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We calculate the frequencies of collective modes of neutrons, protons and electrons in the outer core of neutron stars. The neutrons and protons are treated in a hydrodynamic approximation and the electrons are regarded as collisionless. The coupling of the nucleons to the electrons leads to Landau damping of the collective modes and to significant dispersion of the low-lying modes. We investigate the sensitivity of the mode frequencies to the strength of entrainment between neutrons and protons, which is not well characterized. The contribution of collective modes to the thermal conductivity is evaluated.

I. INTRODUCTION

In the outer core of a neutron star, matter consists of neutrons, protons and electrons, with possibly other minority constituents. The protons are expected to be superconducting with pairs in the $^1S_0$ state and gaps of order 1 MeV. The accuracy of present calculations does not permit an unambiguous answer to the question of whether or not neutrons are superfluid: calculations in which the internucleon interaction is taken to be the same as in free space predict pairing in the $^3P_2$ state [1] while estimates that take into account the modification of the interaction by the medium predict very small gaps [2] and are consistent with there being no gap (for a review of pairing gaps in neutron star matter see [3]). Collective modes have previously been studied in Refs. [4–7], and the possible role of such modes in transporting heat in neutron stars has been discussed in Ref. [8]. In this work, we shall assume that matter consists only of superfluid neutrons, superconducting protons and normal electrons. The superfluid critical temperatures for both neutrons and protons are estimated to be of the order of 1 MeV, which is about two orders of magnitude larger than the typical temperature in the interior of neutron stars, and therefore it is a good first approximation to take the temperature to be zero.

In a recent work [7] collective modes with wavelengths large compared with the Debye screening length for the electrons were studied under the assumption that the electrons could be treated in the hydrodynamic approximation. Here we extend that treatment to shorter wavelengths. When the wavelength is of order or shorter than the Debye screening length for electrons, the modes exhibit dispersion. In addition, modes are damped by collisions between electrons when the electrons are in the hydrodynamic regime and by Landau damping when the electrons are collisionless [9]. On the basis of our results for the damping of collective modes, we calculate the contribution of collective modes to the thermal conductivity of the outer core.

The plan of this paper is as follows. In Section II, we extend the formalism previously used to calculate the velocities of collective modes at long wavelengths [7] to wavelengths shorter than the Debye screening length, when the plasma is no longer locally neutral electrically. Expressions for the real and imaginary parts of mode frequencies are given in Sec. III, and in Sec. IV we discuss the magnitude of the parameter determining the entrainment of neutrons by protons (and vice versa) and give numerical results for the mode frequencies derived using an equation of state based on chiral effective field theory [10]. The thermal conductivity of matter is calculated in Sec. V, where we also comment on possible astrophysical effects. In the Appendix we demonstrate that the approach adopted in this paper, where we work in terms of the nucleon momenta per particle in the condensates, is equivalent to the traditional two-fluid formalism [11–13], where one works in terms of the momentum per particle in the neutron condensate and the velocity of the charged particles.

II. BASIC FORMALISM

We shall assume that the neutrons are superfluid, and that protons are superconducting and we denote the phases of the pair wave functions by $2\phi_n$ and $2\phi_p$, respectively. Here we shall not take into account variations in the spatial orientation of possible anisotropic order parameters. We shall work in terms of the momenta per particle in the neutron and proton condensates, which are given by

$$\mathbf{p}_\alpha = \hbar \boldsymbol{\nabla} \phi_\alpha, \tag{1}$$

where the index $\alpha = n, p$ refers to neutrons and protons. In the conventional treatment of superfluid dynamics, one defines quantities $\hbar \boldsymbol{\nabla} \phi_\alpha / m$, where m is the particle mass. These are referred to as superfluid velocities, even though they are covariant vectors, and therefore do not transform as velocities, which are contravariant vectors. To avoid confusion, we shall work almost exclusively in terms of the $\mathbf{p}_\alpha$. We shall neglect the difference between the neutron and proton rest masses since it is small compared with other effects, such as the contribution of interactions to the mass density, which we also neglect. Our approach is to start from the equations of motion for the nucleon densities and condensate phases, but we shall not impose the condition that the proton and electron densities are equal locally. Rather we shall include the Coulomb interaction between electrons and protons explicitly. For frequencies small compared with the pairing gaps and for wavelengths large compared with the coherence lengths, the neutrons and protons may be described by hydrodynamics (see [14] and references therein), and the Hamiltonian for the system is thus

$$H = \int d^3r\left(E^{\rm nuc}(n_n,n_p)+\frac{n_{nn}}{2m}p_n^2+\frac{n_{nn}}{2m}p_p^2+\frac{n_{np}}{m}\mathbf{p}_n\cdot\mathbf{p}_p\right) + H^e_{\rm kin}+H^{\rm Coul}. \quad (2)$$

Here the superfluid density tensor is given by

$$n_{\alpha\beta} = m\frac{\partial^2 E}{\partial\mathbf{p}_\alpha\partial\mathbf{p}_\beta}, \quad (3)$$

where E is the energy density. Here we include terms of second order in the velocities and neglect higher order contributions. With the help of the conditions [15, 16]

$$n_{pp}+n_{np} = n_p \quad (4)$$

and

$$n_{nn}+n_{np} = n_n, \quad (5)$$

which follow from the Galilean invariance of the neutrons and protons when relativistic effects are neglected, the kinetic energy contributions to the Hamiltonian density take on the form

$$\frac{n_{nn}}{2m}p_n^2+\frac{n_{pp}}{2m}p_p^2+\frac{n_{np}}{2m}\mathbf{p}_n\cdot\mathbf{p}_p = \frac{n_n}{2m}p_n^2+\frac{n_p}{2m}p_p^2-\frac{n_{np}}{2m}(\mathbf{p}_n-\mathbf{p}_p)^2. \quad (6)$$

When the condensation energy associated with pairing is small, the $n_{\alpha\beta}$ may be expressed in terms of Landau parameters [16]:

$$n_{nn} = n_n\frac{m}{m_n^*}\left(1+N_n(0)\frac{f_1^{nn}}{3}\right), \quad (7)$$

$$n_{pp} = n_p\frac{m}{m_p^*}\left(1+N_p(0)\frac{f_1^{pp}}{3}\right), \quad (8)$$

and

$$n_{np} = \frac{m\,k_n^2\,k_p^2}{9\pi^4}f_1^{np}. \quad (9)$$

Here k_α is the Fermi wavenumber, m_α^* the effective mass and $N_\alpha(0) = m_\alpha^* k_\alpha/\pi^2\hbar^2$ the density of states at the Fermi surface of species α, and $f_1^{\alpha\beta}$ is the $l=1$ component of the quasiparticle interaction. The equations of motion for the neutrons and protons may be obtained from Hamilton's equations, since the particle density n_α and $\hbar\phi_\alpha$ are conjugate variables and therefore $\dot{n}_\alpha = \hbar^{-1}\delta H/\delta\phi_\alpha$ and $\dot{\phi}_\alpha = -\hbar^{-1}\delta H/\delta n_\alpha$ [17]. The continuity equations for the number densities of neutrons and protons have the same form as in Ref. [7]:

$$\partial_t n_n+\boldsymbol{\nabla}\cdot\left[\frac{n_n}{m}\mathbf{p}_n-\frac{n_{np}}{m}(\mathbf{p}_n-\mathbf{p}_p)\right] = 0 \quad (10)$$

and

$$\partial_t n_p+\boldsymbol{\nabla}\cdot\left[\frac{n_p}{m}\mathbf{p}_p-\frac{n_{np}}{m}(\mathbf{p}_p-\mathbf{p}_n)\right] = 0. \quad (11)$$

