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Identity Method Reexamined

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This paper discusses the impact of finite particle losses associated with instrumental effects in measurements of moments of produced multiplicities with the Identity Method towards the evaluation of fluctuation measures such as ν_{dyn}. One finds that the identity method remains applicable provided it is modified to determine factorial moments $\langle N(N-1)\rangle$, rather than moments $\langle N^2\rangle$. It is further demonstrated that ν_{dyn} remains robust if detection efficiencies are uniform across the measurement's acceptance. The robustness is lost, however, if detection efficiencies are momentum dependent, although the identity method remains applicable provided detection efficiencies can be determined with sufficient accuracy.

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I. INTRODUCTION

Studies of fluctuations of the relative yield of produced particles in high-energy nucleus-nucleus collisions provide valuable information on the particle production dynamics, the collision system evolution, and might also enable the identification of anomalous behavior signaling deconfinement or the existence of critical behavior [1–3]. Measurements of integral correlations based on the ν_{dyn} fluctuation measure [4, 5], in particular, have received a growing level of interest because this observable provides several advantages experimentally and phenomenologically. It is indeed straightforward to measure thanks to its rather simple definition based on a combination of ratios of second factorial moments to the square of inclusive averages, and because it is nominally robust against particle losses due to instrumental effects. It is also relatively insensitive to collision volume uncertainties and fluctuations and its phenomenological interpretation is thus relatively straightforward.

The ν_{dyn} fluctuation measure has been used to study net-charge fluctuations [6–8] as well as fluctuations of the relative yield of different particle species [9, 10]. Measurements of relative species yield fluctuations typically utilize conventional particle selection techniques based on measurements of specific energy loss and time-of-flight measurements. In the context of this technique, particles must be identified and counted event-by-event to determine the number (multiplicity) of particles of each species of interest, and calculate their first and second factorial moments within the collision dataset. Evidently, measurements of specific energy loss or time-of-flight provide unambiguous particle identification capabilities only across a rather limited kinematic range. Beyond such a range, considerable PID ambiguity typically arises. Ambiguity and signal contamination may be suppressed by using narrower selection cuts but these usually imply significant reductions in detection efficiency. In an effort to avoid signal contamination, ambiguities, and efficiency losses implied by narrow PID selection criteria, authors of Refs. [11–13] have developed a technique known as *identity method* which relies on the probability that a given particle might be of a given type or species based on the value of the PID signal and the estimated probability distribution, hereafter referred to as line shape, of such signal for distinct particle species. The method is straightforward for measurements of single particle spectra but becomes significantly more complicated for the evaluation of second or higher moments of multiplicities. Be that as it may, Ref. [13] presents a well defined and relatively straightforward method for the evaluation of second moments and covariances. The method is quite elegant but unfortunately neglects effects associated with particle losses. It is the purpose of this work to investigate the impact of such losses and whether the method can be modified to account for them.

The impact of particle losses on integral fluctuation measures (i.e., cumulants) has been discussed by several authors and correction formula to account for such losses, with fixed and momentum dependent efficiencies, have been developed [14–17]. It should be pointed out that such corrections are unnecessary in the context of $\nu_{\rm dyn}$ analyses provided the detection efficiency is constant (i.e., uniform across the experimental acceptance) and the identification of particle species unambiguous. In this work, however, I shall consider the application of the identity method in cases where the unambiguous identification of species is not possible and for detection systems in which the efficiency might exhibit a complicated dependence on the particle species, their momentum, azimuth angle, and rapidity. I will show that with sufficient statistics and provided the efficiency can be reliably estimated across the acceptance, the $\nu_{\rm dyn}$ observable, though not robust, can be reliably corrected for particle losses and particle identification ambiguities.

This paper is divided as follows. Section II presents a brief review of the impact of uncorrelated efficiency losses in cases where particle counting is unambiguous and exact. Section III builds on the identity method described in Refs. [11, 13] and presents a discussion of the impact of uncorrelated particle losses on the calculation of the moments of the event-wise identity variables W_p. The method is further expanded in sec. IV to account for momentum dependent efficiencies. This work is summarized in sec. V.

II. MEASURING MULTIPLICITY MOMENTS IN THE PRESENCE OF EFFICIENCY LOSSES

The discussion is formulated in the context of a measurement of the ν_{dyn} observable but the results presented can be straightforwardly extended to other fluctuation observables. The observable ν_{dyn} and its properties were introduced and discussed in detail in Ref. [5]:

$$\nu_{dyn}(N_p, N_q) = \frac{\langle N_p\,(N_p-1)\rangle}{\langle N_p\rangle^2} + \frac{\langle N_q\,(N_q-1)\rangle}{\langle N_q\rangle^2} - 2\frac{\langle N_p N_q\rangle}{\langle N_p\rangle\langle N_q\rangle}, \tag{1}$$

The variables N_p and N_q represent the multiplicities of produced particles, of species of type p and q, respectively, measured event-by-event, within the fiducial volume Ω of the experiment. More generally, one is interested in measuring factorial and cross moments of multiplicities N_p and N_q of particle species p and q, with $p, q = 1, \ldots, K$ denoting K distinct particle species (e.g., 1 =pion, 2 =kaon, 3 =proton), observable and countable event-by-event. These moments are determined by the true (T) joint probability of the K particle species and are herein denoted $P_{\mathrm{T}}(N_1, N_2, \ldots, N_K)$:

$$\begin{aligned}
\langle N_p\rangle &\equiv \sum_{N_1,\ldots,N_p,\ldots,N_K=0}^{\infty} P_{\mathrm{T}}(N_1,\ldots,N_p,\ldots,N_K)N_p, \\
\langle N_p^2\rangle &\equiv \sum_{N_1,\ldots,N_p,\ldots,N_K=0}^{\infty} P_{\mathrm{T}}(N_1,\ldots,N_p,\ldots,N_K)N_p^2, \\
\langle N_p N_q\rangle &\equiv \sum_{N_1,\ldots,N_p,\ldots,N_q,\ldots,N_K=0}^{\infty} P_{\mathrm{T}}(N_1,\ldots,N_p,\ldots,N_q,\ldots,N_K)N_pN_q.
\end{aligned} \tag{2}$$

