, Universidad de Sevilla, Apartado 1065, 41080 Sevilla, Spain.*

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In this paper, we address the problem of the post-prior equivalence in the calculation of inclusive breakup and transfer cross sections. For that, we employ the model proposed by Ichimura, Austern, and Vincent [Phys. Rev. C 32, 431 (1985)], conveniently generalized to include the part of the cross section corresponding the transfer to bound states. We pay particular attention to the case in which the unobserved particle is left in a bound state of the residual nucleus, in which case the theory prescribes the use of a complex potential, responsible for the spreading width of the populated single-particle states. We see that the introduction of this complex potential gives rise to an additional term in the prior cross section formula, not present in the usual case of real binding potentials. The equivalence is numerically tested for the ^{58}Ni(d,pX) reaction.

I. INTRODUCTION

The post-prior equivalence of the transition amplitude for direct nuclear reactions involving different rearrangement channels is a key result of nuclear reaction theory. This result provides two formally equivalent ways of expressing the transition amplitude, depending on whether the main interaction appearing in the transition operator is that based on the initial or final internal Hamiltonians. The result holds for the exact transition amplitude and, in the case of transfer reactions, also for the DWBA limit. In this latter case, the post and prior expressions are formally identical, differing only in the transition operator. This result has indeed been confirmed in practical cases.

In the case of inclusive breakup reactions of the form $a+A \to B^*+b$, where $a = b+x$ and B^* is any $A+x$ state, the problem has deserved attention in the past. Several groups proposed formulae for the calculation of inclusive cross sections using either the post or prior DWBA representations [1–4]. Ichimura, Austern and Vincent(IAV) [3] showed that the post and prior equivalence holds also for these inclusive processes but, in this case, it involves an additional term, not present in the usual transfer process between bound states. This terms arises from the non-orthogonality between the initial ($a+A$) and final ($b+B$) partitions. Although some authors (see e.g. Li, Udagawa and Tamura [5]) regarded this term as nonphysical, in our recent work [6] we showed in practical cases that the inclusion of this term is essential to preserve the post-prior equivalence and to reproduce correctly the experimental data.

The calculations of Ref. [6] were restricted to unbound $x + A$ states (i.e. $E_x > 0$, where E_x is the final relative energy between x and A). However, the $E_x < 0$ case was not considered. This region would correspond to bound states of the residual B system and, hence, the process $a + A \to B + b$ becomes a transfer reaction in the usual sense. In some models, such as the DWBA, the scattering amplitude involves the overlap function between the A and B systems, i.e. $\langle A|B\rangle$. Although these overlaps should be in principle obtained from the many-body wave functions of A and B, they are most commonly approximated by single-particle wave functions calculated in a mean-field potential, with the correct quantum numbers and separation energy, and multiplied by a spectroscopic amplitude. The latter accounts for the fragmentation of single-particle strength due to beyond mean-field correlations. If one is not interested in the population of specific final states, but just in their sum, one may incorporate the effect of this fragmentation by means of a complex potential, whose imaginary part accounts for the spreading width of the single-particle levels into these more complicated configurations. This is the case of the dispersive optical potential, first introduced by Mahaux and Sartor [7] and recently pursued by several groups (see Ref. [8] for a recent review). The use of this dispersive potentials permits a natural extension of the inclusive breakup models to negative energies [9, 10]. A recent work, using the IAV model in prior form [11], has shown that this procedure leads to a smooth transition between the positive and negative E_x values and hence between the breakup and transfer regions. However, the relation between the prior and post formulations for the case of transfer reactions with complex binding potentials has not been established to our knowledge. In particular, it remains to clarify the importance of the non-orthogonality term in this case. Indeed, for real potentials, these results should lead to the well-known post-prior equivalence used in transfer reactions, and the non-orthogonality term should not contribute in this case.