In this paper we shall consider only linear modes. In this case, the time dependence of the phases is given by

$$\hbar\partial_t\phi_n = -\mu_n^{\rm nuc} \quad (12)$$

and

$$\hbar\partial_t\phi_p = -\mu_p^{\rm nuc}-e\Phi, \quad (13)$$

where

$$\mu_\alpha^{\rm nuc} = \frac{\partial E^{\rm nuc}}{\partial n_\alpha} \quad (14)$$

is the nucleon chemical potential in the absence of Coulomb contributions and in the absence of flows. The quantity Φ is the electric potential, which satisfies the Poisson equation,

$$\nabla^2\Phi = -4\pi e(n_p-n_e), \quad (15)$$

since the total charge density is $e(n_p-n_e)$. Therefore the equations of motion for $\mathbf{p}_\alpha$ are

$$\partial_t\mathbf{p}_p = -\boldsymbol{\nabla}\left(\mu_p^{\rm nuc}+e\Phi\right) \quad (16)$$

and

$$\partial_t\mathbf{p}_n = -\boldsymbol{\nabla}\mu_n^{\rm nuc}. \quad (17)$$

To close this set of equations one needs to calculate the response of the electrons to an electric potential. We shall consider small disturbances from the uniform state and therefore one can use linear response theory. If the frequency, ω, of the disturbance is large compared with the electron collision frequency and the wave number is large compared with the inverse mean free path, the electrons may be treated as collisionless. The mean free path of electrons is limited by scattering from electrons, and for calculating the properties of longitudinal modes, the relevant relaxation time is that for viscosity. This has been calculated in detail by Shternin and Yakovlev [18], who demonstrate that the dominant scattering process for electrons at temperatures less than $\sim 10^9$ K is via exchange of transverse photons, rather than the Coulomb interaction. For nuclear matter density and a temperature of 10^8K, their calculations lead to a mean free path for viscosity of $\sim 10^{10}$ fm or 10^{-3} cm. This is intermediate between typical stellar length scales ($\sim$ km) and the wavelength scale of thermal excitations $\sim \hbar v/(k_BT) \approx 10^4(v/c)/T_8$ fm. Here v is the velocity of the mode, k_B is the Boltzmann constant, and

T_8 is the temperature in units of 10^8 K. In this paper, our primary interest is collective modes on a microscopic scale, so we shall assume that wavelengths are short compared with the electron mean free path and frequencies are large compared with the electron collision frequency and we shall henceforth neglect collisions.

In the absence of collisions, the response of electrons in dense matter is given to a very good approximation by the random phase approximation and for ultrarelativistic electrons the result is [19]

$$\delta n_e = \chi_0 e \delta\Phi. \tag{18}$$

Here

$$\begin{aligned}\chi_0 &= \frac{\partial n_e}{\partial \mu_e}\left(1 - \frac{s}{2}\ln\frac{s+1}{s-1}\right) \\ &= \frac{\partial n_e}{\partial \mu_e}\left(1 - \frac{s}{2}\ln\left|\frac{s+1}{s-1}\right| + \mathrm{i}\frac{\pi}{2}s\,\Theta(1-|s|)\right), \end{aligned} \tag{19}$$

with $s = \omega/(ck)$, $\partial n_e/\partial \mu_e = p_e^2/\pi^2\hbar^3 c$ and Θ is the Heaviside step function. This form is valid for $\omega \ll \mu_e$ and $k \ll k_e$. In the absence of an external potential, the electric potential is given by

$$\Phi = \frac{4\pi e}{k^2}(n_p - n_e), \tag{20}$$

and therefore to linear order

$$\delta n_e = \frac{4\pi e^2}{k^2}\chi_0(\delta n_p - \delta n_e) \tag{21}$$

or

$$\delta n_e = \frac{(4\pi e^2/k^2)\chi_0}{1+(4\pi e^2/k^2)\chi_0}\delta n_p = \frac{(4\pi e^2/k^2)\chi_0}{\varepsilon_e(\omega,k)}\delta n_p. \tag{22}$$

Here

$$\varepsilon_e(\omega,k) = 1 + (4\pi e^2/k^2)\chi_0 \tag{23}$$

is the dielectric function of the electrons. Equation (22) enables us to eliminate δn_e from the equations of motion.

III. COLLECTIVE MODES

The properties of collective modes are obtained by solving Eqs. (10) – (13) together with Eqs. (20) and (22). Small longitudinal perturbations about a uniform state in which the nucleons and electrons are stationary satisfy the equation

$$\begin{pmatrix} -m\omega/k & 0 & n_{pp} & n_{np} \\ 0 & -m\omega/k & n_{np} & n_{nn} \\ E_{pp} & E_{np} & -\omega/k & 0 \\ E_{np} & E_{nn} & 0 & -\omega/k \end{pmatrix}\begin{pmatrix}\delta n_p \\ \delta n_n \\ \delta p_p \\ \delta p_n\end{pmatrix} = 0, \tag{24}$$

where

$$E_{pp} = E_{pp}^{\mathrm{nuc}} + \frac{4\pi e^2}{k^2\varepsilon_e(\omega,k)}. \tag{25}$$

Here, $E_{pp}^{\mathrm{nuc}} = \partial\mu_p^{\mathrm{nuc}}/\partial n_p$ and $E_{n\alpha} = \partial\mu_n^{\mathrm{nuc}}/\partial n_\alpha$. The first term on the right side of Eq. (25) represents the contribution from nuclear interactions and the second term is the Coulomb interaction between protons screened by electrons. We have dropped the superscript "nuc" on the other $E_{\alpha\beta}$ because there are no Coulomb contributions. Equation (24) reduces to the result of Ref. [7] if one neglects the frequency dependence of the electron response, and replaces ε_e by its asymptotic long-wavelength form $\simeq k_{TF}^2/k^2$, where $k_{TF} = \sqrt{4\pi e^2\partial n_e/\partial\mu_e}$ is the Thomas–Fermi screening wave number.

