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Phenomenological effective interactions like Skyrme forces are currently used in mean–field calculations in nuclear physics. Mean–field models have strong analogies with the first order of the perturbative many–body problem and the currently used effective interactions are adjusted at the mean–field level. In this work, we analyze the renormalizability of the nuclear many–body problem in the case where the effective Skyrme interaction is employed in its standard form and the perturbative problem is solved up to second order. We focus on symmetric nuclear matter and its equation of state, which can be calculated analytically at this order. It is shown that only by applying specific density dependence and constraints to the interaction parameters could renormalizability be guaranteed in principle. This indicates that the standard Skyrme interaction does not in general lead to a renormalizable theory. For achieving renormalizability, other terms should be added to the interaction and employed perturbatively only at first order.

I. INTRODUCTION

Bulk properties of medium–mass and heavy nuclei are very well described by phenomenological effective interactions treated in the mean–field picture [1]. Despite this success, the necessity of increasing the accuracy of theoretical predictions in some cases has motivated several groups to formulate beyond–mean–field models, which explicitly include more correlations in their formal scheme. Several directions have been explored, for instance: projection techniques in the framework of the generator coordinate method [2–5]; second random–phase approximation (SRPA) calculations with Skyrme and Gogny forces [6, 7], as well as with an interaction derived from a realistic force [8]; particle–vibration coupling (PVC) techniques with the Skyrme interaction [9, 10] and with a relativistic Lagrangian [11]; multiparticle–multihole configuration mixing (mpmhCM) methods with both Skyrme [12] and Gogny [13] forces.

A challenge faced by all these models is how to overcome the overcounting of correlations when conventional forces or Lagrangians are used. Conventional forces and Lagrangians are actually designed for mean–field–based models, and the adjustment of their parameters is performed at this level. Using the same interactions, with the same values of the parameters, in calculations where different types of correlations are explicitly taken into account obviously produces some double counting. In other words, when beyond–mean–field methods are used, the adjustment of the parameters should be done at the same level (at the same order) in the perturbative many–body problem. Otherwise, subtraction procedures should be applied to cancel the overcounted correlations such as, for instance, the subtraction method introduced by Tselyaev [14, 15] and applied to PVC [16–21] and SRPA [22] models. Apart from this general problem, several technical difficulties are encountered in many of these sophisticated models. Let us mention for instance the irregularities and the divergences that may be found in projection calculations [2, 23–25] and the ultraviolet (UV) divergences that are present when zero–range interactions are employed in SRPA, PVC or mpmhCM calculations (these divergences may be eliminated in some specific cases by applying the subtraction procedure mentioned above).

The issue of UV divergences in second–order calculations with the zero–range Skyrme force has been addressed by three of us in the case of nuclear matter using cutoff– and dimensional–regularization techniques [26, 27]. New–generation Skyrme–type interactions have been designed to provide a reasonable equation of state (EOS) for nuclear matter by including first– and second–order contributions in the evaluation of the energy. This approach produces well–defined results that avoid overcounting.

The specific problem of designing new interactions to be used in beyond–mean–field calculations can be viewed as a part of a more general issue: the formulation of an interaction that provides a renormalizable theory order by order in the perturbative many–body problem. Renormalizability means that the theory is independent of the details of high–energy physics and, in particular, the arbitrary regularization procedure. High–energy physics eliminated from loops by the regulator is accounted for in the coefficients of the interactions, which are then cutoff dependent in such a way as to ensure that observables are not. Renormalizability is guaranteed once all interactions allowed by the symmetries of the underlying dynamics are included. The framework to accomplish this is that of effective field theories (EFTs), which has been successfully applied to the physics of light nuclei over the last two decades [28, 29].

Ensuring renormalizability is, in turn, a step towards an even more general objective, that of searching for the correct power counting which indicates the proper hierarchy of allowed interactions. A consistent power counting generates at each order enough interactions so that any remaining regularization dependence can be eliminated with a sufficiently high value for the regulator parameter. Thus, imposing renormalizability is a guide for theory construction, the best-known example being the development of the electroweak theory known as the Standard Model. A nuclear example is provided by Pionless EFT, where the existence of a three–body force in leading order was discovered by demanding renormalizability of the theory's description of the three-body system [30–32]. Unfortunately, the renormalization of Chiral EFT, which extends Pionless EFT to momenta comparable to the pion mass, is not fully understood even in few-nucleon systems [33–35].