Evidently, not all produced particles are properly counted given there are instrumental losses. The multiplicities of measured (M) particles (i.e., actually detected and counted) are denoted using lower case letters, n_p. The instrumental losses are modeled with independent binomial distributions, $B(n_p|N_p, \varepsilon_p)$, $p = 1, \ldots, K$, defined as

$$B(n_p|N_p,\varepsilon_p) = \frac{N_p!}{n_p!(N_p-n_p)!}\varepsilon_p^n\,(1-\varepsilon_p)^{N_p-n_p}\,, \tag{3}$$

where ε_p represent the detection efficiency of particle species p. In general, the efficiencies ε_p differ for species $p = 1, \ldots, K$. The joint probability $P_{\mathrm{M}}(n_1, n_2, \ldots, n_K)$ of the number of observed particles is obtained by summing over all multiplicities the product of the joint probability of produced multiplicities $P_{\mathrm{T}}(N_1, N_2, \ldots, N_K)$ by the probabilities of observing the multiplicities n_p given the produced multiplicities N_p.

$$P_{\mathrm{M}}(n_1,n_2,\ldots,n_K) = \sum_{N_1,N_2,\ldots,N_K=0}^{\infty} P_{\mathrm{T}}(N_1,N_2,\ldots,N_K)B(n_1|N_1,\varepsilon_1)B(n_2|N_2,\varepsilon_2)\times\cdots\times B(n_K|N_K,\varepsilon_K). \tag{4}$$

The moments of the observed multiplicities are then calculated similarly as those of the produced multiplicities and one gets

$$\begin{aligned}
\langle n_p\rangle &\equiv \sum_{n_1,\ldots,n_p,\ldots,n_K=0}^{\infty} P_{\mathrm{M}}(n_1,\ldots,n_p,\ldots,n_K)n_p, \\
\langle n_p^2\rangle &\equiv \sum_{n_1,\ldots,n_p,\ldots,n_K=0}^{\infty} P_{\mathrm{M}}(n_1,\ldots,n_p,\ldots,n_K)n_p^2, \\
\langle n_p n_q\rangle &\equiv \sum_{n_1,\ldots,n_p,\ldots,n_q,\ldots,n_K=0}^{\infty} P_{\mathrm{M}}(n_1\ldots,n_p,\ldots,n_q,\ldots,n_K)n_pn_q.
\end{aligned} \tag{5}$$

It is then straightforward to verify (see for instance Ref. [5]) that the moments of the observed multiplicities (measured) are related to those of the produced multiplicities (true) according to

$$\begin{aligned} \langle n_p \rangle &= \varepsilon_p \langle N_p \rangle \\ \langle n_p^2 \rangle &= \varepsilon_p \left(1 - \varepsilon_p\right) \langle N_p \rangle + \varepsilon_p^2 \langle N_p^2 \rangle \\ \langle n_p n_q \rangle &= \varepsilon_p \varepsilon_q \langle N_p N_q \rangle. \end{aligned} \tag{6}$$

and the measured factorial moments $\langle n_p \left(n_p - 1\right) \rangle$ are then related to the true factorial moments according to

$$\langle n_p \left(n_p - 1\right) \rangle = \varepsilon_p^2 \langle N_p \left(N_p - 1\right) \rangle. \tag{7}$$

The observable ν_{dyn} is thus considered robust because efficiencies for species p and q cancel out of each of the three terms of Eq. 1.

The neglect of particle losses can have a significant impact on measurements of the variance of fluctuations (See also Refs [14, 15] for extensive discussions of the impact on cumulants). To illustrate this impact, consider a system with an average multiplicity of species p of order $\langle N_p \rangle = 100$ and a variance $\langle \Delta N_p^2 \rangle = 90$. The second moment of N_p is thus $\langle N_p^2 \rangle = 10,090$, and $\langle \Delta N_p^2 \rangle / \langle N_p \rangle^2 = 0.009$. Assuming the efficiency is $\varepsilon_p = 0.8$, one finds, using Eq. (6), $\langle n_p \rangle = 80$, $\langle n_p^2 \rangle = 6,473.6$, and $\langle \Delta n_p^2 \rangle / \langle n_p \rangle^2 = 0.0115$, which amounts to a 28% error. However, one verifies that $\langle n(n-1) \rangle / \langle n_p \rangle^2 = \langle N(N-1) \rangle / \langle N_p \rangle^2$ holds perfectly. One thus expect that to the extent that the identity method enables proper unfolding of the PID signal line shape, the moments $\langle n \rangle$ and $\langle n^2 \rangle$ shall then be heavily biased by particle losses, but quantities such as $\langle n(n-1) \rangle / \langle n_p \rangle^2$ shall remain robust and unbiased, that is, independent of particle detection efficiencies. This conclusion is shown to hold, in the next section, if the efficiencies are momentum independent.

III. THE IDENTITY METHOD

The identity method was introduced in Ref. [11] for two species, $p = 1, 2$, and extended in Ref. [12, 13] for $K > 2$ species, i.e., for $p, q = 1, \ldots, K > 2$, and the determination of higher moments. It is based on the realization that it is often not possible, experimentally, to uniquely identify a particle species based on observables such as average energy loss in the gas of a Time Projection Chamber, time-of-flight measurement, or any other techniques aiming at the determination of the mass of measured particles. Indeed, one finds, in general, that there are limited kinematic regimes in which different species can be unambiguously identified (i.e., identified with perfect certainty) based on a particle identification (PID) observable, s. In most situations and kinematic ranges, however, there remains varying degrees of ambiguity. For instance, a given particle might likely be a pion, but there might be a finite probability that it is a kaon or a proton instead. This leads to contamination of the moments $\langle N_q \rangle$ and $\langle N_q(N_q - 1) \rangle$ which may have a rather detrimental impact on the evaluations of correlation observables such as ν_{dyn}. Within the identity method, rather than summing integer counts (e.g., increasing a counter by one unit if the particle is a pion, and zero, otherwise) and neglecting such contaminations, one accounts for ambiguities by summing weights $\omega_p(s)$ for each PID hypothesis. The weights are determined particle-by-particle, and for each hypothesis p, according to the relative frequency of particles of type p for a signal of amplitude s, that is, in the range $[s, s + ds]$,

$$\omega_p(s) \equiv \frac{\rho_p(s)}{\rho(s)}, \tag{8}$$

where

$$\rho(s) \equiv \sum_{p=1}^{K} \rho_p(s). \tag{9}$$

By construction, the sum of the weights of all particles, at a given signal amplitude s, equal unity:

$$\sum_{p=1}^{K} \omega_p(s) = 1. \tag{10}$$

The functions $\rho_p(s)$ represent the **line shapes** of the PID signal s for species $p = 1, \ldots, K$, determined from an average over a large ensemble of events. As per Eq. (1), in Ref.[12], their normalization is defined according to

$$\int \rho_p(s) ds = \langle N_p \rangle, \tag{11}$$

where $\langle N_p \rangle$ represents the mean multiplicity of the particle type p. Accordingly, one can also write

$$\rho_p(s) = \langle N_p \rangle p_p(s), \tag{12}$$

with

$$\int p_p(s) ds = 1, \tag{13}$$

in which the function $p_p(s)$ is a probability density function (p.d.f) representing the probability density that particles of type p produce a signal of amplitude s in the detector (formally in the range $[s, s+ds]$).