The dispersion relation for the collective modes is given by the condition $\det M = 0$, where M is the matrix in Eq. (24). In general, there are three collective modes, corresponding to the three components: two acoustic modes with in-phase and out-of-phase motions of the neutrons and the charged particles, and the plasma oscillation of the electrons, which has a nonzero frequency $\approx (4\pi n_e e^2c^2/\mu_e)^{1/2}$ as $k \to 0$. The present formalism is inadequate to calculate the frequency shift of the electron plasma oscillation due to the motion of neutrons and protons because the frequency is high compared with the gap frequencies and consequently the hydrodynamic approximation for the neutrons and protons is not valid.

A. Uncoupled modes

It is instructive to investigate the case when there is no coupling between neutrons and protons ($E_{np} = 0, n_{np} = 0, n_{pp} = n_p$, and $n_{nn} = n_n$). The dispersion relation for the collective mode of electrons and protons is then

$$\omega^2 = \frac{\Omega_p^2}{\varepsilon_e(\omega,k)} + \frac{n_p E_{pp}^{\mathrm{nuc}}}{m}k^2 \equiv v_{p0}^2 k^2, \tag{26}$$

where $\Omega_p = (4\pi e^2 n_p/m)^{1/2}$ is the proton plasma frequency.

Because of the frequency- and wave number dependence of ε_e, Eq. (26) has two solutions, the electron plasma oscillation and an acoustic mode, which has a velocity small compared with c. To calculate the frequency of the latter mode we may expand χ_0 in powers of s,

$$\chi_0 \simeq \frac{\partial n_e}{\partial\mu_e}\left(1 + \mathrm{i}\frac{\pi}{2}s - s^2\right). \tag{27}$$

As we shall show, to take into account the nonzero effective mass of the electrons it is necessary to retain terms of order s^2. We write the frequency of the mode in terms of its real part ω' and its imaginary part as ω''. In the long-wavelength limit, one finds

$$\left[m + \frac{\mu_e}{3c^2}\left(\frac{\pi^2}{4} - 1\right)\right]\omega'^2 \simeq \frac{\mu_e}{3}k^2 + n_p E_{pp}^{\mathrm{nucl}}k^2. \tag{28}$$

The coefficient of ω^2 in Eq. (28) shows that an electron contributes an amount $(\pi^2/4-1)\mu_e/3c^2 \approx 0.49\mu_e/c^2$ to the effective mass of the charged particles.

If one further neglects E_{pp}^{nuc} and considers the long-wavelength limit $k \to 0$. Eq. (28) is the analog for relativistic electrons of the expression for the sound speed in metals derived by Bohm and Staver [20]. The leading contribution to ω'' is

$$\omega'' \simeq -\frac{\pi}{12}\frac{\mu_e}{mc^2}ck. \tag{29}$$

This is due to Landau damping, the decay of the mode into electron–hole pairs [9].

If collisions between electrons are so frequent that conditions are hydrodynamic, the response function for the electrons has the form

$$\chi = \frac{\partial n_e}{\partial \mu_e}\frac{c_e^2 k^2}{c_e^2 k^2 - \omega^2}, \tag{30}$$

where $c_e = c/\sqrt{3}$ is the speed of hydrodynamic (first) sound in a ultrarelativistic Fermi gas. With this expression for χ, one sees that the effective mass of an electron is μ_e/c^2, or roughly twice the value for the collisionless limit. For $E_{pp}^{\rm nuc} \neq 0$, μ_e in Eqs. (28)-(29) is replaced by $\mu_e + 3E_{pp}^{\rm nuc}/n_p$. Since $\mu_e \sim 100$ MeV, the effects of the electron inertia in neutron star cores are at the 10% level and are therefore much more important than in terrestrial matter. However, the effects are comparable to the deviations of the energy per particle from the rest mass, which we have neglected in this article, where we have treated the nucleons nonrelativistically.

Due to incomplete screening at higher wave numbers, the velocity of the proton–electron mode depends on k. On solving the eigenvalue problem to first order in $\varepsilon_e'' = \Im \varepsilon_e$ one finds

$$\omega_0'^2 = \frac{n_p}{m}\left(E_{pp}^{\rm nuc} + \frac{4\pi e^2}{k^2 + k_{TF}^2}\right)k^2, \tag{31}$$

and

$$\omega_0'' = -\frac{\pi}{12}\frac{\mu_e}{mc^2}\frac{ck}{(1+k^2/k_{TF}^2)^2}, \tag{32}$$

which shows that the velocity and damping of modes is markedly reduced for $k \gtrsim k_{TF}$. Equation (31) shows that both nucleons and electrons contribute to the bulk modulus of the medium.

The frequency of the neutron mode when coupling between neutrons and protons is neglected is given by

$$\omega^2 = \frac{n_n E_{nn}}{m}k^2 \equiv v_{n0}^2 k^2. \tag{33}$$

B. The general case

The solution of the eigenvalue problem for Eq. (24) written in terms of the phase velocities $v = \omega/k$ of the modes has the form

$$v_\pm^2 = \frac{c_s^2}{2} \pm \sqrt{\left(\frac{c_s^2}{2}\right)^2 - \frac{\det[n_{\alpha\beta}]\det[E_{\alpha\beta}]}{m^2}}, \tag{34}$$

where

$$\begin{aligned} c_s^2 &= (E_{pp}n_{pp} + 2E_{np}n_{np} + E_{nn}n_{nn})/m \\ &= v_{p0}^2 + v_{n0}^2 + \frac{n_{np}}{m}(2E_{np} - E_{nn} - E_{pp}). \end{aligned} \tag{35}$$

where v_{p0} and v_{n0} are the velocities of the modes in the absence of coupling between neutrons and protons, given by Eqs. (26) and (33). To bring out the effects of the entrainment, it is useful to calculate mode frequencies in the absence of entrainment,

$$\left(v_\pm^{sc}\right)^2 = \frac{v_{p0}^2 + v_{n0}^2}{2} \pm \sqrt{\left(\frac{v_{p0}^2 - v_{n0}^2}{2}\right)^2 + \frac{n_n n_p E_{np}^2}{m^2}}. \tag{36}$$

The supercript sc indicates that only the scalar coupling between components is present, and the vector coupling that gives rise to entrainment is absent. Equation (34) is algebraically equivalent to the results derived for collective mode velocities in the inner crust of a neutron star in Ref. [13, Sec. III] from the standard two-fluid model. The fact that, in the neutron star crust, the charged particles are not superfluid is irrelevant for the mode frquencies, because in that work the ions and electrons were treated using a hydrodynamic approtach. From Eq. (34) one sees that there is a zero frequency mode if $\det[E_{\alpha\beta}]$ or $\det[n_{\alpha\beta}]$ vanish. This reflects the fact that the uniform system with no flow of protons or neutrons is unstable to formation of a density wave if $\det[E_{\alpha\beta}] = 0$ and to relative flow of the neutrons and protons if $\det[n_{\alpha\beta}] = 0$.