The successes of mean–field models suggest that there should be a controlled expansion around it. However, renormalizability has not yet been extensively explored in the case of phenomenological effective interactions like Gogny and Skyrme forces. In this exploratory study, we focus on the zero–range Skyrme force, which bears formal similarities with the interactions in Pionless EFT. We are thus implicitly assuming that non–relativistic nucleons are the relevant degrees of freedom for the low–energy dynamics of the nuclei of interest. The analysis is performed by including first– and second–order contributions in the EOS of symmetric nuclear matter. The objective is to reveal the implications of demanding renormalizability through a redefinition of the existing parameters at second order. A similar procedure can be followed for more complex forces, higher orders, different isospin asymmetries, and finite nuclei.

II. RENORMALIZATION

We consider the standard Skyrme force [36], which contains central, density–dependent, spin–orbit, and velocity–dependent terms of zero range. The spin–orbit term does not contribute in infinite matter at first order, but in general does provide a second–order contribution to the EOS [37]. The contribution of this term under dimensional regularization can be found in Ref. [38]. For the sake of simplicity, in this first exploratory study of the renormalizability of the problem, we have omitted this term in the interaction as also done in Ref. [27]. In contrast, we keep the density–dependent part of the Skyrme interaction, even though such a term might be problematic in connection to the so–called self–interaction problem. It was recognized already in the 70s [39] that only density–independent contact forces allow one to satisfy specific antisymmetry conditions in the solution of random–phase–approximation equations. The violation of such conditions is associated with a violation of the Pauli principle generated by spurious contributions coming from the interaction of a particle with itself. This self–interaction problem was discussed again more recently [1, 23–25, 40–43]. For instance, a strategy to solve pathologies produced by the self–interaction problem was suggested in Ref. [24] for the beyond–mean–field case of the generator–coordinate method. We retain the density–dependent term because it is known to be necessary to describe well the equilibrium point of symmetric matter not only in first order but also in second order, as discussed in Refs. [27, 38]. Alternative terms such as a real three–body force would yield much more involved calculations. It is also worth mentioning that, by including the rearrangement terms associated with the density–dependent force in the computation of the second–order EOS, as we do here, the Hugenholtz-Van Hove theorem [44] is satisfied.

We define the incoming and outgoing relative momenta, $\vec{k} = (\vec{k}_1 - \vec{k}_2)/2$ and $\vec{k}' = (\vec{k}'_1 - \vec{k}'_2)/2$, where $\vec{k}_i^{(\prime)}$ denotes the momentum of nucleon $i^{(\prime)}$. We also introduce the spin–exchange operator $P_\sigma = (1 + \vec{\sigma}_1 \cdot \vec{\sigma}_2)/2$ in terms of the spin $\vec{\sigma}_i/2$ of nucleon i. We deal only with symmetric nuclear matter, for which the density ρ and the Fermi momentum k_F (the same for neutrons and protons) are related by the relation $k_F = (3\pi^2\rho/2)^{1/3}$. In terms of these quantities, the interaction is written as

$$V(\vec{k},\vec{k}') = t_0(1+x_0P_\sigma) + T_3(1+x_3P_\sigma)k_F^{3\alpha} + \frac{t_1}{2}(1+x_1P_\sigma)\left(\vec{k}'^2+\vec{k}^2\right) + t_2(1+x_2P_\sigma)\vec{k}'\cdot\vec{k}, \tag{1}$$

where α is a real number. The usual Skyrme parameters $t_{0,1,2}$ and $x_{0,1,2,3}$ are present, while the parameter T_3 is defined in terms of the Skyrme parameter t_3 as $T_3 = (2/(3\pi^2))^\alpha t_3/6$. The T_3 term describes the so–called density–dependent part of the interaction, which is necessary to ensure the correct description of the saturation point and of the compressibility modulus of symmetric matter not only at the mean–field level but also at second order [27, 38].

Our regulator is chosen, as in Ref. [27], as a cutoff λ put on the outgoing relative momentum $\vec{k}'$, $\lambda = \tilde{\lambda}/k_F$. Other regulators generate terms of the same form but with different coefficients. Dimensional regularization with standard subtraction procedures sets several of these coefficients to zero and tends to hide a potential lack of renormalizability, one example being the two-body system with resonant p-wave interactions [45, 46]. For this reason, we do not employ such type of renormalization here.