Guided by these considerations, in this paper, we address the post-prior equivalence for transfer reactions of the form of $a + A \to b + B^*$ in presence of complex $x + A$ potentials. For that purpose, we revisit and generalize the IAV model which allows us to describe the breakup ($E_x > 0$) and transfer ($E_x < 0$) regions in the same footing. We will see that, in the extended version of the IAV

* jinl@ohio.edu
† moro@us.es

model, the use of a complex U_{xA} potential leads to different formulas for post and prior representations. As a practical application of the derived formulas, we present calculations for the ^{58}Ni(d,pX) reaction at $E = 80$ MeV.

The paper is organized as follows. In Sec. II we summarize the main formulas of the IAV model in post and prior forms, and outline its relation with the prior form UT model. We show how the IAV model can be naturally extended to final bound states. In Sec. III, the formalism is applied to the ^{58}Ni(d,pX) reaction. Finally, in Sec. IV we summarize the main results.

II. THEORETICAL FRAMEWORK

In this section, we briefly summarize the IAV model, in its post and prior forms, and highlight its connection with the UT model. Further details can be found in Ref. [3] as well as in our previous works [6, 12, 13]. We write the process under study as

$$a(= b + x) + A \to b + B^*, \tag{1}$$

where the projectile a, composed of b+x, collides with the target A, emitting the ejectile b and leaving the residual system B^* $(= x + A)$ in any possible final state compatible with energy and momentum conservation. This includes $x + A$ states with both positive and negative relative energies.

The IAV model, as well as the UT model, treats the particle b as a spectator, meaning that its interaction with the target nucleus is described with an optical potential U_{bA}.

Using the post-form IAV model in DWBA, the inclusive breakup differential cross section, as a function of the detected angle and energy of the fragment b, is given by

$$\left.\frac{d^2\sigma}{d\Omega_b E_b}\right|^{\rm post} = \frac{2\pi}{\hbar v_a}\rho(E_b)\sum_c |\langle \chi_b^{(-)}\Psi_{xA}^{c,(-)}|V_{\rm post}|\chi_a^{(+)}\phi_a\phi_A^0\rangle|^2 \times \delta(E - E_b - E^c), \tag{2}$$

where v_a is the velocity of the incoming particle a, $V_{\rm post} \equiv V_{bx} + U_{bA} - U_{bB}$ is the post-form transition operator, $\rho_b(E_b) = k_b\mu_b/[(2\pi)^3\hbar^2]$ (with μ_b the reduced mass of $b + B$ and k_b their relative wave number), $|\phi_a\rangle$ and $|\phi_A^0\rangle$ are the projectile and target ground states, $\chi_a^{(+)}$ and $\chi_b^{(-)}$ are distorted waves describing the $a - A$ and $b - B$ relative motion by the optical potentials U_{aA} and U_{bB}, respectively, and $\Psi_{xA}^{c,(-)}$ is any possible state of the $x + A$ many body system, with $c = 0$ denoting the x and A ground states. Thus, the $c = 0$ term in Eq. (2) corresponds to the processes in which the target remains in the ground state after the breakup, usually called *elastic breakup* (EBU), whereas the terms $c \neq 0$ correspond to the so-called non-elastic breakup (NEB) contributions.

The theory of IAV allows to perform the sum in a formal way, making use of the Feshbach projection formalism and the optical model reduction, and leading to the following closed form for the NEB differential cross section:

$$\left.\frac{d^2\sigma}{dE_b d\Omega_b}\right|_{\rm IAV}^{\rm post} = -\frac{2}{\hbar v_a}\rho_b(E_b)\langle\psi_x^{\rm post}|W_x|\psi_x^{\rm post}\rangle, \tag{3}$$

where W_x is the imaginary part of the optical potential U_x, which describes $x + A$ elastic scattering. $\psi_x^{\rm post}(\vec{r}_x)$ is a projected x-channel wave function which describes the $x - A$ relative motion for a given outgoing momentum $\vec{k}_b$ of the b particle, and obtained after projection onto the A ground state using the Feshbach formalism. It verifies the equation