Several types of PID signal s may be used, including the average energy loss of a particle determined in an ionization chamber (e.g., a Time Projection Chamber), a particle's time of flight or mass determined from a combination of other observables, etc.

One defines an event-by-event variable W_p, hereafter called event-wise identity variable for species p, as the sum of the weights $\omega_p(s_i)$ calculated for all M particles of an event:

$$W_p = \sum_{i=1}^{M} \omega_p(s_i). \tag{14}$$

In this context, and for notational brevity in the following, it is convenient to introduce a vector notation $\vec{W} = (W_1, W_2, \ldots, W_K)$ to succinctly represent the K event-wise identity variables W_p. The vector $\vec{W}$ can then be viewed as a particle identity vector measured event-by-event. The identity method involves the calculation of moments of this vector and its components. In this work, the discussion is limited to moments $\langle W_p \rangle$, $\langle W_p^2 \rangle$, and $\langle W_p W_q \rangle$, for all relevant species p and q. These moments may be used towards the determination of the moments $\langle N_p \rangle$, $\langle N_p^2 \rangle$, and $\langle N_p N_q \rangle$ by solving a linear equation similar to that derived in Ref. [13]. However, note that the identity method as outlined in Ref. [13] neglects the detector response and does not account for particle losses. Extracted multiplicity moments $\langle N_p \rangle$ and $\langle N_p^2 \rangle$ are consequently potentially biased and the results obtained may thus be unreliable. This oversight is easily remedied and I derive, in this and following section, formula for the extraction of moments that include effects associated with efficiency losses.

Toward this end, one calculates the expectation value of the moments $\langle W_p \rangle$, $\langle W_p^2 \rangle$, and $\langle W_p W_q \rangle$ and show they can be related to the expectation value of the moments $\langle N_p \rangle$, $\langle N_p^2 \rangle$, and $\langle N_p N_q \rangle$ even in the presence of particles losses. However, efficiencies, ε_p, defined by Eq. (3), are needed for each of the particle species p of interest. In general, in a given event, there shall be n_1 particles of type 1, n_2 particles of type 2, and so forth. Assuming there are K species of interest, the variable W_p may then be written

$$\begin{aligned} W_p &= \sum_{i_1=1}^{n_1} \omega_p(s_{i_1}^{(1)}) + \sum_{i_2=1}^{n_2} \omega_p(s_{i_2}^{(2)}) + \cdots + \sum_{i_K=1}^{n_K} \omega_p(s_{i_K}^{(K)}), & (15) \\ &= \sum_{j=1}^{K} \sum_{i_j=1}^{n_j} \omega_p(s_{i_j}^{(j)}), & (16) \end{aligned}$$

which includes K distinct sums consisting of n_1, n_2, ..., and n_K terms. The variables $s_{i_j}^{(j)}$ represent the PID variables that might be observed for particles of species j. In order to calculate the moments, one must sum over all permissible permutations of the multiplicities n_1, n_2, ..., and n_K and all possible values of the variables $s_{i_j}^{(j)}$. Expressing the joint probability $P_{\mathrm{M}}(n_1, n_2, \ldots, n_K)$ according to Eq. (4), the expectation value of $\langle W_p \rangle$ may then be written

$$\begin{aligned} \langle W_p \rangle = & \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} \cdots \sum_{n_K=0}^{N_K} \sum_{N_1=0}^{\infty} \sum_{N_2=0}^{\infty} \cdots \sum_{N_K=0}^{\infty} P_{\mathrm{T}}(N_1, N_2, \ldots, N_K) \\ & \times B(n_1|N_1, \varepsilon_1) B(n_2|N_2, \varepsilon_2) \times \cdots \times B(n_K|N_K, \varepsilon_K) \\ & \times \prod_{i_1=1}^{n_1} \int p_1(s_{i_1}^{(1)}) ds_{i_1}^{(1)} \times \prod_{i_2=1}^{n_2} \int p_2(s_{i_2}^{(2)}) ds_{i_2}^{(2)} \times \cdots \times \prod_{i_K=1}^{n_K} \int p_K(s_{i_K}^{(K)}) ds_{i_K}^{(K)} \\ & \times \left[\sum_{j=1}^{K} \sum_{i'_j=1}^{n_j} \omega_p(s_{i'_j}^{(j)}) \right], \end{aligned} \tag{17}$$

where the functions $p_p(s^{(p)}_{i_p}) \equiv \rho_p(s^{(p)}_{i_p})/\langle n_p\rangle$ represent the probability density of observing PID variable values $s^{(p)}_{i_p}$. Evaluation of the above expression is accomplished by distributing the K terms of W_p and changing the order of the sums. This yields an expression consisting of K distinct sums, i.e., one for each particle species considered in the analysis, as follows:

$$\begin{aligned}
\langle W_p\rangle \;=\; & \sum_{i'_1=1}^{n_1}\sum_{N_1=0}^{\infty}\sum_{N_2=0}^{\infty}\cdots\sum_{N_K=0}^{\infty}\sum_{n_1=0}^{N_1}\sum_{n_2=0}^{N_2}\cdots\sum_{n_K=0}^{N_K} P_{\mathrm{T}}(N_1,N_2,\ldots,N_K) \qquad (18)\\
& \times B(n_1|N_1,\varepsilon_1)B(n_2|N_2,\varepsilon_2)\times\cdots\times B(n_K|N_K,\varepsilon_K)\\
& \times \prod_{i_1=1}^{n_1}\int p_1(s^{(1)}_{i_1})ds^{(1)}_{i_1}\times\prod_{i_2=1}^{n_2}\int p_2(s^{(2)}_{i_2})ds^{(2)}_{i_2}\times\cdots\times\prod_{i_K=1}^{n_K}\int p_K(s^{(K)}_{i_K})ds^{(K)}_{i_K}\times\omega_p(s^{(1)}_{i'_1})\\
& +\sum_{i'_2=1}^{n_2}\sum_{N_1=0}^{\infty}\sum_{N_2=0}^{\infty}\cdots\sum_{N_K=0}^{\infty}\sum_{n_1=0}^{N_1}\sum_{n_2=0}^{N_2}\cdots\sum_{n_K=0}^{N_K} P_{\mathrm{T}}(N_1,N_2,\ldots,N_K)\\
& \times B(n_1|N_1,\varepsilon_1)B(n_2|N_2,\varepsilon_2)\times\cdots\times B(n_K|N_K,\varepsilon_K)\\
& \times \prod_{i_1=1}^{n_1}\int p_1(s^{(1)}_{i_1})ds^{(1)}_{i_1}\times\prod_{i_2=1}^{n_2}\int p_2(s^{(2)}_{i_2})ds^{(2)}_{i_2}\times\cdots\times\prod_{i_K=1}^{n_K}\int p_K(s^{(K)}_{i_K})ds^{(K)}_{i_K}\times\omega_p(s^{2}_{i'_2})\\
& \cdots\\
& +\sum_{i'_K=1}^{n_k}\sum_{N_1=0}^{\infty}\sum_{N_2=0}^{\infty}\cdots\sum_{N_K=0}^{\infty}\sum_{n_1=0}^{N_1}\sum_{n_2=0}^{N_2}\cdots\sum_{n_K=0}^{N_K} P_{\mathrm{T}}(N_1,N_2,\ldots,N_K)\\
& \times B(n_1|N_1,\varepsilon_1)B(n_2|N_2,\varepsilon_2)\times\cdots\times B(n_K|N_K,\varepsilon_K)\\
& \times \prod_{i_1=1}^{n_1}\int p_1(s^{(1)}_{i_1})ds^{(1)}_{i_1}\times\prod_{i_2=1}^{n_2}\int p_2(s^{(2)}_{i_2})ds^{(2)}_{i_2}\times\cdots\times\prod_{i_K=1}^{n_K}\int p_K(s^{(K)}_{i_K})ds^{(K)}_{i_K}\times\omega_p(s^{(K)}_{i'_K}).
\end{aligned}$$

Integrals of the form $\int p_p(s)ds$ yield unity by definition, Eq. (13). Introducing u_{pq} coefficients defined as

$$u_{pq} = \int \omega_p(s)p_q(s)ds, \qquad (19)$$

and carrying first the sums on observed multiplicities and next those on produced multiplicities, one gets

$$\begin{aligned}
\langle W_p\rangle \;&=\; \sum_{N_1=0}^{\infty}\sum_{N_2=0}^{\infty}\cdots\sum_{N_K=0}^{\infty} P_{\mathrm{T}}(N_1,N_2,\ldots,N_K)\left[u_{p1}\varepsilon_1 N_1 + u_{p2}\varepsilon_2 N_2+\cdots+u_{pK}\varepsilon_K N_K\right], \qquad (20)\\
&=\; \sum_{i=1}^{K} u_{pi}\varepsilon_i\langle N_i\rangle. \qquad (21)
\end{aligned}$$

The coefficient products $u_{pi}\varepsilon_i$ nominally form a $K\times K$ square matrix that can be inverted to solve for the moments $\langle N_i\rangle$. However, this requires a priori knowledge of the efficiencies ε_i. It is thus more convenient to factor the efficiencies out of the matrix and define uncorrected multiplicities $N'_p = \varepsilon_p N_p$. Defining a matrix $\mathbf{U}$ with elements $U_{ij} = u_{ij}$, these uncorrected moments $\langle N_i\rangle$ are obtained by inversion of the matrix $\mathbf{U}$:

$$\langle N'_i\rangle = \sum_{j=1}^{K}\left(\mathbf{U}^{-1}\right)_{ij}\langle W_j\rangle. \qquad (22)$$

Dividing by the efficiency of detection of each species, one gets:

$$\langle N_i\rangle = \frac{\langle N'_i\rangle}{\varepsilon_i} = \frac{\left(\mathbf{U}^{-1}\langle\vec{W}\rangle\right)_i}{\varepsilon_i}. \qquad (23)$$

Calculation of the second moments $\langle W_p^2\rangle$ proceeds similarly but one must expand the square of W_p in terms of sums over single particle particle species and pairs of species:

$$
\begin{aligned}
\langle W_p^2\rangle &= \sum_{n_1=0}^{N_1}\sum_{n_2=0}^{N_2}\cdots\sum_{n_K=0}^{N_K}\sum_{N_1=0}^{\infty}\sum_{N_2=0}^{\infty}\cdots\sum_{N_K=0}^{\infty} P_{\rm T}(N_1,N_2,\ldots,N_K) \qquad (24)\\
&\times B(n_1|N_1,\varepsilon_1)B(n_2|N_2,\varepsilon_2)\times\cdots\times B(n_K|N_K,\varepsilon_K)\\
&\times\prod_{i_1=1}^{n_1}\int p_1(s_{i_1}^{(1)})ds_{i_1}^{(1)}\times\prod_{i_2=1}^{n_2}\int p_2(s_{i_2}^{(2)})ds_{i_2}^{(2)}\times\cdots\times\prod_{i_K=1}^{n_K}\int p_K(s_{i_K}^{(K)})ds_{i_K}^{(K)}\\
&\times\left[\sum_{j=1}^{K}\sum_{i'_j=1}^{n_j}\omega_p(s_{i'_j}^{(j)})\right]^2,\\
&= \sum_{n_1=0}^{N_1}\sum_{n_2=0}^{N_2}\cdots\sum_{n_K=0}^{N_K}\sum_{N_1=0}^{\infty}\sum_{N_2=0}^{\infty}\cdots\sum_{N_K=0}^{\infty} p_{\rm T}(N_1,N_2,\ldots,N_K) \qquad (25)\\
&\times B(n_1|N_1,\varepsilon_1)B(n_2|N_2,\varepsilon_2)\times\cdots\times B(n_K|N_K,\varepsilon_K)\\
&\times\prod_{i_1=1}^{n_1}\int p_1(s_{i_1}^{(1)})ds_{i_1}^{1}\times\prod_{i_2=1}^{n_2}\int p_2(s_{i_2}^{(2)})ds_{i_2}^{(2)}\times\cdots\times\prod_{i_K=1}^{n_K}\int p_K(s_{i_K}^{(K)})ds_{i_K}^{(K)}\\
&\times\left\{\sum_{j=1}^{K}\sum_{i'_j=1}^{n_j}\left[\omega_p(s_{i'_j}^{(j)})\right]^2+\sum_{j=1}^{K}\sum_{i'_1\neq i''_1=1}^{n_j}\omega_p(s_{i'_j}^{(j)})\omega_p(s_{i''_j}^{(j)})+\sum_{j\neq j'=1}^{K}\sum_{i'_j=1}^{n_j}\sum_{i''_{j'}=1}^{n_{j'}}\omega_p(s_{i'_j}^{(j)})\omega_p(s_{i''_{j'}}^{(j')})\right\}.
\end{aligned}
$$