The EOS for symmetric matter is given, up to second order, by the diagrams shown in Fig. 1. The upper (lower) line displays first– (second–) order diagrams, while direct (exchange) contributions are shown on the left (right) column. The evaluation of these diagrams gives for the energy per nucleon

$$\frac{E}{A}(k_F,\tilde{\lambda}) = \frac{3\hbar^2}{10m}k_F^2 + \frac{t_0}{4\pi^2}k_F^3 + \frac{T_3}{4\pi^2}k_F^{3+3\alpha} + \frac{\theta_s}{4\pi^2}k_F^5 + \frac{\Delta E^{(2)}}{A}(k_F,\tilde{\lambda}). \tag{2}$$

The first term of Eq. (2) is the kinetic contribution (m is the nucleon mass) and the following three terms are first–order, with

$$\theta_s = \frac{1}{10}\left[3t_1 + t_2(5+4x_2)\right]. \tag{3}$$

The last term of Eq. (2) collects the second–order contributions, which depend on the momentum cutoff. The expression for $\Delta E^{(2)}(k_F,\tilde{\lambda})/A$ in symmetric matter can be found in Ref. [27], including the contributions coming from rearrangement terms in the prescription of Ref. [47]. The asymptotic behavior ($\tilde{\lambda} \gg k_F$), which has a polynomial form in the cutoff, is also given. We have checked that this asymptotic polynomial form practically coincides with the full expressions starting from $\tilde{\lambda} \gtrsim 1$ fm^{-1}. The asymptotic expression can be split into three terms,

$$\frac{\Delta E^{(2)}(k_F,\tilde{\lambda})}{A} = \frac{\Delta E_f^{(2)}(k_F)}{A} + \frac{\Delta E_a^{(2)}(k_F,\tilde{\lambda})}{A} + \frac{\Delta E_d^{(2)}(k_F,\tilde{\lambda})}{A}, \tag{4}$$

where the subscripts f, a, and d stand for "finite", "absorbed", and "divergent" and denote, respectively, the finite part, the contribution where the cutoff dependence can be absorbed with a redefinition of the interaction parameters, and the part that cannot in general be regrouped with mean–field terms and thus diverges when $\tilde{\lambda} \to \infty$.

Denoting by m^* the nucleon effective mass in symmetric matter and using its mean–field expression, as done in Ref. [27],

$$m^* = m\left(1 + \frac{5m}{6\pi^2\hbar^2}k_F^3\theta_s\right)^{-1}, \tag{5}$$

the three contributions in Eq. (4) can be written as

$$\frac{\Delta E_f^{(2)}(k_F)}{A} = \frac{3m^*}{2\pi^4\hbar^2}k_F^4\left[A_0 + A_1T_3k_F^{3\alpha} + A_2T_3^2k_F^{6\alpha} + A_3k_F^2 + A_4T_3k_F^{2+3\alpha} + A_5k_F^4\right], \tag{6}$$

$$\frac{\Delta E_a^{(2)}(k_F,\tilde{\lambda})}{A} = -\frac{m}{8\pi^4\hbar^2}\tilde{\lambda}k_F^3\left[B_0(\tilde{\lambda}) + B_1(\tilde{\lambda})T_3k_F^{3\alpha} + B_2(\tilde{\lambda})k_F^2\right], \tag{7}$$

and

$$\frac{\Delta E_d^{(2)}(k_F,\tilde{\lambda})}{A} = -\frac{m^*}{8\pi^4\hbar^2}\tilde{\lambda}k_F^3\left[C_0T_3^2k_F^{6\alpha} + C_1T_3k_F^{2+3\alpha} + C_2k_F^4\right] + \frac{m^*-m}{m}\frac{\Delta E_a^{(2)}(k_F,\tilde{\lambda})}{A}. \tag{8}$$

The coefficients A_i, $B_i(\tilde{\lambda})$, and C_i are combinations of Skyrme parameters and (for the coefficients B_i) cutoff, which are shown explicitly in the Appendix.

From the k_F dependence in Eq. (7), one sees that $\Delta E_a^{(2)}(k_F,\tilde{\lambda})/A$ may be regrouped with mean–field terms in the EOS. The cutoff dependence can be absorbed in the bare interaction parameters in the form of renormalized parameters (denoted by superscript ^{R})

$$t_0^R = t_0(\tilde{\lambda}) - \frac{m\tilde{\lambda}}{2\pi^2\hbar^2}B_0(\tilde{\lambda}), \tag{9}$$

$$T_3^R = T_3(\tilde{\lambda})\left[1 - \frac{m\tilde{\lambda}}{2\pi^2\hbar^2}B_1(\tilde{\lambda})\right], \tag{10}$$

and

$$\theta_s^R = \theta_s(\tilde{\lambda}) - \frac{m\tilde{\lambda}}{2\pi^2\hbar^2}B_2(\tilde{\lambda}). \tag{11}$$

Assuming that the expansion beyond mean field is well defined, one can directly replace the bare parameters in Eqs. (6) and (8) by the renormalized ones. We indicate this by a superscript ^{R} in the coefficients A and C. With this replacement, we induce a change of the original equations, but such modifications are of higher order (at least third–order induced terms) and may thus be neglected if only terms up to second order are retained in the EOS, as we require here.