In the long-wavelength limit the imaginary part of E_{pp} can be obtained by combining Eqs. (23), (25) and (27) and Taylor-expanding χ_0^{-1}: $E_{pp}'' = E_{ee}'' \approx -(\pi/2)E_{ee}'\omega/ck$, where the real part of E_{ee} is $E_{ee}' = \partial\mu_e/\partial n_e$. Thus, the dispersion relation for the modes, the condition that the determinant of the matrix in Eq. (24) vanish, may be written

$$(v^2 - \tilde{v}_+^2)(v^2 - \tilde{v}_-^2) - \frac{\pi E_{ee}'}{2}\left(E_{nn}\frac{\det n_{\alpha\beta}}{m^2} - \frac{n_{pp}}{m}v^2\right)\frac{v}{c} = 0, \tag{37}$$

where $\tilde{v}_\pm$ are the mode velocities when $E_{ee}'' = 0$. To first order in E_{ee}'' the real parts of the velocities are equal to $v_\pm'$ and the imaginary parts of the velocities are thus given by

$$v_+'' = -\frac{\pi E_{ee}'}{2c}\frac{\left[(n_{pp}/m)v_+^2 - E_{nn}\det n_{\alpha\beta}/m^2\right]}{(v_+^2 - v_-^2)} \tag{38}$$

and

$$v_-'' = -\frac{\pi E_{ee}'}{2c}\frac{\left[E_{nn}\det n_{\alpha\beta}/m^2 - (n_{pp}/m)v_-^2\right]}{(v_+^2 - v_-^2)}, \tag{39}$$

where the $v_\pm$ on the right sides are to be evaluated in the absence of damping.

C. Physical character of the modes

The coupled modes generally involve oscillations of both the neutron and proton densities. On eliminating the momentum variables p_p and p_n from the linear system given in Eq. (24), we find

$$\begin{pmatrix} -mv^2 + n_{nn}E_{nn} + n_{np}E_{np} & n_{nn}E_{np} + n_{np}E_{pp} \\ n_{np}E_{nn} + n_{pp}E_{np} & -mv^2 + n_{np}E_{np} + n_{pp}E_{pp} \end{pmatrix} \times \begin{pmatrix} \delta n_n \\ \delta n_p \end{pmatrix} = 0. \tag{40}$$

Therefore the ratio of the neutron and proton density variations in the eigenmode with velocity $v_\pm$ is given by

$$\left.\frac{\delta n_n}{\delta n_p}\right|_{v_\pm} = \frac{n_{nn}E_{np} + n_{np}E_{pp}}{mv_\pm^2 - n_{nn}E_{nn} - n_{np}E_{np}}. \quad (41)$$

It is convenient to define mixing angles by the equations

$$\gamma_{n+} = \arctan \left.\frac{\delta n_n}{\delta n_p}\right|_{v_+} \quad (42)$$

and

$$\gamma_{p-} = \arctan \left.\frac{\delta n_p}{\delta n_n}\right|_{v_-}. \quad (43)$$

If $\gamma_{n+} = 0$, the mode v_+ involves motion only of the charged particles. When $0 < \gamma_{n+} < \pi/2$, the mode v_+ corresponds to an in-phase oscillation of the proton and neutron densities, and when $-\pi/2 < \gamma_{n+} < 0$, to an out-of-phase oscillation. Analogously, the mode with velocity v_- is a pure neutron oscillation when $\gamma_{p-} = 0$, an in-phase oscillation of the neutron and proton densities when $0 < \gamma_{p-} < \pi/2$, and an out-of-phase oscillation when $-\pi/2 < \gamma_{p-} < 0$.

IV. CALCULATION OF MODE FREQUENCIES

In this section we apply the formalism to calculate mode frequencies in the outer core. For the nucleon contributions to the quantities $E_{\alpha\beta}$ we use the results of Hebeler et al. [10], which were earlier employed in calculations of mode frequencies at long wavelengths [14]. In our numerical calculations we use values of $E^{nuc}_{\alpha\beta}$ based on chiral effective field theory, because we regard these as the best available. However, since f_1^{np} has not been calculated in detail for these interactions, we used for that the SLy4 value, Eq. (44), rather than estimates (51) and (54) based on arguments from nucleon effective masses obtained in chiral EFT models. The other important piece of input is the quantity n_{np} that determines the strength of entrainment. Chamel and Haensel showed that values of n_{np} obtained from Skyrme interactions had a large spread [24], and in Ref. [14] the result for the SLy4 Skyrme interaction,

$$n_{np}^{(a)} = -1.567\,\mathrm{fm}^3 n_n n_p = -0.04012\frac{n_n n_p}{n_0^2}\,\mathrm{fm}^{-3}, \quad (44)$$

which is representative of results found for many other Skyrme interactions, was adopted.

An alternative approach to estimating n_{np} is to use information on nucleon effective masses, which are related to the Landau parameters by the relations [23]

$$\frac{m_n^*}{m} = 1 + \frac{1}{3}\frac{m_n^* k_n}{\pi^2}\left[f_1^{nn} + \left(\frac{k_p}{k_n}\right)^2 f_1^{np}\right] \quad (45)$$

and

$$\frac{m_p^*}{m} = 1 + \frac{1}{3}\frac{m_p^* k_p}{\pi^2}\left[f_1^{pp} + \left(\frac{k_n}{k_p}\right)^2 f_1^{np}\right]. \quad (46)$$

Thus, for symmetric nuclear matter at the saturation density, it follows from Eq. (45) that

$$f_1^{np} = \frac{3\pi^2\hbar^2}{mk_0}\left(1 - \frac{m}{m_0^*}\right) - f_1^{nn}, \quad (47)$$

where $k_0 = (3\pi^2 n_0/2)^{1/3}$ is the Fermi wave number and m_0^* the nucleon effective mass for matter under those conditions. To obtain information about f_1^{nn}, we make use of results for the limiting cases of low neutron densities and pure neutron matter. For low neutron densities, f_1^{nn} tends to zero. Calculations for pure neutron matter indicate that $m_n^*/m \approx 1.0-1.2$, $0.9-1.1$, and $0.8-1.0$ for densities $n = 0.5n_0, n_0$, and $1.5n_0$, respectively [21, 22]. These results are all consistent with an effective mass equal to the bare mass, which would imply that f_1^{nn} for pure neutron matter is zero. Therefore the simplest assumption for the dependence of f_1^{nn} on proton fraction is that it vanishes for all neutron densities, in which case Eq. (47) becomes

$$f_1^{np} = \frac{3\pi^2\hbar^2}{mk_0}\left(1 - \frac{m}{m_0^*}\right), \quad (48)$$

and, from Eq. (9),

$$n_{np} = \left(1 - \frac{m}{m_0^*}\right)\frac{n_0}{2}, \ \text{(symmetric nuclear matter at saturation)}. \quad (49)$$

If one assumes that f_1^{np} is proportional to the product of the neutron and proton Fermi wavenumbers, as it is for the Skyrme interaction, one thus finds

$$n_{np} = 2\left(1 - \frac{m}{m_0^*}\right)\frac{n_n n_p}{n_0}. \quad (50)$$

For symmetric nuclear matter, the nucleon effective mass is typically taken to be $0.7m$ [21, 22], but it could be larger. If one is conservative and takes the value to be $0.8m$, one finds

$$n_{np}^{(b)} = -\frac{1}{2}\frac{n_n n_p}{n_0} = -3.125\ \mathrm{fm}^3 n_n n_p = -0.0800\frac{n_n n_p}{n_0^2}\ \mathrm{fm}^{-3}, \quad (51)$$

roughly twice the value (44) obtained from the SLy4 Skyrme interaction.