$$|\psi_x^{\rm post}\rangle = G_x|\rho\rangle_{\rm post}, \tag{4}$$

where $G_x = 1/(E_x - H_x + i\epsilon)$, with $H_x = T_x + U_x$, $E_x = E - E_b$ and $\langle\vec{r}_x|\rho_{\rm post}\rangle = (\chi_b^{(-)}\vec{r}_x|V_{\rm post}|\phi_a\chi_a^{(+)}\rangle$. Udagawa and Tamura [4] proposed a very similar formula for the same problem, but making use of the prior-form representation. The prior-form $x-$channel wave function reads

$$|\psi_x^{\rm prior}\rangle = G_x|\rho\rangle_{\rm prior}, \tag{5}$$

where $\langle\vec{r}_x|\rho_{\rm prior}\rangle = (\chi_b^{(-)}\vec{r}_x|V_{\rm prior}|\phi_a\chi_a^{(+)}\rangle$. Using this channel wave function, UT proposed the following prior-form inclusive breakup formula

$$\left.\frac{d^2\sigma}{dE_b d\Omega_b}\right|_{\rm UT} = -\frac{2}{\hbar v_a}\rho_b(E_b)\langle\psi_x^{\rm prior}|W_x|\psi_x^{\rm prior}\rangle, \tag{6}$$

which is analogous to the IAV formula, Eq. (3) .

Despite their formal analogy, the UT and IAV expressions give rise to different predictions for NEB cross section. This discrepancy led to a long-standing dispute between these two groups. The problem has been also reexamined recently [12, 14]. These works have concluded that the UT formula is incomplete, and must be supplemented with additional terms, as we show below. The comparison of the IAV and UT models with experimental data supports this interpretation [12–17]

In general, the NEB will contain also contributions coming from the population of states below the breakup threshold of the $x + A$ system ($E_x < 0$). One would like to have a common framework to describe transfer to both continuum as well as bound states. For that purpose, Udagawa and co-workers [9] proposed to extend the complex potential to negative energies. Then, the bound states of the system are simulated by the eigenstates in this complex potential, whose imaginary part is associated with the spreading width of the single-particle states. The latter accounts for the fragmentation of these states into more complicated configurations due to the residual interactions. The method has been recently reexamined by Potel *et al.* [11, 14], who have provided an

efficient implementation of this idea. The key point is the realization that the Green function $G_x(r_x, r'_x)$ in Eqs. (4) and (5) can be expressed for both $E_x > 0$ and $E_x < 0$ cases. Proceeding in this way, the application of Eq. (3) to positive and negative energies is formally analogous.

The post-prior equivalence for transfer reactions with real binding potentials is well known in the literature [18, 19]. However, the post-prior equivalence with complex binding potentials has never been investigated to our knowledge.

The relation between the post and prior formulae was first established by IAV [3].

To relate the post and prior inclusive breakup cross sections, we note that

$$V_{\rm post} = V_{\rm prior} + (V_{bx} + T_{bx} + U_{aA} + T_{aA}) - (U_{xA} + T_{xA} + U_{bB} + T_{bB}). \quad (7)$$

We consider a definite final state of the $x - A$ system, denoted $|\phi_n\rangle$ and evaluate the difference

$$\langle\phi_n|\psi_x^{\rm post}\rangle - \langle\phi_n|\psi_x^{\rm prior}\rangle = \frac{\langle\phi_n\chi_b^{(-)}|(V_{bx} + T_{bx} + U_{aA} + T_{aA}) - (U_{xA} + T_{xA} + U_{bB} + T_{bB})|\chi_a^{(+)}\phi_a\rangle}{E_x - H_x}. \quad (8)$$

The first term in parenthesis can be replaced by E when acting on $|\chi_a^{(+)}\phi_a\rangle$. The second parenthesis, acting on $\langle\phi_n\chi_b^{(-)}|$ gives also the total energy E, provided H_x is Hermitian (i.e. U_x real). In that case, we have

$$\langle\phi_n\chi_b^{(-)}|V_{\rm post}|\chi_a^{(+)}\phi_a\rangle = \langle\phi_n\chi_b^{(-)}|V_{\rm prior}|\chi_a^{(+)}\phi_a\rangle, \quad (9)$$

which corresponds to the well-known post-prior equivalence for transfer reactions.