Equation (8) shows that the cutoff dependence in $\Delta E_d^{(2)}(k_F,\tilde{\lambda})/A$ cannot be similarly handled. Adoption of the mean–field effective mass (5) brings a k_F depen-

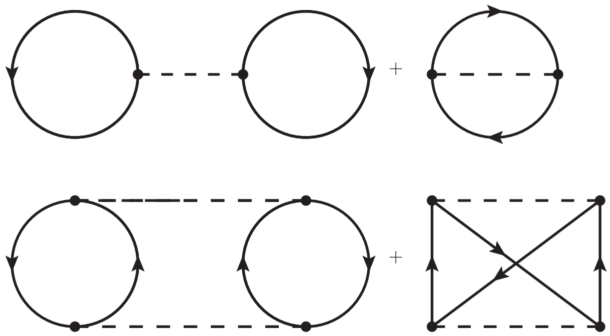

Figure 1: First–order (upper line) and second–order (lower line) contributions to the energy of nuclear matter. Direct (exchange) terms are shown on the left (right). Particles and holes are denoted by oriented solid lines and the interaction by a dashed line.

dence in the denominator that cannot be absorbed in density–independent parameters. This forces us to set

$$\theta_s^R = 0 \tag{12}$$

to eliminate the last term in Eq. (8). It also follows that $m^* = m$ throughout Eqs. (6) and (8). The remaining contributions in Eq. (8) must now be handled by imposing renormalizability. The most dangerous dependence is $\propto k_F^7$, which cannot be modified by a choice of α and has a positive coefficient, $C_2 \geq 0$. Because $C_0 > 0$, the $k_F^{3+6\alpha}$ dependence is also constrained.

The first possibility is that the divergent terms cancel among themselves. This requires $\alpha = 2/3$, in which case all three divergent terms have the same k_F dependence. Then, Eq. (8) vanishes if and only if

$$C_0^R (T_3^R)^2 + C_1^R T_3^R + C_2^R = 0. \tag{13}$$

However, we found that, even without the constraint $\theta_s^R = 0$, the discriminant of the above equation is always less or equal to 0. Thus, no real set of parameters satisfies Eq. (13), except for the trivial case $t_1^R = t_2^R = T_3^R = 0$. In this case, the only divergence is absorbed through Eq. (9), and the resulting EOS is finite. The renormalized EOS for symmetric nuclear matter evaluated up to second order is given by

$$\frac{E}{A}(k_F) = \frac{3\hbar^2}{10m} k_F^2 + \frac{t_0^R}{4\pi^2} k_F^3 + \frac{3m}{2\pi^4\hbar^2} A_0^R k_F^4. \tag{14}$$

As shown in the Appendix, A_0^R contains x_0, which in symmetric matter does not contribute in first order. We can take $A_0^R \geq 0.069(t_0^R)^2$ as an independent parameter so that Eq. (14) has only two parameters to adjust, if we interpret m as the in-vacuum nucleon mass. Unless m is taken as a free, negative parameter, this "t_0 model" does not lead to saturation at mean–field level, meaning that the saturation point requires a second–order contribution comparable to first order.

For a more meaningful model, not all divergent terms vanish, and thus must be absorbed in mean–field terms. The only possibility for the k_F^7 divergence is the T_3 term (in which case $\alpha = 4/3$), but then the $k_F^{3+6\alpha}$ divergence cannot be eliminated. Thus we must consider the "$t_0 - t_3$ model" where $t_1^R = t_2^R = 0$ and

$$C_2^R = 0, \qquad C_1^R = 0. \tag{15}$$

This in turn implies $B_2(\tilde{\lambda}) = 0$, and $A_3^R = A_4^R = A_5^R = 0$. In this model, saturation is obtained at first order thanks to the T_3 term.

Now, only for specific values of $\alpha \neq 0$ can we eliminate the $k_F^{3+6\alpha}$ divergent term.

