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We analyze the power counting of the peripheral singlet partial waves in nucleon-nucleon scattering. In agreement with conventional wisdom, we find that pion exchanges are perturbative in the peripheral singlets. We quantify from the effective field theory perspective the well-known suppression induced by the centrifugal barrier in the pion-exchange interactions. By exploring perturbation theory up to fourth order, we find that the one-pion-exchange potential in these channels is demoted from leading to subleading order by a given power of the expansion parameter that grows with the orbital angular momentum. We discuss the implications of these demotions for few-body calculations: though higher partial waves have been known for a long time to be irrelevant in these calculations (and are hence ignored), here we explain how to systematize the procedure in a way that is compatible with the effective field theory expansion.

The derivation of nuclear forces from first principles is one of the central problems of nuclear physics. In fact the nuclear potential – though not an observable – is the theoretical object from which we usually derive the properties of nuclei and calculate their reaction rates. With the recent onset of *ab initio* calculations, the direct link between nuclear forces and nuclear structure can be established for the first time, at least for light nuclei. Yet the understanding of nuclear forces from quantum chromodynamics (QCD) remains incomplete.

Nowadays, among the strategies to explain nuclear observables from QCD the most promising are lattice QCD and effective field theory (EFT). The first approach tries to calculate nuclear properties directly from QCD. Lattice QCD, though close to, is still not able to explain light nuclei in the physical world, i.e. for the physical pion mass. First calculations for pion masses larger than the physical one are promising [1] and it is apparent that $m_\pi = 140\,\text{MeV}$ will be reached soon. The second approach – the EFT formulation – establishes the connection between nuclear physics and QCD indirectly, avoiding the requirement of complex lattice QCD calculations (see Refs. [2, 3] for reviews). Nuclear EFT is the renormalization group evolution of QCD at low energies and thus equivalent to QCD. The idea is to write down the most general Lagrangian involving the low-energy fields and respecting the low-energy symmetries of QCD, in particular chiral symmetry. In this regard nuclear EFT simply amounts to rewriting the QCD Lagrangian in terms of a different set of fields, nucleons and pions instead of quarks and gluons.

A key ingredient of nuclear EFT is the existence of a separation of scales. Pions and nucleons are no longer valid degrees of freedoms at momenta above a characteristic hard scale M of the order of $0.5-1.0\,\text{GeV}$. Yet most nuclear processes of interest happen at a softer scale Q of the order of $100-200\,\text{MeV}$, which can be identified with the typical momenta of nucleons inside nuclei or with the pion mass. Moreover on a first approximation naive dimensional analysis (NDA) suggests that the dimensionful couplings of the EFT might be natural when expressed in units of the hard scale M. From this point of view M can be interpreted as the natural QCD mass scale for hadrons $M \sim M_{\text{QCD}}$ (of the order of the rho or the nucleon mass, not to be confused with the QCD perturbative scale Λ_{QCD}) or $4\pi f_\pi$. As a consequence we can write the scattering amplitude in EFT as a power series in terms of the ratio between the soft-scale Q and the hard-scale M:

$$T = \sum_\nu T^{(\nu)}\left(\frac{Q}{M}\right)^\nu. \tag{1}$$

This property goes under the name of power counting and is probably the most useful feature of EFT. Power counting implies that we can organize the calculations to achieve a given goal of accuracy and lies at the core of the systematics of the EFT approach.

We stress that the existence of a power counting depends on *renormalizability*. EFT expresses the interactions among its fields as a low-energy expansion. While at low momenta Q each term in this expansion is smaller than the previous one, at high momenta M the contrary happens and eventually the EFT interactions diverge in the ultraviolet. This does not pose a conceptual problem because EFT is only expected to work at low energies. But it poses a technical one: the ultraviolet divergences threaten to destroy the low-energy EFT expansion via loop corrections. The way to deal with this issue is *renormalization*: we introduce a cut-off Λ to regularize the loops, include counterterms to absorb the Λ-dependence and

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finally check that the low-energy observables do not depend on the choice of Λ. Cut-off independence, i.e. renormalizability, is therefore a necessary condition to have power counting in EFT. Renormalizability and power counting are intimately intertwined, a feature that is emphasized for instance in Wilsonian renormalization [4]. If renormalizability is absent at a certain order we have to include new contact interactions until cut-off independence is restored, which in turn constrains the power counting. If for some reason our prejudices about power counting do not lead to a renormalizable EFT, they have to be revisited and amended.

Traditional EFT knowledge is perturbative, chiral perturbation theory (ChPT) being a good example. The understanding of the renormalization group analysis of non-perturbative physics – as is the case in nuclear physics – is recent in comparison. But the first formulation of nuclear EFT, the Weinberg counting [5, 6], predates these developments. To deal with the non-perturbative features of nuclear physics, the Weinberg counting applies not directly to the scattering amplitude but rather to the nuclear potential:

$$V_{\text{EFT}} = V^{(0)} + V^{(1)} + V^{(2)} + \dots . \qquad (2)$$

The rationale is that, though power counting is not obvious for the scattering amplitude in the non-perturbative case, it is for the potential. Thus the idea is to iterate the EFT potential, or at least the leading order (LO) piece, in the Lippmann-Schwinger (or Schrödinger) equation with the implicit expectation that the resulting T-matrix follows the EFT expansion

$$T_{\text{EFT}} = T^{(0)} + T^{(1)} + T^{(2)} + \dots . \qquad (3)$$

Taking into account the simplicity of the approach outlined by Weinberg it comes as no surprise the enthusiasm with which it has been followed, leading to the development of two-nucleon potentials with $\chi^2/\text{d.o.f.} \sim 1$ [7–9].