However, when U_x is complex we can not perform the last step. Instead, we may rewrite (8) as

$$|\psi_x^{\rm post}\rangle - |\psi_x^{\rm prior}\rangle = \frac{(\chi_b^{(-)}|E_x - H_x|\chi_a^{(+)}\phi_a\rangle}{E_x - H_x} = (\chi_b^{(-)}|\chi_a^{(+)}\phi_a\rangle, \quad (10)$$

The function $|\psi_x^{\rm NO}\rangle = (\chi_b^{(-)}|\chi_a^{(+)}\phi_a\rangle$ is the so-called non-orthogonality (NO) overlap, also referred to in the literature as Hussein-McVoy term. Upon replacement of this relation into Eq. (3) we finally get

$$\frac{d^2\sigma}{d\Omega_b E_b}\Big|_{\rm IAV} = \frac{d^2\sigma}{d\Omega_b E_b}\Big|_{\rm UT} + \frac{d^2\sigma}{d\Omega_b E_b}\Big|_{\rm NO} + \frac{d^2\sigma}{d\Omega_b E_b}\Big|_{\rm IN} \quad (11)$$

where the first term is the UT prior-form formula of the NEB cross section, Eq. (6),

$$\frac{d^2\sigma}{dE_b d\Omega_b}\Big|_{\rm NO} = -\frac{2}{\hbar v_a}\rho_b(E_b)\langle\psi_x^{\rm NO}|W_x|\psi_x^{\rm NO}\rangle, \quad (12)$$

is the non-orthogonality term and

$$\frac{d^2\sigma}{dE_b d\Omega_b}\Big|_{\rm IN} = -\frac{4}{\hbar v_a}\rho_b(E_b)Re[\langle\psi_x^{\rm NO}|W_x|\psi_x^{\rm prior}\rangle], \quad (13)$$

is the interference term.

Equation (11) shows that the IAV post-form formula and UT prior-form formula are not equivalent. The latter needs to be supplemented with two additional terms, which stem from the non-orthogonality of the initial and final states. Recent calculations comparing these expressions with experimental data support the interpretation of IAV [6, 12].

We also note that the relations (10) and (11) hold for both $E_x < 0$ and $E_x > 0$ cases. That is, even in the transfer to bound states the prior form expression requires the inclusion of the non-orthogonality term. Only when the $x - A$ potential is real the contribution of these terms vanish in the DWBA expression. It is one of the purposes of this work to assess the validity of (11) in actual calculations for complex U_x.

It is enlightening to consider the simple case in which W_x is taken as a constant. In this case, if one inserts the Green's operator into Eq. (3), the double differential cross section for transferring particle x to bound states in post-form results

$$\frac{{\rm d}^2\sigma}{{\rm d}E{\rm d}\Omega}\Big|_{\rm IAV}^{\rm post} = -\frac{2}{\hbar v_a}\rho_b(E_b)\langle\rho|G_x^\dagger W_x G_x|\rho\rangle_{\rm post} \quad (14)$$