Introducing the coefficients $u_{pj}^{(2)}$ defined as

$$u_{pj}^{(2)} = \int \omega_p^2(s)p_j(s)ds, \qquad (26)$$

the integrals and sums of Eq. (25) reduce to

$$\langle W_p^2\rangle = \sum_{j=1}^{K}\langle N_j\rangle\varepsilon_j u_{pj}^{(2)}+\sum_{j=1}^{K}\langle N_j(N_j-1)\rangle\varepsilon_j^2\left(u_{pj}\right)^2+\sum_{j\neq j'=1}^{K}\langle N_jN_{j'}\rangle\varepsilon_j\varepsilon_{j'}u_{pj}u_{pj'}. \qquad (27)$$

Note that if terms in equal powers of N_j are regrouped, as in Ref. [13], one ends up with a term in $\langle N_j\rangle$ with a coefficient proportional to a sum of linear and quadratic powers of the efficiency. It is thus more appropriate to keep the above expression as is, given it is the factorial moments that are required for the calculation of ν_{dyn} and they feature a simple square dependence on the detection efficiency.

The calculation of the covariance $\langle W_pW_q\rangle$ proceeds in a similar fashion. Introducing functions, u_{pqj}, defined as

$$u_{pqj} = \int \omega_p(s)\omega_q(s)p_j(s)ds, \qquad (28)$$

one obtains

$$\langle W_pW_q\rangle = \sum_{j=1}^{K}\langle N_j\rangle\varepsilon_j u_{pqj}+\sum_{j=1}^{K}\langle N_j(N_j-1)\rangle u_{pj}u_{qj}\varepsilon_j^2+\sum_{j\neq j'=1}^{K}\langle N_jN_{j'}\rangle\varepsilon_j\varepsilon_{j'}u_{pj}u_{qj'}. \qquad (29)$$

Equations (27,29) express the second moments $\langle W_p^2\rangle$ and cross-moments $\langle W_pW_q\rangle$ in terms of the moments $\langle N_p\rangle$, $\langle N_p(N_p-1)\rangle$, and $\langle N_pN_q\rangle\rangle$ in the presence of particle losses with efficiencies ε_p and ε_q. Next, one seeks to invert these expressions to obtain formula for the moments $\langle N_p(N_p-1)\rangle$ and $\langle N_pN_q\rangle\rangle$ in terms of $\langle W_p^2\rangle$ and $\langle W_pW_q\rangle$. Proceeding as in Eq. (23), one can absorb the efficiencies into the first moments, second order factorial moments, and covariance by defining

$$
\begin{aligned}
\langle N'_p\rangle &\equiv \langle N_p\rangle\varepsilon_p,\\
\langle N_p(N_p-1)'\rangle &\equiv \langle N_p(N_p-1)\rangle\varepsilon_p^2,\\
\langle N'_pN'_q\rangle &\equiv \langle N_pN_q\rangle\varepsilon_p^2. \qquad (30)
\end{aligned}
$$

Expressions for the moments $\langle W_p^2\rangle$ and $\langle W_p W_q\rangle$ thus reduce to

$$\begin{aligned}
\langle W_p^2\rangle &= \sum_{j=1}^{K}\langle N_j'\rangle u_{pj}^{(2)} + \sum_{i=1}^{K}\langle N_j(N_j-1)'\rangle\,(u_{pj})^2 + \sum_{j\neq j'=1}^{K}\langle N_j' N_{j'}'\rangle u_{pj}u_{pj'}, && (31)\\
\langle W_p W_q\rangle &= \sum_{j=1}^{K}\langle N_j'\rangle u_{pqj} + \sum_{j=1}^{K}\langle N_j(N_j-1)'\rangle u_{pj}u_{qj} + \sum_{j\neq j'=1}^{K}\langle N_j' N_{j'}'\rangle u_{pj}u_{qj'}, && (32)
\end{aligned}$$

which defines a system of $(K^2+K)/2$ linear equations. Proceeding similarly as in Ref. [13], one first defines two "b" coefficients

$$\begin{aligned}
b_p &= \langle W_p^2\rangle - \sum_{j=1}^{K} u_{pj}^{(2)}\langle N_j'\rangle, && (33)\\
b_{pq} &= \langle W_p W_q\rangle - \sum_{j=1}^{K} u_{pqj}\langle N_j'\rangle, && (34)
\end{aligned}$$

with $p<q$, and four sets of "a" coefficients

$$\begin{aligned}
a_p^j &= (u_{pj})^2, && 1\le p,j\le K; && (35)\\
a_p^{jj'} &= 2u_{pj}u_{pj'}, && 1\le p\le K;\quad 1\le j<j'\le K;\\
a_{pq}^j &= u_{pj}u_{qj}, && 1\le p<q\le K;\quad 1\le j\le K;\\
a_{pq}^{jj'} &= u_{pj}u_{qj'}+u_{pj'}u_{qj}, && 1\le p<q\le K; 1\le j<j'\le K.
\end{aligned}$$

One next define the $K+K(K-1)/2$-vectors $\mathbf{N}$ and $\mathbf{B}$ as

$$\mathbf{N} = \begin{pmatrix} \langle N_1(N_1-1)'\rangle \\ \vdots \\ \langle N_K(N_K-1)'\rangle \\ \left\langle {N_1}'{N_2}'\right\rangle \\ \vdots \\ \left\langle {N_{k-1}}'{N_K}'\right\rangle \end{pmatrix},\ \mathbf{B} = \begin{pmatrix} b_1 \\ \vdots \\ b_K \\ b_{12} \\ \vdots \\ b_{(K-1)K} \end{pmatrix} \tag{36}$$

and the $(K+K(K-1)/2)\times(K+K(K-1)/2)$ matrix $\mathbf{A}$ as

$$\mathbf{A} = \begin{pmatrix} a_1^1 & \cdots & a_1^K & a_1^{12} & \cdots & a_1^{(K-1)K} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_K^1 & \cdots & a_K^K & a_K^{12} & \cdots & a_K^{(k-1)k} \\ a_{12}^1 & \cdots & a_{12}^K & a_{12}^{12} & \cdots & a_{12}^{(K-1)K} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{12}^K & \cdots & a_{(K-1)K}^K & a_{(K-1)K}^{12} & \cdots & a_{(K-1)K}^{(K-1)K} \end{pmatrix}. \tag{37}$$

Eqs. (31,32) may then be written $\mathbf{AN}=\mathbf{B}$, which is solved by inversion of $\mathbf{A}$:

$$\mathbf{N} = \mathbf{A}^{-1}\mathbf{B}. \tag{38}$$

Note that while this expression is of the same form as that obtained in Ref. [13], the definitions of both $\mathbf{N}$ and $\mathbf{B}$ are quite different and the procedure outlined in this work is thus distinct from the original identity method.