Yet there is a fly in the ointment: it is not clear whether the scattering amplitudes follow a power counting in Weinberg's proposal. In fact the situation is rather the contrary. Several works have shown once and again [10–20] that the Weinberg counting is not renormalizable. As the cut-off is increased above $\Lambda \sim 0.5\,\text{GeV}$, the scattering amplitudes show all kinds of fascinating behavior with the notable exception of cut-off independence. The most natural interpretation of the previous results is that renormalization is lost in the Weinberg counting. Not only that, the connection with QCD disappears because this approach does not make for a proper EFT. We mention in passing that a few authors have proposed to reinterpret what renormalization means in order to accommodate the previous observations in the Weinberg counting [21, 22]. This is adequate as long as one does not insist on a clear power counting (beyond the vague impression that calculations get better at higher orders) and a direct connection to QCD. However this is not the solution for an appropriate nuclear EFT. Rather we should strive to get scattering amplitudes that are unambiguously renormalizable and have a power counting.

As a matter of fact there exists already a consistent version of nuclear EFT [12, 23–28]. It is renormalizable, can describe the scattering amplitudes for $k < M$, converges well and power counting is realized at the level of observables. Its foundation relies on a better understanding of the renormalization of non-perturbative physics and singular interactions [11–14, 29, 30]. The key improvements over the original Weinberg proposal are the non-perturbative renormalization of the LO amplitudes and the addition of subleading order contributions as perturbations. At LO the main difference with the Weinberg counting lies in the promotion of a series of *P*- and *D*-wave counterterms to LO in triplet partial waves for which the tensor force is attractive, a change proposed in Ref. [12]. At subleading orders there are more counterterms than in Weinberg counting, for instance in the attractive triplets that already received a counterterm at LO. The convergence of the EFT expansion is acceptable and the description of the data too, but it has not achieved yet a $\chi^2/\text{d.o.f.} \sim 1$ as in the Weinberg approach. However this is expected if we take into account that the calculations of Refs. [24–28] are still one order below the most advanced ones in the Weinberg approach [7–9].

One prediction of the original Weinberg counting is that the one-pion-exchange (OPE) potential – the LO nuclear potential – is *always* non-perturbative. Yet this is in contradiction with common wisdom in nuclear physics. Pion exchanges have been known for a long time to be perturbative in peripheral partial waves [31, 32]. While this is easy to understand in terms of the repulsive centrifugal barrier for high angular momenta ($l \gg k/m_\pi$), the analogous explanation in terms of power counting has traditionally remained elusive. From time to time this has been discussed in the literature [12, 23, 25, 27], but rather as an afterthought. This peripheral demotion already happens for moderate angular momenta $l \sim k/m_\pi$. Different arbitrary angular momentum cut-offs for when to consider OPE perturbative (or not) have been proposed, usually in the range $l = 2 - 4$. However, taking into account that the importance of this matter is rather tangential in two-body calculations, it comes as no surprise the disinterest for this particular problem [1]. The task of power counting should be to quantify the size of the peripheral wave suppression to systematically include it in EFT calculations.

The answer to this question is not only of academic interest, but has practical implications too. The main applications are few-body calculations, which usually require the inclusion of contributions arising from two-body partial waves up to a critical value of the orbital angular momentum (typically $l \geq 5$ or $j \geq 5$ in the three-nucleon system [33]). Yet the choice of a maximum angular momentum is driven by numerical considerations rather than by the constraints that the EFT expansion imposes on the accuracy of physical observables.

The purpose of this manuscript is to translate the quantum-mechanical peripheral-wave suppression into a power-counting demotion of the OPE potential. This will allow to take the decisions regarding which two-body partial waves to ignore based on power-counting arguments. That means im-

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[1] Actually power counting takes into account the peripheral suppression in the particular case of the contact-range interactions, where the l-wave counterterms are less relevant than the S-wave ones by a factor of Q^{2l} in naive dimensional analysis.

proving the systematics of these calculations or even simplifying them at the lowest orders where probably very few partial waves need to be included. Here we limit ourselves to the spin-singlet waves where OPE is not singular and thus can be defined without counterterms.

The manuscript is structured as follows. In Section I we will compare the non-perturbative OPE predictions for the phase shifts in the singlets with their perturbative expansion. This will allow us to check up to what point OPE is perturbative. In Section II we will find a power-counting explanation for the peripheral demotion of central OPE, which will later be checked against numerical calculations of the expansion parameter of central OPE. Finally we will present our conclusions in Section III.

I. PERTURBATIVE PIONS IN PERIPHERAL SINGLET WAVES

In this section we will analyze whether the OPE potential is perturbative in the peripheral singlet waves, that is, the $^1P_1$, $^1D_2$, $^1F_3$, ... partial waves. The approach is straightforward: we will compare the amplitudes that we obtain after the full iteration of OPE with the perturbative ones. We will work out the phase shifts up to fourth order in perturbation theory. The outcome of these lengthy calculations is that the OPE potential is definitely perturbative in all the singlet waves with orbital angular momentum $l \geq 1$ (the $l = 0$ case will be dealt with in a separate manuscript). The interpretation in terms of traditional power counting is that the OPE potential is subleading (where LO is reserved for interactions that have to be iterated to reproduce the non-perturbative physics of the S-waves).

A. Formalism

Here we will determine the perturbative character of OPE in the peripheral singlets by means of a direct calculation. In first place we will solve the Schrödinger equation with the OPE potential to obtain the non-perturbative phase shift δ_l, where l indicates the orbital angular momentum. In a second step we will expand the phase shift δ_l perturbatively as

$$\delta_l = \delta_l^{[1]} + \delta_l^{[2]} + \delta_l^{[3]} + \ldots, \qquad (4)$$

where the numbers in brackets indicate the number of insertions of the OPE potential. Comparing the right- and left-hand sides of the equation above, we can check the convergence of the perturbative series.