1. If $\alpha = -1/6$ the $k_F^{3+6\alpha}$ divergent term can be absorbed into a renormalized–mass term by a choice of the bare mass, that is,

$$\frac{1}{m^R} = \frac{1}{m(\tilde{\lambda})} \left\{ 1 - \frac{5\tilde{\lambda} m^2(\tilde{\lambda})}{12\pi^4\hbar^4} (T_3^R)^2 \left[\left(\frac{175}{192} \right)^2 + x_3^2 \right] \right\}. \tag{16}$$

Since we keep only terms up to second order in the EOS, we can directly replace m by m^R in Eqs. (6) and (7) and neglect the induced higher–order contributions. The finite A_2 term in Eq. (6) is now $\propto k_F^3$ and can also be absorbed in t_0^R, that is, we modify Eq. (9) to

$$t_0^R = t_0(\tilde{\lambda}) - \frac{m\tilde{\lambda}}{2\pi^2\hbar^2} B_0(\tilde{\lambda}) + \frac{6m^R}{\pi^2\hbar^2} A_2^R (T_3^R)^2. \tag{17}$$

The EOS becomes

$$\frac{E}{A}(k_F) = \frac{3\hbar^2}{10m^R} k_F^2 + \frac{T_3^R}{4\pi^2} k_F^{5/2} + \frac{t_0^R}{4\pi^2} k_F^3 + \frac{3m^R}{2\pi^4\hbar^2} \left(A_1^R T_3^R k_F^{7/2} + A_0^R k_F^4 \right). \tag{18}$$

The pure second–order terms A_1^R and A_0^R are proportional to t_0^R and $(t_0^R)^2$, respectively, but they also depend on the spin coefficients $x_{0,3}$. In symmetric nuclear matter, where $x_{0,3}$ do not appear in first order, we can treat A_1^R and $A_0^R \geq 0.069(t_0^R)^2$ as independent parameters.

2. If $\alpha = 1/3$ the $k_F^{3+6\alpha}$ divergent term and the finite A_1 term in Eq. (6) can be absorbed into θ_s^R, changing Eq. (11) into

$$\theta_s^R = \theta_s(\tilde{\lambda}) - \frac{m\tilde{\lambda}}{2\pi^2\hbar^2} C_0^R (T_3^R)^2 + \frac{6m}{\pi^2\hbar^2} A_1^R T_3^R. \quad (19)$$

This is a fine-tuned scenario where at least one of the bare $t_1(\tilde{\lambda})$ and $t_2(\tilde{\lambda})$ parameters is not zero, but their renormalized values are. Although unusual, it is a scenario similar to Pionless EFT at the unitarity limit, where the bare coefficient of the non-derivative two-body contact interaction absorbs a linear divergence but its inverse renormalized value is zero [48]. Now, the finite A_0 term in Eq. (6) is $\propto k_F^4$ and can be absorbed in T_3^R by replacing Eq. (10) with

$$T_3^R = T_3(\tilde{\lambda}) \left[1 - \frac{m\tilde{\lambda}}{2\pi^2\hbar^2} B_1(\tilde{\lambda})\right] + \frac{6m}{\pi^2\hbar^2} A_0^R. \quad (20)$$

The EOS is in this case

$$\frac{E}{A}(k_F) = \frac{3\hbar^2}{10m}k_F^2 + \frac{t_0^R}{4\pi^2}k_F^3 + \frac{T_3^R}{4\pi^2}k_F^4 + \frac{3m}{2\pi^4\hbar^2}A_2^R (T_3^R)^2 k_F^6. \quad (21)$$

Again, here we can take $A_2^R > 0$ as an independent parameter.

Note that for the $t_0 - t_3$ models we chose to absorb finite terms in Eqs. (17) and (20), just as in Eq. (19). This is done consistently with the underlying assumption that there is an expansion around the mean field. As a consequence, the k_F^3 and k_F^4 terms in Eqs. (18) and (21), respectively, are unconstrained. Had we not absorbed these terms, there would be additional pieces $\propto k_F^3$ and $\propto k_F^4$ with a fixed dependence on, respectively, $(T_3^R)^2$ and $(t_0^R)^2$, which further constrain the EOS. The difference between absorbing these finite terms or not provides an estimate of the error stemming from our assumption of convergence around the mean field.

There is no other possibility to eliminate the divergent terms while retaining an expansion where second order does not overcome first order. As pointed out above, assuming higher orders provide smaller contributions, the coefficients of the second–order terms can be replaced by their finite, renormalized values. One cannot, for example, cancel the divergent terms against other second–order contributions.

III. FITS

Many Skyrme parametrizations exist. Typical values of the parameter α range from 1/6, for instance in the Saclay-Lyon forces [49, 50], up to 1, for instance in SIII [51]. Not all of the three values of α that provide a renormalizable force fall in this range, but we do not discard $\alpha = -1/6$ immediately. Instead we judge the phenomenological promise of the three possibilities by fitting a successful EOS. Because of the overcounting problem raised earlier, it is not expected that Eqs. (14), (18), and (21) will lead to a reasonable EOS when one uses parameters extracted at mean–field level. However, if the expansion converges, changes in the renormalized parameters (but not the bare parameters, which depend on the arbitrary cutoff) should be relatively small.