Three remarks are in order. First, since the calculation of b_p and b_{pq} requires knowledge of $\langle N_j'\rangle$, one must first solve Eq. (21) before attempting the solution of Eq. (38). Second, once the moments $\langle N_p(N_p-1)'\rangle$ and $\langle N_p' N_q'\rangle$ are obtained from Eq. (38), it is then unnecessary to correct them for efficiencies towards the determination of ν_{dyn} using

$$\nu_{dyn} = \frac{\langle N_p\,(N_p-1)'\rangle}{\langle N_p'\rangle^2} + \frac{\langle N_q\,(N_q-1)'\rangle}{\langle N_q'\rangle^2} - 2\frac{\langle N_p' N_q'\rangle}{\langle N_p'\rangle\langle N_q'\rangle}, \tag{39}$$

since the efficiencies cancel out term by term in this expression. Finally, it should be clear that for the purpose of a measurement of ν_{dyn}, the identity method as formulated in Ref. [13] shall produce proper results because the method is linear and thus produces ratios $\langle n(n-1)\rangle/\langle n\rangle^2$ that are robust even though the moments $\langle n^2\rangle$ feature a non factorizable dependence on the detection efficiency. However, the method outlined in this section presents the advantage of yielding factorial moments $\langle n(n-1)\rangle$ which have a simpler dependence on the efficiency, and it is straightforward, as shown in the following section, to extend the method to account for efficiency dependencies on the particle momentum or direction.

IV. THE IDENTITY METHOD WITH SEVERAL $p_\perp$ BINS

The method outlined in the previous section assumes that the line shape of the PID signal s is independent of the momentum and direction of the particles. In practice, for instance, the energy loss of a particle does depend on its momentum and the dE/dx line shape is then a function of the particle momentum. This in turn implies that the probabilities $p_p(s)$ are also dependent on the momentum of the particles. The identity method analysis must then be carried out in fine bins of momentum and one must also consider how detection efficiencies may change with the particle momentum and direction (i.e., vs. $p_\perp$, rapidity, and azimuth angle). The calculation technique used in the previous section remains applicable provided one assumes there is a definite (albeit unknown a priori) probability to find particles in specific bins of $p_\perp$, rapidity, and azimuth angle. In the following, the calculation is carried with finite momentum binning exclusively, but the technique can be extended to account for binning in other coordinates.

In order to account for particle production in R momentum bins, one replaces the probability distribution $P_{\rm M}(n_1, n_2, \ldots, n_K)$ by a new function $P_{\rm M}(n_{11}, n_{12}, \ldots, n_{1R}, n_{21}, \ldots, n_{2R}, \ldots, n_{K1}, \ldots, n_{KR})$, in which the variables $n_{i\alpha}$, with $i = 1, \ldots, K$, $\alpha = 1, \ldots, R$, denote the number of particles of species i produced in momentum bin α. Hereafter, roman letters index particle species and greek letters index momentum bins. Equations 5 must be extended to include momentum bin dependencies. Introducing the shorthand

$$\vec{n} \equiv (n_{11}, n_{12}, \ldots, n_{1R}, n_{21}, \ldots, n_{2R}, \ldots, n_{K1}, \ldots, n_{KR}), \tag{40}$$

as well as the sum notation

$$\sum_{\vec{n}} \equiv \sum_{n_{11}=0}^{\infty} \cdots \sum_{n_{1R}=0}^{\infty} \sum_{n_{21}=0}^{\infty} \cdots \sum_{n_{2R}=0}^{\infty} \cdots \sum_{n_{K1}=0}^{\infty} \cdots \sum_{n_{KR}=0}^{\infty}, \tag{41}$$

the moments of the multiplicities $n_{i\alpha}$ can be calculated for each species p and each $p_\perp$ bin α, according to

$$\begin{aligned} \langle n_{p\alpha}\rangle &\equiv \sum_{\vec{n}} P_{\rm M}(\vec{n}) n_{p\alpha}, \\ \langle n_{p\alpha}^2\rangle &\equiv \sum_{\vec{n}} P_{\rm M}(\vec{n}) n_{p\alpha}^2, \\ \langle n_{p\alpha} n_{q\beta}\rangle &\equiv \sum_{\vec{n}} P_{\rm M}(\vec{n}) n_{p\alpha} n_{q\beta}. \end{aligned} \tag{42}$$

For fluctuations analyses, one seeks the moments of multiplicities n_p consisting of the sum of the $n_{p\alpha}$ across all $p_\perp$ bins, i.e.,

$$n_p = \sum_{\alpha=1}^{R} n_{p\alpha}. \tag{43}$$

The first moment $\langle n_p\rangle$ is trivially obtained as a sum of the first moments $\langle n_{p\alpha}\rangle$

$$\langle n_p\rangle = \left\langle \sum_{\alpha=1}^{R} n_{p\alpha} \right\rangle = \sum_{\alpha=1}^{R} \langle n_{p\alpha}\rangle \tag{44}$$

Second moments and covariances require one sums all relevant momentum bin combinations

$$\langle n_p^2\rangle = \left\langle \left(\sum_{\alpha=1}^{R} n_{p\alpha} \right)^2 \right\rangle = \sum_{\alpha=1}^{R} \langle n_{p\alpha}^2\rangle + \sum_{\alpha\neq\alpha'=1}^{R} \langle n_{p\alpha} n_{p\alpha'}\rangle \tag{45}$$

$$\langle n_p n_q\rangle = \left\langle \left(\sum_{\alpha=1}^{R} n_{p\alpha} \right) \left(\sum_{\alpha'=1}^{R} n_{q\alpha'} \right) \right\rangle = \sum_{\alpha,\alpha'=1}^{R} \langle n_{p\alpha} n_{q\alpha'}\rangle. \tag{46}$$