We now estimate the changes in this result due to a nonzero value of f_1^{nn}. Let us denote the effective mass of neutrons in pure neutron matter at density n_0 by m^*_{NM}. It then follows from Eq. (45) that

$$(f_1^{nn})_{\mathrm{NM}} = \frac{3\pi^2\hbar^2}{mk_{\mathrm{NM}}}\left(1 - \frac{m}{m^*_{\mathrm{NM}}}\right), \quad (52)$$

where $k_{\mathrm{NM}} = (3\pi^2 n_0)^{1/3}$ is the Fermi wave number of neutron matter. If f_1^{nn} scales with neutron density in the same way as it does for a Skyrme interaction ($\propto k_n^2$), this gives for this quantity for symmetric nuclear matter at density n_0,

$$f_1^{nn} = \frac{1}{2}\frac{3\pi^2\hbar^2}{mk_0}\left(1 - \frac{m}{m^*_{\mathrm{NM}}}\right). \quad (53)$$

Therefore, from Eq. (47), one finds

$$f_1^{np} = \frac{3\pi^2\hbar^2}{mk_0}\left(1 - \frac{m}{m_0^*} - \frac{1}{2}\left[1 - \frac{m}{m_{\rm NM}^*}\right]\right). \qquad (54)$$

This shows that for $m_0 = 0.8\,m$, a neutron effective mass of $1.1m$ rather than m for pure neutron matter would increase the magnitude of f_1^{np}, and hence also that of n_{np}, by $\sim 20\%$.

Throughout these arguments, we have assumed that the Landau parameters scale with Fermi momentum as they would for a Skyrme interaction. How good this assumption should be investigated using microscopic calculations. Results for low-density Fermi gases exhibit other scalings with density, e.g., for a spin-1/2 one component Fermi gas, f_1 scales as the Fermi wave number, not its square. This is due to the fact that this term comes from the induced interaction. However, because scattering lengths for nucleon–nucleon scattering are so large, we do not expect the scaling behavior predicted by the low-density gas to be relevant at most densities of interest in neutron stars.

Using the effective masses discussed above, we have evaluated the entrainment parameters defined in Eqs. (7), (8), and (9) and these are shown in Table I for characteristic densities in the neutron star outer core. The table also shows the derivatives of the energy density $E_{\alpha\beta}$ obtained using the expression for the energy density described in Ref. [10].

n_B(fm^{-3})	n_{pp}(fm^{-3})	n_p(fm^{-3})	n_{nn}(fm^{-3})	n_n(fm^{-3})	${n_{np}}^{(a)}$(fm^{-3})	${n_{np}}^{(b)}$(fm^{-3})	$E_{pp}^{\rm nuc}$ (GeV fm^3)	E_{nn} (GeV fm^3)	E_{np} (GeV fm^3)
0.08	2.96×10^{-3}	2.64×10^{-2}	7.768×10^{-2}	7.736×10^{-2}	-3.205×10^{-4}	-6.391×10^{-4}	1.294	0.1886	-0.5729
0.16	9.597×10^{-3}	7.746×10^{-3}	1.541×10^{-1}	1.523×10^{-1}	-1.851×10^{-3}	-3.692×10^{-3}	1.247	0.2684	-0.1891
0.24	1.678×10^{-2}	1.236×10^{-2}	2.321×10^{-1}	2.277×10^{-1}	-4.418×10^{-3}	-8.811×10^{-3}	1.325	0.3352	0.0811
0.32	2.237×10^{-2}	1.513×10^{-2}	3.121×10^{-1}	3.048×10^{-1}	-7.239×10^{-3}	-1.444×10^{-2}	1.433	0.3895	0.2972

TABLE I: Entrainment parameter and thermodynamic derivatives. The derivatives $E_{\alpha\beta}$ are calculated from an equation of state based on chiral effective field theory [10]. The entrainment parameter ${n_{np}}^{(a)}$ is calculated from Fermi liquid theory and Skyrme energy functional SLy4, Eq. (44), while ${n_{np}}^{(b)}$ is derived from considerations of nucleon effective masses, Eq. (51).

We write the mode velocity in terms of its real part, v', and its imaginary part v'' and results for these quantities are shown in Figs. 1 and 2, respectively. Only Landau damping is included in calculating the imaginary parts. This is valid when $k < 2\Delta/v'$, since for larger wave numbers pair-breaking is possible.

Equation (34) determines the real and imaginary parts of the eigenmode velocities and from them one may obtain the mean free paths,

$$l_k = v'\tau_k = \frac{1}{2}\frac{v'}{v''}\frac{1}{k} = \frac{1}{4\pi}\frac{v'}{v''}\lambda = \frac{1}{2}\frac{(v')^2}{v''}\frac{1}{\omega_k}, \qquad (55)$$

where $\tau_k = 1/2kv''$ is the relaxation time of the hydrodynamic motion. The factor of $1/2$ comes from the fact that the relaxation of the hydrodynamic motion is linked to damping of *intensity* of the mode, which is the amplitude squared, while the amplitude is exponentially damped on time scale of $1/kv''$. At nuclear matter density and at long wavelengths for a typical thermal phonon frequency $\omega = 3k_BT/\hbar$ with $T = 10^8$ K, the mean free path of the predominantly neutron mode is 3.022×10^{-9} cm for the case of no entrainment, 3.248×10^{-9} cm for $n_{np}^{(a)}$ and 3.321×10^{-9} cm for $n_{np}^{(b)}$, while for the predominantly charged particle mode the corresponding results are 7.003×10^{-10} cm, 6.454×10^{-10} cm, and 6.492×10^{-10} cm. The mean free paths for a thermal excitation vary as $1/T$ and numerically they are much shorter than those due to phonon–phonon scattering, which diverge as a higher inverse power of T at low temperatures [25].