At this point we find it worth commenting that this kind of direct comparison can only be used in the singlets. As it is well-known, the OPE potential comprises central and tensor pieces. The former is regular, and hence the phase shifts and their perturbative expansion are well-defined. But the latter is singular: to determine whether the tensor piece is perturbative or not requires renormalization. In the singlet channels only the central piece of the OPE potential is involved in the calculations. This is what makes possible the simple analysis we pursue in this section.

The calculations are done as follows. For the phase shifts, we begin with the reduced Schrödinger equation:

$$-u_l'' + \left[2\mu V(r) + \frac{l(l+1)}{r^2} \right] u_l(r) = k^2 u_l(r), \qquad (5)$$

where V is the OPE potential. We solve it with a regular boundary condition at the origin, i.e. $u(0) = 0$. We can extract the phase shifts from the asymptotic behavior of the wave function at large distances ($m_\pi r \gg 1$):

$$u_l(r; k) \rightarrow \hat{\jmath}_l(kr) - \tan \delta_l \hat{y}_l(kr), \qquad (6)$$

where $\hat{\jmath}_l(x) = x j_l(x)$, $\hat{y}_l(x) = x y_l(x)$ are reduced spherical Bessel functions. For obtaining the non-perturbative phase shifts, we will match the asymptotic expression above to the reduced wave function u_l at the infrared cut-off $R = 20$ fm.

In contrast, if we are interested in the perturbative phase shifts, the asymptotic matching will not be the method of choice. We will find much more useful instead the following integral representation of the phase shift:

$$\tan \delta_l = -\frac{2\mu}{k} \int_0^\infty u_k(r) \, V(r) \, \hat{\jmath}_l(kr) \, dr, \qquad (7)$$

in which the wave function u_k is normalized as in Eq. (6). The advantage of the integral formula above is that it is better suited for a perturbative calculation: with a wave function of order n we can calculate the phase shift at order $n + 1$. A derivation of this formula can be found in Appendix A.

If the potential is weak enough, we can expand the reduced wave function perturbatively as follows

$$u_l(r; k) = u_l^{[0]} + u_l^{[1]} + u_l^{[2]} + u_l^{[3]} + \ldots, \qquad (8)$$

where the brackets indicate the number of OPE insertions. The differential equations followed by the $u_l^{[n]}$'s are then

$$u_l^{[0]''} + \left[k^2 - \frac{l(l+1)}{r^2} \right] u_l^{[0]}(r) = 0, \qquad (9)$$

$$u_l^{[n]''} + \left[k^2 - \frac{l(l+1)}{r^2} \right] u_l^{[n]}(r) = 2\mu V(r) u_l^{[n-1]}(r)$$
$$\text{for } n \geq 1, \qquad (10)$$

which we obtain from plugging the perturbative expansion of the wave function into the reduced Schrödinger equation and rearranging the terms.

We solve the set of differential equations iteratively, starting with $n = 0$ for which we take $u_l^{[0]}(r; k) = \hat{\jmath}_l(kr)$, i.e. the free solution. If we now expand the integral expression for the phase shifts (Eq. (7)) perturbatively, we end up with the Born approximation

$$\delta_l^{[1]} = -\frac{2\mu}{k} \int_0^\infty u_l^{[0]}(k; r) \, V(r) \, \hat{\jmath}_l(kr) \, dr. \qquad (11)$$

For $n \geq 1$ the only subtlety is finding a suitable boundary condition for $u_l^{[n]}$. Actually, the easiest thing to do is to integrate from infinity to the origin. The reason is that the asymptotic form of $u_l^{[n]}(k; r)$ is readily available from expanding Eq. (6)

perturbatively. At first order we find the asymptotic boundary condition

$$u_l^{[1]}(r;k) \; \rightarrow \; -\delta^{[1]} \, \hat{y}_l(kr) \,, \qquad (12)$$

from which we can integrate $u_l^{[1]}$ for arbitrary r. Then we can calculate the second order contribution to the phase shifts by inserting $u_l^{[1]}$ in the perturbative expansion of Eq. (7), yielding

$$\delta_l^{[2]} \; = \; -\frac{2\mu}{k} \int_0^\infty u_l^{[1]}(r;k) \, V(r) \, \hat{j}_l(kr) \, dr \,. \qquad (13)$$

For the higher orders we simply repeat the previous steps. The asymptotic boundary conditions and the expressions for the $n = 3, 4$ contributions to the phase shifts read:

$$u_l^{[2]} \; \rightarrow \; -\delta^{[2]} \, \hat{y}_l(kr) \,, \qquad (14)$$

$$\delta_l^{[3]} \; = \; -\frac{2\mu}{k} \int_0^\infty u_l^{[2]} \, V(r) \, \hat{j}_l(kr) \, dr - \frac{1}{3}\delta_l^{[1]\,3} \,, \qquad (15)$$

$$u_l^{[3]} \; \rightarrow \; -\left[\delta^{[3]} + \frac{1}{3}\delta^{[1]\,3}\right] \hat{y}_l(kr) \,, \qquad (16)$$

$$\delta_l^{[4]} \; = \; -\frac{2\mu}{k} \int_0^\infty u_l^{[3]} \, V(r) \, \hat{j}_l(kr) \, dr - \delta_l^{[1]\,2} \, \delta_l^{[2]} \,. \qquad (17)$$

Expressions for orders $n > 4$ can be obtained analogously.

B. Perturbative OPE

Now we evaluate the previous perturbative differential equations and integrals to calculate the phase shifts. We remind that the form of the central piece (the only one we need in the singlets) of the OPE potential is given by

$$V(r) = \frac{\sigma \, \tau}{3} \, \frac{1}{M_N \Lambda_{NN}} \, m_\pi^3 \, \frac{e^{-m_\pi r}}{m_\pi r} \,, \qquad (18)$$

where $\sigma = \vec{\sigma}_1 \cdot \vec{\sigma}_2$ ($= -3$ for the singlets), $\tau = \vec{\tau}_1 \cdot \vec{\tau}_2$, M_N is the nucleon mass and Λ_{NN} is the characteristic momentum scale of OPE, defined as

$$\Lambda_{NN} = \frac{16\pi f_\pi^2}{M_N \, g_A^2} \sim 300 \,\text{MeV} \,, \qquad (19)$$

with $g_A = 1.26$ and $f_\pi = 92.4$ MeV. By *characteristic scale* we mean the following: we naively expect the OPE to behave perturbatively for momenta $k < \Lambda_{NN}$, while for higher momenta it becomes non-perturbative [34–36]. As we will see, this is not the case in the peripheral singlets.