Evidently, our discussion of efficiency losses applies for each momentum bin individually. One can then write

$$\begin{aligned}
\langle n_{p\alpha}\rangle &= \varepsilon_{p\alpha}\langle N_{p\alpha}\rangle && (47)\\
\langle n_{p\alpha}\left(n_{p\alpha}-1\right)\rangle &= \varepsilon_{p\alpha}^2\langle N_{p\alpha}\left(N_{p\alpha}-1\right)\rangle && (48)\\
\langle n_{p\alpha}n_{q\beta}\rangle &= \varepsilon_{p\alpha}\varepsilon_{q\beta}\langle N_{p\alpha}N_{q\beta}\rangle, && (49)
\end{aligned}$$

where the variables $n_{p\alpha}$ and $N_{p\alpha}$ represent the measured and true numbers of particles of species p in momentum bin α, respectively. A proper calculation of the moments $\langle N_p\rangle$, $\langle N_p\left(N_p-1\right)\rangle$ and $\langle N_pN_q\rangle$ shall then require efficiency corrections $p_\perp$-bin by $p_\perp$-bin, if the efficiencies $\varepsilon_{p\alpha}$ depend on α, i.e., the momentum of the particles.

$$\begin{aligned}
\langle N_p\rangle &= \sum_{\alpha=1}^{R}\frac{\langle n_{p\alpha}\rangle}{\varepsilon_{p\alpha}} && (50)\\
\langle N_p\left(N_p-1\right)\rangle &= \sum_{\alpha=1}^{R}\frac{\langle n_{p\alpha}\left(n_{p\alpha}-1\right)\rangle}{\varepsilon_{p\alpha}^2}+\sum_{\alpha\neq\alpha'=1}^{R}\frac{\langle n_{p\alpha}n_{p\alpha'}\rangle}{\varepsilon_{p\alpha}\varepsilon_{p\alpha'}} && (51)\\
\langle N_pN_q\rangle &= \sum_{\alpha,\alpha'=1}^{R}\frac{\langle n_{p\alpha}n_{q\alpha'}\rangle}{\varepsilon_{p\alpha}\varepsilon_{q\alpha'}}. && (52)
\end{aligned}$$

Equations (50-52) are general and can be applied to traditional cut analyses or with the $p_\perp$ dependent identity method discussed next.

To apply the identity method in cases involving multiple $p_\perp$ bins, one must obtain expressions for the moments $\langle n_{p\alpha}\rangle$, $\langle n_{p\alpha}\left(n_{p\alpha}-1\right)\rangle$, and $\langle n_{p\alpha}n_{q\alpha'}\rangle$ in terms of identity variables determined for each species and each momentum bin. One thus defines

$$W_{p\alpha}=\sum_{i=1}^{n}\omega_{p\alpha}(s_i)\Theta_\alpha(s_i), \qquad (53)$$

in which the sum proceeds over all (accepted) particles of an event. The function $\omega_{p\alpha}(s_i)$ represents the probability of the i-th particle being of species p when observed in $p_\perp$ bin α, and the function $\Theta_\alpha(s_i)$ is unity if the i-th particle is within the $p_\perp$ bin α and null otherwise. Calculations of the expectation value of the moments of $W_{p\alpha}$ proceed as in sec. III but are carried out for specific $p_\perp$ bins α (β). The first moments are

$$\langle W_{p\alpha}\rangle=\sum_{j=1}^{K}\langle n_{p\alpha}\rangle u_{pj,\alpha}=\sum_{j=1}^{K}\langle N_{p\alpha}\rangle u_{pj,\alpha}\varepsilon_{i\alpha}, \qquad (54)$$

in which the coefficients $u_{pj,\alpha}$ are calculated according to

$$u_{pj,\alpha}=\int\omega_{p\alpha}(s)p_{j\alpha}(s)ds, \qquad (55)$$

where $p_{j\alpha}(s)$ represents the probability of observing a PID signal s for a particle of species j in momentum bin α.

Four second order moments must be considered. They are denoted $\langle W_{p\alpha}^2\rangle$, $\langle W_{p\alpha}W_{p\beta}\rangle$, $\langle W_{p\alpha}W_{q\alpha}\rangle$, and $\langle W_{p\alpha}W_{q\beta}\rangle$, with $p<q$ and $\alpha\neq\beta$. Calculation of these moments yields

$$\begin{aligned}
\langle W_{p\alpha}^2\rangle &= \sum_{j=1}^{K}\langle N_{j\alpha}\rangle\varepsilon_{j\alpha}u_{pj,\alpha}^{(2)}+\sum_{j=1}^{K}\langle N_{j\alpha}(N_{j\alpha}-1)\rangle\varepsilon_{j\alpha}^2\left(u_{pj,\alpha}\right)^2+\sum_{j\neq j'=1}^{K}\langle N_{j\alpha}N_{j'\alpha}\rangle\varepsilon_{j\alpha}\varepsilon_{j'\beta}u_{pj,\alpha}u_{pj',\alpha} && (56)\\
\langle W_{p\alpha}W_{p\beta}\rangle &= \sum_{j,j'=1}^{K}\langle N_{j\alpha}N_{j'\beta}\rangle\varepsilon_{j\alpha}\varepsilon_{j'\beta}u_{pj,\alpha}u_{pj',\beta} && (57)\\
\langle W_{p\alpha}W_{q\alpha}\rangle &= \sum_{j=1}^{K}\langle N_{j\alpha}\rangle\varepsilon_{j\alpha}u_{pqj,\alpha}+\sum_{j=1}^{K}\langle N_{j\alpha}(N_{j\alpha}-1)\rangle\varepsilon_{j\alpha}^2u_{pj,\alpha}u_{qj,\alpha}+\sum_{j\neq j'=1}^{K}\langle N_{j\alpha}N_{j'\alpha}\rangle\varepsilon_{j\alpha}\varepsilon_{j'\alpha}u_{pj,\alpha}u_{qj',\alpha} && (58)\\
\langle W_{p\alpha}W_{q\beta}\rangle &= \sum_{j,j'=1}^{K}\langle N_{j\alpha}N_{j'\beta}\rangle\varepsilon_{j\alpha}\varepsilon_{j'\beta}u_{pj,\alpha}u_{qj',\beta}. && (59)
\end{aligned}$$

with coefficients

$$u^{(2)}_{pj,\alpha} = \int \omega_{p\alpha}(s)^2 p_{j\alpha}(s) ds, \tag{60}$$

$$u_{pqj,\alpha} = \int \omega_{p\alpha}(s)\omega_{q\alpha}(s) p_{j\alpha}(s) ds. \tag{61}$$

By construction, the cross-terms are symmetric under interchanges of the indices p and q and indices α and β:

$$\begin{aligned} \langle W_{p\alpha}W_{p\beta}\rangle &= \langle W_{p\beta}W_{p\alpha}\rangle \\ \langle W_{p\alpha}W_{q\alpha}\rangle &= \langle W_{q\alpha}W_{p\alpha}\rangle \\ \langle W_{p\alpha}W_{q\beta}\rangle &= \langle W_{q\beta}W_{p\alpha}\rangle. \end{aligned}$$

There are thus $K \times R$ independent terms of the form $\langle W^2_{p\alpha}\rangle$, $K \times R(R-1)/2$ of the form $\langle W_{p\alpha}W_{p\beta}\rangle$, $K(K-1)/2 \times R$ of the form $\langle W_{p\alpha}W_{q\alpha}\rangle$, and $K(K-1)/2 \times R(R-1)/2$ of the form $\langle W_{p\alpha}W_{q\beta}\rangle$. The relation between the second order moments of W_p and the second order moments of the multiplicities N_p may then be viewed as a system of $Q = (K + K(K-1)/2) \times (R + R(R-1)/2)$ independent linear equations.