For weak mixing, $E_{np}^2 \ll E_{nn}E_{pp}$ and $n_{np}^2 \ll n_nn_p$, we can obtain from Eqs. (38) and (39) simple expressions for the damping of the modes. For the predominantly charged-particle mode one finds

$$v_p'' \simeq -\frac{\pi}{4mc}n_pE_{ee} \qquad (56)$$

and the mean free path is given by

$$l_p(\omega) = \frac{2}{\pi}\frac{E_{pp}}{E_{ee}}\frac{c}{\omega}$$
$$\simeq 1.66\times10^{-8}\left(\frac{\mu_e}{100\text{ MeV}}\right)^2\left(\frac{E_{pp}}{10^4\text{ MeV fm}^3}\right)\left(\frac{\text{keV}}{\hbar\omega}\right)\text{ cm}. \qquad (57)$$

Here and in the remainder of this section E_{pp} and E_{ee} refer to the real parts of these quantities. For the predominantly neutron mode the corresponding results are

$$v_n'' \simeq -\frac{\pi\zeta^2}{4mc}n_nE_{ee} = \zeta^2\frac{n_n}{n_p}v_p'', \qquad (58)$$

where

$$\zeta = \frac{n_pE_{pn} + n_{np}E_{nn}}{n_pE_{pp} - n_nE_{nn}}, \qquad (59)$$

and

$$l_n(\omega) = \frac{2}{\pi\zeta^2}\frac{E_{nn}}{E_{ee}}\frac{c}{\omega}$$
$$\simeq \frac{1.66\times10^{-8}}{\zeta^2}\left(\frac{\mu_e}{100\ \mathrm{MeV}}\right)^2\left(\frac{E_{nn}}{10^4\ \mathrm{MeV\ fm}^3}\right)\left(\frac{\mathrm{keV}}{\hbar\omega}\right)\ \mathrm{cm}. \tag{60}$$

The results have limited utility for the situation considered in this paper because there is an avoided crossing of the two sound speeds, as one can see from Fig. 3. In the vicinity of the avoided crossing, the mixing is strong. Another complicating feature is that the quantity $n_pE_{pn} + n_{np}E_{nn}$ that enters in the expression for ζ passes through zero at a density close to that of the avoided crossing. This results in a vanishing of the damping of the lower frequency mode (and a divergence of the mean free path) at a particular density, as may be seen in Fig. 4.

In earlier work, Bedaque and Reddy [6], obtained a rough estimate of these mean free paths by neglecting effects due to entrainment. The range of values they predicted is compatible with the detailed estimates presented here. It is also worth noting that in the neutron star inner crust, where phonons of the neutron superfluid mix with the longitudinal lattice phonons, the predominantly neutron mode with thermal energy was found to have a mean free path in the range 10^{-6}–10^{-3} cm, much larger that that associated with lattice phonon modes, suggesting that heat transport due to phonons could play a role [8]. However, revised estimates which included the effects due to entrainment lead to larger mixing and stronger damping of the neutron mode [12].

V. ASTROPHYSICAL CONSIDERATIONS AND CONCLUDING REMARKS

The dispersion and decay of collective modes in superfluid and superconducting dense matter can influence observable aspects of neutron star evolution. At long wavelengths, these modes are relevant to understanding neutron star seismology. Short wavelength modes can contribute to the heat capacity and thermal conductivity, both of which play a role in the thermal evolution. In the following we shall explore some of the implications of the results obtained in the preceding section for neutron star phenomenology.

The temperatures of interest in isolated neutron stars (older than a few hundred years) are expected to be in the range 10^7-10^9 K. In the core where nucleons condense to form Cooper pairs, the relevant contributions to the heat capacity are due to electrons and collective modes. The collective mode contribution to the heat capacity can be obtained from the dispersion relations we calculated in the preceding section. At low temperature when the typical momentum $k \ll k_{TF}$, the dispersion relations are approximately linear with $\omega \simeq v'_\pm k$, and the associated heat capacity for each of the collective modes is given

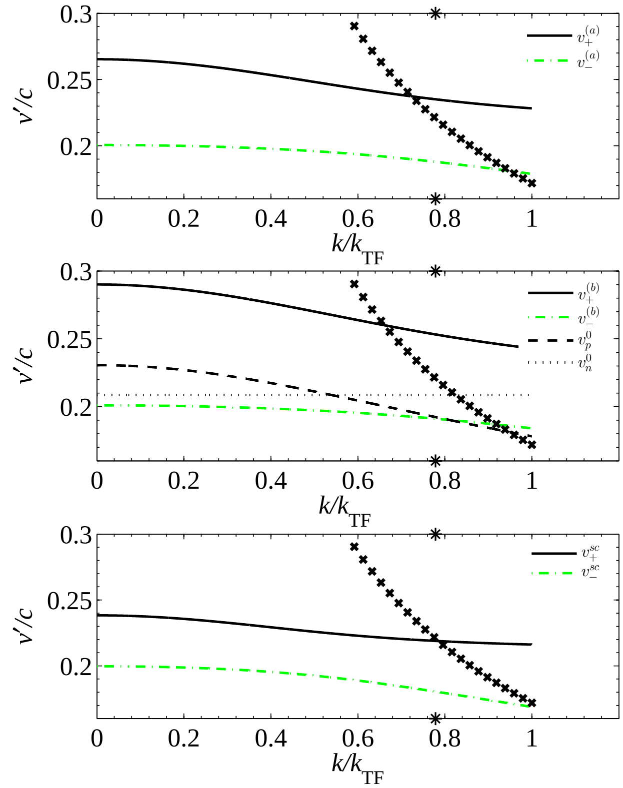

FIG. 1: (Color online) The real part of the mode velocity v in the collisionless regime at nuclear saturation density 0.16 fm^{-3}, as a function of wave number. The velocities $v_\pm^{(a,b)}$, Eq. (34), are calculated with $n_{np}^{(a)}$, Eq. (44), or $n_{np}^{(b)}$, Eq. (51). The velocities $v_\pm^{sc}$ are given by Eq. (36), and the velocities v_n^0 and v_p^0 of the uncoupled modes are given in Eqs. (26) and (33). The crosses correspond to the onset of damping by breaking of Cooper pairs. This occurs for $\omega/k \geq 2\Delta/\hbar k$, and the crosses correspond to a neutron or proton gap $\Delta = 1$ MeV. The stars mark the wave number corresponding to the inverse size of the Cooper pairs of neutrons in the BCS approximation, $\xi_n^{-1} = \pi m\Delta_n/\hbar k_n$, where the neutron gap Δ_n is taken to be 1 MeV; the inverse size of the proton Cooper pairs lies outside the domain of the figure, $\xi_p^{-1} > k_{\mathrm{TF}}$.

by

$$C_V^\pm = \frac{2\pi^2}{15}\left(\frac{k_BT}{\hbar v'_\pm}\right)^3 k_B\,,$$
$$\simeq 1.5\times10^{13}\left(\frac{0.1\ c}{v'_\pm}\right)^3 T_8^3\ \frac{\mathrm{ergs}}{\mathrm{cm}^3\ \mathrm{K}}\,. \tag{61}$$

In contrast the electron contribution in the core

$$C_V^e = \pi^2\, n_e\,\frac{k_B^2T}{\mu_e}\,,$$
$$\simeq 5.2\times10^{17}\left(\frac{\mu_e}{100\ \mathrm{MeV}}\right)^2 T_8\ \frac{\mathrm{ergs}}{\mathrm{cm}^3\ \mathrm{K}}\,, \tag{62}$$

is significantly larger. The contribution from the collective modes is small because of the higher power of the temperature and the relatively high velocities of the modes, $\gtrsim 0.1c$.

