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Parity- and Time Reversal-Violating Pion Nucleon Couplings: Higher Order Chiral Matching Relations

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Abstract

Parity- and time reversal-violating (PVTV) pion-nucleon couplings govern the magnitude of long-range contributions to nucleon and atomic electric dipole moments. When these couplings arise from chiral symmetry-breaking CP-violating operators, such as the QCD θ-term or quark chromo-electric dipole moments, one may relate hadronic matrix elements entering the PVTV couplings to nucleon and pion mass shifts by exploiting the corresponding chiral transformation properties at leading order (LO) in the chiral expansion. We compute the higher-order contributions to the lowest order relations arising from chiral loops and next-to-next-to leading order (NNLO) operators. We find that for the QCD θ-term the higher order contributions are analytic in the quark masses, while for the quark chromoelectric dipole moments and chiral symmetry-breaking four-quark operators, the matching relations also receive non-analytic corrections. Numerical estimates suggest that for the isoscalar PVTV pion-nucleon coupling, the higher order corrections may be as large as $\sim 20\%$, while for the isovector coupling, more substantial corrections are possible.

I. INTRODUCTION

The study of P- and T-violating (PVTV) interactions can be traced back to the 1950s when Purcell and Ramsey proposed searching for the existence of a permanent electric dipole moment (EDM) of neutron [1]. Today, the subject attracts considerable attention as it is known that CP-violation[1] is one of the necessary ingredients for explaining the imbalance between the amount of matter and antimatter of the current universe [2]. The Standard Model (SM) allows CP-violating interactions through the complex phase in the Cabibbo-Kobayashi-Maskawa (CKM) matrix [3] but it is insufficient to account for the total observed asymmetry [4–6]. Therefore, alternative sources of CP-violation (CPV) are required.

Assuming that the extra degrees of freedom (DOFs) associated with the beyond Standard Model (BSM) CPV are heavy, they can be integrated out of the theory at low energy to obtain effective operators of higher dimensions that consist solely of SM DOFs. The PVTV components of these effective operators will in turn generate PVTV low-energy observables, such as EDMs. Current experiments set upper limits on EDMs, including those of the electron ($8.7 \times 10^{-29} e$ cm, 90% C.L.) [7], mercury atom ($7.4 \times 10^{-30} e$ cm, 95% C.L.) [8] and neutron ($3.0 \times 10^{-26} e$ cm, 90% C.L.) [9, 10]. These upper limits imply upper bounds of the magnitudes of Wilson coefficients of the PVTV effective operators. When hadrons are involved, translating EDM limits onto these operator bounds is highly non-trivial. The matching between low-energy hadronic observables and the Wilson coefficients of operators in quark-gluon sector involves various hadronic matrix elements that are difficult to evaluate from first-principles due to the non-perturbative nature of Quantum Chromodynamics (QCD) at low energy.

In this work we are particularly interested in the PVTV pion-nucleon coupling constants $\bar{g}_\pi^{(i)}$, where $i = 0, 1, 2$ denotes the isospin. [11–15]. The $\bar{g}_\pi^{(i)}$ govern the strength of of long-range (pion-exchange) contributions to atomic EDMs as well as to those of the proton and neutron (see, *e.g.*, [15, 16]). These interactions can be induced by various PVTV effective operators at the quark-gluon/photon level such as the θ-term, the quark EDM and chromo-EDM, the Weinberg three-gluon operator and various four-quark operators. In particular, if a specific PVTV effective operator breaks chiral symmetry, then its P and T-

[1] Implying T-violation assuming that CPT is conserved

conserving (PCTC) counterpart will also generate corrections to pion and nucleon masses. Consequently, there exist matching formulae that relate the induced $\bar{g}_\pi^{(i)}$ and these mass corrections simply due to chiral symmetry [11, 12, 14, 15, 17]. In terms of the SO(4) representation of Chiral Perturbation Theory (ChPT), the statement above reflects the fact that the and PVTV components of the effective operator belong to different components of a single SO(4) representation; therefore, their hadronic matrix elements are related through the Wigner-Eckart Theorem. This idea is practically beneficial as one may then extract the PVTV hadronic matrix elements for the $\bar{g}_\pi^{(i)}$ from the study of parity- and time reversal conserving (PCTC) hadronic matrix elements that are the pion and nucleon mass shifts. The latter may be obtained by lattice gauge theory or other phenomenological approaches. For example, application to the θ-term with a lattice value of nucleon mass shift yields $\bar{g}_\pi^{(0)} \approx (0.0155 \pm 0.0025)\bar{\theta}$ [18].

It is important to ask how robust these relations are when taking into account possible higher order contributions involving chiral loops and higher order terms in the chiral Lagrangian characterized by addition low-energy constants (LEC's). As a matter of principle as well as for purposes of numerical precision, one must include these corrections when appealing the matching relations. For instance, Ref. [18] studied the higher-order effects to the matching formulae induced by a QCD θ-term in a three-flavor ChPT. They found that when the matching relation of $\bar{g}_\pi^{(0)}$ is expressed in terms of the nucleon mass splitting then its form is preserved by the chiral loop correction. Consequently, the corrections to the LO matching relations are analytic in the quark masses.

In this work, we extend the study of Ref. [18] to cover all effective operators up to dimension 6 that include only the first generation quarks based on 2-flavor Heavy Baryon Chiral Perturbation Theory (HBChPT) in the $\mathrm{SU(2)_L} \times \mathrm{SU(2)_R}$ representation. First, we perform a general study of how these operators break chiral symmetry using the spurion method. This allows us to implement the effects of the chiral symmetry breaking (CSB) operators to the chiral Lagrangian in a straightforward manner and obtain the tree-level matching formulae. Next, we study the chiral one-loop corrections to both $\bar{g}_\pi^{(i)}$ and the hadron mass shifts. The results are expressed in the most general form so that they may be straightforwardly applied to any specific effective operator. Based on the general formalism above, we study higher-order effects to the matching formulae induced by the complex quark mass term (induced by the QCD θ-term), the chromo-EDM and the left-right four quark (LR4Q) operator that are the only three effective operators in the quark sector that contain both PCTC and PVTV components simultaneously and at the same time break chiral symmetry.

Given the length of this paper, it is useful to summarize here our main results point the reader to respective sections for details. For convenience, we summarize these features in The basic messages we wish to convey to the in Table I, whose content we now proceed to explain. First, matching relations exist only for $\bar{g}_\pi^{(i)}$ induced by chiral non-invariant operators that posses both PCTC and PVTV components. For these sources, $\bar{g}_\pi^{(i)}$ can be expressed in terms of mass shifts for nucleon and pion. For chirally invariant sources, their low-energy PCTC and PVTV effects are not related by any chiral symmetry and are, therefore, mutually independent. Hence, there exist no matching relations between $\bar{g}_\pi^{(i)}$ and hadron mass shifts induced by these sources. It is also interesting to notice that $\bar{g}_\pi^{(0)}$ depends on the $I = 1$ nucleon mass shifts, while $\bar{g}_\pi^{(1)}$ which has $I = 1$ depends on the $I = 0$ nucleon mass shifts.

Next, we consider higher-order effects, including both one-loop corrections as well as contributions from higher-order LECs. For the loop correction, we find that in many cases a one-loop diagram that corrects $\bar{g}_\pi^{(i)}$ will have a corresponding diagram with similar structure that corrects the nucleon mass shift (*e.g.*, Figs. 1a and 2a). Furthermore, the CSB vertices in these diagrams are related by the treel-level matching relations. As a consequence, the one-loop corrections to $\bar{g}_\pi^{(i)}$ and nucleon mass shifts induced by these diagrams satisfy the same tree-level matching relation. There exist exceptions to this rule, arising from one or more of the following situations: (1) when the tree-level matching involves (Δm_π^2), the shift of squared pion mass due to the extra operators, the loop corrections to this term do not have counterparts that correct $\bar{g}_\pi^{(i)}$; (2) the naïve matching between the PVTV tree-pion coupling $\bar{g}_{\pi\pi\pi}^{(1)}$ and (Δm_π^2) is spoiled by vacuum alignment, so that diagrams involving insertion of these operators do not satisfy the tree-level matching, and (3) there are several corrections to the $I = 0$ nucleon mass that do not require extra CSB operators (such as in Fig. 2h, 2i and 2k) so there are no corresponding diagrams that contribute to $\bar{g}_\pi^{(i)}$. With these observations in mind, we show that the tree-level matching for $\bar{g}_\pi^{(0)}$ induced by the θ-term is preserved under one-loop correction, confirming the result from Ref. [18], while matchings induced by other operators such as dipole operators and four-quark operators are not respected by loop corrections. On the other hand, contributions from higher-order terms in the chiral Lagrangian do not respect the original matching relations in general, implying a dependence of the matching relations on the associated LECs.

Finally, we estimate the numerical size of higher-order corrections to the tree-level matching relation using experimental and lattice-calculated hadron mass parameters as inputs. For each operator CSB operator $\mathcal{O}$, if the tree-level matching relation has the form $F_\pi \bar{g}_\pi^{(i)} = f^{(i)}$ where $F_\pi \approx 186$MeV is the pion decay constant and $f^{(i)}$ is a function of hadronic mass parameters as well as the PVTV Wilson coefficients, we may characterize the correction to the LO matching relation as:

$$F_\pi \bar{g}_\pi^{(i)} = f^{(i)} \cdot (1 + \delta_{\text{loop}}^{(i)} + \delta_{\text{LEC}}^{(i)}) \tag{1}$$

where $\delta_{\text{loop}}^{(i)}$ and $\delta_{\text{LEC}}^{(i)}$ are relative deviations due to one-loop correction and higher-order LECs respectively. We are particularly interested at $\delta_{\text{loop}}^{(i)}$ because $\delta_{\text{LEC}}^{(i)}$ does not involve chiral logs and is therefore suppressed by usual chiral power counting. In principle, one could write down explicit expressions for $\{\delta_{\text{loop}}^{(i)}\}$ as we shall present in the following sections, however their numerical values cannot be determined because they involve the isoscalar and isovector nucleon mass shifts $\{(\Delta m_N)_{\mathcal{O}}, (\delta m_N)_{\mathcal{O}}\}$ induced by the operator $\mathcal{O}$ which is not the quark mass operator (except for the case of θ-term). Therefore, in the numerical estimation of $\{\delta_{\text{loop}}^{(i)}\}$ we shall simply set them to zero. The result is summarized in Table I where we find that the tree-level matching formula for $\bar{g}_\pi^{(0)}$ is relatively robust numerically under loop corrections regardless of choice of the underlying operator while the status for $\bar{g}_\pi^{(1)}$ in general receive large loop corrections. On the other hand, the quantity $\delta_{\text{LEC}}^{(i)}$ involves unknown LECs and can only be estimated at present based on rough dimensional arguments. Of course, such estimation may never pretend to be any trustworthy prediction to the actual numerical values of the LECs; in particular, as pointed out in [19], it makes no prediction to their signs. It therefore only serves to provide a rough estimation to the order of magnitude of the uncertainty brought up by the LECs. We find that the impact of the LECs on the $\bar{g}_\pi^{(0)}$-matching can be as large as $(10-20)\%$ while their effect on the $\bar{g}_\pi^{(1)}$-matching is usually not much larger than 1%.

Our discussion of this study organized as follows. In Sec. II we give introduce a spurion formalism and give a general discussion from the possible forms of the spurion that encode the explicit CSB effects of the effective operators up to dimension 6. In Sec. III we write down the most general form of PVTV operators as well as PCTC and CSB operators that could contribute to the loop corrections for $\bar{g}_\pi^{(i)}$ and the mass shifts of the pion and nucleon.

Operator	$\bar{g}_\pi^{(0)}$ matching	$\bar{g}_\pi^{(1)}$ matching	$\delta^{(0)}_{\rm loop}$	$\delta^{(1)}_{\rm loop}$
θ-term	LO	NNLO	0	N/A
chromo-MDM/EDM	LO	LO	0.021	-3.1
LR4Q	LO	LO	-0.12	-3.2
Chiral-invariant operators	N/A	N/A	N/A	N/A

Table I: Numerical estimates of the one-loop contribution to the deviation of the tree-level matching formulae. The numerical values of $\delta^{(i)}_{\rm loop}$ are evaluated at the renormalization scale $\mu = 1$GeV assuming all the nucleon mass shifts induced by non-quark-mass operators are zero. Columns two and three indicate whether the leading matching relation arises at LO, NNLO, or not at all.

These loop corrections are then computed in Sec. IV in their most general form. Based on these results, we perform case-by-case study of the matching formulae for $\bar{g}_\pi^{(i)}$ induced by different effective operators, including both loop and LEC contributions, in Sec. V. Finally, we shall draw our conclusions in Sec. VI.

II. CHIRAL SYMMETRY AND THE SPURION METHOD

It is well known that a massless two-flavor QCD obeys the $\mathrm{SU(2)_L \times SU(2)_R}$ chiral symmetry defined by the following transformation on the quark field:

$$Q_R \to V_R Q_R, \quad Q_L \to V_L Q_L \tag{2}$$

where $\{V_R, V_L\}$ are 2×2 unitary matrices. Chiral symmetry is explicitly broken in ordinary QCD only by the quark mass terms. However, when we consider effects from BSM physics there may be additional higher-dimensional operators that break the symmetry as well. In general, these symmetry-breaking terms can always be expressed as products of Q_R, Q_L with some constant matrices (or products of matrices) in such as way that if these matrices would transform with a specific way under the chiral rotation then the corresponding terms would be chirally invariant. These matrices, known as spurions, are used to describe the explicit CSB effects in the low-energy effective theory of QCD because we expect the latter to obey the same symmetry breaking pattern as QCD itself.

Here we present the most general form of QCD spurion that encodes the effects from all effective CSB operators up to dimension 6 that involve only the light quarks and massless gauge bosons. Our choice of operators are those that obey the SM gauge symmetry at high energy (see Ref. [20] for a complete list of operators). They then undergo electroweak symmetry breaking (EWSB) where the neutral Higgs is replaced by its vacuum expectation value (VEV). These operators can be divided into two categories, namely the quark bilinears and the four-quark operators. Operators in different categories in general take different form of spurions.

A. Quark bilinears

At dimension four the only CSB terms are the quark Yukawa coupling terms that then undergo EWSB to give rise to the quark masses. At the same time, a non-vanishing QCD θ-term may then be rotated away using the axial anomaly to be replaced by complex phases in the quark masses (this procedure will be reviewed in Sec. V A). The resulting Lagrangian will take the general form

$$-\bar{Q}_R X Q_L + h.c. \quad , \tag{3}$$

where X is a complex 2×2 diagonal matrix in flavor space and the term would be chirally invariant if X would transform as $X \to V_R X V_L^\dagger$ under chiral rotation.

At dimension six the only CSB bilinear operators of quarks are the $\psi^2 H^3$ operators and the dipole-like operators [2]. On the one hand, the $\psi^2 H^3$ operators reduce to complex quark mass terms after EWSB so we do not need to discuss them separately. On the other hand, the dipole operators have the general form

$$\bar{Q}_L \sigma^{\mu\nu} T^A H d_R V^A_{\mu\nu} \tag{4}$$

where T^A is a generator of any one of the SM gauge groups and $V^A_{\mu\nu}$ are the corresponding field strength tensor (a similar structure appears for up-type quarks with $d_R \to u_R$ and $H_j \to \epsilon_{jk} H_k^*$). After EWSB, the dipole operators reduce to the dimension five forms

$$\bar{q}_L \sigma^{\mu\nu} \frac{\lambda^a}{2} q_R G^a_{\mu\nu}, \quad \bar{q}_L \sigma^{\mu\nu} q_R F_{\mu\nu}, \quad \bar{q}_L \sigma^{\mu\nu} q_R Z_{\mu\nu}, \quad \bar{u}_L \sigma^{\mu\nu} d_R W^+_{\mu\nu} \quad . \tag{5}$$

[2] Another operator of the form $i(\tilde{H}^\dagger D_\mu H)\bar{u}_R \gamma^\mu d_R$, with $\tilde{H}_j \equiv \epsilon_{jk} H_k^*$ will be classified as a four-quark operator after the W-boson is integrated out.

We can neglect the last three operators because their effects in the generation of pure hadronic operators will be suppressed with respect to the first either by the electromagnetic coupling strength or inverse powers of the heavy gauge boson masses. The remaining operators are the flavor-diagonal quark chromo-magnetic dipole moment(cMDM)/chromo-electric dipole moment(cEDM).

In terms of the chiral spurion, the cMDM and CEDM operators take the form

$$\bar{Q}_R \sigma^{\mu\nu} X \frac{\lambda^a}{2} Q_L G^a_{\mu\nu} \quad , \tag{6}$$

where again X is a complex 2×2 diagonal matrix. We then conclude that the quark bilinears appearing in Eqs.(3,6) imply the same form of the spurion, namely:

$$X = a + b\tau_3 \tag{7}$$

where $\{a, b\}$ are complex numbers. If the spurion would transform as $X \to V_R X V_L^\dagger$ under $\mathrm{SU(2)_L} \times \mathrm{SU(2)_R}$ then the Lagrangian would be chirally invariant. Furthermore, any PVTV effects are contained in the imaginary part of a and b.

When the spurion method is applied to the baryon sector of the chiral Lagrangian, it is convenient to define the following quantities:

$$\tilde{X}_\pm \equiv u^\dagger X u^\dagger \pm u X^\dagger u \tag{8}$$

where the subscript "+" ("-") denotes that the matrix is Hermitian (anti-Hermitian) and u is a matrix function of pion fields defined in Appendix A. They "transform" under chiral rotation as $\tilde{X}_\pm \to K \tilde{X}_\pm K^\dagger$. One advantage of this notation is that it allows us to construct Lagrangian of which PVTV effects come entirely from the spurion matrix X. For instance, $\tilde{X}_+$ is parity-even and $\tilde{X}_-$ is parity-odd if X is a real matrix because $u \leftrightarrow u^\dagger$ under P. Therefore, in LO effective Lagrangian, the spurion involved should be $\tilde{X}_+$ and not $\tilde{X}_-$ because we require the Lagrangian to be P (and T)-even when the matrix X is real.

B. Four-quark operators

Next we study the most general form of spurion fields induced by dim-6 four quark operators. As explained at the beginning of the section, these operators encode effects of BSM physics at high scale which is assumed to obey the Standard Model $\mathrm{SU(2)_L} \times \mathrm{U(1)_Y}$

symmetry, so they are constructed using the $\mathrm{SU}(2)_\mathrm{L}$ doublet field Q_L as well as the singlet fields $\{u_R, d_R\}$. Following the notations in Ref. [20], these operators can be grouped into the following categories:

1. *$(\bar{L}L)(\bar{L}L)$:*

   The two independent operators could be chosen as

$$\bar{Q}_L\gamma^\mu Q_L\bar{Q}_L\gamma_\mu Q_L, \quad \bar{Q}_L\gamma^\mu\tau^i Q_L\bar{Q}_L\gamma_\mu\tau^i Q_L. \tag{9}$$

They are both chirally invariant so they do not give rise to any non-trivial spurion.

2. *$(\bar{R}R)(\bar{R}R)$:*

   There are four independent operators in this category that can be chosen as

$$\bar{u}_R\gamma^\mu u_R\bar{u}_R\gamma_\mu u_R, \quad \bar{d}_R\gamma^\mu d_R\bar{d}_R\gamma_\mu d_R, \quad \bar{u}^i_R\gamma^\mu u_R\bar{d}_R\gamma_\mu d_R, \quad \bar{u}^i_R\gamma^\mu\frac{\lambda^a}{2}u_R\bar{d}_R\gamma_\mu\frac{\lambda^a}{2}d_R. \tag{10}$$

These operators break chiral symmetry as X_R or $X_R\otimes X_R$ where the spurion matrix $X_R = \tau_3$. Chiral symmetry would be preserved if the spurion matrix would transform as $X_R \to V_R X_R V_R^\dagger$ under chiral rotation. Here, the notation $A\otimes B$ means that the matrices A and B appear simultaneously in a quark bilinear or a four-quark operator, *e.g.* $\bar{Q}ABQ$ or $\bar{Q}AQ\bar{Q}BQ$.