We have previously commented that the perturbative expansion is finite and well-defined in the singlets. Yet we will keep a finite cut-off in the calculations. In particular we regularize the OPE potential as follows

$$V(r; r_c) = V(r) \, \theta(r - r_c) \,, \qquad (20)$$

which simply amounts to changing the lower limit of the perturbative integrals from $r = 0$ to $r = r_c$. The reasons for having a cut-off are: i) in actual EFT calculations (including counterterms and the two-pion exchange potential, which we have ignored in this work) a finite cut-off is a must, ii) the computational requirements of doing perturbation theory to high order decrease significantly with a finite cut-off [2]. We have chosen $r_c = 0.3$ fm as a reference value for the cut-off. For this cut-off the perturbative calculations with OPE have already converged (as a matter of fact, there are only tiny differences in the results for $r_c < 1$ fm). Besides, $r_c = 0.3$ fm is also significantly below the standard cut-off ranges employed in previous EFT calculations in r-space [24, 25]. For an S-wave the equivalence of the previous r-space cut-off with a sharp p-space cut-off is $\Lambda = 1033$ MeV, where we have used the relationship $\Lambda \, r_c = \pi/2$. This relationship can be obtained by comparing the running of the S-wave counterterm C_0 in a pionless theory for a delta-shell r-space regulator r_c and a sharp cut-off p-space regulator Λ (see Ref. [15] for details). The extension of the previous relationship for peripheral partial waves is trivial, yielding the numerical values $\Lambda = 1590$ MeV for a P wave, $\Lambda = 2127$ MeV for a D wave and higher values for $l \geq 3$. For checking purposes we explicitly show the cut-off dependence of the isoscalar partial waves in Fig. 1, where it can be appreciated that the more peripheral the wave the weaker the cut-off dependence. We have chosen to show the isoscalar channels because it is for these channels that central OPE is strongest, yielding more cut-off dependence. Yet, with the exception of the $^1P_1$ channel for $r_c \geq 1$ fm (already a relatively large cut-off), the cut-off dependence ranges from rather mild ($^1F_3$) to negligible ($^1H_5$).

We obtain the phase shifts shown in Figure 2, where we can appreciate that the perturbative expansion is converging extremely quickly even for the $^1P_1$ wave, the lowest partial wave considered. The perturbative series is more convergent the higher the partial wave: the tree level (that is, the Born approximation) phase shifts already match the non-perturbative ones with a precision of a fraction of a degree, with the exception of the $^1P_1$ wave. All this indicates that the convergence parameter of the perturbative expansion is certainly smaller (i.e. a faster convergence) than that of the EFT for the particular case of the singlet waves.

II. THE PERIPHERAL WAVE DEMOTION

In this section we will discuss the role of angular momentum for the power counting of the singlets. The aim is to ex-

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[2] We are using a variable step integration method for the set of coupled differential equations that define the perturbative expansion, Eqs. (9) and (10). Owing to the number of differential equations involved it is increasingly difficult to solve it for small cut-offs when i) we increase the perturbative order for which we integrate and ii) we increase the angular momentum, particularly at low momenta. The chosen cut-off is actually on the limit of what we can compute at fourth order, yet it is more than enough for a nuclear EFT calculation.
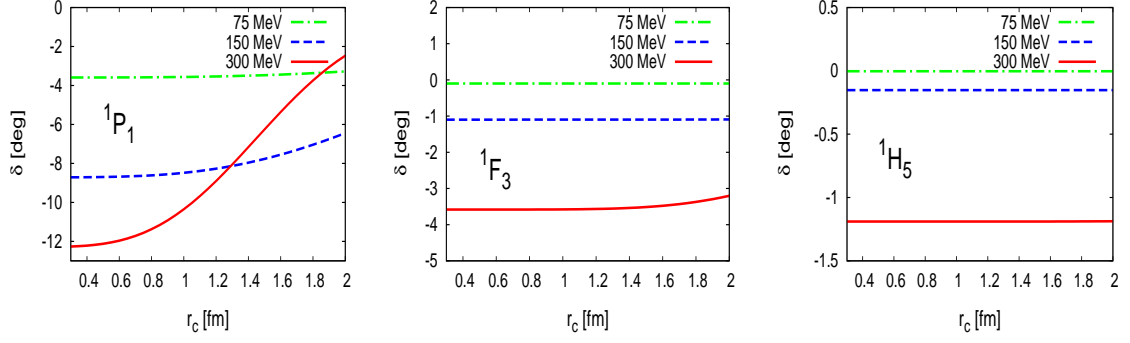
FIG. 1: Cut-off dependence of the non-perturbative phase shifts for the OPE potential in the singlet peripheral waves $^1P_1$, $^1F_3$ and $^1H_5$. We show the isoscalar partial waves because for them the OPE potential is three times stronger than in the isovector partial waves. Hence the cut-off dependence is also expected to be more evident in the isoscalar channels. We show the $r_c = 0.3 - 2.0$ fm cut-off range for the c.m. momenta $k_{\text{c.m.}} = 75, 150$ and 300 MeV.
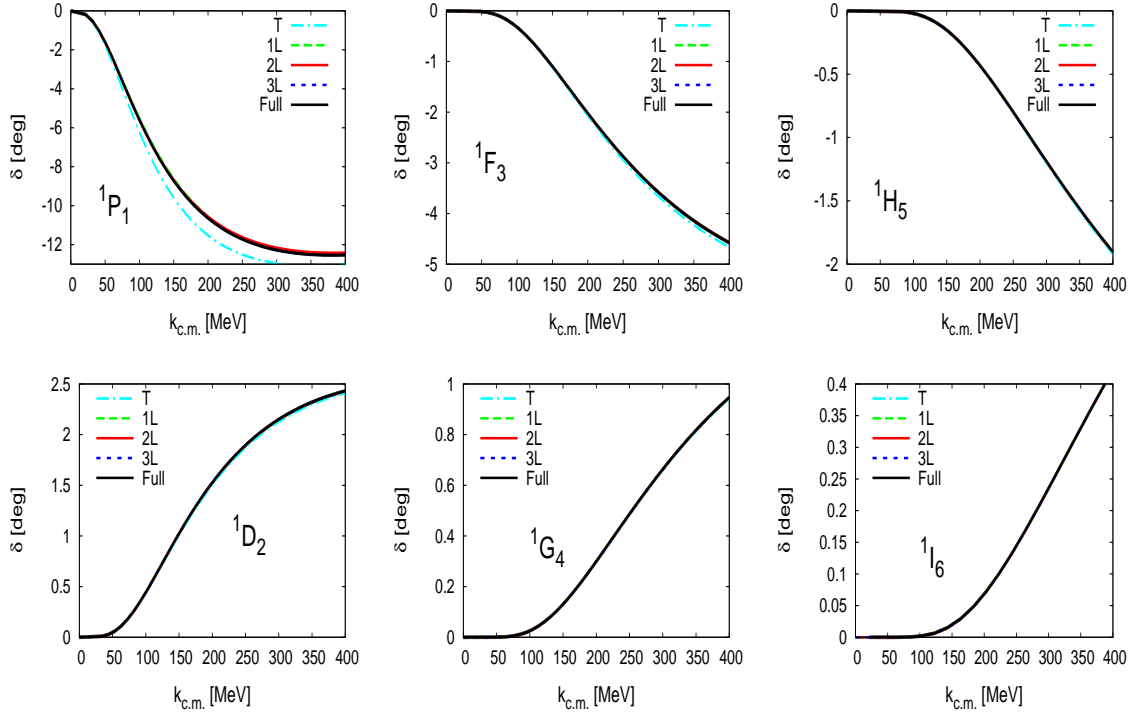