We have performed χ^2 fits of our EsOS by choosing the SLy5 [49, 50] mean–field EOS as a benchmark. We fit in each case $N = 18$ energies E_i, in the range of densities between 0 and 0.3 fm^{-3}, to the SLy5 mean–field reference points $E_{i,ref}$, with

$$\chi^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(\frac{E_i - E_{i,ref}}{0.01 \times E_{i,ref}}\right)^2. \quad (22)$$

The fitted parameters and the associated χ^2 values are listed in Tables I, II, and III. In the case $\alpha = -1/6$, a renormalized mass enters in the EOS and we have treated it as a free parameter to adjust, as we do for other renormalized parameters. For the other cases, we first perform the fit with $m = 939$ MeV, and if a satisfactory result cannot be obtained, we turn the mass into a free parameter. For the cases $\alpha = 2/3$ and $\alpha = 1/3$ only the magnitudes of $x_{0,3}^R$ can be determined.

As one can see from Table I, the fits obtained for the case $t_1^R = t_2^R = T_3^R = 0$ with $\alpha = 2/3$ are totally unsuccessful. The magnitude of x_0^R is found to be extremely small. Adjusting the mass produces a smaller χ^2, which is however still very large. More alarming, the mass becomes negative. The failure of this fit was anticipated by the fact that for $m > 0$ even the saturation point requires large second–order contributions.

m	t_0^R	$\lvert x_0^R\rvert$	χ^2
(MeV)	(MeV fm^3)		
939	-358.16	$< 10^{-4}$	346850
-969.55	212.28	$< 10^{-4}$	15989

Table I: Parameter sets obtained by fitting the renormalized second–order EOS for the case $t_1^R = t_2^R = T_3^R = 0$ with $\alpha = 2/3$, Eq. (14), to the SLy5 mean–field EOS. The upper line refers to the fit where the mass is taken at its in-vacuum value, while for the lower line the mass is adjusted.

m^R	t_0^R	T_3^R	x_0^R	x_3^R	χ^2
(MeV)	(MeV fm^3)	(MeV fm$^{5/2}$)			
591.9	793.15	-1570.8	1.465	-0.1759	< 0.1

Table II: Parameter set obtained by fitting the renormalized second–order EOS for the case $t_1^R = t_2^R = 0$ with $\alpha = -1/6$, Eq. (18), to the SLy5 mean–field EOS. The renormalized mass is treated as a free parameter.

In contrast, for $t_1^R = t_2^R = 0$ with $\alpha = -1/6$, the fit is excellent, as can be seen also from Fig. 2, where the fit outcome is compared to the SLy5 curve. This fit is interesting for several reasons. First, the renormalized mass is somewhat smaller than in-vacuum, but not so much so that it necessarily invalidates the non–relativistic approximation. Second, this value of α inverts the roles of t_0^R and T_3^R in first–order saturation, which now requires $T_3^R < 0$ and $t_0^R > 0$. The second–order fit is qualitative stable in the sense that these signs do not change. Third, the magnitudes, $|T_3^R| \sim (50\,\mathrm{MeV})^{-3/2}$ and $t_0^R \sim (100\,\mathrm{MeV})^{-2}$, have very natural sizes. Fourth, the values of $x_{0,3}$ are not particularly small or large, just as in the usual mean–field EsOS. Despite these interesting features, second–order effects are significant. To see this, we have also performed a fit with the extra constraint that results from not absorbing the finite term in Eq. (17). In this case the fit deteriorates significantly, as shown in Fig. 2. The $\chi^2 = 2319$ is large, with opposite signs for m^R, T_3^R, and t_0^R and negligible $x_{0,3}$. Although this fit requires a large change in parameters when going from first to second order, the difference between the two fits should be considered as a conservative estimate of the error band from higher–order effects. Clearly, the goodness of the fit based on Eq. (18) is not representative of what can be obtained in a systematic expansion of this model around the mean field.

m	t_0^R	T_3^R	$\lvert x_3^R\rvert$	χ^2
(MeV)	(MeV fm^3)	(MeV fm^4)		
939	-1244.1	247.11	$< 10^{-4}$	1364
23845	-580.16	46.248	$< 10^{-2}$	188

Table III: Parameter sets obtained by fitting the renormalized second–order EOS for the case $t_1^R = t_2^R = 0$ with $\alpha = 1/3$, Eq. (21), to the SLy5 mean–field EOS. The upper line refers to the fit where the mass is taken at its in-vacuum value, while for the lower line the mass is adjusted.