3. *$(\bar{L}L)(\bar{R}R)$:*

   There are four independent operators in this category that can be chosen as

$$\bar{Q}_L\gamma^\mu Q_L\bar{u}_R\gamma_\mu u_R, \quad \bar{Q}_L\gamma^\mu\frac{\lambda^a}{2}Q_L\bar{u}_R\gamma_\mu\frac{\lambda^a}{2}u_R, \quad \bar{Q}_L\gamma^\mu Q_L\bar{d}_R\gamma_\mu d_R, \quad \bar{Q}_L\gamma^\mu\frac{\lambda^a}{2}Q_L\bar{d}_R\gamma_\mu\frac{\lambda^a}{2}d_R. \tag{11}$$

These operators break chiral symmetry through a single spurion matrix X_R.

4. *$(\bar{L}R)(\bar{L}R)$:*

   There are two operators in this category, namely

$$\varepsilon^{ij}\bar{Q}^i_L u_R\bar{Q}^j_L d_R, \quad \varepsilon^{ij}\bar{Q}^i_L\frac{\lambda^a}{2}u_R\bar{Q}^j_L\frac{\lambda^a}{2}d_R. \tag{12}$$

Both operators are chirally invariant: for instance, the first operator can be rewritten as $\varepsilon^{ij}\varepsilon^{i'j'}\bar{Q}_L^i Q_R^{i'}\bar{Q}_L^j Q_R^{j'}/2$ so its $\mathrm{SU(2)_L}$ and $\mathrm{SU(2)_R}$-invariance are explicit. Meanwhile, they allow complex Wilson coefficients that give rise to PVTV physics. We can then define their "spurion" simply as a complex number.

5. Induced Left-Right Four Quark (LR4Q) Operator

Finally there is another four quark operator that arises from $i(\tilde{H}^\dagger D_\mu H)\bar{u}_R\gamma^\mu d_R$. When the $W^\pm$ boson contained in D_μ is exchanged with the LH charge changing quark current, one obtains the following four-quark operator after EWSB:

$$\begin{aligned} & c_{4q}\bar{d}_L\gamma^\mu u_L\bar{u}_R\gamma_\mu d_R + h.c. \\ = & -\frac{2}{3}(c_{4q}\bar{Q}_R\frac{1+\tau_3}{2}Q_L\bar{Q}_L\frac{1-\tau_3}{2}Q_R + c_{4q}^*\bar{Q}_R\frac{1-\tau_3}{2}Q_L\bar{Q}_L\frac{1+\tau_3}{2}Q_R) \\ & -4(c_{4q}\bar{Q}_R\frac{1+\tau_3}{2}\frac{\lambda^a}{2}Q_L\bar{Q}_L\frac{1-\tau_3}{2}Q_R + c_{4q}^*\bar{Q}_R\frac{1-\tau_3}{2}\frac{\lambda^a}{2}Q_L\bar{Q}_L\frac{1+\tau_3}{2}Q_R) \end{aligned} \tag{13}$$

where the right hand side is obtained using a Fierz transformation. We see that this operator breaks the chiral symmetry as $c_{4q}X_{RL}\otimes X_{LR} + c_{4q}^* X_{LR}^\dagger \otimes X_{RL}^\dagger$ where $X_{RL} = (1+\tau_3)/2$, $X_{LR} = (1-\tau_3)/2$ and would be chirally invariant if $X_{RL}\to V_R X_{RL} V_L^\dagger$ and $X_{LR}\to V_L X_{LR} V_R^\dagger$ under a chiral rotation. One observes that the part of the operator proportional to $\mathrm{Re}c_{4q}$ is has the structure $\bar{Q}_R Q_L \bar{Q}_L Q_R - \bar{Q}_R\tau_3 Q_L\bar{Q}_L\tau_3 Q_R$ (and terms with λ_a-insertions), so it is PCTC with isospin 0 or 2. Meanwhile, the part proportional to $\mathrm{Im}c_{4q}$ has the structure $\bar{Q}_R Q_L\bar{Q}_L\tau_3 Q_R - \bar{Q}_R\tau_3 Q_L\bar{Q}_L Q_R$ (and terms with λ_a-insertions). It is PVTV and with isospin 1.

Up to this point we have discussed all the possible operators up to dimension six that would break the QCD chiral symmetry. A complete list of spurions induced by these operators can be found in Table II. It is important to note that only the complex quark mass term, the dipole-like operators, and the LR4Q operator are chirally non-invariant and contain both PCTC and PVTV components. These three types of operators will be relevant in the discussion of the matching formula for the $\bar{g}_\pi^{(i)}$ in the upcoming sections.

Operators	Spurion	Constant value	"Transformation Rule"
Quark bilinears	X	$a + b\tau_3$	$X \to V_R X V_L^\dagger$
Four Quark:	a	a	$a \to a$
$(\bar{L}L)(\bar{L}L)$, $(\bar{R}R)(\bar{R}R)$,	X_R	τ_3	$X_R \to V_R X_R V_R^\dagger$
$(\bar{L}L)(\bar{R}R)$, $(\bar{L}R)(\bar{L}R)$	$X_R \otimes X_R$	$\tau_3 \otimes \tau_3$	
Induced LR4Q	$c_{4q} X_{RL} \otimes X_{LR}$	$X_{RL} = (1+\tau_3)/2$,	$X_{RL} \to V_R X_{RL} V_L^\dagger$,
	$+c_{4q}^* X_{LR}^\dagger \otimes X_{RL}^\dagger$	$X_{LR} = (1-\tau_3)/2$	$X_{LR} \to V_L X_{LR} V_R^\dagger$

Table II: Complete list of spurions that enter the chiral Lagrangian. For each spurion, we show the constant value it takes during its implementation in the Lagrangian (third column), and how it would need to transform in order to leave the Lagrangian chirally invariant (fourth column). Among all the operators, only the quark bilinears and the induced LR4Q operator are chirally non-invariant and at the same time contain both PCTC and PVTV components.

III. CHIRAL SYMMETRY BREAKING OPERATORS IN A LINEAR REPRESENTATION

Insertions of the spurion fields we discussed in Sec. II into the chiral Lagrangian will give rise to CSB operators consisting of baryons and pions. Among them, the leading PVTV $NN\pi$ operators and the hadron mass operators are of greatest importance because their Wilson coefficients will enter the matching formulae for the $\bar{g}_\pi^{(i)}$ that is the focus this work. At the same time, the existence of such operators automatically implies the presence of a whole series of CSB operators with higher powers of pions whose operator coefficients are related by chiral symmetry. The relation, however, depends on the explicit form of spurion. Consequently, it is not practical to write down a single CSB Lagrangian containing terms with an arbitrary number of pion fields without specifying the form of spurion.

Nevertheless, a subset of CSB operators with higher powers of pion fields (*e.g.* $NN\pi\pi$, $NN\pi\pi\pi$ and $\pi\pi\pi\pi$ operators) must be included in this work because they contribute to $\bar{g}_\pi^{(i)}$ and hadron mass parameters at one-loop and will therefore modify the matching formulae from their tree-level expressions. For this purpose, it will be convenient to express the Goldstone bosons (*i.e.* pions in our case) in the CSB operators in linear, in stead of non-linear, representation. By doing so we pay the price of losing the manifest chiral structure of each term. On the other hand results of the loop corrections will be completely general and independent of any particular choice of spurion. Eventually, when we need to apply the general result to specific effective operators (spurions) we simply refer back to the non-linear representation, expand each term in powers of pion fields, and match the coefficients with those in the general linear representation.

We will also include the Δ-baryons as explicit DOFs since the nucleon-Δ mass splitting vanishes in the large-N_c limit [21] and since inclusion of Δs is generally required in order to respect $1/N_c$ power counting. As far as this work is concerned, the Δ-baryons only appear as virtual particles in loop corrections to the $\bar{g}_\pi^{(i)}$ and nucleon masses.

A. PVTV operators

Following the foregoing discussion , we proceed to write down all possible forms of lowest-order PVTV operators involving nucleons, pions and Δ-baryons in the linear representation of Goldstone bosons that are relevant to this work. For the coefficient of these operators we adopt the following unified notation, namely: the coefficient $\bar{g}_K^{(I,j)}$ is the real coefficient of the j-th PVTV operator of type K with isospin I (the superscript j will however be suppressed if there is only one operator with isospin I). Because the $\bar{g}_\pi^{(i)}$ can only have isospin $I = 0, 1, 2$, for the renormalization of these operators at leading order we only need to consider all PVTV operators with $I = 0, 1, 2$. Furthermore, we choose to parameterize $\bar{g}_K^{(I,j)}$ in such a way that all of them are dimensionless by the inclusion of appropriate powers of F_π in front of each operator.

1. *$NN\pi$ operators*

The PVTV $NN\pi$ operators are defined as [16]:

$$\mathcal{L} = \bar{g}_\pi^{(0)} \bar{N} \vec{\tau} \cdot \vec{\pi} N + \bar{g}_\pi^{(1)} \pi_0 \bar{N} N - 3\bar{g}_\pi^{(2)} \mathcal{I}^{ab} \pi_a \bar{N} \tau_b N \tag{14}$$

where $\mathcal{I} = (1/3)\mathrm{diag}(1\ \ 1\ \ -2)$ is needed to combine two isospin triplets into an $I = 2$ quantity.

2. $\Delta\Delta\pi$ operators

The PVTV $\Delta\Delta\pi$ operators have the general form $\bar{T}^a_\mu T^{b\mu}\pi$, where T^a_μ is the field representation of the Δ-baryon as explained in Appendix A. They can be chosen as

$$\mathcal{L} = \bar{g}^{(0)}_{\Delta\Delta\pi}\bar{T}^a_\mu\vec{\tau}\cdot\vec{\pi}T^{a\mu} + i\bar{g}^{(1,1)}_{\Delta\Delta\pi}\epsilon^{abc}\pi_b[\bar{T}^a_\mu\tau_c T^{3\mu} - \bar{T}^3_\mu\tau_c T^{a\mu}] + i\bar{g}^{(1,2)}_{\Delta\Delta\pi}\epsilon^{ab3}\bar{T}^a_\mu\vec{\tau}\cdot\vec{\pi}T^{b\mu}$$
$$+\bar{g}^{(2,1)}_{\Delta\Delta\pi}\mathcal{I}^{ab}[\pi_c\bar{T}^c_\mu\tau_a T^{b\mu} + \pi_c\bar{T}^b_\mu\tau_a T^{c\mu} - \pi_a\bar{T}^c_\mu\tau_c T^{b\mu} - \pi_a\bar{T}^b_\mu\tau_c T^{c\mu}] + \bar{g}^{(2,2)}_{\Delta\Delta\pi}\mathcal{I}^{ab}\pi_b\bar{T}^c_\mu\tau_a T^{c\mu}. \quad (15)$$

Note that the T^a_μ ($a = 1, 2, 3$) denote a set of three two-component vectors in isospin space, satisfying $\tau^a T^a_\mu = 0$. This representation allows us to write down all expressions in terms of quantities such as τ^a and $\mathcal{I}^{ab}$ where the indices run from 1 to 3.

3. $\pi\pi\pi$ operators

The PVTV 3-pion operators should look like $\pi\pi\pi$ as operators with derivatives are of higher order. In particular, the only operator relevant to us is the $I = 1$ operator (the others have $I = 3$):

$$\mathcal{L} = \bar{g}^{(1)}_{\pi\pi\pi}F_\pi\vec{\pi}^2\pi_0. \quad (16)$$

It is T-odd because the neutral pion field changes sign under T under our conventions for the pion-nucleon interactions.

4. $NN\pi\pi\pi$ operators

The PVTV $NN\pi\pi\pi$ operators can be chosen as

$$\mathcal{L} = \frac{\bar{g}^{(0)}_{NN3\pi}}{F_\pi^2}\bar{N}\vec{\tau}\cdot\vec{\pi}N\vec{\pi}^2 + \frac{\bar{g}^{(1)}_{NN3\pi}}{F_\pi^2}\bar{N}N\pi_0\vec{\pi}^2 + \frac{\bar{g}^{(2,1)}_{NN3\pi}}{F_\pi^2}\mathcal{I}^{ab}\pi_a\pi_b\bar{N}\vec{\tau}\cdot\vec{\pi}N + \frac{\bar{g}^{(2,2)}_{NN3\pi}}{F_\pi^2}\mathcal{I}^{ab}\pi_b\bar{N}\tau_a N\vec{\pi}^2. \quad (17)$$

B. PCTC operators

Following the same line of thought as in the previous subsection, we shall construct all relevant PCTC CSB operators that contribute to the loop correction to hadron mass shifts. Again these operators are defined using a linear representation of the Goldstone bosons.

1. $\pi\pi$ operators

There are only two kinds of CSB $\pi\pi$ operators that are the isospin invariant ($I = 0$) and isospin-breaking ($I = 2$) mass terms respectively:

$$\mathcal{L} = -\frac{1}{2}(\Delta m_\pi^2)\vec{\pi}^2 - \frac{3}{2}(\delta m_\pi^2)\mathcal{I}^{ab}\pi^a\pi^b. \tag{18}$$

Here we define (Δm_π^2) such that it does not include the LO-contribution from the quark mass (i.e. the well-known $(m_\pi^2)_0 = 2B_0\bar{m}$ contribution in ChPT, as we shall also discuss in Sec. V A). That is, we shall include only $(m_\pi^2)_0$ in the pion propagator while the (Δm_π^2) and (δm_π^2) defined above appear only in the form of two-pion vertex in Feynman diagrams, as depicted in Fig. 2. Similar argument applies for the quantities (Δm_Δ), (δm_Δ) and $(\delta\tilde{m}_\Delta)$ which we shall define below: they appear only in the form of $\Delta - \Delta$ vertex, while the Δ-propagator contains only δ_Δ, namely the nucleon-delta mass splitting in the chiral limit, as defined in Eq. (B1).

2. $\pi\pi\pi\pi$ operators

There are two four-pion operators up to $I = 2$. They can be written as:

$$\mathcal{L} = g_{4\pi}^{(0)}(\vec{\pi}^2)^2 + g_{4\pi}^{(2)}\vec{\pi}^2\mathcal{I}^{ab}\pi_a\pi_b. \tag{19}$$

Again, we define $g_{4\pi}^{(0)}$ such that it does not include the LO-contribution from the quark mass.

3. NN operators

Again there are only two kinds of CSB NN operators, corresponding to the nucleon σ-term and the mass splitting term. We write them as

$$\mathcal{L} = (\Delta m_N)\bar{N}N + \frac{(\delta m_N)}{2}\bar{N}\tau_3 N. \tag{20}$$

Even though the operator $\bar{N}N$ is chirally invariant, it can still be obtained through an insertion of a spurion (e.g. from the isospin-invariant part of the quark mass matrix) so it must included for completeness.

4. $NN\pi\pi$ operators

We are only interested in the $I = 0, 1, 2$ operators that are

$$\mathcal{L} = \frac{g^{(0)}_{NN\pi\pi}}{F_\pi}\bar{N}N\vec{\pi}^2 + \frac{g^{(1,1)}_{NN\pi\pi}}{F_\pi}\bar{N}\tau_3 N\vec{\pi}^2 + \frac{g^{(1,2)}_{NN\pi\pi}}{F_\pi}\bar{N}\vec{\tau}\cdot\vec{\pi}N\pi_0 + \frac{g^{(2)}_{NN\pi\pi}}{F_\pi}\mathcal{I}^{ab}\pi_a\pi_b\bar{N}N. \qquad (21)$$

There is another $I = 2$ operator but it is T-odd.

5. $\Delta\Delta$ operators

There are four kinds of $\Delta\Delta$ operators corresponding to four mass terms. However here we are only interested at the $I = 0, 1, 2$ operators. They are:

$$\mathcal{L} = (\Delta m_\Delta)\bar{T}^a_\mu T^{a\mu} + \frac{(\delta m_\Delta)}{2}\bar{T}^a_\mu \tau_3 T^{a\mu} + 3(\delta\tilde{m}_\Delta)\mathcal{I}^{ab}\bar{T}^a_\mu T^{b\mu} \qquad (22)$$

Again we define (Δm_Δ) such that it does not include the original residual mass δ_Δ in the chiral limit. Similar to $\bar{N}N$, the operator $\bar{T}^a_\mu T^{a\mu}$ is chirally invariant yet it can still be induced by a spurion-insertion so we need to include this term.

IV. ONE-LOOP CORRECTION TO $\bar{g}^{(i)}_\pi$ AND HADRON MASS SHIFTS

With all the relevant operators defined in Sec. III it is now straightforward to compute the most general one-loop corrections to both $\bar{g}^{(i)}_\pi$ and the hadron mass shifts. To obtain the total result one needs to compute both the one-particle irreducible (1PI) diagrams and the wavefunction renormalization graphs. The latter are quite standard and are summarized in Appendix D.

Terms in the chiral effective Lagrangian at low energy are arranged according to increasing powers of E, a typical small energy scale in the theory. A valid power expansion in HBChPT requires $E/(2\pi F_\pi), E/m_N \ll 1$. Following usual conventions [22], forms such as ∂_μ, m_π and the $\Delta - N$ mass splitting δ_Δ count as $O(E^1)$ while the light quark mass m_q and other quantities linearly proportional to m_q count as $O(E^2)$ because we shall see later that $m_\pi^2 \sim m_q$. Based on such power counting, there are seven types of 1PI diagrams that contribute to the correction of $\bar{g}^{(i)}_\pi$ up to NNLO and they are summarized in the first seven diagrams

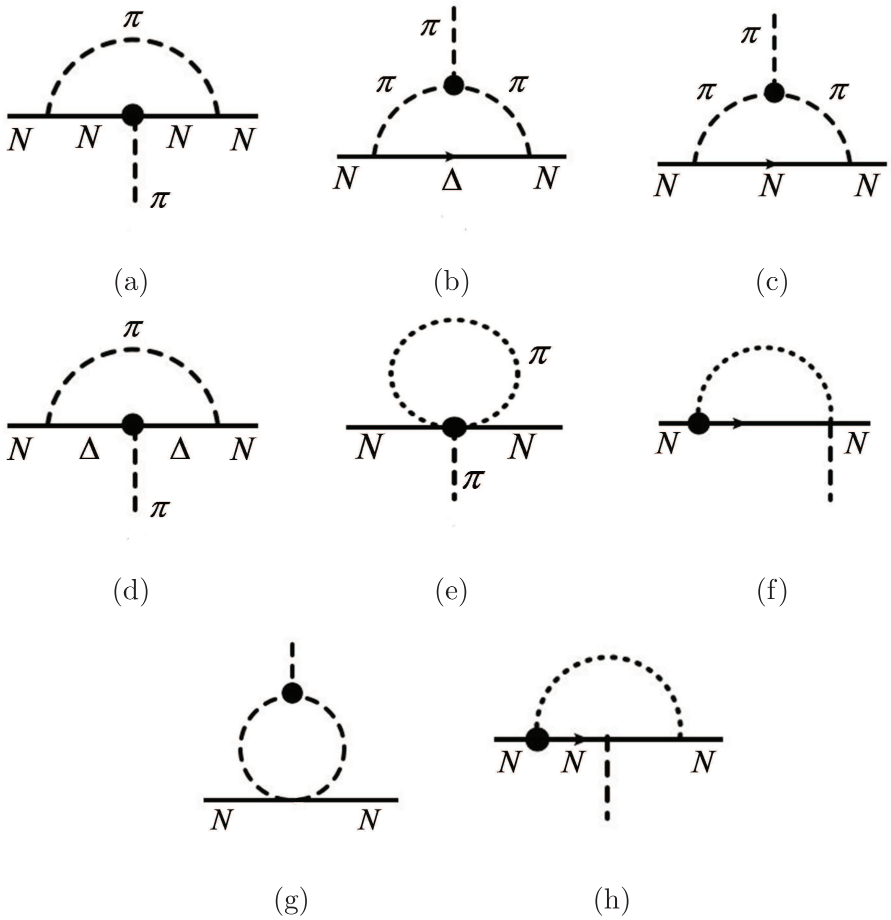

Figure 1: Loop diagrams that give rise to PVTV pion-nucleon interactions. Each circular vertex denotes a PVTV interaction vertex defined in Sec. III A. Diagram (g) involves $O(E^2)$ $NN\pi\pi$ coupling. The last diagram does not contribute to $\bar{g}_\pi^{(i)}$ due to the derivative nature of the chiral-invariant pion-nucleon coupling.