Proceeding as in sec. III, one defines "b" coefficients according to

$$b_{p,\alpha\alpha} = \langle W^2_{p\alpha}\rangle - \sum_{j=1}^{K} \langle N_{p\alpha}\rangle \varepsilon_{j\alpha} u^2_{pj,\alpha}, \tag{62}$$

$$b_{p,\alpha\beta} = \langle W_{p\alpha}W_{p\beta}\rangle, \tag{63}$$

$$b_{pq,\alpha\alpha} = \langle W_{p\alpha}W_{q\alpha}\rangle - \sum_{j=1}^{K} \langle N_{j\alpha}\rangle \varepsilon_{j\alpha} u_{pqj,\alpha}, \tag{64}$$

$$b_{pq,\alpha\beta} = \langle W_{p\alpha}W_{q\beta}\rangle, \tag{65}$$

where $p < q$ and $\alpha \neq \beta$. The "a" coefficients are next defined according to

$$a^{j}_{p,\alpha} = (u_{pj,\alpha})^2 \varepsilon^2_{j\alpha} \tag{66}$$

$$a^{jj'}_{p,\alpha} = u_{pj,\alpha} u_{pj',\alpha} \varepsilon_{j\alpha}\varepsilon_{j'\alpha} \tag{67}$$

$$a^{j}_{pq,\alpha} = u_{pj,\alpha} u_{qj,\alpha} \varepsilon^2_{j\alpha} \tag{68}$$

$$a^{jj'}_{pq,\alpha} = u_{pj,\alpha} u_{qj',\alpha} \varepsilon_{j\alpha}\varepsilon_{j'\alpha} \tag{69}$$

$$a^{jj'}_{pq,\alpha\beta} = u_{pj,\alpha} u_{qj',\beta} \varepsilon_{j\alpha}\varepsilon_{j'\beta}. \tag{70}$$

The column vector $\vec{B}$, matrix $\vec{A}$, and column vector $\vec{N}$ may then written

$$\mathbf{B} = \begin{pmatrix} b_{p,\alpha\alpha} \\ b_{p,\alpha\beta} \\ b_{pq,\alpha\alpha} \\ b_{pq,\alpha\beta} \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} a^{j}_{p,\alpha} & a^{jj'}_{p,\alpha} & 0 \\ 0 & 0 & a^{jj'}_{p,\alpha\beta} \\ a^{j}_{pq,\alpha\alpha} & a^{jj'}_{pq,\alpha\alpha} & 0 \\ 0 & 0 & a^{jj'}_{pq,\alpha\beta} \end{pmatrix} \quad \mathbf{N} = \begin{pmatrix} \langle N_{j,\alpha}(N_{j,\alpha}-1)\rangle \\ \langle N_{j,\alpha}N_{j',\alpha}\rangle \\ \langle N_{j,\alpha}N_{j',\beta}\rangle \end{pmatrix} \tag{71}$$

in which each of the elements are themselves vectors or matrices with indices p, q spanning all values $1 \leq p < q \leq K$ and indices α and β spanning all values $1 \leq \alpha \neq \beta \leq R$. For instance, in the case of $b_{p,\alpha}$, α spans all values 1 to R while p spans all values from 1 to K. However, in the case of the other b coefficients, the values spanned should satisfy $\alpha \neq \beta$ and $p < q$. Equations (56-59) may then be written $\mathbf{B} = \mathbf{AN}$ and can be solved by inversion of the matrix $\mathbf{A}$:

$$\mathbf{N} = \mathbf{A}^{-1}\mathbf{B} \tag{72}$$

This equation provides statistical estimates of the $p_\perp$ dependent moments $\langle N_{j,\alpha}(N_{j,\alpha}-1)\rangle$, $\langle N_{j,\alpha}N_{j',\alpha}\rangle$ and $\langle N_{j,\alpha}N_{j',\beta}\rangle$. These can then be combined to obtain the integral correlators $\langle N_j(N_j-1)\rangle$ and $\langle N_jN_{j'}\rangle$ according to

$$\langle N_p (N_p - 1)\rangle = \sum_{\alpha=1}^{R} \langle N_{p\alpha} (N_{p\alpha} - 1)\rangle, \tag{73}$$

$$\langle N_p N_q\rangle = \sum_{\alpha,\alpha'=1}^{R} \langle N_{p\alpha} N_{q\alpha}\rangle. \tag{74}$$

It is important to note that both **A** and **B** are now explicitly dependent on the detection efficiencies $\varepsilon_{j\alpha}$. Given the efficiencies are $p_\perp$ dependent, efficiency coefficients must be indeed included explicitly in the expressions of **A** and **B**. The robustness of ratios $\langle N_{j,\alpha}(N_{j,\alpha} - 1)\rangle/\langle N_{j,\alpha}\rangle^2$ is thus effectively lost. The identity method remains nonetheless applicable provided the coefficients $u_{pj,\alpha}$, $u^{(2)}_{pj,\alpha}$, $u_{pqj,\alpha}$, and the efficiencies $\varepsilon_{j\alpha}$ can be evaluated with sufficient precision.

V. SUMMARY

I first discussed the impact of finite particle losses associated with instrumental effects in measurements of moments of produced multiplicities with the Identity Method towards the evaluation of fluctuation measures such as ν_{dyn}. The original identity method produces moments $\langle n^2\rangle$ with a complicated dependence on the detection efficiency while the procedure outlined in this work yields factorial moments $\langle n(n-1)\rangle$ that feature a simple square dependence on the efficiency. However, both the original and modified identity methods shall yield robust, i.e., efficiency independent results, for the fluctuation observable ν_{dyn} as long as particle detection efficiencies are momentum independent. I further showed that the modified method outlined in this work provides for a straightforward albeit somewhat tedious extension to experimental cases where detection efficiencies are strongly dependent on the momentum of particles.

The treatment of particle losses discussed in this work can and should be applied to measurements of higher moments discussed in Ref. [12].

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