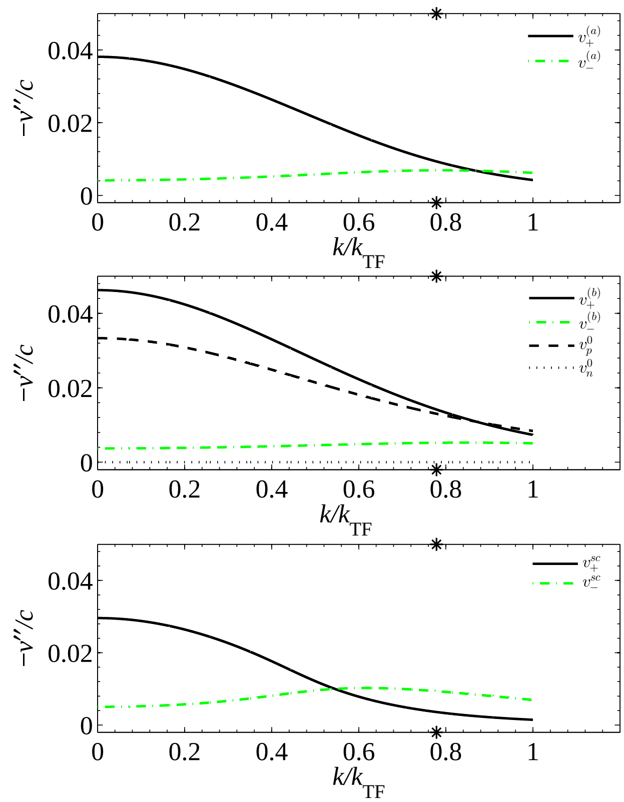

FIG. 2: (Color online) The imaginary part of the mode velocity v as function of wave number, at nuclear saturation density. The modes are labeled as in Fig. 1. The stars mark the wave number corresponding to the inverse size of the Cooper pairs of neutrons in BCS approximation with the neutron gap 1 MeV.

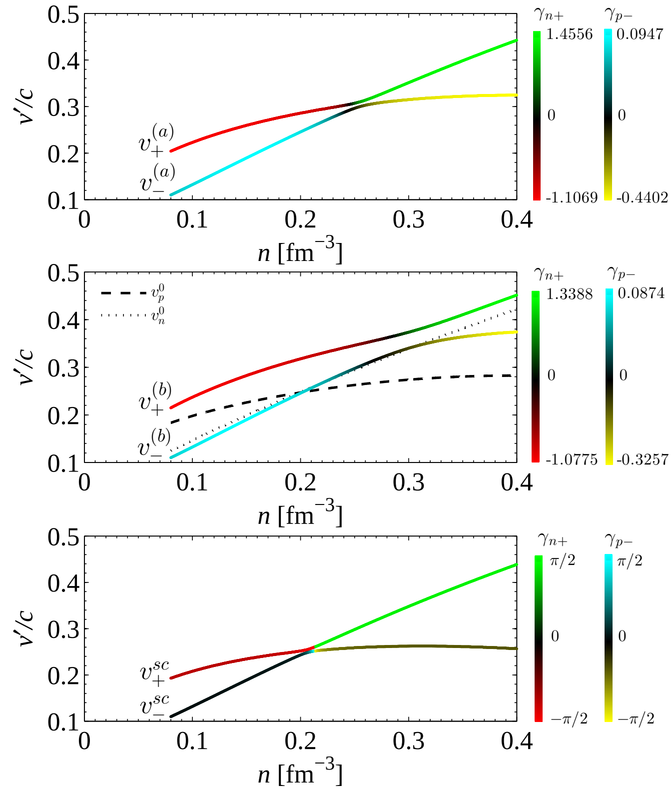

FIG. 3: (Color online) The real part of the collective mode velocities as function of baryon density, at long wavelengths in the collisionless limit. The modes are labeled as in Fig. 1. The low-density boundaries of the plots correspond to the density of the crust–core boundary, ≈ 0.08 fm^{-3} [10]. The color indicates the value of the mixing angles defined in Eqs. (42) and (43).

The coefficient of thermal conduction for a single species of bosonic mode with a dispersion relation ω_k is given by standard kinetic theory

$$\kappa = \frac{\hbar}{3T}\sum_{\mathbf{k}} \omega_k^2 \left(\frac{\partial \omega_k}{\partial k}\right)^2 \left(-\frac{\partial n_k}{\partial \omega_k}\right) \tau_k, \qquad (63)$$

where $n_k = 1/(\mathrm{e}^{\hbar\omega_k/k_B T} - 1)$ is the Bose distribution function, $\tau_k = 1/(2\omega_k'')$ is the relaxation time of the mode and k_B is the Boltzmann constant. For wave numbers small compared with the electron screening wave number, the velocity of the modes may be taken to be constant and, on performing the sum in Eq. (63), one finds the simple result

$$\begin{aligned}\kappa &= \frac{\zeta(3)}{2\pi^2}\frac{(k_B T/\hbar)^2}{|v''|}k_B \\ &= 4.807\times 10^{10}\frac{T_8^2}{|v''|/c}\mathrm{erg}/(\mathrm{cm\,sec\,K}), \qquad (64)\end{aligned}$$

where v'' is the imaginary part of the mode velocity. The thermal conductivity does not depend on the real part of the velocity of the mode, and consequently the modes with the smallest v'' contribute most to heat transport.

It is interesting to compare thermal conductivity of bosonic excitations in superfluid neutrons given in Eq. (64) with other contributions to the thermal conductivity. Thermal conductivity of fermionic excitations in superfluid neutrons and the electron contributions were calculated in [28], and found to be of the order of 10^{24} erg/(cm sec K), which shows that the bosonic excitations provide negligible contribution to the thermal conductivity.

The frequency spectrum of the longitudinal modes that can be excited in the core is sensitive to the real part of the velocity in the long-wavelength limit and will in general differ from the sound velocity defined by $\sqrt{\partial P/\partial \rho}$, where P is total pressure and ρc^2 is the total energy density. In the parlance of stellar oscillations these modes are referred to as pressure modes, or simply p-modes. The existence of two acoustic modes in the long-wavelength limit is a unique feature of the superfluid-superconducting multi-component system that we have studied here, and it differs qualitatively from what is expected in a normal fluid where only one mode exists, with velocity c_s. Further, the difference in mode velocities is small and this may lead to unique neutron star seismology.