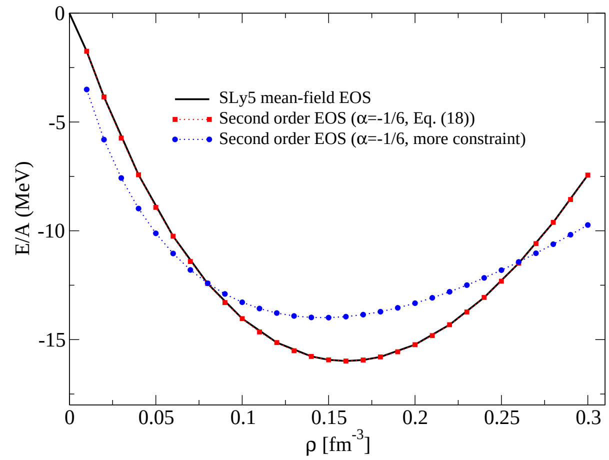

Figure 2: EOS for the case $t_1^R = t_2^R = 0$ with $\alpha = -1/6$. The fitted second–order EOS Eq. (18) (red squares) is compared to the benchmark EOS (black solid line). Also shown (blue circles) is a more constrained fit, where t_0^R is not redefined according to Eq. (17), which gives an estimate of the potential size of higher–order effects.

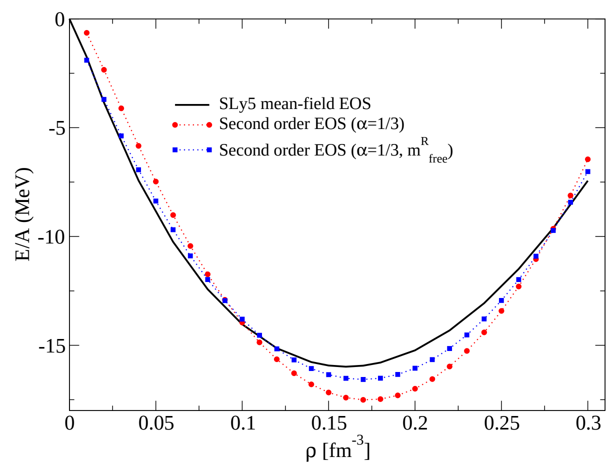

Figure 3: EOS for the case $t_1^R = t_2^R = 0$ with $\alpha = 1/3$. The fitted second–order EOS, Eq. (21), with $m = 939$ MeV (red circles) and with m as a free parameter (blue squares) are compared to the benchmark EOS (black solid line).

The remaining case, $t_1^R = t_2^R = 0$ with $\alpha = 1/3$, produces fits of intermediate quality, shown in Fig. 3. With the in–vacuum mass, the best-fit EOS is qualitatively correct, but overbinds with too-large saturation density. The χ^2 is reduced considerably by allowing m to increase to a large value, but the fit still overbinds. The signs of t_0^R and T_3^R are the ones needed for saturation in first order, but $|x_3|$ is very small. When we perform a more constrained fit without absorbing a finite term in Eq. (20), the χ^2 soars to $\chi^2 > 6000$, again indicating large changes with order.

IV. CONCLUSION

It is interesting that we were able to find renormalized EsOS in second order based on Skyrme forces with $\alpha = -1/6$ and $\alpha = 1/3$, which could potentially serve for a description of finite nuclei. Of course, before a claim of phenomenological success can be made, other nuclear–matter properties (such as the neutron–matter EOS and the density-dependent symmetry energy) need to be investigated. Moreover, in both cases there are indications that higher–order effects will be significant. Both are specific scenarios in the simple $t_0 - t_3$ model, but with unexpected renormalization features. For $\alpha = 1/3$, the renormalization requirements (15) and (19) imply that the derivative interactions at first order decrease as the cutoff increases, representing a fine tuning in the S–wave energy corrections and in the P–wave interaction. For $\alpha = -1/6$, the mass has to be renormalized according to Eq. (16), a standard occurrence in quantum field theory but not in approaches to nuclear matter. It is not a coincidence that the renormalization requirements involve the singular two-derivative two–body terms and the term in Eq. (1) that depends explicitly on the density and is ascribed to few–body forces. At least one of these terms is required for saturation and, it is believed, both are needed for a good fit at the mean–field level. It is not at all obvious that the requirement of renormalization can be fulfilled at higher orders with such a constrained set of interactions.