FIG. 2: Convergence of the perturbative expansion of the phase shifts for the OPE potential in the singlet peripheral partial waves $^1P_1$, $^1D_2$, $^1F_3$, $^1G_4$, $^1H_5$ and $^1I_6$. We compare the full, non-perturbative OPE phase shifts – the black solid line in the figures – with the perturbative ones computed at increasing orders, where "T" stands for "tree level" (i.e. the Born approximation) and "1L", "2L", "3L" for the one-, two- and three-loop calculation (i.e. second, third and fourth order perturbation theory). The calculations have been performed with a finite cut-off of $r_c = 0.3$ fm.

plain the fast convergence rate of the perturbative series for central OPE in the singlets. As already seen, the iteration of the OPE potential is suppressed in the peripheral waves with respect to the expectations of standard power counting. However, the problem is how to incorporate the peripheral suppression into the EFT expansion. For that we have to quantify the size of this effect. We will discuss a series of ideas to

find the factor by which the iteration of the OPE potential is suppressed in the higher partial waves.

A. Quantum Mechanical Suppression

We begin by considering the peripheral suppression of a finite-range potential in standard quantum mechanics. The arguments are straightforward and relatively well-known, but we find it convenient to repeat them for illustrative purposes. This type of suppression is only apparent at momenta far below the inverse of the range of the potential, i.e. $k \ll m_\pi$ for nuclear forces. This means that it will help to explain the power counting in pionless theories. However, the application to pionful theories will be limited to very peripheral waves.

First we begin by considering the matrix element of a central potential in the partial wave l, which is given by

$$\langle p', l|V|p, l\rangle = \frac{4\pi}{M_N} \int r^2 dr \, j_l(p'r) M_N V(r) j_l(pr),$$

(21)

where $j_l(x)$ is the standard spherical Bessel function and V the potential in coordinate space. In the Born approximation the size of this matrix element for $p = p'$ is proportional to the scattering amplitude. If we compare

$$V_l(p) = \langle p, l|V|p, l\rangle$$

(22)

for different angular momenta we can obtain a baseline estimation of the peripheral suppression factor. We can do this by calculating the ratio of V_l against a reference partial wave l_0. A natural choice is the P wave as it is the smallest angular momenta considered in this work. There is the complication that even (odd) partial waves are isoscalars (isovectors). This can be circumvented by taking into account the isospin factors $\tau = \vec{\tau}_1 \cdot \vec{\tau}_2$ into the definition of the ratio

$$R_l(k) = \frac{\tau_{l_0}}{\tau_l} \frac{V_l(k)}{V_p(k)},$$

(23)

where $l_0 = 1$, $V_p = V_{^1P_1}$, and $\tau = 1$ or -3 depending on the total isospin of a particular channel. We have calculated the inverse of this ratio in Figure 3, where the choice of the inverse is for illustrating the suppression in a more obvious way. As can be seen as the angular momentum increases the suppression becomes much bigger, especially at low energies.