Perturbations either during accretion, thermonuclear x-ray bursts and superbursts, and neutron stars mergers may excite modes in the core but it is unclear if neutron star seismology can be probed observationally. One model of quasi-periodic oscillations observed in the tails of magnetar giant flares in-

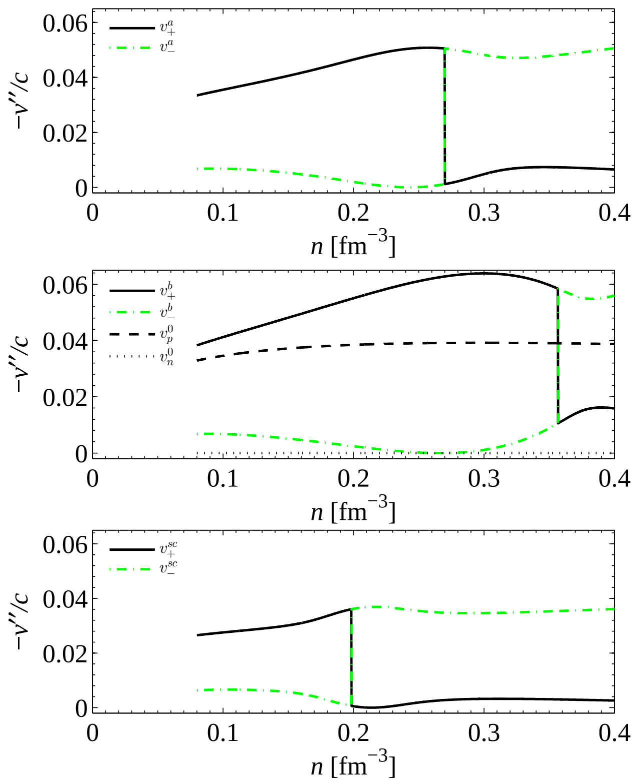

FIG. 4: (Color online) The imaginary part of the collective mode velocities as functions of baryon density. The modes are labeled as in Fig. 1.

vokes a coupling between internal modes of the neutron star crust and the global magnetic field to provide a mechanism by which surface emission can be modulated at characteristic frequencies [29]. It would be interesting to explore if a similar coupling between the magnetic field and core p-modes can lead to observable effects.

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Appendix A: Equivalence of two approaches to superfluid dynamics

In the traditional approach to describing superfluidity [26] one works in terms of the so-called superfluid velocity, $\mathbf{v}_s = \boldsymbol{\nabla}\phi/m$, which is really a momentum, and the velocity of the normal component. This has previously been applied to the superfluid neutrons in the inner crust of neutron stars [11–13]. In the present context, when both neutrons and protons are superfluid, it is natural to treat them symmetrically, as we have done in the text. However, as we shall demonstrate, the equations of motion are equivalent to those in the traditional formalism. For simplicity we consider the case of small perturbations about a uniform state with both components at rest. We express the equations of motion (16) and (17) in terms of the average velocity of the protons, which is the proton current density divided by the proton number density:

$$\bar{\mathbf{v}}_p \equiv \frac{n_{pp}}{n_p}\frac{\mathbf{p}_p}{m} + \frac{n_{pn}}{n_p}\frac{\mathbf{p}_n}{m}. \tag{A1}$$

The continuity equations (11) and (10) for nucleons therefore become

$$\partial_t n_p + \boldsymbol{\nabla}\cdot(n_p \bar{\mathbf{v}}_p) = 0, \tag{A2}$$

and

$$\partial_t n_n + \boldsymbol{\nabla}\cdot\left[\left(n_{nn} - \frac{n_{np}^2}{n_{pp}}\right)\frac{\mathbf{p}_n}{m} + \frac{n_p n_{np}}{n_{pp}}\bar{\mathbf{v}}_p\right] = 0. \tag{A3}$$

Equations (A2) and (A3) have the form to be expected for a two-fluid model, even though in the present case the two components are both superfluid. The neutron current density is given by

$$\mathbf{j}_n = \left(n_{nn} - \frac{n_{np}^2}{n_{pp}}\right)\frac{\mathbf{p}_n}{m} + \frac{n_p n_{np}}{n_{pp}}\bar{\mathbf{v}}_p, \tag{A4}$$

and therefore the superfluid neutron density in the notation of Refs. [11, 13] is given by

$$n_n^s = n_{nn} - \frac{n_{np}^2}{n_{pp}} \tag{A5}$$

while the density of neutrons entrained with the protons is given by

$$n_n^n = \frac{n_p n_{np}}{n_{pp}}. \tag{A6}$$

In the traditional approach, the superscript s refers to the superfluid component and n to the normal component. In the present case, both components are superfluid, and the superscript n refers to the component whose dynamics is described in terms of the average velocity of the component, rather than the momentum per particle in the condensate. The equation of motion for $\mathbf{p}_n$, Eq. (17) has the usual form as in the two-fluid model. We turn now to the equation of motion for $\bar{\mathbf{v}}_p$ and for simplicity we consider the linearized equation for small deviations from a state in which $\mathbf{p}_n$ and $\bar{\mathbf{p}}_p$ vanish. From Eqs. (16) and (17) it follows that

$$\partial_t \bar{\mathbf{v}}_p = \frac{n_{pp}}{m n_p}\partial_t \mathbf{p}_p + \frac{n_{pn}}{m n_p}\partial_t \mathbf{p}_n \tag{A7}$$

$$= -\frac{n_{pp}}{m n_p}\boldsymbol{\nabla}\left(\mu_p^{\rm nuc} + e\Phi\right) - \frac{n_{pn}}{m n_p}\boldsymbol{\nabla}\mu_n^{\rm nuc}. \tag{A8}$$

The total normal mass density is given by

$$\rho^n = m(n_p + n_n^n). \tag{A9}$$

From Eq. (A6) one thus sees that

$$\rho^n = mn_p\left(1+\frac{n^{np}}{n^{pp}}\right) = m\frac{n_p^2}{n^{pp}}, \tag{A10}$$

where the latter result is a consequence of the condition (4) for Galilean invariance. Thus Eq. (A8) may be written in the form

$$\rho^n \partial_t \bar{\mathbf{v}}_p = -n_p \boldsymbol{\nabla}\left(\mu_p^{\rm nuc} + e\Phi\right) - n_n^n \boldsymbol{\nabla}\mu_n^{\rm nuc}. \tag{A11}$$

This result is consistent with [13, Eq. (25)] for the inner crust if one there neglects the shear rigidity of the solid. Equation (A11) represents a generalization of earlier results [11–13] in that charge neutrality is not enforced locally.

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