A more general renormalization would be achieved only if all the cutoff-dependent second–order terms could be regrouped with first–order terms without extra constraints. For this, additional terms should be added to the interaction. From the k_F dependence of the terms in Eq. (8), we can recognize which terms should be added to the interaction to provide the same k_F dependence in the EOS. For example, in the case $\alpha = 1$ (which at mean–field level is a proxy for the three–body force), for the k_F^7 terms one would need a two–body term of the type $\vec{\nabla}^4\delta(\vec{r}_1 - \vec{r}_2)$; for the k_F^8 dependence, a three–body term of the type $\vec{\nabla}^2\delta(\vec{r}_1 - \vec{r}_2)\delta(\vec{r}_2 - \vec{r}_3)$; and, finally, for the k_F^9 dependence, a four–body term $\delta(\vec{r}_1 - \vec{r}_2)\delta(\vec{r}_2 - \vec{r}_3)\delta(\vec{r}_1 - \vec{r}_4)$.

The inclusion of such additional terms would provide of course a much more complicated interaction and calculations would become more difficult to perform in practice. More importantly, if these additional terms are treated on the same footing as the terms in Eq. (1), the higher–order contributions from these additional terms will generate further cutoff dependence. The situation is familiar in field theory, where it is recognized that renormalization requires all possible interactions allowed by the symmetries. (For work in this direction, see Ref. [52].) In this case, to have any predictive power, one should be able to argue that some "sub–leading" terms should be included in first order only when "leading" terms are included in second order. For $\alpha = 1$, this could be the case for the four–derivative two–body, two–derivative three–body, and no-derivative four–body terms.

As we have shown here, the requirement of renormalizability constrains the form of the interactions allowed at different orders in the expansion beyond mean field. It calls for a more general study where a systematic analysis of the correct power counting within the perturbative many–body problem with effective interactions is performed. To our knowledge, this aspect, which we reserve for future work, has not been addressed so far in the framework of the energy–density functional theories based on Skyrme interactions. Once this is done, renormalizability could be investigated in the context of potentially better–grounded interactions, such as those that include pion effects in addition to the most general short–range interactions (see, *e.g.*, Ref. [53] and references therein).

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Appendix

Defining the combinations of Skyrme parameters

$$
\begin{aligned}
d_0 &= t_0^2\left(1 + x_0^2\right) \geq 0, \\
d_1 &= 2t_0\left[1 + \frac{3}{16}\alpha(\alpha+3) + x_0x_3\right], \\
d_2 &= \left[1 + \frac{3}{16}\alpha(\alpha+3)\right]^2 + x_3^2 > 0, \\
e_0 &= t_0t_1(1 + x_1x_0), \\
e_1 &= t_1\left[1 + \frac{3}{16}\alpha(\alpha+3) + x_1x_3\right], \\
h_1 &= t_1^2\left(1 + x_1^2\right) \geq 0, \\
h_2 &= t_2^2\left[1 + x_2^2 + 4\left(1 + x_2\right)^2\right] \geq 0,
\end{aligned}
$$

we can write the coefficients A_i, $B_i(\tilde{\lambda})$, and C_i appearing in Eqs. (6), (7), and (8), respectively, as

$$\begin{aligned}
&A_0 = d_0 I_1, \quad A_1 = d_1 I_1, \quad A_2 = d_2 I_1, \\
&A_3 = e_0 I_2, \quad A_4 = e_1 I_2, \quad A_5 = h_1 I_3 + h_2 I_4, \\
&B_0(\tilde{\lambda}) = d_0 + \frac{\tilde{\lambda}^2}{3} e_0 + \frac{\tilde{\lambda}^4}{20} h_1, \\
&B_1(\tilde{\lambda}) = d_1 + \frac{\tilde{\lambda}^2}{3} e_1, \\
&B_2(\tilde{\lambda}) = \frac{3}{5} e_0 + \frac{\tilde{\lambda}^2}{90} \left(\frac{27}{4} h_1 + h_2 \right), \\
&C_0 = d_2, \quad C_1 = \frac{3}{5} e_1, \quad C_2 = \frac{1}{70} \left(9h_1 + h_2 \right),
\end{aligned}$$

where

$$I_1 = \frac{11 - 2\ln 2}{140}, \qquad I_2 = \frac{167 - 24\ln 2}{1890},$$
$$I_3 = \frac{4943 - 564\ln 2}{166320}, \qquad I_4 = \frac{1033 - 156\ln 2}{498960}.$$

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