in Fig.1. Together with the wavefunction renormalization, they give

$$
\begin{aligned}
\delta(\bar{g}_\pi^{(0)})_{\rm loop} &= \frac{4g_A^2\bar{g}_\pi^{(0)}}{F_\pi^2}I_a - \frac{40g_{\pi N\Delta}^2\bar{g}_{\Delta\Delta\pi}^{(0)}}{9F_\pi^2}I_d + (\frac{4\bar{g}_\pi^{(0)}}{F_\pi^2} + \frac{5\bar{g}_{NN3\pi}^{(0)}}{F_\pi^2})I_e + (Z_N-1)\bar{g}_\pi^{(0)} \\
&+(\sqrt{Z_\pi}-1)_{\rm loop}\bar{g}_\pi^{(0)} \\
\delta(\bar{g}_\pi^{(1)})_{\rm loop} &= -\frac{12g_A^2\bar{g}_\pi^{(1)}}{F_\pi^2}I_a + (\frac{16g_{\pi N\Delta}^2\bar{g}_{\Delta\Delta\pi}^{(1,1)}}{3F_\pi^2} + \frac{8g_{\pi N\Delta}^2\bar{g}_{\Delta\Delta\pi}^{(1,2)}}{3F_\pi^2})I_d - \frac{40g_A^2\bar{g}_{\pi\pi\pi}^{(1)}}{F_\pi}I_c - \frac{80g_{\pi N\Delta}^2\bar{g}_{\pi\pi\pi}^{(1)}}{3F_\pi}I_b \\
&+\frac{5\bar{g}_{NN3\pi}^{(1)}}{F_\pi^2}I_e + \frac{5m_\pi^2\bar{g}_{\pi\pi\pi}^{(1)}}{4\pi^2F_\pi^2}((\gamma_1+4\gamma_2)(L'+1)-2\gamma_2) + (Z_N-1)\bar{g}_\pi^{(1)} \\
&+(\sqrt{Z_\pi}-1)_{\rm loop}\bar{g}_\pi^{(1)} \\
\delta(\bar{g}_\pi^{(2)})_{\rm loop} &= \frac{4g_A^2\bar{g}_\pi^{(2)}}{F_\pi^2}I_a + (\frac{8g_{\pi N\Delta}^2\bar{g}_{\Delta\Delta\pi}^{(2,1)}}{9F_\pi^2} + \frac{40g_{\pi N\Delta}^2\bar{g}_{\Delta\Delta\pi}^{(2,2)}}{27F_\pi^2})I_d - (\frac{2\bar{g}_\pi^{(2)}}{F_\pi^2} + \frac{2\bar{g}_{NN3\pi}^{(2,1)}+5\bar{g}_{NN3\pi}^{(2,2)}}{3F_\pi^2})I_e \\
&+(Z_N-1)\bar{g}_\pi^{(2)} + (\sqrt{Z_\pi}-1)_{\rm loop}\bar{g}_\pi^{(2)} \qquad (23)
\end{aligned}
$$

with the loop integral functions $\{I_a\}$ defined in Appendix C. Throughout this paper, the UV-divergence of the loop integral expressed in terms of the quantities L and L' that are defined as

$$L' \equiv L + \ln(\frac{\mu}{m_\pi})^2 \equiv \frac{2}{4-d} - \gamma + \ln 4\pi + \ln(\frac{\mu}{m_\pi})^2. \qquad (24)$$

Similarly, we shall study the one-loop corrections to the hadron mass shifts, i.e. (Δm_π^2), (Δm_N) and (δm_N) defined in Eqs. (18) and (20). The relevant 1PI-diagrams are given in Fig. 2. In the nucleon sector, the most general one-loop corrections to the nucleon sigma term and mass splitting (defined in Eq. (20)) are given by

$$
\begin{aligned}
\delta(\Delta m_N)_{i,{\rm loop}} &= (Z_N-1)(\Delta m_N)_i - \frac{12(\Delta m_N)_i g_A^2}{F_\pi^2}I_a - \frac{8g_{\pi N\Delta}^2(\Delta m_\Delta)_i}{F_\pi^2}I_d + \frac{3(g_{NN\pi\pi}^{(0)})_i}{F_\pi}I_e \\
&+\frac{12(\Delta m_\pi^2)_i g_A^2}{F_\pi^2}I_c + \frac{8g_{\pi N\Delta}^2(\Delta m_\pi^2)_i}{F_\pi^2}I_b - \frac{3m_\pi^2(\Delta m_\pi^2)_i}{8\pi^2F_\pi^3}((\gamma_1+4\gamma_2)(L'+1)-2\gamma_2) \\
&+\delta_{iq}\{\frac{12g_A^2}{F_\pi^2}I_f + \frac{8g_{\pi N\Delta}^2}{F_\pi^2}I_g - \frac{3m_\pi^4}{16\pi^2F_\pi^3}((\gamma_1+4\gamma_2)(L'+1)+\frac{1}{2}\gamma_1)\} \\
\delta(\delta m_N)_{i,{\rm loop}} &= (Z_N-1)(\delta m_N)_i + \frac{4g_A^2(\delta m_N)_i}{F_\pi^2}I_a - \frac{40g_{\pi N\Delta}^2(\delta m_\Delta)_i}{9F_\pi^2}I_d \\
&+\frac{6(g_{NN\pi\pi}^{(1,1))})_i + 2(g_{NN\pi\pi}^{(1,2)})_i}{F_\pi}I_e. \qquad (25)
\end{aligned}
$$

Here the subscript i denotes the specific choice of effective operator that induces the spurion field, e.g. $i=q$ (quark mass), c (quark cMDM/cEDM) and $4q$ (LR4Q). In the pion sector, we will concentrate on the isospin-singlet pion mass shift. For the case of θ-term and dipole operators, this is the only pion mass shift that comes into play. For the case of LR4Q,

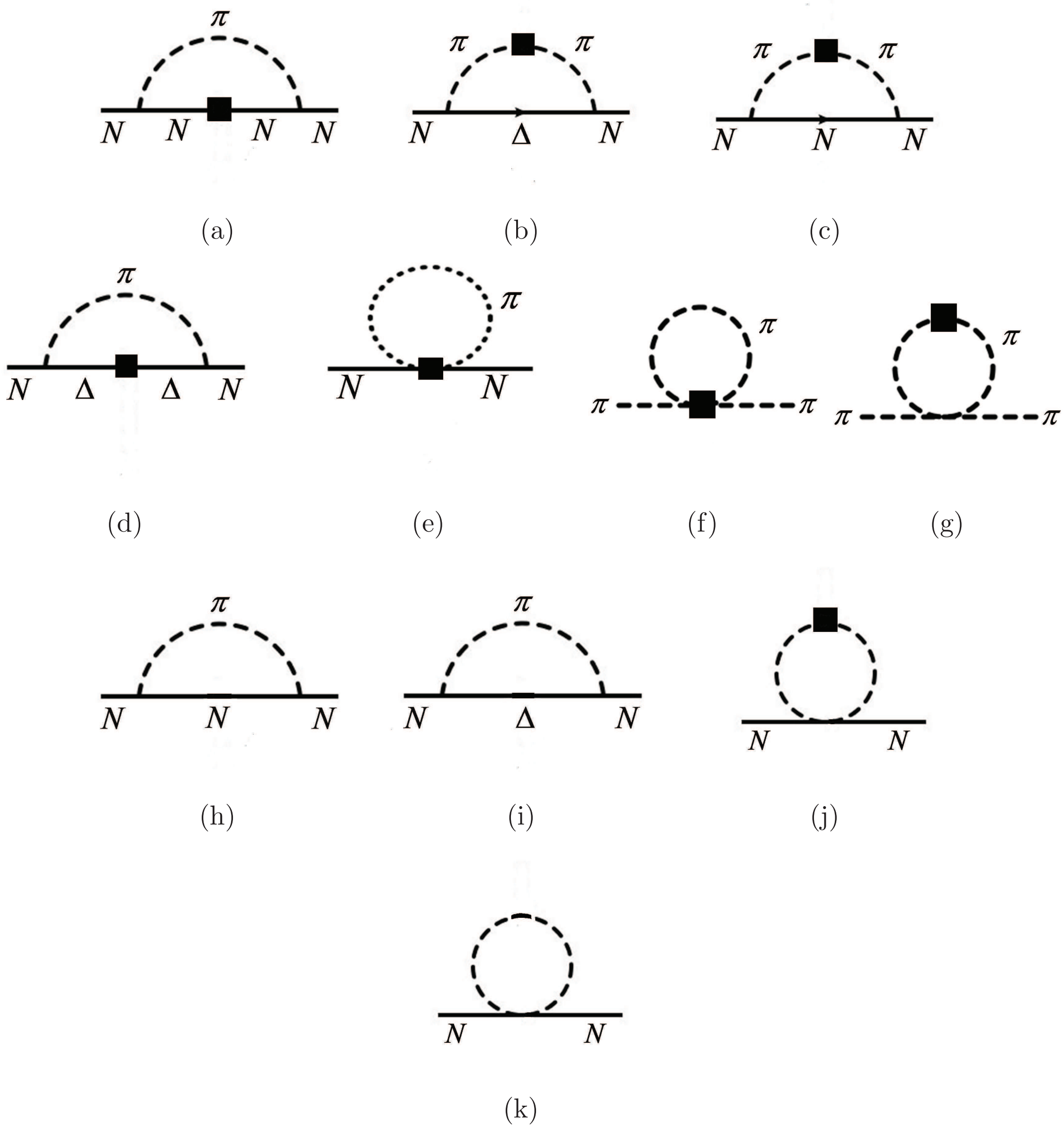

Figure 2: The non-vanishing one-loop amputated diagrams contribution to the hadron mass shifts. Each square denotes a PCTC CSB-interaction defined in Sec. III B. Diagram (j) and (k) involve $O(E^2)$ $NN\pi\pi$ interaction vertex.

although $I = 2$ pion mass shift is also generated, but it is not independent from its isosinglet counterpart. Therefore, we are allowed to choose only the $I = 0$ pion mass shift to enter the matching relations. Its loop correction reads:

$$\delta(\Delta m_\pi^2)_{i,\text{loop}} = -\frac{11(\Delta m_\pi^2)_i m_\pi^2}{24\pi^2 F_\pi^2}(L' + \frac{8}{11}) + \frac{5m_\pi^2(g_{4\pi}^{(0)})_i}{4\pi^2}(L' + 1). \tag{26}$$

V. DISCUSSION OF THE MATCHING RELATIONS OF $\bar{g}_\pi^{(i)}$

With all the preparations in Sec. II-IV we are now in a position to discuss the matching relations between the $\bar{g}_\pi^{(i)}$ and the hadron mass shifts induced by various effective operators. It is obvious that a necessary condition for these matching relations to exist is that the underlying operator should possess both PCTC and PVTV components simultaneously. This simple observation greatly reduces the amount of relevant operators to four types, namely (1) the complex quark mass operator, (2) the dipole-like operators, (3) the LR4Q operator and (4) the two chirally invariant $(\bar{L}R)(\bar{L}R)$-type operators. We shall apply our general formalism in Sec. II-IV to study the influences of these four types of operators separately.

A. Quark mass and QCD θ-term

We shall start with the review of previous studies of the P and T-violation generated by the QCD θ-term (see, e.g. Ref. [12, 18, 23] and references therein). The QCD Lagrangian with a non-zero θ-term takes the following form:

$$\mathcal{L}(\{q_{iR},q_{iL}\}) = \sum_i [\bar{q}_i i\not{D} q_i - \bar{q}_{iR} M_0 q_{iL} - \bar{q}_{iL} M_0 q_{iR}] - \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} - \bar{\theta}\frac{g_s^2}{32\pi^2} G^a_{\mu\nu}\tilde{G}^{a\mu\nu} \tag{27}$$

where $M_0 = \mathrm{diag}(m_1\, m_2\, ...)$ is the real quark mass matrix. Due to the axial anomaly, if we perform an axial rotation $q_i \to e^{i\theta_i\gamma_5} q_i$ to the quark field q_i, the Lagrangian will change as

$$\mathcal{L}(\{q_{iR},q_{iL}\}) \to \mathcal{L}(\{e^{i\theta_i} q_{iR}, e^{-i\theta_i} q_{iL}\}) + \sum_i \theta_i \frac{g_s^2}{16\pi^2} G^a_{\mu\nu}\tilde{G}^{a\mu\nu}. \tag{28}$$

Therefore, for a two-flavor QCD with $Q \equiv (u\, d)^T$, we may perform the following rotation to eliminate the θ-term:

$$Q \to e^{\frac{i}{2}(\frac{\bar{\theta}}{2} - \alpha\tau_3)\gamma_5} Q \tag{29}$$

Here α is so far a free parameter that will be fixed later by the requirement of vacuum stability. The resulting Lagrangian looks like

$$\mathcal{L} = \bar{Q} i\not{D} Q - \bar{Q}_R X_q Q_L - \bar{Q}_L X_q^\dagger Q_R - \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} \tag{30}$$

where now X_q is the complex quark mass matrix that acts as a spurion as described in Sec. II:

$$X_q = \bar{m} e^{-i\frac{\bar{\theta}}{2}} \{\cos\alpha - i\varepsilon\sin\alpha + (-\varepsilon\cos\alpha + i\sin\alpha)\tau_3\} \tag{31}$$

with $\bar{m} = (m_u + m_d)/2$ and $\varepsilon = (m_d - m_u)/(m_u + m_d)$. Throughout this paper we shall take $\bar{m} \approx 3.6\text{MeV}$ and $\varepsilon \approx 0.33$ from lattice calculation [24] whenever their values are needed.

1. *Tree-level matching*

Now we would like to generate PVTV operators in the chiral Lagrangian by appropriate insertions of X_q. In the pure pionic sector, the leading operator at $O(E^2)$ with an X_q-insertion is:

$$\begin{aligned}\frac{F_0^2 B_0}{8}\mathrm{Tr}[X_q U^\dagger + U X_q^\dagger] &= \frac{F_0^2 B_0 \bar{m}}{2}(\cos\alpha\cos\frac{\bar{\theta}}{2} - \varepsilon\sin\alpha\sin\frac{\bar{\theta}}{2})(1 - \frac{2\vec{\pi}^2}{F_0^2} + ...) \\ &\quad + F_0^2 B_0 \bar{m}(\sin\alpha\cos\frac{\bar{\theta}}{2} + \varepsilon\cos\alpha\sin\frac{\bar{\theta}}{2})(1 - \frac{2\vec{\pi}^2}{3F_0^2} + ...)\frac{\pi_0}{F_0}. \quad (32)\end{aligned}$$

where F_0 is just F_π in the chiral limit.

Note that the existence of the π_0 term makes the vacuum unstable as one may lower the energy of the system indefinitely by keep creating neutral pions from the vacuum. To avoid that, we simply impose the "vacuum alignment" condition that says the value of α should be chosen such that the π_0 term vanishes [25, 26]. For the case that the θ-term is the only source of T-violation, the vacuum alignment condition is simply $\alpha \approx -\varepsilon\bar{\theta}/2$ assuming $\bar{\theta}$ is small. The complex quark mass matrix X_q then turns into:

$$X_q = M_0 - i\frac{\bar{m}}{2}(1 - \varepsilon^2)\bar{\theta}. \quad (33)$$

After imposing the alignment condition we obtain $m_\pi^2 = 2B_0\bar{m}$ where m_π^2 is defined as the squared mass of charged pions $\pi_\pm$ (which leads to $B_0 \approx 2.7\text{GeV}$ if we take the lattice value for $\bar{m}$ [24]). The mass splitting between charged and neutral pion occurs at $O(E^4)$ level. Also, note that the requirement of vacuum alignment kills the π_0 term as well as all other terms that have an odd number of pions. Therefore the term in Eq. (32) does not give any T-violating operator. Instead, we obtain an isospin-invariant mass term for the pion triplet. Also, an important feature one observes is that, after vacuum alignment the PVTV term in $\tilde{X}_{q+}$ is an isoscalar (recall the definition of $\tilde{X}_\pm$ in Eq. (8) and the subscript q denotes the contribution from the complex quark mass matrix). That implies, at the leading order of m_π^2 expansion, the PVTV interactions induced by the θ-term are all isoscalars.

In the nucleon sector, the leading term that gives a non-zero T-violating effect is[3]:

$$c_1 \bar{N}\tilde{X}_{q+}N + c_1' \mathrm{Tr}[\tilde{X}_{q+}]\bar{N}N \;=\; 2\bar{m}(c_1+2c_1')(\bar{N}N+...) - 2\bar{m}\varepsilon c_1(\bar{N}\tau_3 N+...) \\ -\frac{2\bar{m}(1-\varepsilon^2)\bar{\theta}c_1}{F_0}(1-\frac{2\vec{\pi}^2}{3F_0^2}+...)\bar{N}\vec{\tau}\cdot\vec{\pi}N. \tag{34}$$

where $+\cdots$ denotes terms additional pion fields. This simply corresponds to the pion-nucleon Lagrangian with chiral index $\Delta = 1$ by Mereghetti et al [12]. The first term contributes to the nucleon sigma term $(\Delta m_N)_q = 2\bar{m}(c_1 + 2c_1')$ while the second term contributes to the nucleon mass splitting $(\delta m_N)_q = -4\bar{m}\varepsilon c_1$. The third term contributes to $\bar{g}_\pi^{(0)}$ and $\bar{g}_{NN3\pi}^{(0)}$ with $\bar{g}_\pi^{(0)} = -2\bar{m}(1-\varepsilon^2)\bar{\theta}c_1/F_0$ and $\bar{g}_{NN3\pi}^{(0)} = -\frac{2}{3}\bar{g}_\pi^{(0)}$ (it is interesting to note that, in the SO(4)-representation of ChPT, e.g. in Ref. [12], the relation between $\bar{g}_{NN3\pi}^{(0)}$ and $\bar{g}_\pi^{(0)}$ is apparently different: $\bar{g}_{NN3\pi}^{(0)} = -\bar{g}_\pi^{(0)}$. However one is able to show their equivalence using the equation of motion (EOM) . Now since both $(\delta m_N)_q$ and $\bar{g}_\pi^{(0)}$ depend linearly on c_1, one may relate them as:

$$F_\pi \bar{g}_\pi^{(0)} = \frac{1-\varepsilon^2}{2\varepsilon}(\delta m_N)_q \bar{\theta}. \tag{35}$$

Notice that we have made use of the fact that $F_\pi = F_0$ at leading order. This replacement is crucial so that the same equation holds even when higher-order corrections to the pion decay constant are included, as we shall discuss later. Eq. (35) is exactly the tree-level matching relation between $\bar{g}_\pi^{(0)}$ and $(\delta m_N)_q$. The same procedure is used to determine all other matching relations at tree level so we may skip the intermediate steps when we introduce them later.

In the Δ-sector we have an analogous leading term that gives T-violation:

$$c_2 \bar{T}^i_\mu \tilde{X}_{q+} T^{i\mu} + c_2' \mathrm{Tr}[\tilde{X}_{q+}]\bar{T}^i_\mu T^{i\mu} \tag{36}$$

and we have an analogous coefficient matching:

$$F_\pi \bar{g}_{\Delta\Delta\pi}^{(0)} \;=\; \frac{1-\varepsilon^2}{2\varepsilon}(\delta m_\Delta)_q \bar{\theta} \tag{37}$$

2. Loop correction

Next we shall consider the one-loop correction to the left-hand-side (LHS) and right-hand-side (RHS) of the tree-level matching relation (35). The loop correction to $F_\pi \bar{g}_\pi^{(0)}$ and

[3] The operator coefficients are related to those in Bernard, Kaiser and Meißner [27] by $c_1 = 2B_0 c_5^{\mathrm{BKM}}$, $c_1' = 2B_0 c_1^{\mathrm{BKM}}$.