Using the explicit form of G_x and introducing the identity, $\sum_n |\phi_n\rangle\langle\phi_n| = I$, we get[1]

$$\frac{{\rm d}^2\sigma}{{\rm d}E{\rm d}\Omega}\Big|_{\rm IAV}^{\rm post} = \sum_n \omega_n \frac{d\sigma_n}{d\Omega}\Big|_{\rm post}, \quad (15)$$

where we have introduced the notation

$$\omega_n = -\frac{1}{\pi}\frac{\Gamma_n}{(E_x - E_n)^2 + \Gamma_n^2}, \quad (16)$$

with $\Gamma_n = \langle\phi_n|W_x|\phi_n\rangle = W_x$ and

$$\frac{d\sigma_n}{d\Omega}\Big|_{\rm post} = \frac{2\pi}{\hbar v_a}\rho_b(E_b)\big|\langle\phi_n|\rho\rangle_{\rm post}\big|^2 = \frac{2\pi}{\hbar v_a}\rho_b(E_b)|\langle\phi_n\chi_b^{(-)}|V_{\rm post}|\chi_a^{(+)}\phi_a\rangle|^2. \quad (17)$$

[1] Note that the expression of the identity operator used here does not not hold in general for general non-Hermitian potentials, but it remains valid in the case of constant W, as assumed here.

It should be noted that the single-differential cross section given by Eq. (17) is nothing but the DWBA cross section of transfer cross sections. Equation (16) shows that, for constant W_x, the energy distribution of the double differential cross section follows a Breit-Wigner (Lorentzian) shape for the bound states $E_n < 0$, and Γ_n represents the spreading width of the single-particle levels generated by the V_x potential.

Notice that, in the limit case $W \to 0$, $\omega_n \to \delta(E_x - E_n)$ and hence $\mathrm{d}^2\sigma/\mathrm{d}E\mathrm{d}\Omega|_{\mathrm{post}}$ is zero everywhere except at the pole energies $E_x = E_n$.

Finally, we note that a similar result to that of Eqs. (15) and (16) can be obtained also for a non-constant imaginary potential, provided $W_x \ll V_x$ [14]. In this case, perturbation theory leads to a equation formally analogous to Eqs. (15)-(16), with $\Gamma_n = \langle \phi_n | W_x | \phi_n \rangle$, with the ϕ_n functions corresponding to the eigenstates of the Hamiltonian $T_x + V_x$.

III. CALCULATIONS

We consider the reaction ^{58}Ni(d,pX) at $E_d^{\mathrm{lab}} = 80$ MeV. This reaction was also considered in our previous work [12], where we compared with the inclusive breakup data from Ref. [20] using the original IAV model and considering only the $E_x > 0$ region for the n-^{58}Ni residual system. Here, we extend these calculations to negative energies ($E_x < 0$), studying the effect of the imaginary part of the U_x potential, and comparing the post and prior results.

In the present calculations, we consider for the $n-p$ interaction the simple Gaussian form of Ref. [21]. The deuteron and proton distorted waves are generated with the same optical potentials used in Ref. [12]. The neutron-^{58}Ni potential is extrapolated to negative energies by simply fixing its real and imaginary parts to their values at $E_n = 1$ MeV for $E_n \leq 1$ MeV, that is, $U_x(E_x < 1\ \mathrm{MeV}) = U_x(E_x = 1\ \mathrm{MeV})$. The bin procedure is used to average the distorted wave χ_b over small momentum intervals to evaluate the post-form formula. Although this is not required for the prior-form formula, the same averaging procedure is adopted in that case for consistency with the post-form results.

In Fig. 1(a) we present the post and prior calculations for the angle-integrated differential cross section of the outgoing protons as a function of their center-of-mass energy. The black thick solid line is the post-form calculation obtained with the IAV post-form model and the red thin solid line is the UT prior-form calculation. It is seen that there is a significant difference between these two calculations, for both $E_n < 0$ and $E_n > 0$ regions. When adding the NO (dotted line) and IN (dot-dashed) terms to the UT result, one obtains an excellent agreement with the IAV-post result both at positive and negative neutron energies. This demonstrates, for the first time to our knowledge, the post-prior equivalence of the transfer cross section leading to bound states, in presence