The next step is to quantify this effect by means of a more concrete argument. For instance we can expand Eq. (21) at low momenta. For $pr \ll \sqrt{l + 1/2}$ we can expand j_l as

$$j_l(pr) = \frac{(pr)^l}{(2l + 1)!!} \left[1 - \frac{1}{2} \frac{(pr)^2}{2l + 3} + O\left((pr)^4\right) \right].$$

(24)

If we take into account that for radii $m_\pi r > 1$ the OPE potential falls off exponentially, then for momenta $p, p' \ll \sqrt{l + 1/2} \, m_\pi$:

$$
\begin{aligned}
\langle p', l|V|p, l\rangle &= \frac{4\pi}{M_N} \frac{p^l p'^l}{(2l + 1)!!^2} \Big[\int r^{2l+2} dr \, M_N V(r) \\
&\quad - \frac{1}{2} \frac{p'^2 + p^2}{2l + 3} \int r^{2l+4} dr \, M_N V_C(r) \\
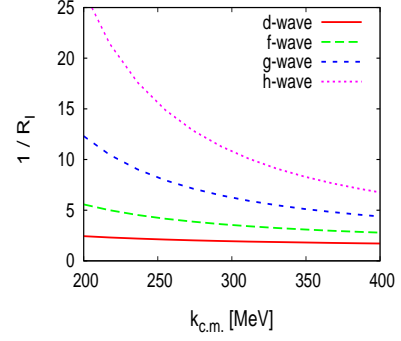&\quad + O(p'^4, p^4) \Big],
\end{aligned}
$$

(25)

FIG. 3: Inverse of the ratio of the the diagonal potential V_l in momentum space between the l-wave and the p-wave. (see the main text for a detailed explanation of the ratio).

where we have plugged the expansion of the spherical Bessel function into the integral. As can be seen, the power law piece of the matrix element is in fact consistent with the behavior of the lowest-order l-wave counterterm in pionless theory (Q^{2l}), in agreement with naive expectations. In addition, if we take into account the exponential tail of OPE it is apparent that

$$\int r^{2k} dr \, M_N \, V(r) \sim \frac{1}{\Lambda_{NN}} \frac{1}{m_\pi^{2k-2}},$$

(26)

which means that every p^k and p'^k within the expansion of the potential is divided by a m_π^k. That is, we find the expected breakdown scale of the pionless power counting.

This begs the question about whether the previous argument can be used to justify the peripheral demotion of OPE, and in principle the answer is yes: if $\sqrt{l + 1/2} \, m_\pi \gg M$ the argument applies over all the range of validity of the pionful theory. Using $M \sim 0.5 - 1.0 \, \text{GeV}$ this happens for $l \gg 12 - 50$, for which a demotion of OPE similar to the one found in pionless EFT will begin to show up. However this is of little value for practical purposes. Though we can show that a peripheral demotion indeed exists it is only apparent for extremely high angular momenta, far beyond where the partial-wave expansion is truncated in three-body calculations. This means that we have to resort to a different type of argument for analyzing the demotion at moderate angular momenta. We will do this in the next section.

B. Power Counting Suppression

Here we consider the peripheral wave suppression from the power-counting perspective. The idea is to find a relationship between the scales of a two-body system and the angular momentum. The arguments we present are in principle tailored for the particular case of the OPE potential in the singlet channels, in which case the potential is regular and we do not have to take into account the issue of cut-off dependence.

We begin by writing the OPE potential in the form

$$\langle p, l|V|p', l\rangle = \frac{4\pi}{M_N} \frac{1}{\Lambda_{NN}} f_l(\frac{p}{m_\pi}, \frac{p'}{m_\pi}), \qquad (27)$$

which underlines the scales that enter into the problem: Λ_{NN} is the natural momentum scale of OPE and we factor out the $4\pi/M_N$ that always appears in non-relativistic scattering. In the formula above f_l is a dimensionless function defined as

$$f_l(\frac{p}{m_\pi}, \frac{p'}{m_\pi}) = \int dr\, r^2\, j_l(pr) W_C(r) j_l(p'r), \qquad (28)$$

where W_C – the central piece of the OPE potential after removing the $1/(M_N \Lambda_{NN})$ factor – is given by

$$W_C(r) = \frac{\sigma\tau}{3} m_\pi^3 \frac{e^{-m_\pi r}}{m_\pi r}. \qquad (29)$$

The scale Λ_{NN} determines the strength of the OPE potential in the singlets, modulo the spin and isospin factors which we have included in the dimensionless function f_l. How to count Λ_{NN} determines whether OPE is perturbative or not. If we iterate the OPE potential, we can argue on dimensional grounds that

$$\langle V_l G_0 V_l\rangle \sim \left(\frac{4\pi}{M_N \Lambda_{NN}}\right) \times \left(\frac{M_N Q}{4\pi}\right) \times \left(\frac{4\pi}{M_N \Lambda_{NN}}\right)$$
$$\sim \left(\frac{4\pi}{M_N \Lambda_{NN}}\right) \times \left(\frac{Q}{\Lambda_{NN}}\right), \qquad (30)$$

where V_l stands for the l-wave projection of the OPE potential and Q is either the external momentum k or the pion mass m_π. From this estimation, we can see that the decision of iterating OPE or not depends on the dimensionless ratio Q/Λ_{NN}.

At this point we have the choices $\Lambda_{NN} \sim Q$ (Weinberg) and $\Lambda_{NN} \sim M$ (Kaplan-Savage-Wise, or KSW). The first requires the iteration of OPE at LO and the second not. The numerical value of $\Lambda_{NN} \sim 300\,\text{MeV}$ lies in between of what one could consider a soft and a hard scale. As a matter of fact, none of the previous counting conventions works for all partial waves: on one hand we have the $^3S_1$ and $^3P_0$ triplets where OPE is thought to be non-perturbative, while on the other we have the peripheral singlets where the explicit calculations of Section I show that OPE is perturbative and probably demoted even with respect to the $\Lambda_{NN} \sim M$ scenario.

The previous mismatch between scaling expectations and numerical calculations lies in the dimensionless functions and numerical factors conforming the potential. Usually we naively assume that these dimensionless factors are of $O(1)$ and do not affect the counting. But if that were the case OPE would be either perturbative or non-perturbative in all partial waves. If we take into account these factors, we can have a clearer idea of what *soft* and *hard* means for Λ_{NN}.