$(\delta m_N)_q$ are given in Eq. (E1) and (E3) in Appendix E respectively. The former is expressed in terms of $\bar{g}_\pi^{(0)}$ and $\bar{g}_{\Delta\Delta\pi}^{(0)}$ that can be related to $(\delta m_N)_q$ and $(\delta m_\Delta)_q$ by Eq. (35) and (37) respectively because any correction is of higher order in power counting. With these we obtain

$$\delta(F_\pi \bar{g}_\pi^{(0)})_{\text{loop}} = \frac{1-\varepsilon^2}{2\varepsilon}\bar{\theta}\delta(\delta m_N)_{q,\text{loop}}, \tag{38}$$

i.e., the tree-level matching formula (35) for $\bar{g}_\pi^{(0)}$ is preserved at one-loop.

3. LECs and the higher-order matching formula

Higher-order terms in the chiral Lagrangian must be introduced to cancel the UV-divergence in the loop corrections and make the full expression μ-independent. Apart from the baryon wavefunction and F_π renormalization that are well-known, for the case of the θ-term CPV source we only need the $O(E^4)$ terms that involve two insertions of $\tilde{X}_{q\pm}$. Such terms in the pure pionic sector are introduced in Appendix D. In the nucleon sector we have:

$$\begin{aligned}\mathcal{L}_N^{O(E^4)} &= F_\pi^{-3}B_0^2\{f_1\text{Tr}[\tilde{X}_{q+}^2]\bar{N}N + f_2\text{Tr}[\tilde{X}_{q+}]\bar{N}\tilde{X}_{q+}N + f_3\bar{N}\tilde{X}_{q+}^2N \\ &\quad + f_4\text{Tr}[\tilde{X}_{q-}^2]\bar{N}N + f_5\text{Tr}[\tilde{X}_{q-}]\bar{N}\tilde{X}_{q-}N + f_6\bar{N}\tilde{X}_{q-}^2N\} + ...\end{aligned} \tag{39}$$

Details of the $O(E^4)$ contribution to $F_\pi\bar{g}_\pi^{(i)}$ and the hadron masses are summarized in Appendix E 1. After some rearrangement we are able to match the final result with Eq. (1) as:

$$F_\pi\bar{g}_\pi^{(0)} = \frac{1-\varepsilon^2}{2\varepsilon}\bar{\theta}(\delta m_N)_q(1+\delta_{\text{LEC}}^{(0)}) \tag{40}$$

where the relative deviation from the LECs is given by:

$$\delta_{\text{LEC}}^{(0)} = -\frac{4m_\pi^4\varepsilon}{(\delta m_N)_q F_\pi^3}(f_5^r + f_6^r) - \frac{64m_\pi^2\varepsilon^2}{F_\pi^2}(2L_7^r + L_8^r) \tag{41}$$

Throughout this paper, we use the superscript "r" to represent a renormalized quantity of which the infinite value $L+1$ is subtracted from the corresponding bare quantity following the Gasser-Leutwyler subtraction scheme [28], i.e. a bare quantity A and its renormalized value A^r are related by $A = A^r + B(L+1)$ where B is a finite number. The absence of $\delta_{\text{loop}}^{(0)}$ shows that the LO-matching formula is modified at higher order but the modification is analytic in the quark masses (*i.e.*, not logarithmic with respect to pion masses). This relation has already been studied under the SU(3) version of ChPT in Ref. [18].

One may perform a quick estimation of the size of $\delta^{(0)}_{\rm LEC}$ using lattice results and dimensional analysis arguments. First, for the contribution from L^r_i, we note that it contains a large prefactor 64 but is also suppressed by the square of the isospin breaking parameter ε. Furthermore, we have $L^r_{7,8} \sim 10^{-3}$ from meson data fits [29]. That gives a contribution of order 10^{-3} to $\delta^{(0)}_{\rm LEC}$ that is very small. On the other hand, the impact of f^r_i is less transparent because the sizes of $f^r_{5,6}$ are not well-determined. Here we shall estimate their order of magnitude based on chiral power counting. For instance, one may compare the contribution of $O(p^2)$ (i.e. linear to $\bar{m}$) and $O(p^4)$ (i.e. quadratic to $\bar{m}$) contribution to the nucleon mass; they are proportional to $\bar{m}c_i$ and $F_\pi^{-3}B_0^2\bar{m}^2 f_i$ respectively, as one is able to read off from Eqs. (34) and (39)[4]. Chiral power-counting suggests that the latter should be suppressed with respect to the former by a factor of order $(m_\pi/2\pi F_\pi)^2$. This implies

$$F_\pi^{-3}B_0^2\bar{m}^2 f_i \sim (m_\pi/2\pi F_\pi)^2 c_i \bar{m} \tag{42}$$

which leads to $f_i \sim 3\times 10^{-3}$ for $c_i \sim 1$. This gives $\delta^{(0)}_{\rm LEC} \sim 0.1$ which is a 10% correction to the tree-level matching relation.

B. The dipole-like operators

Next we shall study the effect of the leading flavor-diagonal dipole-like operator namely the quark cMDM/cEDM. As discussed in Sec. II, the spurion for this operator is simply identical to the one for the complex quark mass. The Lagrangian in the quark-gluon level reads

$$\begin{aligned}\mathcal{L} &= \sum_{q=u,d} g_s \tilde{d}^M_q \bar{q}\sigma^{\mu\nu}\frac{\lambda^a}{2}G^a_{\mu\nu}q - i\sum_{q=u,d} g_s \tilde{d}_q \bar{q}\sigma^{\mu\nu}\gamma_5\frac{\lambda^a}{2}G^a_{\mu\nu}q \\ &= g_s\bar{Q}_R\sigma^{\mu\nu}\frac{\lambda^a}{2}G^a_{\mu\nu}X_c Q_L + h.c\end{aligned} \tag{43}$$

where $\tilde{d}^M_q$ and $\tilde{d}_q$ are the cMDM and cEDM of the quark q respectively. The matrix X_c acts as the spurion for cMDM/cEDM as described in Sec. II, and is given by

$$X_c = \frac{1}{2}(\tilde{d}^M_0 + i\tilde{d}_0) + \frac{1}{2}(\tilde{d}^M_1 + i\tilde{d}_1)\tau_3 \tag{44}$$

[4] The factor $F_\pi^{-3}B_0^2$ is just due to the definition of the coefficients of the $O(p^4)$ counterterms f_i so that they are dimensionless.

where $\tilde{d}_0(\tilde{d}_0^M) = \tilde{d}_u(\tilde{d}_u^M) + \tilde{d}_d(\tilde{d}_d^M)$ and $\tilde{d}_1(\tilde{d}_1^M) = \tilde{d}_u(\tilde{d}_u^M) - \tilde{d}_d(\tilde{d}_d^M)$ are the isoscalar and isovector cEDM (cMDM) respectively.

1. *Tree-level matching*

The implementation of the spurion X_c into the chiral Lagrangian works exactly in exactly the same way as the complex quark mass. In the pionic sector the only operator at lowest order is:

$$\beta F_0^5 \mathrm{Tr}[X_c U^\dagger + U X_c^\dagger] \;=\; 2\beta F_0^5 \tilde{d}_0^M (1 - \frac{2\vec{\pi}^2}{F_0^2} + ...) + 4\beta F_0^5 \tilde{d}_1 (1 - \frac{2\vec{\pi}^2}{3F_0^2} + ...) \frac{\pi_0}{F_0} \tag{45}$$

where β is a dimensionless constant. This operator will again induce a pion tadpole term that makes the vacuum unstable. In order to cancel this term, we have to include the term with an X_q-insertion given in Eq. (32) and choose an appropriate value for the free parameter α to eliminate the π_0 term induced by Eq. (45). Assuming no θ-term, we obtain a pion mass shift $(\Delta m_\pi^2)_c = 8\beta F_0^3 \tilde{d}_0^M$ as well as the vacuum alignment condition

$$\alpha \approx (-4\beta F_0^3 \tilde{d}_1)/(B_0 \bar{m}) = -(\Delta m_\pi^2)_c \tilde{d}_1/(m_\pi^2 \tilde{d}_0^M) \tag{46}$$

. The non-zero value of α leads to an interesting consequence, namely: in order to study the effect of T-violation induced by the cEDM, it is not enough to consider only terms with X_c insertions. One needs to include all the terms with X_q insertions as given in Sec. V A because the quark mass picks up a complex phase α even without the existence of a θ-term. Also, α is related but should not be confused with the so-called "induced θ-term" introduced by Pospelov and Ritz [30]. To see their relation, we take the complex quark mass matrix X_q defined in Eq. (31) (without $\bar{\theta}$ just for simplicity) and expand it to the first power in α. After plugging in the explicit expression of α one immediately sees that X_q takes the following form

$$X_q = M_0 + i\frac{\bar{m}}{2}(1-\varepsilon^2)\bar{\theta}_{\mathrm{ind}} - i\frac{2(\Delta m_\pi^2)_c}{\tilde{d}_0^M}\frac{\bar{m}}{m_\pi^2}\tilde{\mathrm{D}} \tag{47}$$

where M_0 is the real quark mass matrix and $\tilde{\mathrm{D}} = \mathrm{diag}(\tilde{d}_u\ \tilde{d}_d)$ is the cEDM matrix. The second term in Eq. (47) has the same form as the second term of Eq. (33) and defines an "induced θ angle" whose value is given by

$$\bar{\theta}_{\mathrm{ind}} = 2(\Delta m_\pi^2)_c(\tilde{d}_0 + \varepsilon \tilde{d}_1)/(m_\pi^2(1-\varepsilon^2)\tilde{d}_0^M) \quad . \tag{48}$$

In the presence of a θ-term one needs only to replace $\bar{\theta}_{\rm ind}$ by $\bar{\theta}_{\rm ind} - \bar{\theta}$ (see Eq. (33)). Furthermore, if we assume Peccei-Quinn mechanism [31] then $\bar{\theta}$ simply relaxes to $\bar{\theta}_{\rm ind}$.

We may now construct other PVTV chiral operators induced by the quark cEDM remembering that they can be generated by either the X_c or the X_q-insertion. It should be pointed out that the tree-level matching relations we present below are already well-studied previously using the chiral SO(4) formalism [11]. Here we recast the analysis using the $\mathrm{SU}(2)_{\rm L} \times \mathrm{SU}(2)_{\rm R}$ formalism and also generalize it to include the Δ-resonances to show how the same physics works under different representations. Also, our method has the advantage that it can be generalized more easily to the three-flavor case in order to study the role of the strange quark in the matching relations. In the nucleon sector the leading CSB operators are

$$c_1 \bar{N}\tilde{X}_{q+}N + c_1' \mathrm{Tr}[\tilde{X}_{q+}]\bar{N}N + \tilde{c}_1 F_0^2 \bar{N}\tilde{X}_{c+}N + \tilde{c}_1' F_0^2 \mathrm{Tr}[\tilde{X}_{c+}]\bar{N}N. \tag{49}$$

Following the same logic as in Sec. V A 1, one finds the following tree-level matching relations:

$$\begin{aligned} F_\pi \bar{g}_\pi^{(0)} &= -(\delta m_N)_q \frac{(\Delta m_\pi^2)_c}{m_\pi^2}\frac{\tilde{d}_1}{\tilde{d}_0^M} + (\delta m_N)_c \frac{\tilde{d}_0}{\tilde{d}_1^M} \\ F_\pi \bar{g}_\pi^{(1)} &= 2\left[-(\Delta m_N)_q \frac{(\Delta m_\pi^2)_c}{m_\pi^2} + (\Delta m_N)_c\right]\frac{\tilde{d}_1}{\tilde{d}_0^M} \end{aligned} \tag{50}$$

as well as $\bar{g}_{NN3\pi}^{(0,1)} = -2\bar{g}_\pi^{(0,1)}/3$. One observes that $\bar{g}_\pi^{(0)}$ depends on both $\tilde{d}_0$ and $\tilde{d}_1$ while $\bar{g}_\pi^{(1)}$ depends only on $\tilde{d}_1$. However, if we take lattice calculations [32, 33] that give $(\Delta m_N)_q \approx -37$MeV and $(\delta m_N)_q \approx 2.26$MeV then we find that $\bar{g}_\pi^{(1)}$ is about 30 times more sensitive to $\tilde{d}_1$ than $\bar{g}_\pi^{(0)}$ provided that there is no accidental cancelation between the two terms in $\bar{g}_\pi^{(1)}$.

In the Δ sector, the most general terms at leading order are:

$$c_2 \bar{T}^i_\mu \tilde{X}_{q+} T^{i\mu} + c_2' \mathrm{Tr}[\tilde{X}_{q+}]\bar{T}^i_\mu T^{i\mu} + \tilde{c}_2 F_0^2 \bar{T}^i_\mu \tilde{X}_{c+} T^{i\mu} + \tilde{c}_2' F_0^2 \mathrm{Tr}[\tilde{X}_{c+}]\bar{T}^i_\mu T^{i\mu} \tag{51}$$

that lead to analogous tree-level matching relations[5]:

$$\begin{aligned} F_\pi \bar{g}_{\Delta\Delta\pi}^{(0)} &= -(\delta m_\Delta)_q \frac{(\Delta m_\pi^2)_c}{m_\pi^2}\frac{\tilde{d}_1}{\tilde{d}_0^M} + (\delta m_\Delta)_c \frac{\tilde{d}_0}{\tilde{d}_1^M} \\ F_\pi \bar{g}_{\Delta\Delta\pi}^{(1,1)} &= -2\left[-(\Delta m_\Delta)_q \frac{(\Delta m_\pi^2)_c}{m_\pi^2} + (\Delta m_\Delta)_c\right]\frac{\tilde{d}_1}{\tilde{d}_0^M} \\ &= F_\pi \bar{g}_{\Delta\Delta\pi}^{(1,2)} \end{aligned} \tag{52}$$

[5] One can easily show that $i\epsilon^{abc}\pi_b[\bar{T}^a_\mu \tau_c T^{3\mu} - \bar{T}^3_\mu \tau_c T^{a\mu}] + i\epsilon^{ab3}\bar{T}^a_\mu \vec{\tau}\cdot\vec{\pi} T^{b\mu} = -\pi_0 \bar{T}^i_\mu T^{i\mu}$.

2. Loop correction

The one-loop corrections to the LHS and RHS of the tree-level matching relations (50) can be inferred from Eq. (E8) and (E9) in Appendix E. After some straightforward rearrangement, one obtains:

$$\begin{aligned}
\delta(F_\pi\bar{g}_\pi^{(0)})_{\rm loop} &= \delta\left[-(\delta m_N)_q\frac{(\Delta m_\pi^2)_c}{m_\pi^2}\frac{\tilde{d}_1}{\tilde{d}_0^M}+(\delta m_N)_c\frac{\tilde{d}_0}{\tilde{d}_1^M}\right]_{\rm loop}-(\delta m_N)_q\frac{\tilde{d}_1}{\tilde{d}_0^M}\frac{(\Delta m_\pi^2)_c}{8\pi^2F_\pi^2}L' \\
\delta(F_\pi\bar{g}_\pi^{(1)})_{\rm loop} &= 2\delta\left[-(\Delta m_N)_q\frac{(\Delta m_\pi^2)_c}{m_\pi^2}+(\Delta m_N)_c\right]_{\rm loop}\frac{\tilde{d}_1}{\tilde{d}_0^M} \\
&\quad +2(\Delta m_\pi^2)_c\frac{\tilde{d}_1}{\tilde{d}_0^M}\left[-\frac{(\Delta m_N)_q}{8\pi^2F_\pi^2}L'-4\{\frac{3g_A^2}{F_\pi^2}(I_c-\frac{I_f}{m_\pi^2})+\frac{2g_{\pi N\Delta}^2}{F_\pi^2}(I_b-\frac{I_g}{m_\pi^2})\}\right] \\
&\quad +\frac{3m_\pi^2(\Delta m_\pi^2)_c}{8\pi^2F_\pi^3}\frac{\tilde{d}_1}{\tilde{d}_0^M}\left((\gamma_1+4\gamma_2)(L'+1)-\frac{1}{2}\gamma_1-4\gamma_2\right) \qquad (53)
\end{aligned}$$

i.e., the one-loop corrections do not obey the tree-level matching relations and the induced mismatch between the LHS and RHS of the relations are proportional to $(\Delta m_\pi^2)_c$. For the case of $I=0$, the tree-level matching is preserved by the loop correction only in the $\tilde{d}_1\to 0$ limit.

3. LECs and the higher-order matching formula

The relevant LECs that are needed to cancel the UV-divergences in the loop diagrams are in the $O(E^4)$ Lagrangian and the $O(E^2\tilde{d})$ terms in the chiral Lagrangian. The former have already been discussed in Sec. V A 3 so we shall concentrate on the latter. In the pionic sector the relevant $O(E^2\tilde{d})$ Lagrangian is

$$\begin{aligned}
\mathcal{L}_\pi^{O(E^2\tilde{d})} &= 2B_0F_\pi^3\{G_1{\rm Tr}[X_qU^\dagger+UX_q^\dagger]{\rm Tr}[X_cU^\dagger+UX_c^\dagger]+G_2{\rm Tr}[X_qU^\dagger-UX_q^\dagger]{\rm Tr}[X_cU^\dagger \\
&\quad -UX_c^\dagger]+G_3{\rm Tr}[U^\dagger X_cU^\dagger X_q+UX_c^\dagger UX_q^\dagger]\}+F_\pi^3\{G_4{\rm Tr}[\partial_\mu U\partial^\mu U^\dagger]{\rm Tr}[X_cU^\dagger+UX_c^\dagger] \\
&\quad +G_5{\rm Tr}[\partial_\mu U^\dagger\partial^\mu U(X_c^\dagger U+U^\dagger X_c)]\}. \qquad (54)
\end{aligned}$$

In the equation above, the E^2 factor comes either from a factor of X_q or two derivatives. In particular, the G_4 and G_5 terms are required as they cancel divergences of both F_π and Z_π that receive extra loop corrections due to the generation of $(\Delta m_\pi^2)_c$. In the nucleon sector, the $O(E^2\tilde{d})$ Lagrangian can be chosen as

$$\begin{aligned}\mathcal{L}_N^{O(E^2\tilde{d})} &= 2B_0\{g_1\mathrm{Tr}[\tilde{X}_{q+}\tilde{X}_{c+}]\bar{N}N + g_2\mathrm{Tr}[\tilde{X}_{q+}]\bar{N}\tilde{X}_{c+}N + g_3\mathrm{Tr}[\tilde{X}_{c+}]\bar{N}\tilde{X}_{q+}N \\ &+g_4\bar{N}\{\tilde{X}_{q+},\tilde{X}_{c+}\}N + g_5\mathrm{Tr}[\tilde{X}_{q-}\tilde{X}_{c-}]\bar{N}N + g_6\mathrm{Tr}[\tilde{X}_{q-}]\bar{N}\tilde{X}_{c-}N \\ &+g_7\mathrm{Tr}[\tilde{X}_{c-}]\bar{N}\tilde{X}_{q-}N + g_8\bar{N}\{\tilde{X}_{q-},\tilde{X}_{c-}\}N\}. \qquad (55)\end{aligned}$$