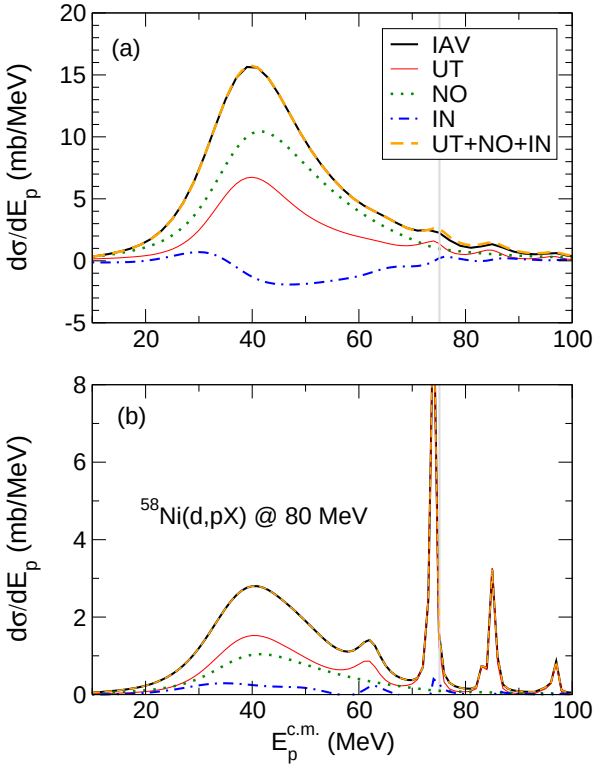

FIG. 1. (Color online) (a) Angle-integrated proton energy spectra ^{58}Ni(d,pX) at $E_d = 80$ MeV. The thick solid line is the post-form calculation (IAV model). The thin solid, dotted and dot-dashed lines are the UT, NO, IN terms contributing to the prior-form cross section and the dashed line is their sum. The vertical line indicates the threshold ($E_n = 0$) energy. (b) Same as panel (a), but with the imaginary part of the n-^{58}Ni reduced by a factor of 10. See text for details.

of complex binding potentials. This result has implications if, for example, a dispersive optical model potential is to be used to describe the bound states of the residual B system [17].

The fact that the difference between the IAV and UT results at negative neutron energies originates from the use of a complex neutron potential is illustrated in panel (b), where the imaginary part of this potential is reduced by a factor of 10. As expected, for $E_n > 0$ this leads to a reduction of the NEB cross section [c.f. Eq. (3)]. For $E_n < 0$ the much weaker absorption leads to an almost perfect agreement between the IAV and UT formulas, which is the usual post-prior equivalence for transfer reactions. We note also that, for this weak-absorption case, the differential cross sections displays marked peaks at the position of the bound states and resonances of the neutron-^{58}Ni potential. In particular, a very narrow $\ell = 4$ resonance is found near the neutron threshold. Therefore, the role of the imaginary part is to increase NEB cross section, but also to smear the contribution of the bound states and resonances. This is also apparent from Eq. (14) which, in the case of constant W_x, predicts

a Lorentzian shape with a width given by $\Gamma = W_x$. To conclude this section, we notice that, even in the limit of small W_x, the IAV and UT results differ for $E_n > 0$. In this case, the addition of the NO and IN terms is essential to restore the post-prior equivalence, as shown in our previous work [6].

IV. SUMMARY AND CONCLUSIONS

In summary, we have addressed the problem of the post-prior equivalence in the calculation of inclusive transfer reactions of the form $A(a,b)B$, were B is any bound state of the $x + A$ system. For that, we have considered the post-form inclusive breakup model proposed by Ichimura, Austern and Vincent (IAV) [1–3], conveniently extended to negative (bound states) of the $x + A$ system. We have also considered the prior-form model of Udagawa and Tamura (UT) [4].