To be more specific, let us consider the l-wave T-matrix of the OPE potential in the absence of contact-range physics. If we define the on-shell T-matrix as:

$$T_l(k) = \langle k, l|T|k, l\rangle$$
$$= \langle k, l|V|k, l\rangle + \langle k, l|VG_0V|k, l\rangle + \dots, \qquad (31)$$

then – according to the naive analysis of Eq. (30) – we should have the expansion

$$T_l(k) = \frac{4\pi}{M_N \Lambda_{NN}} \sum_n^\infty t_n^{(l)}(\frac{k}{m_\pi})\left(\frac{Q}{\Lambda_{NN}}\right)^n, \qquad (32)$$

where Q is either k or m_π and n refers to the number of loops. We expect the coefficients/functions $t_n^{(l)}$ to be of $O(1)$. On dimensional grounds the $t_n^{(l)}(x)$ are functions of k/m_π, as can be deduced from Eq. (27). The dependence on the angular momentum quantum number l enters via the specific value of the coefficients. As a consequence, provided that the naturalness hypothesis for $t_n^{(l)}$ is correct, the convergence radius of the series is independent of l.

On the contrary, if the convergence of perturbative OPE depends on the partial wave, then the form of the loop expansion must take a different form

$$T_l(k) = \frac{4\pi}{M_N \Lambda_{NN}} \sum_n^\infty t'^{(l)}_n(\frac{k}{m_\pi})\left(\frac{Q}{b_l \Lambda_{NN}}\right)^n, \qquad (33)$$

where, as in the previous case, n refers to the number of loops and the coefficients $t'^{(l)}_n$ are $O(1)$. However we include now a factor b_l to account for the different expansion parameter and convergence radius in each partial wave l. This factor means that Q/Λ_{NN} is not the ratio to check when discussing the power counting of OPE, but rather $Q/(b_l \Lambda_{NN})$.

The only problem left is to determine the expansion parameter of the power series for $T_l(k)$. From complex analysis we know that the radius of convergence of this series is given by the pole of $T_l(k)$ that is closest to the threshold. However the OPE potential is relatively weak, which means that the poles of the T-matrix will be far from threshold and not easy to find. With this in mind we can resort to a trick to circumvent this difficulty. Instead of finding the poles of $T_l(k)$ for the physical value of Λ_{NN}, we can change the scale Λ_{NN} up to the point that there is a pole at $k = 0$, i.e. a bound state at threshold.

We will give the name $\Lambda^*_{NN}(l)$ to the critical value of Λ_{NN} that generates a bound state at threshold in the l-wave. For $\Lambda^*_{NN}(l)$ the perturbative series at $k = 0$ (the location of the threshold bound state) diverges:

$$T_l^*(0) = \frac{4\pi}{M_N \Lambda^*_{NN}(l)} \sum_n^\infty t'^{(l)}_n(0)\left(\frac{Q}{b_l \Lambda^*_{NN}(l)}\right)^n$$
$$\rightarrow \infty, \qquad (34)$$

where the notation T_l^* indicates that this is the T-matrix for $\Lambda^*_{NN}(l)$. This means that the expansion parameter is one for the previous series

$$\frac{Q}{b_l \Lambda^*_{NN}(l)} = 1. \qquad (35)$$

Consequently, for the physical Λ_{NN} we have at $k = 0$:

$$T_l(0) = \frac{4\pi}{M_N \Lambda_{NN}} \sum_{n,m}^\infty t'^{(l)}_n(0)\left(\frac{\Lambda^*_{NN}(l)}{\Lambda_{NN}}\right)^n, \qquad (36)$$

from which it is clear that the expansion parameter is simply $\Lambda^*_{NN}(l)/\Lambda_{NN}$.

For finite momenta $k \neq 0$ the analysis can be extended easily, though the conclusions are not as clear-cut as for $k = 0$. We begin by disentangling the $Q = m_\pi, k$ power series explicitly in the perturbative expansion of the T-matrix

$$
\begin{aligned}
T_l(k) &= \frac{4\pi}{M_N\Lambda_{NN}}\sum_n^\infty t'^{(l)}_n(\frac{k}{m_\pi})\left(\frac{Q}{b_l\Lambda_{NN}}\right)^n \\
&= \frac{4\pi}{M_N\Lambda_{NN}}\sum_n^\infty t'^{(l)}_n(\frac{k}{m_\pi})\sum_{r+s=n}c^{(l)}_{r,s}\frac{m_\pi^r k^s}{b_l\Lambda^n_{NN}}
\end{aligned} \quad (37)
$$

where $c^{(l)}_{r,s}$ are coefficients that differentiate the contributions coming from powers of m_π and k. The point is that we can rearrange the previous expansion as

$$
T_l(k) = \frac{4\pi}{M_N\Lambda_{NN}}\sum_m^\infty t''^{(l)}_n(\frac{k}{m_\pi})\left(\frac{\Lambda^*_{NN}(l)}{\Lambda_{NN}}\right)^n, \quad (38)
$$

where the new coefficients $t''^{(l)}_n$ are defined as

$$
t''^{(l)}_n(\frac{k}{m_\pi}) = t'^{(l)}_n(\frac{k}{m_\pi})\sum_{r+s=n}c^{(l)}_{r,s}\left(\frac{k}{m_\pi}\right)^s. \quad (39)
$$

If the coefficients $c^{(l)}_{r,s}$ are of order one, the expansion parameter of the perturbative series will still be $\Lambda^*_{NN}(l)/\Lambda_{NN}$ for $k \leq m_\pi$. Meanwhile for $k > m_\pi$ the expansion parameter is of the order $k/m_\pi \times \Lambda^*_{NN}(l)/\Lambda_{NN}$, which amount to assuming that m_π and k play the same role in the expansion. However we cannot discard the possibility of relative numerical factors between the expansions in powers of m_π and k. For instance, it could happen that the $c^{(l)}_{r,s} \sim 1/2^s$ or that $c^{(l)}_{r,s} \sim 2^s$, giving a different convergence radius in terms of k than in terms of m_π. Though this makes no difference at the conceptual level, this effect could have a moderate impact when estimating the peripheral demotion of OPE. We will briefly discuss this at the end of the section, but we can advance that the impact is going to be negligible. Part of the reason lies in the fact that the OPE for $k \geq m_\pi$ is very similar to the Coulomb potential, which happens to be always perturbative except for low energies. In terms of the perturbative expansion this translates into the coefficients $c^{(l)}_{r,s}$ having a behavior with respect to s that is extremely suppressed (e.g. $1/s!$).