With all these, one can straightforwardly deduce the modified matching formula for $\bar{g}_\pi^{(i)}$. While all details are given in Appendix E 2, the final outcome is:

$$\begin{aligned}F_\pi\bar{g}_\pi^{(0)} &= \left(-(\delta m_N)_q\frac{(\Delta m_\pi^2)_c}{m_\pi^2}\frac{\tilde{d}_1}{\tilde{d}_0^M} + (\delta m_N)_c\frac{\tilde{d}_0}{\tilde{d}_1^M}\right)(1+\delta_{\mathrm{loop}}^{(0)}+\delta_{\mathrm{LEC}}^{(0)}) \\ F_\pi\bar{g}_\pi^{(1)} &= 2\left(-(\Delta m_N)_q\frac{(\Delta m_\pi^2)_c}{m_\pi^2} + (\Delta m_N)_c\right)\frac{\tilde{d}_1}{\tilde{d}_0^M}(1+\delta_{\mathrm{loop}}^{(1)}+\delta_{\mathrm{LEC}}^{(1)}) \qquad (56)\end{aligned}$$

where the relative corrections $\{\delta^{(i)}\}$ due to loop and higher order LECs are given by

$$\begin{aligned}\delta_{\mathrm{loop}}^{(0)} &= \frac{m_\pi^2}{8\pi^2F_\pi^2}\left(-1+\ln(\frac{\mu}{m_\pi})^2\right)\left(1-\frac{m_\pi^2}{(\delta m_N)_q}\frac{(\delta m_N)_c}{(\Delta m_\pi^2)_c}\frac{\tilde{d}_0}{\tilde{d}_1^M}\frac{\tilde{d}_0^M}{\tilde{d}_1}\right)^{-1} \\ \delta_{\mathrm{loop}}^{(1)} &= \left(\frac{3g_A^2m_\pi^3}{16\pi(\Delta m_N)_qF_\pi^2} + \frac{8g_{\pi N\Delta}^2m_\pi^2}{(\Delta m_N)_qF_\pi^2}(I_b^r-\frac{I_g^r}{m_\pi^2}) - \frac{3m_\pi^4}{16\pi^2(\Delta m_N)_qF_\pi^3}((\gamma_1+4\gamma_2)\ln(\frac{\mu}{m_\pi})^2\right. \\ &\left.-\frac{1}{2}\gamma_1-4\gamma_2) + \frac{m_\pi^2}{8\pi^2F_\pi^2}(-1+\ln(\frac{\mu}{m_\pi})^2)\right)\left(1-\frac{m_\pi^2}{(\Delta m_N)_q}\frac{(\Delta m_N)_c}{(\Delta m_\pi^2)_c}\right)^{-1} \qquad (57)\end{aligned}$$

and

$$
\begin{aligned}
\delta^{(0)}_{\rm LEC} \;=\; & \Bigg(-\frac{4m_\pi^4\varepsilon}{3(\delta m_N)_q F_\pi^3}[f_2^r + f_3^r + 4(f_5^r + f_6^r)] + \frac{8m_\pi^4 \tilde{d}_0^M}{(\delta m_N)_q (\Delta m_\pi^2)_c}[\varepsilon(\frac{1}{3} - \frac{\tilde{d}_0^M}{\tilde{d}_1^M}\frac{\tilde{d}_0}{\tilde{d}_1})(g_3^r + g_4^r) \\
& -\frac{\tilde{d}_0}{\tilde{d}_1}(g_7^r + g_8^r) + \frac{\varepsilon}{3}(g_6^r + g_8^r)] - \frac{16F_\pi m_\pi^2 \tilde{d}_0^M}{(\Delta m_\pi^2)_c}[(2G_1^r + G_3^r - 2G_4^r - G_5^r) \\
& -\frac{2}{3}\varepsilon(\frac{\tilde{d}_1^M}{\tilde{d}_0^M} - \frac{3}{2}\frac{\tilde{d}_0}{\tilde{d}_1})(2G_2^r + G_3^r)] - \frac{64m_\pi^2}{F_\pi^2}\varepsilon^2(2L_7^r + L_8^r)\Bigg) \times \\
& \left(1 - \frac{m_\pi^2}{(\delta m_N)_q}\frac{(\delta m_N)_c}{(\Delta m_\pi^2)_c}\frac{\tilde{d}_0}{\tilde{d}_1^M}\frac{\tilde{d}_0^M}{\tilde{d}_1}\right)^{-1} \\
\delta^{(1)}_{\rm LEC} \;=\; & \Bigg(\frac{m_\pi^4}{(\Delta m_N)_q F_\pi^3}[2(1+\varepsilon^2)f_1^r + 2f_2^r + (1+\varepsilon^2)f_3^r + 4(1+\varepsilon^2)f_4^r + 2(\varepsilon^2 - 1)f_5^r \\
& +2(1+\varepsilon^2)(f_5^r + f_6^r)] + \frac{4m_\pi^4\tilde{d}_0^M}{(\Delta m_N)_q(\Delta m_\pi^2)_c}[\varepsilon\frac{\tilde{d}_0}{\tilde{d}_1}(g_1^r + g_4^r + g_5^r + g_6^r + g_7^r + g_8^r) \\
& -(g_5^r + g_8^r) - \varepsilon\frac{\tilde{d}_1^M}{\tilde{d}_0^M}(g_1^r + g_4^r)] - \frac{16F_\pi m_\pi^2\tilde{d}_0^M}{(\Delta m_\pi^2)_c}[(2G_1^r + G_3^r - 2G_4^r - G_5^r) \\
& -\frac{2}{3}\varepsilon(\frac{\tilde{d}_1^M}{\tilde{d}_0^M} - \frac{3}{2}\frac{\tilde{d}_0}{\tilde{d}_1})(2G_2^r + G_3^r)] - \frac{64m_\pi^2}{F_\pi^2}\varepsilon^2(2L_7^r + L_8^r)\Bigg) \times \\
& \left(1 - \frac{m_\pi^2}{(\Delta m_N)_q}\frac{(\Delta m_N)_c}{(\Delta m_\pi^2)_c}\right)^{-1} \qquad (58)
\end{aligned}
$$

respectively. The functions $\{I_i^r\}$ are just the renormalized version of the loop functions $\{I_i\}$ defined in Appendix C following the Gasser-Leutwyler subtraction scheme [28].

One may perform a numerical estimation of the loop corrections to the tree-level matching relations upon neglecting the unknown matrix elements $(\Delta m_N)_c$ and $(\delta m_N)_c$. In the isoscalar channel, we find that $\delta^{(0)}_{\rm loop} \approx 0.021$ (taking $\mu = 1$GeV for the renormalization scale) therefore one has good convergence. On the hand, in the isovector channel we have $\delta^{(0)}_{\rm loop} \approx -3.1$ that does not show any sign of convergence . The reason is that $(\Delta m_N)_q \approx -37$MeV is much smaller than δ_Δ and m_π, so terms in Eq. (58) such as

$$
(\delta_\Delta/(\Delta m_N)_q)\ln(\frac{\mu}{m_\pi})^2, \quad (\delta_\Delta/(\Delta m_N)_q)(\delta_\Delta/m_\pi)^2\ln(\frac{\mu}{m_\pi})^2, \quad (m_\pi/(\Delta m_N)_q)\ln(\frac{\mu}{m_\pi})^2
$$

may overcome the usual chiral suppression. This implies that the matching formula for $I = 1$ cEDM has very limited practical use. Fortunately, there is a recent study by de Vries et al suggesting that the effect of $\delta^{(i)}_{\rm loop}$ can be completely get rid of by re-expressing the tree-level matching relations (50) in terms of derivative operators [34].

The impact of higher-order LECs encoded in $\delta^{(i)}_{\text{LEC}}$ can also be studied following the power-counting argument in Sec. V A 3. To make the discussion tractable, let us assume $\tilde{d}^M_0 \sim \tilde{d}^M_1$, $\tilde{d}_0 \sim \tilde{d}_1$ and take the denominators in $\delta^{(i)}_{\text{LEC}}$ to be $O(1)$. The contribution from L^r_i is negligible as we discussed before. The contribution from f^r_i to $\delta^{(0)}_{\text{LEC}}$ is around 0.1, similar to the case of θ-term, while its contribution to $\delta^{(1)}_{\text{LEC}}$ is expected to be much smaller because it is divided by $(\Delta m_N)_q$ instead of $(\delta m_N)_q$. New LECs appeared in Eq. (58) are $\{G^r_i\}$ and $\{g^r_i\}$. The estimation of their sizes involves two steps: first, the contribution from the $O(\tilde{d})$ Lagrangian (45) and (49) to the pion and nucleon masses can be estimated using Weinberg's counting rule [35]. Then, the effects from $\{G_i\}$ and $\{g_i\}$ are expected to receive a further $(m_\pi/2\pi F_\pi)^2$ suppression due to chiral power-counting. This implies, using Eq. (E12) and (E13):

$$\begin{aligned}
(\Delta m^2_\pi)^r_{c,ct} &\sim 16 m^2_\pi F_\pi \tilde{d}^M_i G^r_i \sim (\frac{m_\pi}{2\pi F_\pi})^2 (\Delta m^2_\pi)_c \sim (\frac{m_\pi}{2\pi F_\pi})^2 \frac{(2\pi F_\pi)^3 \tilde{d}^M_i}{4\pi} \\
(\Delta m_N)^r_{c,ct} &\sim 4 m^2_\pi \tilde{d}^M_i g^r_i \sim (\frac{m_\pi}{2\pi F_\pi})^2 (\Delta m_N)_c \sim (\frac{m_\pi}{2\pi F_\pi})^2 \frac{(2\pi F_\pi)^2 \tilde{d}^M_i}{4\pi}
\end{aligned} \tag{59}$$

which gives $G^r_i \sim 0.03$ and $g^r_i \sim 0.02$. Applying them to Eq. (58), we find that the contributions from g^r_i to $\delta^{(0)}_{\text{LEC}}$ is around 0.2 and all other effects are of order 10^{-2}. Hence the only potentially-large LEC corrections to the tree-level matching relations are the f^r_i and g^r_i-correction to $\bar{g}^{(0)}_\pi$.

Finally we shall mention briefly about the quark MDM/EDM operator $\bar{q}_R \sigma^{\mu\nu} q_L F_{\mu\nu}$. Although its form is analogous to that of the quark cMDM/cEDM, it involves an interaction with photon that in turn introduces another CSB quantity, namely the quark charge matrix. As a consequence, the chiral structure of the resulting hadronic operators is much more complicated. Interested readers are referred to Ref. [11] for more discussion.

C. LR4Q

The last type of chirally non-invariant operator that contains both PCTC and PVTV components simultaneously is the LR4Q operator. The form of its corresponding spurion is already explained in Sec. II so we shall go straight its application.

1. Tree-level matching

In the pionic sector, the only LO-operator we can write down is

$$\mathcal{L} = \rho F_0^6 (c_{4q} \mathrm{Tr}[U^\dagger X_{RL}] \mathrm{Tr}[U X_{LR}] + c_{4q}^* \mathrm{Tr}[U^\dagger X_{LR}^\dagger] \mathrm{Tr}[U X_{RL}^\dagger]) \tag{60}$$

where $X_{RL} = (1+\tau_3)/2$, $X_{LR} = (1-\tau_3)/2$ are the LR4Q spurion matrices defined after Eq. (13), c_{4q} is the complex Wilson coefficient of the LR4Q operator and ρ is a real dimensionless number. Again, this Lagrangian induces a pion tadpole term that must be removed by including the term with an X_q-insertion and choosing the value of the free parameter α appropriate. This imposes the vacuum alignment condition $\alpha \approx -8\rho F_0^4 \mathrm{Im}c_{4q}/(B_0 \bar{m}) = -16\rho F_0^4 \mathrm{Im}c_{4q}/m_\pi^2$. Therefore, to study the T-odd effect from the LR4Q operator, we need to also include the contribution from X_q with the value of α chosen above. With these, the relevant quantities one could extract from Eq. (60) are:

$$\begin{aligned} (\Delta m_\pi^2)_{4q} &= \frac{64\rho F_0^4 \mathrm{Re}c_{4q}}{3} \\ (\delta m_\pi^2)_{4q} &= -\frac{16\rho F_0^4 \mathrm{Re}c_{4q}}{3} = -\frac{1}{4}(\Delta m_\pi^2)_{4q} \\ \bar{g}_{\pi\pi\pi}^{(1)} &= -16\rho F_0^2 \mathrm{Im}c_{4q} = -\frac{3(\Delta m_\pi^2)_{4q}}{4F_0^2} \frac{\mathrm{Im}c_{4q}}{\mathrm{Re}c_{4q}}. \end{aligned} \tag{61}$$

There are several differences compared to the case of quark bilinears. Firstly, there exists an $I = 2$ pion mass term (recall its definition in Sec. III B) because the spurion for the LR4Q contains all $I = 0, 1, 2$ components while the spurion for quark bilinears only have $I = 0$ and $I = 1$ pieces. Secondly, we find a non-vanishing PVTV three-pion coupling $\bar{g}_{\pi\pi\pi}^{(1)}$.In the case of the quark bilinears, the dipole operators and complex quark mass have the same spurion structure. Consequently, if one chooses the parameter α such that the tadpole contributions from the complex quark mass and dipole operators cancel (vacuum alignment), the corresponding contributions to $\bar{g}_{\pi\pi\pi}^{(1)}$ also cancel. In contrast, the spurion structure for the LR4Q operator differs from that for the complex quark mass, so the three pion term is not eliminated together with the pion tadpole.

In the nucleon sector, the leading operators are [6]:

$$c_1 \bar{N} \tilde{X}_{q+} N + c_1' \mathrm{Tr}[\tilde{X}_{q+}] \bar{N} N + \tilde{\tilde{c}}_1 F_0^3 \{c_{4q} \mathrm{Tr}[U^\dagger X_{RL}] \mathrm{Tr}[U X_{LR}] + c_{4q}^* \mathrm{Tr}[U^\dagger X_{LR}^\dagger] \mathrm{Tr}[U X_{RL}^\dagger]\} \bar{N} N. \tag{62}$$

[6] One can show that, other structures such as $c_{4q} \mathrm{Tr}[U^\dagger X_{RL}] \bar{N} u X_{LR} u N + c_{4q}^* \mathrm{Tr}[U^\dagger X_{LR}^\dagger] \bar{N} u X_{RL}^\dagger u N + h.c.$ are not independent from the $\tilde{\tilde{c}}_1$ structure we just wrote down.

Again, following the same logic as in Sec. V A 1, they lead to the following tree-level matching relations [14, 36–38]:

$$\begin{aligned} F_\pi \bar{g}_\pi^{(0)} &= -\frac{3(\Delta m_\pi^2)_{4q}}{4m_\pi^2}(\delta m_N)_q \frac{\mathrm{Im}c_{4q}}{\mathrm{Re}c_{4q}} \\ F_\pi \bar{g}_\pi^{(1)} &= \left[-\frac{3(\Delta m_\pi^2)_{4q}}{2m_\pi^2}(\Delta m_N)_q + 4(\Delta m_N)_{4q}\right]\frac{\mathrm{Im}c_{4q}}{\mathrm{Re}c_{4q}} \end{aligned} \tag{63}$$

as well as $\bar{g}_{NN3\pi}^{(0)} = -\frac{2}{3}\bar{g}_\pi^{(0)}$ and $F_\pi \bar{g}_{NN3\pi}^{(1)} = (\frac{(\Delta m_\pi^2)_{4q}}{m_\pi^2}(\Delta m_N)_q - \frac{32}{3}(\Delta m_N)_{4q})\frac{\mathrm{Im}c_{4q}}{\mathrm{Re}c_{4q}}$. In particular, one observes that before considering the vacuum alignment the only PVTV $NN\pi$ operator is $\bar{g}_\pi^{(1)}$. Including the vacuum alignment contribution gives

$$\bar{g}_\pi^{(0)}/\bar{g}_\pi^{(1)}\big|_{\mathrm{vac}} = (\delta m_N)_q/2(\Delta m_N)_q \quad . \tag{64}$$

Taking the lattice inputs for $(\Delta m_N)_q$ and $(\delta m_N)_q$ gives $\bar{g}_\pi^{(0)}/\bar{g}_\pi^{(1)} \approx -0.03$, *i.e.*, the $I = 1$ component is the dominant piece as long as there is no accidental cancellation between the direct and vacuum alignment contribution to $\bar{g}_\pi^{(1)}$. This is consistent with observations in Ref. [14, 37]. This is because the $I = 0, 2$ components of the LR4Q operator is PCTC while the $I = 1$ component is PVTV (see the discussion in Sec. II B 5). Finally, in the Δ-sector the leading operators are

$$\begin{aligned} & c_2 \bar{T}^i_\mu \tilde{X}_{q+} T^{i\mu} + c_2' \mathrm{Tr}[\tilde{X}_{q+}]\bar{T}^i_\mu T^{i\mu} + \tilde{\tilde{c}}_2 F_0^3 \{c_{4q}\mathrm{Tr}[U^\dagger X_{RL}]\mathrm{Tr}[U X_{LR}] \\ & + c_{4q}^* \mathrm{Tr}[U^\dagger X_{LR}^\dagger]\mathrm{Tr}[U X_{RL}^\dagger]\}\bar{T}^i_\mu T^{i\mu} \end{aligned} \tag{65}$$

that lead to the following tree-level matching:

$$\begin{aligned} F_\pi \bar{g}_{\Delta\Delta\pi}^{(0)} &= -\frac{3(\Delta m_\pi^2)_{4q}}{4m_\pi^2}(\delta m_\Delta)_q \frac{\mathrm{Im}c_{4q}}{\mathrm{Re}c_{4q}} \\ F_\pi \bar{g}_{\Delta\Delta\pi}^{(1,1)} &= \left[\frac{3(\Delta m_\pi^2)_{4q}}{2m_\pi^2}(\Delta m_\Delta)_q - 4(\Delta m_\Delta)_{4q}\right]\frac{\mathrm{Im}c_{4q}}{\mathrm{Re}c_{4q}} \\ &= F_\pi \bar{g}_{\Delta\Delta\pi}^{(1,2)} \end{aligned} \tag{66}$$

2. Loop correction

The one-loop corrections to the LHS and RHS of the tree-level mataching relations (63) can be inferred from Eq. (E15) and (E16) in Appendix E. After straightforward rearrangement on obtains:

$$\begin{aligned}
\delta(F_\pi \bar{g}_\pi^{(0)})_{\rm loop} &= \delta\left[-\frac{3(\Delta m_\pi^2)_{4q}}{4m_\pi^2}(\delta m_N)_q\right]_{\rm loop} + (\delta m_N)_q \frac{3(\Delta m_\pi^2)_{4q}}{8\pi^2 F_\pi^2}\frac{{\rm Im}c_{4q}}{{\rm Re}c_{4q}}(L' + \frac{5}{4}) \\
\delta(F_\pi \bar{g}_\pi^{(1)})_{\rm loop} &= \frac{{\rm Im}c_{4q}}{{\rm Re}c_{4q}}\delta\left[-\frac{3}{2}(\Delta m_N)_q\frac{(\Delta m_\pi^2)_{4q}}{m_\pi^2} + 4(\Delta m_N)_{4q}\right]_{\rm loop} \\
&+\frac{3}{2}(\Delta m_\pi^2)_{4q}\frac{{\rm Im}c_{4q}}{{\rm Re}c_{4q}}\left[\frac{(\Delta m_N)_q}{2\pi^2 F_\pi^2}(L' + \frac{5}{4}) - 4\{\frac{3g_A^2}{F_\pi^2}(I_c - \frac{I_f}{m_\pi^2}) \right. \\
&\left. +\frac{2g_{\pi N\Delta}^2}{F_\pi^2}(I_b - \frac{I_g}{m_\pi^2})\}\right] + \frac{9m_\pi^2(\Delta m_\pi^2)_{4q}}{32\pi^2 F_\pi^3}\frac{{\rm Im}c_{4q}}{{\rm Re}c_{4q}}((\gamma_1 + 4\gamma_2)(L'+1) - \frac{1}{2}\gamma_1 - 4\gamma_2).
\end{aligned} \tag{67}$$

Again, one clearly sees that the tree-level matching relations (63) are not obeyed by one-loop corrections. Analogous to the case of dipole operators, the mismatch between LHS and RHS of the matching relations are proportional to $(\Delta m_\pi^2)_{4q}$.