We have shown that the equivalence between the post-form (IAV) and prior-form (UT) expressions holds only for real $x - A$ potentials. For complex interaction, the non-orthogonality (NO) term is indispensable. Once this term is included, the post-prior equivalence is restored.

To assess this equivalence at a numerical level, we have performed calculations for the ^{58}Ni(d,pX) reaction at 80 MeV. We find that, when a complex potential is used for the $x - A$ system, the IAV and UT results significantly disagree, both for the unbound ($E_x > 0$) and bound ($E_x < 0$) regions. Inclusion of the NO term gives an excellent agreement between the post and prior cross sections. We have also verified that, as the imaginary part of U_x is reduced, the UT result approaches the IAV one, thus recovering the well-known post-prior equivalence of the DWBA formula.

We believe that the present results are relevant because they extend a fundamental property of the transition amplitude, namely, the post-prior equivalence, to the case of non-Hermitian binding potentials. In particular, the results will be useful in the context of the exclusive or inclusive transfer reactions with dispersive optical potentials, currently under development.

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[1] N. Austern and C. M. Vincent, Phys. Rev. C **23**, 1847 (1981).
[2] A. Kasano and M. Ichimura, Phys. Lett. **115B**, 81 (1982).
[3] M. Ichimura, N. Austern, and C. M. Vincent, Phys. Rev. C **32**, 431 (1985).
[4] T. Udagawa and T. Tamura, Phys. Rev. C **24**, 1348 (1981).
[5] X.-H. Li, T. Udagawa, and T. Tamura, Phys. Rev. C **30**, 1895 (1984).
[6] J. Lei and A. M. Moro, Phys. Rev. C **92**, 061602 (2015).
[7] C. Mahaux and R. Sartor, Phys. Rev. Lett. **57**, 3015 (1986).
[8] W. Dickhoff, R. Charity, and M. Mahzoon, Journal of Physics G: Nuclear and Particle Physics **44**, 033001 (2017).
[9] T. Udagawa, Y. J. Lee, and T. Tamura, Phys. Rev. **C39**, 47 (1989).
[10] T. Udagawa, Y. Lee, and T. Tamura, Physics Letters B **196**, 291 (1987).
[11] G. Potel, F. M. Nunes, and I. J. Thompson, in *Proceedings, 14th International Conference on Nuclear Reaction Mechanisms: Varenna, Italy, June 15-19 Jun, 2015* (2015) pp. 155–162, arXiv:1510.02727 [nucl-th].
[12] J. Lei and A. M. Moro, Phys. Rev. C **92**, 044616 (2015).
[13] J. Lei and A. M. Moro, Phys. Rev. C **95**, 044605 (2017).
[14] G. Potel, F. M. Nunes, and I. J. Thompson, Phys. Rev. C **92**, 034611 (2015).
[15] A. M. Moro and J. Lei, Few-Body Systems **57**, 319 (2016).
[16] B. V. Carlson, R. Capote, and M. Sin, Few-Body Systems **57**, 307 (2016).
[17] G. Potel, G. Perdikakis, B. V. Carlson, M. C. Atkinson, W. H. Dickhoff, J. E. Escher, M. S. Hussein, J. Lei, W. Li, A. O. Macchiavelli, A. M. Moro, F. M. Nunes, S. D. Pain, and J. Rotureau, The European Physical Journal A **53**, 178 (2017).
[18] G. Satchler, *Direct nuclear reactions*, International series of monographs on physics (Clarendon Press, 1983).
[19] N. Glendenning, *Direct nuclear Reactions* (Elsevier Science, 2012).
[20] J. R. Wu, C. C. Chang, and H. D. Holmgren, Phys. Rev. C **19**, 370 (1979).
[21] N. Austern, Y. Iseri, M. Kamimura, M. Kawai, G. Rawitscher, and M. Yahiro, Physics Reports **154**, 125 (1987).