3. *LECs and the higher-order matching formula*

The $O(E^2 c_{4q})$ terms in pion and nucleon sector can be chosen as:

$$\begin{aligned}
\mathcal{L}_\pi^{O(E^2 c_{4q})} &= 2B_0 F_\pi^4\{K_1(c_4 q{\rm Tr}[X_q^\dagger X_{RL}]{\rm Tr}[UX_{LR}] + c_{4q}^*{\rm Tr}[X_q^\dagger X_{LR}^\dagger]{\rm Tr}[UX_{RL}^\dagger]) \\
&+K_2{\rm Tr}[U^\dagger X_q](c_{4q}{\rm Tr}[U^\dagger X_{RL}]{\rm Tr}[UX_{LR}] + c_{4q}^*{\rm Tr}[U^\dagger X_{LR}^\dagger]{\rm Tr}[UX_{RL}^\dagger])\} \\
&+F_\pi^4\{K_3 c_{4q}{\rm Tr}[\partial_\mu U^\dagger X_{RL}]{\rm Tr}[\partial^\mu U X_{LR}] \\
&+K_4 c_{4q}{\rm Tr}[\partial_\mu U \partial^\mu U^\dagger]{\rm Tr}[U^\dagger X_{RL}]{\rm Tr}[UX_{LR}]\} + h.c.
\end{aligned} \tag{68}$$

$$\begin{aligned}
\mathcal{L}_N^{O(E^2 c_{4q})} &= 2B_0 F_\pi\{h_1(c_{4q}{\rm Tr}[X_q^\dagger X_{RL}]{\rm Tr}[UX_{LR}] + c_{4q}^*{\rm Tr}[X_q^\dagger X_{LR}^\dagger]{\rm Tr}[UX_{RL}^\dagger])\bar{N}N \\
&+h_2{\rm Tr}[U^\dagger X_q](c_{4q}{\rm Tr}[U^\dagger X_{RL}]{\rm Tr}[UX_{LR}] + c_{4q}^*{\rm Tr}[U^\dagger X_{LR}^\dagger]{\rm Tr}[UX_{RL}^\dagger])\bar{N}N \\
&+h_3(c_{4q}{\rm Tr}[X_q^\dagger X_{RL}]\bar{N}uX_{LR}uN + c_{4q}^*{\rm Tr}[X_q^\dagger X_{LR}^\dagger]\bar{N}uX_{RL}^\dagger uN) \\
&+h_4(c_{4q}{\rm Tr}[U^\dagger X_{RL}U^\dagger X_q]\bar{N}uX_{LR}uN + c_{4q}^*{\rm Tr}[U^\dagger X_{LR}^\dagger U^\dagger X_q]\bar{N}uX_{RL}^\dagger uN) \\
&+h_5(c_{4q}{\rm Tr}[U^\dagger X_{RL}]\bar{N}\{u^\dagger X_q u^\dagger, uX_{LR}u\}N \\
&+c_{4q}^*{\rm Tr}[U^\dagger X_{LR}^\dagger]\bar{N}\{u^\dagger X_q u^\dagger, uX_{RL}^\dagger u\}N)\} + h.c.
\end{aligned} \tag{69}$$

respectively. One can thus straightforwardly deduce the matching formula for $\bar{g}_\pi^{(i)}$ precise to $O(E^2 c_{4q})$. With details provided in Appendix E 3, the final result turns out to be

$$
\begin{aligned}
F_\pi \bar{g}_\pi^{(0)} &= -\frac{3}{4}\frac{(\Delta m_\pi^2)_{4q}}{m_\pi^2}(\delta m_N)_q \frac{\mathrm{Im} c_{4q}}{\mathrm{Re} c_{4q}}(1+\delta_{\mathrm{loop}}^{(0)}+\delta_{\mathrm{LEC}}^{(0)}) \\
F_\pi \bar{g}_\pi^{(1)} &= \left[-\frac{3(\Delta m_\pi^2)_{4q}}{2m_\pi^2}(\Delta m_N)_q + 4(\Delta m_N)_{4q}\right]\frac{\mathrm{Im} c_{4q}}{\mathrm{Re} c_{4q}}(1+\delta_{\mathrm{loop}}^{(1)}+\delta_{\mathrm{LEC}}^{(1)})
\end{aligned} \tag{70}
$$

with

$$
\begin{aligned}
\delta_{\mathrm{loop}}^{(0)} &= -\frac{m_\pi^2}{2\pi^2 F_\pi^2}\left[\frac{1}{4}+\ln(\frac{\mu}{m_\pi})^2\right] \\
\delta_{\mathrm{loop}}^{(1)} &= \left[\frac{3g_A^2 m_\pi^3}{16\pi(\Delta m_N)_q F_\pi^2}+\frac{8g_{\pi N\Delta}^2 m_\pi^2}{(\Delta m_N)_q F_\pi^2}(I_b^r-\frac{I_g^r}{m_\pi^2})-\frac{3m_\pi^4}{16\pi^2(\Delta m_N)_q F_\pi^3}((\gamma_1+4\gamma_2)\ln(\frac{\mu}{m_\pi})^2\right. \\
&\left.-\frac{1}{2}\gamma_1-4\gamma_2)-\frac{m_\pi^2}{2\pi^2 F_\pi^2}(\frac{1}{4}+\ln(\frac{\mu}{m_\pi})^2)\right]\left[1-\frac{8}{3}\frac{m_\pi^2}{(\Delta m_N)_q}\frac{(\Delta m_N)_{4q}}{(\Delta m_\pi^2)_{4q}}\right]^{-1}
\end{aligned} \tag{71}
$$

as well as

$$
\begin{aligned}
\delta_{\mathrm{LEC}}^{(0)} &= -\frac{4m_\pi^4\varepsilon}{3F_\pi^3(\delta m_N)_q}[f_2^r+f_3^r+4(f_5^r+f_6^r)]-\frac{16F_\pi m_\pi^4 \mathrm{Re} c_{4q}\varepsilon}{9(\delta m_N)_q(\Delta m_\pi^2)_{4q}}[3h_3^r+5h_4^r-2h_5^r] \\
&-\frac{16F_\pi^2 m_\pi^2 \mathrm{Re} c_{4q}}{3(\Delta m_\pi^2)_{4q}}(-K_1^r+6K_2^r+K_3^r-6K_4^r)-\frac{64m_\pi^2}{F_\pi^2}\varepsilon^2(2L_7^r+L_8^r) \\
\delta_{\mathrm{LEC}}^{(1)} &= \left[\frac{m_\pi^4}{F_\pi^3(\Delta m_N)_q}[2(1+\varepsilon^2)f_1^r+2f_2^r+(1+\varepsilon^2)f_3^r+4(1+\varepsilon^2)f_4^r+2(\varepsilon^2-1)f_5^r\right. \\
&+2(1+\varepsilon^2)(f_5^r+f_6^r)]+\frac{8}{3}\frac{F_\pi m_\pi^4 \mathrm{Re} c_{4q}}{(\Delta m_N)_q(\Delta m_\pi^2)_{4q}}[2h_1^r+h_3^r-h_4^r+2h_5^r] \\
&\left.-\frac{16}{3}\frac{F_\pi^2 m_\pi^2 \mathrm{Re} c_{4q}}{(\Delta m_\pi^2)_{4q}}(-K_1^r+6K_2^r+K_3^r-6K_4^r)-\frac{64m_\pi^2}{F_\pi^2}\varepsilon^2(2L_7^r+L_8^r)\right]\times \\
&\left[1-\frac{8}{3}\frac{m_\pi^2}{(\Delta m_N)_q}\frac{(\Delta m_N)_{4q}}{(\Delta m_\pi^2)_{4q}}\right]^{-1}.
\end{aligned} \tag{72}
$$

Similarly, we may estimate the magnitude of the loop correction to the tree-level matching relations. Upon neglecting the unknown matrix elements $(\Delta m_N)_{4q}$ and $(\delta m_N)_{4q}$, we have $\delta_{\mathrm{loop}}^{(0)} = -0.12$ and $\delta_{\mathrm{loop}}^{(1)} = -3.2$ respectively, which implies a moderate convergence in the isoscalar channel and the non-convergence in the isovector channel; the $\bar{g}_\pi^{(1)}$ matching formula for LR4Q is therefore not useful in practice. Meanwhile, the magnitudes of the LEC corrections $\delta_{\mathrm{LEC}}^{(i)}$ can be estimated following the procedure outlined at the end of Sec. V B 3. The contributions from f_i^r and L_i^r are similar with the case of cEDM, while for the new LECs labeled as $\{K_i^r\}$ and $\{h_i^r\}$, we estimate their sizes again using the Weinberg counting rule. Eq. (E19) and (E20) then give:

$$
\begin{aligned}
(\Delta m_\pi^2)^r_{4q,ct} &\sim 16 m_\pi^2 F_\pi^2 \mathrm{Re}c_{4q} K_i^r \sim (\frac{m_\pi}{2\pi F_\pi})^2 (\Delta m_\pi^2)_{4q} \sim (\frac{m_\pi}{2\pi F_\pi})^2 \frac{(2\pi F_\pi)^4 \mathrm{Re}c_{4q}}{(4\pi)^2} \\
(\Delta m_N)^r_{4q,ct} &\sim 4 m_\pi^2 F_\pi \mathrm{Re}c_{4q} h_i^r \sim (\frac{m_\pi}{2\pi F_\pi})^2 (\Delta m_N)_{4q} \sim (\frac{m_\pi}{2\pi F_\pi})^2 \frac{(2\pi F_\pi)^3 \mathrm{Re}c_{4q}}{(4\pi)^2}
\end{aligned}
\tag{73}
$$

leading to $K_i^r \sim 0.02$ and $h_i^r \sim 0.01$. With these, we find that the h_i^r-contribution to $\delta^{(0)}_{\mathrm{LEC}}$ is around 0.08 while all other contributions are of the order 10^{-2}. Hence the conclusion we may draw here is again similar to the case of cEDM.

D. The $(\bar{L}R)(\bar{L}R)$ operators

Finally, let us discuss the last class of four-quark operators that could be T-odd, namely the $(\bar{L}R)(\bar{L}R)$-type operators we introduced in Sec. II B 4. Since they are chirally invariant their "spurion" is nothing but a complex a complex number a. Therefore, when their effects are implemented into the chiral Lagrangian, terms proportional to a and a^* can in principle both appear with independent coefficients (e.g. via terms like $(\alpha_1 a + \alpha_2 a^*)\hat{O} + h.c.$ where α_1 and α_2 are unrelated coefficients) so there is no definite matching formula between the PCTC and PVTV observables. Similar considerations apply for other chirally invariant operators, such as the Weinberg three-gluon operator. Therefore, we focus only on the chirally non-invariant operators in this paper.

VI. CONCLUSION

The computation of hadronic matrix elements induced by effective quark-gluon operators that are relevant for tests of fundamental symmetries is a non-trivial task. Among them, the PVTV pion-nucleon couplings $\bar{g}_\pi^{(i)}$ that contribute to nucleon and atomic EDMs are of particular interest in this paper. These operators can be induced by PVTV effective operators that are either chirally invariant nor non-invariant . The latter class is interesting theoretically because the PCTC and PVTV components of the CSB operator can be grouped into a single spurion field that enters the effective chiral Lagrangian. Consequently, there exist matching relations between the $\bar{g}_\pi^{(i)}$ induced by the spurion field and various PCTC and CSB observables, such as the pion mass and the nucleon mass shifts that are induced by the same spurion field. These relationships are analogous to the relationships between matrix elements of different components of a vector due to the Wigner-Eckart theorem. The relations between PVTV and PCTC hadronic matrix elements are extremely useful because one could utilize studies of the PCTC hadronic observables (say, through lattice) to obtain their PVTV counterparts.

A caveat to the use of this formalism is that the matching formulae are derived at tree-level and may receive non-negligible higher order corrections from loop diagrams and/or higher order terms in the chiral Lagrangian. In order to study the higher order effects, we have performed a general classifications of relevant operators that could generate loop corrections to $\bar{g}_\pi^{(i)}$ and CSB observables, and we have calculated the most general loop corrections to those quantities. We then applied this general formalism to study the loop corrections to the matching formulae induced by all relevant effective operators (of the lightest generation) up to dimension 6. In general, we found that the matching relations for $\bar{g}_\pi^{(0)}$ are relatively stable as the loop corrections lead to at most $\mathcal{O}(10\%)$ modifications. On the other hand, the robustness of the $\bar{g}_\pi^{(1)}$ matching formulae is more complicated as the corrections depend strongly on the ratio $F_\pi(\Delta m_N)_\mathcal{O}/(\Delta m_\pi^2)_\mathcal{O}$. We also find that, the inclusion of Δ-resonances in the loop diagrams does not spoil the matching relation of $\bar{g}_\pi^{(0)}$ but does affect the $\bar{g}_\pi^{(1)}$-matching significantly. For the impact of higher-order terms in the chiral Lagrangian, we find that the largest effects arise from the corresponding LECs in the nucleon sector, which may give rise to a (10-20)% modification of the matching relation for $\bar{g}_\pi^{(0)}$. Contributions from the LECs in the pion sector are in general not much larger than 1%.

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Appendix A: Chiral Building Blocks in $\mathrm{SU(2)_L \times SU(2)_R}$

Here we summarize the building blocks of $\mathrm{SU(2)_L \times SU(2)_R}$ ChPT that are required to construct a chirally invariant Lagrangian and implement the effect of chiral symmetry breaking. For most of the notations and conventions we follow the pedagogical article by Scherer [39]. For the pion decay constant we take $F_\pi = 186\,\mathrm{MeV}$ following the convention [16]. Although this is not the standard convention in the literature using $\mathrm{SU(2)_L \times SU(2)_R}$ ChPT, it allows us to compare our results more easily with previous work on the study of $\bar{g}_\pi^{(i)}$ that mostly adopt the SO(4) representation.

1. $U = \exp\{i\frac{2\pi_a\tau_a}{F_0}\}$ transforms as $U \to V_R U V_L^\dagger$ under $\mathrm{SU(2)_L \times SU(2)_R}$.

2. $u = \sqrt{U}$ transforms as $u \to V_R u K^\dagger = K u V_L^\dagger$ where $K = K(V_R, V_L, U)$ is a unitary matrix. It reduces to isospin transformation matrix when $V_R = V_L$.

3. The SU(2) nucleon field: $N = (p \;\; n)^T$ transforms as $N \to KN$.

4. The chiral axial vector $u_\mu = i u^\dagger(\partial_\mu U)u^\dagger$ is a Hermitian and traceless operator. It transforms as $u_\mu \to K u_\mu K^\dagger$.

5. The Δ-resonance field T^i_μ transforms as $T^i_\mu \to \tilde{K}^{ij} K T^i_\mu$, where:

$$\tilde{K}^{ij} = \delta^{ij} + \epsilon^{ijk}\theta^k_V + \frac{1}{F_0}(\pi^i\theta^j_A - \pi^j\theta^i_A) + O(\theta^2, \pi^{(2)}) \tag{A1}$$

   satisfying $\tilde{K}^{ij}\tilde{K}^{ij'} = \delta^{jj'}$. For $\mathrm{SU(2)_V}$, the matrix $\tilde{K}^{ij}$ simply reduces to the transformation matrix of an isospin triplet. Also, in order to eliminate the spin-1/2 and isospin-1/2 components, T^i_μ is subject to the following constraints:

$$\begin{aligned} \gamma^\mu T^i_\mu &= 0 \\ \tau^i T^i_\mu &= 0. \end{aligned} \tag{A2}$$

   In particular, the first relation, combining with the (relativistic) free-field EOM $(i\partial\!\!\!/ - m_\Delta)T^i_\mu = 0$, gives $\partial_\mu T^{i\mu} = 0$ that reduces to $v^\mu T^i_\mu = 0$ in the HBChPT formalism. In terms of the physical Δ-fields, T^i_μ can be expressed as:

$$
\begin{aligned}
T^1_\mu &= \frac{1}{\sqrt{2}} \begin{pmatrix} \Delta^{++} - \frac{1}{\sqrt{3}}\Delta^0 \\ \frac{1}{\sqrt{3}}\Delta^+ - \Delta^- \end{pmatrix}_\mu \\
T^2_\mu &= \frac{i}{\sqrt{2}} \begin{pmatrix} \Delta^{++} + \frac{1}{\sqrt{3}}\Delta^0 \\ \frac{1}{\sqrt{3}}\Delta^+ + \Delta^- \end{pmatrix}_\mu \\
T^3_\mu &= -\sqrt{\frac{2}{3}} \begin{pmatrix} \Delta^+ \\ \Delta^0 \end{pmatrix}_\mu .
\end{aligned} \tag{A3}
$$

Details of the inclusion of Δ-resonance in ChPT can be found in Ref. [22].

6. $\omega^i_\mu = \frac{1}{2}\mathrm{Tr}[\tau^i u_\mu]$ is a Hermitian operator that transforms as $\omega^i_\mu \to \tilde{K}^{ij}\omega^j_\mu$.

Appendix B: Relevant Chirally Invariant Lagrangian

Here we write down the PCTC, chirally invariant Lagrangian involving pion, nucleon and Δ-resonance fields that is relevant to our work, expanded to $O(E^2)$ in ChPT using the $\mathrm{SU(2)_L} \times \mathrm{SU(2)_R}$ formalism. It is given by [22, 39, 40]:

$$
\begin{aligned}
\mathcal{L} &= \frac{F_0^2}{16}\mathrm{Tr}[\partial_\mu U \partial^\mu U^\dagger] + \bar{N} i v \cdot \mathcal{D} N + g_A \bar{N} u_\mu S^\mu N \\
&\quad + F_0^{-1}\bar{N}[\gamma_1 (v\cdot u)^2 + \gamma_2 u \cdot u]N \\
&\quad - \bar{T}^\mu_i [iv \cdot \mathcal{D}^{ij} - \delta_\Delta] T_{j\mu} + g_{\pi N \Delta}[\bar{T}^\mu_i \omega^i_\mu N + \bar{N}\omega^i_\mu T^\mu_i] + ...
\end{aligned} \tag{B1}
$$

where $\mathcal{D}_\mu$ and $\mathcal{D}^{ij}_\mu$ are the chiral covariant derivatives on the nucleon and Δ respectively while the $\Delta - N$ mass-splitting is given by $\delta_\Delta = m_\Delta - m_N$. In the absence of external fields, we have $\mathcal{D}_\mu = \partial_\mu + \frac{1}{2}\{u^\dagger, \partial_\mu u\}$ and $\mathcal{D}^{ij}_\mu = \delta^{ij}\mathcal{D}_\mu - \frac{i}{2}\epsilon^{ijk}\mathrm{Tr}[\tau^k\{u^\dagger, \partial_\mu u\}]$. The value of the Δ-nucleon-pion coupling constant is given by $g_{\pi N\Delta} \approx 1.05$ according to [22]. Fitting to scattering observables yields [39]:

$$
\gamma_1 \approx 0.621, \gamma_2 \approx -0,984 \tag{B2}
$$

Also, we have dropped the terms that are not needed in our work, e.g. coupling term of the form $\bar{N}[S^\mu, S^\nu]u_\mu u_\nu N$ as well as the $\Delta\Delta\pi$ interaction terms.

The free propagators of pion, nucleon and Δ-resonance are

$$\begin{aligned} iD_\pi(k) &= \frac{i}{k^2 - m_{\pi,0}^2 + i\epsilon} \\ iS_N(k) &= \frac{i}{v\cdot k + i\epsilon} \\ iD_\Delta(k)^{ij}_{\mu\nu} &= \frac{-i}{v\cdot k - \delta_\Delta + i\epsilon} P^{3/2}_{\mu\nu}\xi^{ij}_{3/2} \end{aligned} \tag{B3}$$

respectively, where

$$\begin{aligned} P^{3/2}_{\mu\nu} &= g_{\mu\nu} - v_\mu v_\nu + \frac{4}{d-1} S_\mu S_\nu \\ \xi^{ij}_{3/2} &= \frac{2}{3}\delta^{ij} - \frac{i}{3}\epsilon^{ijk}\tau^k \end{aligned} \tag{B4}$$

are the projection operators for spin-3/2 and isospin-3/2 respectively.

Appendix C: Loop Integral Functions

Here we define several loop integral functions $\{I_i\}$ that always appear when one calculate loop diagrams given in Fig.1 and 2:

$$
\begin{aligned}
I_a &\equiv \mu^{4-d}\int \frac{d^d l}{(2\pi)^d} S\cdot l \frac{i}{-v\cdot l + i\epsilon}\frac{i}{-v\cdot l + i\epsilon} S\cdot l \frac{i}{l^2 - m_\pi^2 + i\epsilon} \\
&= \frac{3m_\pi^2}{64\pi^2}(L' + \frac{1}{3}) \qquad \text{(C1)}\\
I_b &\equiv \mu^{4-d}\int \frac{d^d l}{(2\pi)^d} l_\alpha \frac{-i}{-v\cdot l - \delta_\Delta + i\epsilon} P_{3/2}^{\alpha\beta} l_\beta \frac{i}{l^2 - m_\pi^2 + i\epsilon}\frac{i}{l^2 - m_\pi^2 + i\epsilon} \\
&= \frac{\delta_\Delta}{8\pi^2}L' + \frac{1}{8\pi^2}\left[\delta_\Delta - 2\sqrt{\delta_\Delta^2 - m_\pi^2}\ln\frac{\delta_\Delta + \sqrt{\delta_\Delta^2 - m_\pi^2}}{m_\pi}\right] \qquad \text{(C2)}\\
I_c &\equiv \mu^{4-d}\int \frac{d^d l}{(2\pi)^d} S\cdot l \frac{i}{-v\cdot l + i\epsilon} S\cdot l \frac{i}{l^2 - m_\pi^2 + i\epsilon}\frac{i}{l^2 - m_\pi^2 + i\epsilon} \\
&= \frac{3m_\pi}{64\pi} \qquad \text{(C3)}\\
I_d &\equiv \mu^{4-d}\int \frac{d^d l}{(2\pi)^d} l_\alpha \frac{-i}{-v\cdot l - \delta_\Delta + i\epsilon} P_{3/2}^{\alpha\beta} g_{\beta\delta}\frac{-i}{-v\cdot l - \delta_\Delta + i\epsilon} P_{3/2}^{\delta\rho} l_\rho \frac{i}{l^2 - m_\pi^2 + i\epsilon} \\
&= \frac{1}{8\pi^2}(2\delta_\Delta^2 - m_\pi^2)L' + \frac{\delta_\Delta^2}{4\pi^2} - \frac{\delta_\Delta}{2\pi^2}\sqrt{\delta_\Delta^2 - m_\pi^2}\ln\frac{\delta_\Delta + \sqrt{\delta_\Delta^2 - m_\pi^2}}{m_\pi} \qquad \text{(C4)}\\
I_e &\equiv \mu^{4-d}\int \frac{d^d l}{(2\pi)^d}\frac{i}{l^2 - m_\pi^2} \\
&= -\frac{m_\pi^2}{16\pi^2}(L' + 1) \qquad \text{(C5)}\\
I_f &\equiv i\mu^{4-d}\int \frac{d^d l}{(2\pi)^d} S\cdot l \frac{i}{-v\cdot l + i\epsilon} S\cdot l \frac{i}{l^2 - m_\pi^2 + i\epsilon} \\
&= \frac{m_\pi^3}{32\pi} \qquad \text{(C6)}\\
I_g &\equiv i\mu^{4-d}\int \frac{d^d l}{(2\pi)^d} l_\mu \frac{-i}{-v\cdot l - \delta_\Delta + i\epsilon} P_{3/2}^{\mu\nu} l_\nu \frac{i}{l^2 - m_\pi^2 + i\epsilon} \\
&= \frac{\delta_\Delta}{12\pi^2}(-\delta_\Delta^2 + \frac{3}{2}m_\pi^2)L' + \frac{1}{72\pi^2}\left[2\delta_\Delta(6m_\pi^2 - 5\delta_\Delta^2) + 12(\delta_\Delta^2 - m_\pi^2)^{3/2}\ln\frac{\delta_\Delta + \sqrt{\delta_\Delta^2 - m_\pi^2}}{m_\pi}\right] \\
&\qquad \text{(C7)}
\end{aligned}
$$

with the divergent quantity L' defined in Eq. (24).

In Table III we give the numerical values of the renormalized loop functions I_i^r where the divergent quantity $L+1$ is subtracted and the renormalization scale μ is taken to be 1 GeV.

Appendix D: Relevant One-Loop Corrections in Ordinary ChPT

Here we summarize some important results for ordinary ChPT at one loop that are necessary in our work. First we introduce the relevant $O(E^3)$ and $O(E^4)$ Lagrangian that

I_a^r	I_b^r	I_c^r	I_d^r	I_e^r	I_f^r	I_g^r
303MeV^2	5.66MeV	2.08MeV	2590MeV^2	-486MeV^2	27000MeV^3	-273000MeV^3

Table III: Numerical values of the renormalized loop functions with $\mu = 1$ GeV.

are crucial in the cancellation of one-loop UV-divergence. They are [28, 41]:

$$\begin{aligned}
\mathcal{L}_{O(E^3)} &= \frac{B_{20}}{(2\pi F_\pi)^2} B_0 \mathrm{Tr}[\tilde{X}_{q+}] \bar{N} iv \cdot \mathcal{D} N + ... \\
\mathcal{L}_{O(E^4)} &= 2B_0 L_4 \mathrm{Tr}[\partial_\mu U \partial^\mu U^\dagger] \mathrm{Tr}[X_q U^\dagger + U X_q^\dagger] + 2B_0 L_5 \mathrm{Tr}[\partial_\mu U^\dagger \partial^\mu U (U^\dagger X_q + X_q^\dagger U)] \\
&\quad +4B_0^2 L_6 \mathrm{Tr}[X_q U^\dagger + U X_q^\dagger]^2 + 4B_0^2 L_7 \mathrm{Tr}[X_q U^\dagger - U X_q^\dagger]^2 \\
&\quad +4B_0^2 L_8 \mathrm{Tr}[U X_q^\dagger U X_q^\dagger + U^\dagger X_q U^\dagger X_q] + ...
\end{aligned} \tag{D1}$$

The loop and LEC corrections to Z_N and Z_π are given by:

$$\begin{aligned}
Z_N - 1 &= \frac{9g_A^2 m_\pi^2}{16\pi^2 F_\pi^2}\left[-\frac{2}{3} + \ln(\frac{\mu}{m_\pi})^2\right] - \frac{g_{\pi N\Delta}^2}{\pi^2 F_\pi^2}[(2\delta_\Delta^2 - m_\pi^2)L' \\
&\quad +2\delta_\Delta^2 - 4\delta_\Delta\sqrt{\delta_\Delta^2 - m_\pi^2}\ln\frac{\delta_\Delta + \sqrt{\delta_\Delta^2 - m_\pi^2}}{m_\pi}] - \frac{2B_{20}^r m_\pi^2}{\pi^2 F_\pi^2} \\
\sqrt{Z_\pi} - 1 &= -\frac{m_\pi^2}{12\pi^2 F_\pi^2}(L'+1) - \frac{16 m_\pi^2}{F_\pi^2}(2L_4 + L_5) \\
&= \frac{4}{3F_\pi^2} I_e - \frac{16 m_\pi^2}{F_\pi^2}(2L_4 + L_5).
\end{aligned} \tag{D2}$$

It is worth pointing out that the wavefunction renormalization of the nucleon field Z_N is finite as its infinity is absorbed by the LEC B_{20}. Meanwhile, the wavefunction renormalization of the pion field Z_π remains infinite as the LEC $2L_4 + L_5$ is not used to subtract the infinity in Z_π. This is not an issue because Z_π is not a physical observable. On the other hand, the pion decay constant F_π is a physical observable; therefore its renormalization must be finite. The same combination of LECs $2L_4 + L_5$ is used to subtract the divergence entering F_π instead of Z_π. It gives:

$$F_\pi = F_0\left[1 + \frac{m_\pi^2}{4\pi^2 F_\pi^2}(L'+1) + \frac{16 m_\pi^2}{F_\pi^2}(2L_4 + L_5)\right] = F_0\left[1 - \frac{4}{F_\pi^2} I_e^r + \frac{16 m_\pi^2}{F_\pi^2}(2L_4^r + L_5^r)\right]. \tag{D3}$$

Also, the one-loop correction to the squared charged-pion mass is useful:

$$m_\pi^2 = m_{\pi,0}^2\left[1 + \frac{2}{F_\pi^2} I_e^r - \frac{32 m_\pi^2}{F_\pi^2}(2L_4^r + L_5^r - 4L_6^r - 2L_8^r)\right]. \tag{D4}$$

Finally, since the LEC B_{20} is only used for Z_N and not other quantities (as far as this work is concerned), we will not distinguish the loop and LEC contribution to Z_N in the main text. Rather, Z_N is simply taken as a finite quantity.

Appendix E: Some important details in Section V

Here we collect some important intermediate results – including loop corrections, LEC corrections and implications of higher-order vacuum alignment – that are crucial in order to derive the main conclusions in Sec. V yet are too long to be put in the main text.

1. θ-term

First we consider the one-loop renormalization to $F_\pi \bar{g}_\pi^{(0)}$. Eq. (23) together with Sec. V A 1 gives

$$\delta(F_\pi \bar{g}_\pi^{(0)})_{\text{loop}} = F_\pi \left[(Z_N - 1)\bar{g}_\pi^{(0)} + (-\frac{2}{F_\pi^2} I_e + \frac{4g_A^2}{F_\pi^2} I_a)\bar{g}_\pi^{(0)} - \frac{40 g_{\pi N\Delta}^2 \bar{g}_{\Delta\Delta\pi}^{(0)}}{9F_\pi^2} I_d \right]. \quad \text{(E1)}$$

Similarly, the nucleon sigma term and mass splitting (recall their definitions in Sec. III B) also receive loop corrections from the CSB operators we wrote down in Sec. V A 1. With the identification $(g_{NN\pi\pi}^{(0)})_q = -2(\Delta m_N)_q/F_\pi$, $(g_{NN\pi\pi}^{(1,1)})_q = 0$ and $(g_{NN\pi\pi}^{(1,2)})_q = -(\delta m_N)_q/F_\pi$, we obtain:

$$\begin{aligned}
\delta(\Delta m_N)_{q,\text{loop}} &= (Z_N - 1)(\Delta m_N)_q - (\frac{12 g_A^2}{F_\pi^2} I_a + \frac{6}{F_\pi^2} I_e)(\Delta m_N)_q - \frac{8 g_{\pi N\Delta}^2 (\Delta m_\Delta)_q}{F_\pi^2} I_d \\
&\quad + \frac{12 g_A^2}{F_\pi^2} I_f + \frac{8 g_{\pi N\Delta}^2}{F_\pi^2} I_g - \frac{3 m_\pi^4}{16\pi^2 F_\pi^3}\left((\gamma_1 + 4\gamma_2)(L' + 1) + \frac{1}{2}\gamma_1 \right) \quad \text{(E2)} \\
\delta(\delta m_N)_{q,\text{loop}} &= (Z_N - 1)(\delta m_N)_q + (-\frac{2}{F_\pi^2} I_e + \frac{4 g_A^2}{F_\pi^2} I_a)(\delta m_N)_q - \frac{40 g_{\pi N\Delta}^2 (\delta m_\Delta)_q}{9F_\pi^2} I_d. \quad \text{(E3)}
\end{aligned}$$

Here we include loop corrections for both $(\Delta m_N)_q$ and $(\delta m_N)_q$ even though the tree-level matching relation only involves the latter in the case of θ-term, because the former will appear in the matching relations induced by cEDM and LR4Q.

Next we consider consequences of the introduction of $O(E^4)$ Lagrangian. First, the $O(E^4)$ Lagrangian in pion sector defined in Eq. (D1) leads to a modification of the vacuum-alignment condition:

$$\alpha = -\frac{\varepsilon\bar{\theta}}{2}\left[1 + \frac{64 m_\pi^2}{F_\pi^2}(1 - \varepsilon^2)(2L_7^r + L_8^r) \right] \quad \text{(E4)}$$

where L_i^r are the renormalized $O(E^4)$ LECs in the pion sector. This gives rise to an extra $O(E^4)$ contribution to $\bar{g}_\pi^{(i)}$:

$$\begin{aligned}
\delta(\bar{g}_\pi^{(0)})_v &= -\frac{32(\delta m_N)_q m_\pi^2 \varepsilon(1-\varepsilon^2)(2L_7^r+L_8^r)}{F_\pi^3}\bar{\theta} = \delta(\bar{g}_\pi^{(0)})_v^r \\
\delta(\bar{g}_\pi^{(1)})_v &= -\frac{64(\Delta m_N)_q m_\pi^2 \varepsilon(1-\varepsilon^2)(2L_7^r+L_8^r)}{F_\pi^3}\bar{\theta} = \delta(\bar{g}_\pi^{(1)})_v^r
\end{aligned} \tag{E5}$$

The subscript "v" denotes the contribution from higher-order vacuum alignment.

In the nucleon sector, the $O(E^4)$ Lagrangian (39) provides LEC-contributions to the nucleon sigma term and mass-splitting:

$$\begin{aligned}
\delta(\Delta m_N)_{q,ct} &= \frac{m_\pi^4}{F_\pi^3}\left[2(1+\varepsilon^2)f_1 + 2f_2 + (1+\varepsilon^2)f_3\right] \\
\delta(\delta m_N)_{q,ct} &= -\frac{4\varepsilon m_\pi^4}{F_\pi^3}(f_2+f_3).
\end{aligned} \tag{E6}$$

The subscript "ct" denotes direct contributions from higher-order LECs (which also play the role of counterterms, hence the naming of the subscript). Meanwhile, the LEC contributions for $\bar{g}_\pi^{(i)}$ are given by:

$$\begin{aligned}
\delta(F_\pi\bar{g}_\pi^{(0)})_{ct} &= \frac{2(\varepsilon^2-1)(f_2+f_3+f_5^r+f_6^r)m_\pi^4}{F_\pi^3}\bar{\theta} \\
\delta(F_\pi\bar{g}_\pi^{(1)})_{ct} &= -\frac{2\varepsilon(\varepsilon^2-1)(2f_1^r+f_3^r+2f_4^r+2f_5^r+f_6^r)m_\pi^4}{F_\pi^3}\bar{\theta}.
\end{aligned} \tag{E7}$$

Note that the LECs for F_π and $\sqrt{Z_\pi}-1$ will always cancel each other so they never appear in $\delta(F_\pi\bar{g}_\pi^{(i)})_{ct}$ (see Appendix D).

2. dipole-operators

First, the one-loop renormalization to $F_\pi\bar{g}_\pi^{(i)}$ is:

$$\begin{aligned}
\delta(F_\pi\bar{g}_\pi^{(0)})_{\text{loop}} &= F_\pi\left[\left(\frac{4g_A^2}{F_\pi^2}I_a - \frac{2}{F_\pi^2}I_e\right)\bar{g}_\pi^{(0)} - \frac{40g_{\pi N\Delta}^2\bar{g}_{\Delta\Delta\pi}^{(0)}}{9F_\pi^2}I_d + (Z_N-1)\bar{g}_\pi^{(0)}\right] \\
\delta(F_\pi\bar{g}_\pi^{(1)})_{\text{loop}} &= F_\pi\left[-\left(\frac{12g_A^2}{F_\pi^2}I_a + \frac{6}{F_\pi^2}I_e\right)\bar{g}_\pi^{(1)} + \frac{8g_{\pi N\Delta}^2\bar{g}_{\Delta\Delta\pi}^{(1,1)}}{F_\pi^2}I_d + (Z_N-1)\bar{g}_\pi^{(1)}\right].
\end{aligned} \tag{E8}$$

Meanwhile, with $(g_{4\pi}^{(0)})_c = (\Delta m_\pi^2)_c/6F_\pi^2$, $(g_{NN\pi\pi}^{(0)})_c = -2(\Delta m_N)_c/F_\pi$, $(g_{NN\pi\pi}^{(1,1)})_c = 0$ and $(g_{NN\pi\pi}^{(1,2)})_c = -(\delta m_N)_c/F_\pi$ (recall their definitions in Sec. III B), the one-loop renormalization to hadron mass parameters is:

$$
\begin{aligned}
\delta(\Delta m_N)_{c,\text{loop}} &= (Z_N-1)(\Delta m_N)_c - (\Delta m_N)_c(\frac{6}{F_\pi^2}I_e + \frac{12g_A^2}{F_\pi^2}I_a) - \frac{8g_{\pi N\Delta}^2(\Delta m_\Delta)_c}{F_\pi^2}I_d \\
&+\frac{12g_A^2(\Delta m_\pi^2)_c}{F_\pi^2}I_c + \frac{8g_{\pi N\Delta}^2(\Delta m_\pi^2)_c}{F_\pi^2}I_b - \frac{3m_\pi^2(\Delta m_\pi^2)_c}{8\pi^2F_\pi^3}((\gamma_1+4\gamma_2)(L'+1) \\
&-2\gamma_2) \\
\delta(\delta m_N)_{c,\text{loop}} &= (Z_N-1)(\delta m_N)_c + (-\frac{2}{F_\pi^2}I_e + \frac{4g_A^2}{F_\pi^2}I_a)(\delta m_N)_c - \frac{40g_{\pi N\Delta}^2(\delta m_\Delta)_c}{9F_\pi^2}I_d \\
\delta(\Delta m_\pi^2)_{c,\text{loop}} &= -\frac{(\Delta m_\pi^2)_c m_\pi^2}{4\pi^2F_\pi^2}(L'+\frac{1}{2}).
\end{aligned}
\tag{E9}
$$

Next we consider consequences of the introduction of $O(E^2\tilde{d})$ LECs. First, $O(E^2\tilde{d})$ terms in the pion sector (54), together with the $O(E^4)$ pion Lagrangian, modify the vacuum-alignment formula as

$$
\begin{aligned}
\alpha &= -\frac{(\Delta m_\pi^2)_{c,0}}{(m_\pi^2)_0}\frac{\tilde{d}_1}{\tilde{d}_0^M} + \frac{64(\Delta m_\pi^2)_c}{F_\pi^2}[(2L_6^r+L_8^r)+\varepsilon^2(2L_7^r+L_8^r)]\frac{\tilde{d}_1}{\tilde{d}_0^M} \\
&-16F_\pi[(2G_1^r+G_3^r)\tilde{d}_1 - \varepsilon(2G_2^r+G_3^r)\tilde{d}_0].
\end{aligned}
\tag{E10}
$$

Note that the divergent pieces of the LECs cancel each other, leaving α finite. This leads to extra vacuum-alignment contribution to $\bar{g}_\pi^{(i)}$:

$$
\begin{aligned}
\delta(\bar{g}_\pi^{(0)})_v &= \frac{64(\delta m_N)_q(\Delta m_\pi^2)_c}{F_\pi^3}[(2L_6^r+L_8^r)+\varepsilon^2(2L_7^r+L_8^r)]\frac{\tilde{d}_1}{\tilde{d}_0^M} \\
&-16(\delta m_N)_q[(2G_1^r+G_3^r)\tilde{d}_1 - \varepsilon(2G_2^r+G_3^r)\tilde{d}_0] \\
&= \delta(\bar{g}_\pi^{(0)})_v^r \\
\delta(\bar{g}_\pi^{(1)})_v &= \frac{128(\Delta m_N)_q(\Delta m_\pi^2)_c}{F_\pi^3}[(2L_6^r+L_8^r)+\varepsilon^2(2L_7^r+L_8^r)]\frac{\tilde{d}_1}{\tilde{d}_0^M} \\
&-32(\Delta m_N)_q[(2G_1^r+G_3^r)\tilde{d}_1 - \varepsilon(2G_2^r+G_3^r)\tilde{d}_0] \\
&= \delta(\bar{g}_\pi^{(1)})_v^r
\end{aligned}
\tag{E11}
$$

Eq. (54) also provides LEC-contributions for the $I=0$ and $I=2$ pion mass shift:

$$
\begin{aligned}
\delta(\Delta m_\pi^2)_{c,ct} &= \frac{32}{3}m_\pi^2F_\pi[3(2G_1+G_3)\tilde{d}_0^M - \varepsilon(2G_2^r+G_3^r)\tilde{d}_1^M] \\
&-16\tilde{d}_0^M m_\pi^2F_\pi(2G_4+G_5) - \frac{32m_\pi^2}{F_\pi^2}(2L_4+L_5)(\Delta m_\pi^2)_c \\
\delta(\delta m_\pi^2)_{c,ct} &= \frac{32}{3}m_\pi^2F_\pi\varepsilon(2G_2^r+G_3^r)\tilde{d}_1^M.
\end{aligned}
\tag{E12}
$$

Note that there is no loop contribution to $(\delta m_\pi^2)_c$ so the corresponding LECs must be finite, therefore $2G_2 + G_3 = 2G_2^r + G_3^r$. Also notice that terms like $2L_4 + L_5$ and $2G_4 + G_5$ come from the wavefunction renormalization.

In the nucleon sector, the introduction of $O(E^2\tilde{d})$ Lagrangian gives the following consequences. First, it modifies the cMDM-induced nucleon mass shifts:

$$\begin{aligned}
\delta(\Delta m_N)_{c,ct} &= 4m_\pi^2[(\tilde{d}_0^M - \varepsilon\tilde{d}_1^M)(g_1^r + g_4^r) + \tilde{d}_0^M(g_2 + g_3)] \\
\delta(\delta m_N)_{c,ct} &= 8m_\pi^2[(g_2 + g_4)\tilde{d}_1^M - \varepsilon(g_3^r + g_4^r)\tilde{d}_0^M].
\end{aligned} \tag{E13}$$

Also, combining with the $O(E^4)$ LECs and the leading-order vacuum alignment, we obtain the LEC-contributions for $\bar{g}_\pi^{(i)}$:

$$\begin{aligned}
\delta(F_\pi\bar{g}_\pi^{(0)})_{ct} &= \frac{16(\Delta m_\pi^2)_c m_\pi^2\varepsilon}{3F_\pi^3}\frac{\tilde{d}_1}{\tilde{d}_0^M}[f_2 + f_3 + f_5^r + f_6^r] \\
&\quad + \frac{8m_\pi^2}{3}[3\tilde{d}_0(g_2 + g_4 + g_7^r + g_8^r) - \varepsilon\tilde{d}_1(g_3^r + g_4^r + g_6 + g_8)] \\
\delta(F_\pi\bar{g}_\pi^{(1)})_{ct} &= -\frac{4(\Delta m_\pi^2)_c m_\pi^2}{F_\pi^3}\frac{\tilde{d}_1}{\tilde{d}_0^M}[2(1+\varepsilon^2)f_1 + 2f_2 + (1+\varepsilon^2)f_3 + 2(1+\varepsilon^2)f_4 + (\varepsilon^2 - 1)f_5 \\
&\quad + (1+\varepsilon^2)(f_5^r + f_6^r)] + 8m_\pi^2[\tilde{d}_1(g_1^r + g_2 + g_3 + g_4^r + g_5 + g_8) \\
&\quad - \varepsilon\tilde{d}_0(g_1^r + g_4^r + g_5^r + g_6^r + g_7^r + g_8^r)] \\
\delta(F_\pi\bar{g}_\pi^{(2)})_{ct} &= \frac{4(\Delta m_\pi^2)_c m_\pi^2\varepsilon}{3F_\pi^3}\frac{\tilde{d}_1}{\tilde{d}_0^M}[f_2^r + f_3^r + f_5^r + f_6^r] - \frac{8m_\pi^2\varepsilon}{3}\tilde{d}_1(g_3^r + g_4^r + g_6^r + g_8^r)
\end{aligned} \tag{E14}$$

3. LR4Q

First, the one-loop renormalization to $F_\pi\bar{g}_\pi^{(i)}$ is:

$$\begin{aligned}
\delta(F_\pi\bar{g}_\pi^{(0)})_{\mathrm{loop}} &= F_\pi\left[(\frac{4g_A^2}{F_\pi^2}I_a - \frac{2}{F_\pi^2}I_e)\bar{g}_\pi^{(0)} - \frac{40g_{\pi N\Delta}^2\bar{g}_{\Delta\Delta\pi}^{(0)}}{9F_\pi^2}I_d + (Z_N - 1)\bar{g}_\pi^{(0)}\right] \\
\delta(F_\pi\bar{g}_\pi^{(1)})_{\mathrm{loop}} &= F_\pi\left[(-\frac{12g_A^2}{F_\pi^2}I_a - \frac{8}{3F_\pi^2}I_e)\bar{g}_\pi^{(1)} + \frac{8g_{\pi N\Delta}^2\bar{g}_{\Delta\Delta\pi}^{(1,1)}}{F_\pi^2}I_d - (\frac{40g_A^2}{F_\pi}I_c + \frac{80g_{\pi N\Delta}^2}{3F_\pi}I_b)\bar{g}_{\pi\pi\pi}^{(1)}\right. \\
&\quad \left. + \frac{5\bar{g}_{NN3\pi}^{(1)}}{F_\pi^2}I_e - \frac{15m_\pi^2(\Delta m_\pi^2)_{4q}}{16\pi^2F_\pi^4}\frac{\mathrm{Im}c_{4q}}{\mathrm{Re}c_{4q}}((\gamma_1 + 4\gamma_2)(L' + 1) - 2\gamma_2) + (Z_N - 1)\bar{g}_\pi^{(1)}\right].
\end{aligned} \tag{E15}$$

Meanwhile, for LR4Q-induced hadron mass parameters, with the identification that $(g_{NN\pi\pi}^{(0)})_{4q} = -16(\Delta m_N)_{4q}/3F_\pi$, $(g_{4\pi}^{(0)})_{4q} = 2(\Delta m_\pi^2)_{4q}/3F_\pi^2$ and $(g_{4\pi}^{(2)})_{4q} = 2(\delta m_\pi^2)_{4q}/F_\pi^2$ at tree-level, we obtain:

$$\begin{aligned}
\delta(\Delta m_N)_{4q,\mathrm{loop}} &= (Z_N-1)(\Delta m_N)_{4q} - \Big(\frac{12g_A^2}{F_\pi^2}I_a + \frac{16}{F_\pi^2}I_e\Big)(\Delta m_N)_{4q} - \frac{8g_{\pi N\Delta}^2(\Delta m_\Delta)_{4q}}{F_\pi^2}I_d \\
&\quad + \Big(\frac{12g_A^2}{F_\pi^2}I_c + \frac{8g_{\pi N\Delta}^2}{F_\pi^2}I_b\Big)(\Delta m_\pi^2)_{4q} - \frac{3m_\pi^2(\Delta m_\pi^2)_{4q}}{8\pi^2F_\pi^3}((\gamma_1+4\gamma_2)(L'+1)-2\gamma_2) \\
\delta(\Delta m_\pi^2)_{4q,\mathrm{loop}} &= \frac{3(\Delta m_\pi^2)_{4q}m_\pi^2}{8\pi^2F_\pi^2}\Big(L'+\frac{4}{3}\Big) \\
\delta(\delta m_\pi^2)_{4q,\mathrm{loop}} &= \frac{3(\delta m_\pi^2)_{4q}m_\pi^2}{4\pi^2F_\pi^2}\Big(L'+\frac{2}{3}\Big).
\end{aligned} \tag{E16}$$

Next we consider consequences of the introduction of $O(E^2c_{4q})$ LECs. First, $O(E^2c_{4q})$ terms in the pion sector (68), together with the $O(E^4)$ pion Lagrangian, modify the vacuum-alignment condition as:

$$\begin{aligned}
\alpha &= -\frac{3}{4}\frac{(\Delta m_\pi^2)_{4q,0}}{(m_\pi^2)_0}\frac{\mathrm{Im}c_{4q}}{\mathrm{Re}c_{4q}} + \frac{48(\Delta m_\pi^2)_{4q}}{F_\pi^2}[(2L_6^r+L_8^r)+\varepsilon^2(2L_7^r+L_8^r)]\frac{\mathrm{Im}c_{4q}}{\mathrm{Re}c_{4q}} \\
&\quad -16F_\pi^2\mathrm{Im}c_{4q}(K_1^r+4K_2^r) + \frac{15(\Delta m_\pi^2)_{4q}}{32\pi^2F_\pi^2}\frac{\mathrm{Im}c_{4q}}{\mathrm{Re}c_{4q}}(L+1).
\end{aligned} \tag{E17}$$

Unlike the case of the θ-term and cEDM, the angle α here is UV-divergent. It does not cause any problem though since α itself is not a physical observable. The modified vacuum alignment condition leads to extra contributions to $\bar{g}_\pi^{(0)}$ and $\bar{g}_\pi^{(1)}$:

$$\begin{aligned}
\delta(\bar{g}_\pi^{(0)})_v &= \frac{48(\delta m_N)_q(\Delta m_\pi^2)_{4q}}{F_\pi^3}[(2L_6+L_8)+\varepsilon^2(2L_7^r+L_8^r)]\frac{\mathrm{Im}c_{4q}}{\mathrm{Re}c_{4q}} \\
&\quad -16(\delta m_N)_qF_\pi\mathrm{Im}c_{4q}(K_1+4K_2) \\
&= \delta(\bar{g}_\pi^{(0)})_v^r + \frac{15(\Delta m_\pi^2)_{4q}(\delta m_N)_q}{32\pi^2F_\pi^3}\frac{\mathrm{Im}c_{4q}}{\mathrm{Re}c_{4q}}(L+1) \\
\delta(\bar{g}_\pi^{(1)})_v &= \frac{96(\Delta m_N)_q(\Delta m_\pi^2)_{4q}}{F_\pi^3}[(2L_6+L_8)+\varepsilon^2(2L_7^r+L_8^r)]\frac{\mathrm{Im}c_{4q}}{\mathrm{Re}c_{4q}} \\
&\quad -32(\Delta m_N)_qF_\pi\mathrm{Im}c_{4q}(K_1+4K_2) \\
&= \delta(\bar{g}_\pi^{(0)})_v^r + \frac{15(\Delta m_\pi^2)_{4q}(\Delta m_N)_q}{16\pi^2F_\pi^3}\frac{\mathrm{Im}c_{4q}}{\mathrm{Re}c_{4q}}(L+1).
\end{aligned} \tag{E18}$$

that are also UV-divergent. This should not bother us because these divergences, together with the divergences from the one-loop corrections to $F_\pi\bar{g}_\pi^{(i)}$, will be canceled by the $O(E^2c_{4q})$ LECs as we shall discuss later.

Finally, Eq. (68) also contributes to the pion mass shifts:

$$
\begin{aligned}
\delta(\Delta m_\pi^2)_{4q,ct} &= -\frac{32m_\pi^2}{F_\pi^2}(2L_4+L_5)(\Delta m_\pi^2)_{4q} - 16F_\pi^2\mathrm{Re}c_{4q}(2K_4-\frac{1}{3}K_3)m_\pi^2 \\
&\quad +\frac{16}{3}F_\pi^2 m_\pi^2\mathrm{Re}c_{4q}(3K_1+22K_2) \\
\delta(\delta m_\pi^2)_{4q,ct} &= -\frac{32m_\pi^2}{F_\pi^2}(2L_4+L_5)(\delta m_\pi^2)_{4q} - \frac{16}{3}F_\pi^2\mathrm{Re}c_{4q}K_3 m_\pi^2 - \frac{64}{3}F_\pi^2 m_\pi^2\mathrm{Re}c_{4q}K_2.
\end{aligned}
\tag{E19}
$$

The introduction of $O(E^2 c_{4q})$ LECs in nucleon sector contributes to the LR4Q-induced nucleon mass shifts:

$$
\begin{aligned}
\delta(\Delta m_N)_{4q,ct} &= 2F_\pi m_\pi^2\mathrm{Re}c_{4q}[2h_1+4h_2+h_3+h_4+2h_5] \\
\delta(\delta m_N)_{4q,ct} &= 4\varepsilon F_\pi m_\pi^2\mathrm{Re}c_{4q}[h_3+h_4-2h_5].
\end{aligned}
\tag{E20}
$$

Next, combining with the $O(E^4)$ LECs and the leading-order vacuum alignment, we obtain the LEC-contributions for $\bar{g}_\pi^{(i)}$:

$$
\begin{aligned}
\delta(F_\pi\bar{g}_\pi^{(0)})_{ct} &= \frac{4(\Delta m_\pi^2)_{4q}m_\pi^2\varepsilon}{F_\pi^3}\frac{\mathrm{Im}c_{4q}}{\mathrm{Re}c_{4q}}[f_2+f_3+f_5^r+f_6^r] + \frac{4F_\pi\mathrm{Im}c_{4q}m_\pi^2\varepsilon}{3}[3h_3+5h_4-2h_5] \\
\delta(F_\pi\bar{g}_\pi^{(1)})_{ct} &= -\frac{3(\Delta m_\pi^2)_{4q}m_\pi^2}{F_\pi^3}\frac{\mathrm{Im}c_{4q}}{\mathrm{Re}c_{4q}}[2(1+\varepsilon^2)f_1+2f_2+(1+\varepsilon^2)f_3+2(1+\varepsilon^2)f_4 \\
&\quad +(\varepsilon^2-1)f_5+(1+\varepsilon^2)(f_5^r+f_6^r)] + 4F_\pi\mathrm{Im}c_{4q}m_\pi^2[2h_1+8h_2+h_3+3h_4+2h_5] \\
\delta(F_\pi\bar{g}_\pi^{(2)})_{ct} &= \frac{(\Delta m_\pi^2)_{4q}m_\pi^2\varepsilon}{F_\pi^3}\frac{\mathrm{Im}c_{4q}}{\mathrm{Re}c_{4q}}[f_2+f_3+f_5^r+f_6^r] + \frac{8F_\pi\mathrm{Im}c_{4q}m_\pi^2\varepsilon}{3}[h_4-h_5]
\end{aligned}
\tag{E21}
$$

As we noticed before, one may choose the values of the combinations $3h_3+5h_4-2h_5$, $2h_1+8h_2+h_3+3h_4+2h_5$ and h_4-h_5 to subtract out the UV-divergences from $\delta(F_\pi\bar{g}_\pi^{(i)})_{\mathrm{loop}} + F_\pi\delta(\bar{g}_\pi^{(i)})_v$ together with the residual UV-divergences coming from the LECs f_i in $\delta(F_\pi\bar{g}_\pi^{(i)})_{ct}$. Details of such subtractions are given in Table IV.

Appendix F: Divergence Subtraction and Renormalized LECs

In this section we summarize the divergence subtractions by the counter terms. Following the Gasser-Leutwyler subtraction scheme, the relation between the bare and renormalized LEC is given by

$$
A = A^r + \frac{B}{\pi^2}(L+1) \tag{F1}
$$

where A is the bare LEC, A^r is the renormalized LEC and B is a finite quantity. Also, since any physical result must be μ-independent, the renormalized LEC A^r must be μ-dependent in the following way:

$$A^r(\mu') = A^r(\mu) + \frac{B}{\pi^2} \ln \left(\frac{\mu'}{\mu} \right)^2 \tag{F2}$$

in order to cancel the μ-dependence in the divergent loop integral.

The values of the finite quantity B for different combinations of LECs are summarized in Table IV.

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A	B
$2L_4 + L_5$	$-\frac{1}{64}$
$2L_6 + L_8$	$-\frac{3}{512}$
$2L_7 + L_8$	0
B_{20}	$\frac{9\pi^2 g_A^2}{32} + \frac{\pi^2 g_{\pi N\Delta}^2}{2}\frac{m_\pi^2 - 2\delta_\Delta^2}{m_\pi^2}$
$2f_1 + f_3, 2f_4 + f_5, f_5 + f_6$	0
f_2	$\frac{F_\pi(c_1+2c_1')}{2B_0}(\frac{9g_A^2}{16} - \frac{3}{8}) + \frac{F_\pi g_{\pi N\Delta}^2(c_2+2c_2')}{2B_0}\frac{2\delta_\Delta^2 - m_\pi^2}{m_\pi^2} + \frac{F_\pi g_{\pi N\Delta}^2 \delta_\Delta}{6m_\pi^2}\frac{2\delta_\Delta^2 - 3m_\pi^2}{m_\pi^2} + \frac{3(\gamma_1+4\gamma_2)}{32}$
$f_2 + f_3$	$-\frac{F_\pi c_1}{2B_0}(\frac{3g_A^2}{16} + \frac{1}{8}) + \frac{5F_\pi g_{\pi N\Delta}^2 c_2}{18B_0}\frac{2\delta_\Delta^2 - m_\pi^2}{m_\pi^2}$
$2G_1 + G_3$	$-\frac{3\beta}{16}$
$2G_2 + G_3$	0
$2G_4 + G_5$	$-\frac{\beta}{4}$
$g_1 + g_4, g_3 + g_4$ $g_5 + g_6, g_7 + g_8$	0
$g_2 + g_3$	$\frac{\tilde{c}_1+2\tilde{c}_1'}{4}(\frac{9g_A^2}{16} - \frac{3}{8}) + \frac{g_{\pi N\Delta}^2(\tilde{c}_2+2\tilde{c}_2')}{4}\frac{2\delta_\Delta^2 - m_\pi^2}{m_\pi^2} - \frac{2g_{\pi N\Delta}^2 \beta F_\pi \delta_\Delta}{m_\pi^2} + \frac{3\beta(\gamma_1+4\gamma_2)}{4}$
$g_2 + g_4$	$-\frac{\tilde{c}_1}{4}(\frac{3g_A^2}{16} + \frac{1}{8}) + \frac{5g_{\pi N\Delta}^2 \tilde{c}_2}{36}\frac{2\delta_\Delta^2 - m_\pi^2}{m_\pi^2}$
$g_6 + g_8$	$-\frac{2F_\pi \beta c_1}{B_0}(\frac{3g_A^2}{16} + \frac{1}{8}) + \frac{10F_\pi g_{\pi N\Delta}^2 \beta c_2}{9B_0}\frac{2\delta_\Delta^2 - m_\pi^2}{m_\pi^2}$
$2f_4 - f_5$ $-\frac{1}{4\beta}(g_5 + g_8)$	$-\frac{F_\pi(c_1+2c_1')}{2B_0}(\frac{9g_A^2}{16} - \frac{3}{8}) - \frac{F_\pi g_{\pi N\Delta}^2(c_2+2c_2')}{2B_0}\frac{2\delta_\Delta^2 - m_\pi^2}{m_\pi^2}$ $-\frac{F_\pi g_{\pi N\Delta}^2 \delta_\Delta}{6m_\pi^2}\frac{4\delta_\Delta^2 - 3m_\pi^2}{m_\pi^2} - \frac{3(\gamma_1+4\gamma_2)}{16}$
$2K_4 - \frac{1}{3}K_3$	$-\frac{2\rho}{3}$
K_3	$-\frac{\rho}{4}$
$3K_1 + 22K_2$	$-\frac{11\rho}{2}$
K_2	$-\frac{\rho}{4}$
$2h_1 + 4h_2 + h_3 + h_4 + 2h_5$	$\tilde{\tilde{c}}_1(\frac{9g_A^2}{16} - 1) + g_{\pi N\Delta}^2 \tilde{\tilde{c}}_2 \frac{2\delta_\Delta^2 - m_\pi^2}{m_\pi^2} - \frac{32g_{\pi N\Delta}^2 \rho}{3}\frac{F_\pi \delta_\Delta}{m_\pi^2} + 4\rho(\gamma_1 + 4\gamma_2)$
$h_3 + h_4 - 2h_5$	0
$3h_3 + 5h_4 - 2h_5$	$\frac{32F_\pi \rho c_1}{B_0}(\frac{3g_A^2}{64} + \frac{1}{2}) - \frac{40F_\pi g_{\pi N\Delta}^2 \rho c_2}{9B_0}\frac{2\delta_\Delta^2 - m_\pi^2}{m_\pi^2}$
$2f_4 - f_5$ $-\frac{1}{16\rho}(2h_1 + 8h_2 + h_3$ $+3h_4 + 2h_5)$	$-\frac{1}{6}(\frac{9g_A^2}{16} - 1)(\frac{3F_\pi(c_1+2c_1')}{B_0} + \frac{3\tilde{\tilde{c}}_1}{4\rho})$ $-\frac{g_{\pi N\Delta}^2}{6}(\frac{2\delta_\Delta^2 - m_\pi^2}{m_\pi^2})(\frac{3F_\pi(c_2+2c_2')}{B_0} + \frac{3\tilde{\tilde{c}}_2}{4\rho})$ $-\frac{F_\pi g_{\pi N\Delta}^2 \delta_\Delta}{6m_\pi^2}\frac{4\delta_\Delta^2 - 11m_\pi^2}{m_\pi^2} - \frac{\gamma_1+4\gamma_2}{2}$
$h_4 - h_5$	$\frac{4F_\pi \rho c_1}{B_0}(\frac{3g_A^2}{16} + \frac{1}{8}) - \frac{20F_\pi g_{\pi N\Delta}^2 \rho c_2}{9B_0}\frac{2\delta_\Delta^2 - m_\pi^2}{m_\pi^2}$

Table IV: Infinity subtraction by the LECs.