For quantizing the power counting demotion, we compare the expansion parameter of perturbative OPE with the expansion parameter of nuclear EFT,

$$
\frac{Q}{b_l\Lambda_{NN}} = \frac{\Lambda^*_{NN}(l)}{\Lambda_{NN}} = \left(\frac{Q}{M}\right)^{\nu_{\text{OPE}}(l)}. \quad (40)
$$

Putting the pieces together the actual order of OPE in the l-wave singlet is $N^{\nu_{\text{OPE}}(l)}$LO, instead of LO as in Weinberg or NLO as in KSW countings. It is important to mention here that the existence of a peripheral demotion $\nu_{\text{OPE}}(l)$ hangs on the fact that the scale separation in nuclear EFT is imperfect. That is, the clear peripheral demotion we observe depends on the fact that the separation of scales in nuclear EFT

is not particularly good. If we use the values $Q = m_\pi$ and $M \sim 0.5-1.0\,\text{GeV}$, the expansion parameter lies theoretically between $1/7$ and $1/3$. However, we mention in passing that concrete calculations [24–28] and analyses [37–39] in pionful EFT suggest an expansion parameter closer to $1/3$ than $1/7$.

In Fig. 4 we calculate the ratio $\Lambda_{NN}/\Lambda^*_{NN}$ for the peripheral singlets as a function of the angular momentum l. For convenience we consider the angular momentum as a continuous variable [3] and include the actual peripheral waves as particular points along this curve. In the isoscalar partial waves the ratio is negative, the reason being that the OPE potential is repulsive in isoscalar singlets. The specific values of the $\Lambda_{NN}/\Lambda^*_{NN}$ ratios leading to a bound state at threshold are shown in Table I, where we also list the effective order at which OPE enters in each peripheral singlet. The power counting is normalized in agreement with the conventions of the previous discussion: LO corresponds to a potential that has to be iterated to all orders (for instance, the lowest-order contact interaction in the $^1S0$ partial wave) and NLO to the size of the OPE potential in the KSW counting. Even for the $^1P_1$ partial wave OPE is slightly demoted with respect to KSW. Table I only shows partial waves for which OPE is not much demoted beyond N^4LO (considering the average demotion). The reason for this is because contributions above this order are unlikely to enter in any practical EFT calculation in the near future. The chiral nuclear potential has not been used beyond leading three-pion exchange (or subsubleading TPE) in full EFT calculations [4]. This piece of the potential enters at N^4LO in a Weinberg-inspired counting [5] and at N^5LO in a KSW-inspired one (for which the convergence is expected to be slower, leading to a stronger peripheral demotion). Thus is does not seem necessary to go beyond that point.

The spread in the demotion depends on the expansion parameter taken: for $1/7$ the demotion is relatively mild – the lowest estimation in Table I –, while for $1/3$ it is much more obvious. For instance, in the $^1P_1$ case $N^{1.0}$LO corresponds to $1/7$ and $N^{1.7}$LO to $1/3$. If we take into account the observation that the actual expansion parameter seems to be closer to $1/3$, then the larger estimations for the demotion are expected to be more accurate than the lower ones. However overestimating the demotion can lead to the underestimation of the theoretical errors in a calculation. In this sense it might be more cautious to use a value in the middle.

Apart from the uncertainty in Q/M there is a second source of error in Table I, namely the interplay between the k and m_π expansions that we have previously discussed at the qualitative level. Addressing this problem formally is beyond the

---

[3] That is, we take l real – instead of integer – and determine the location of the threshold bound states from the asymptotic condition $u_l(\infty) = 0$. In practice we take an infrared cut-off of $R = 40\,\text{fm}$, though the results are stable for $R > 10\,\text{fm}$.

[4] Though recently the EFT potential has been calculated one order further and used in first-order perturbation theory for peripheral nucleon-nucleon scattering, see Ref. [40].

[5] This corresponds to N^3LO in the traditional notation used in Ref. [9], which skips one order because the Q/M contribution vanishes.

scope of this manuscript and in fact it has never been done in the literature for a pionful EFT expansion. So instead of analyzing in detail the perturbative expansion, we will explore the demotion with an alternative definition of $\Lambda_{NN}^*(l)$, since this scale is up to a certain extent condition-dependent. Previously we defined $\Lambda_{NN}^*(l)$ as the Λ_{NN} for which there is a bound state at threshold. As a consequence, the ratio $\Lambda_{NN}^*(l)/\Lambda_{NN}$ corresponds to the expansion parameter of the T-matrix at zero energy. However, the existence of a low-lying virtual state or resonance also calls for the iteration of the potential, the required strength of the potential being weaker than in the bound state case. Therefore we release the definition of the scale $\Lambda_{NN}^*(l, k_p)$ as the value of Λ_{NN} for which there is a pole at $k = k_p$ (either a bound/virtual state or a resonance, which means that k_p is a complex value), this new scale is related to the convergence of the T-matrix for the physical value of Λ_{NN} at the location of the pole

$$T_l(k_p) \;=\; \frac{4\pi}{M_N \Lambda_{NN}} \sum_{n}^{\infty} t'^{(l)}_n (\frac{k_p}{m_\pi}) \left(\frac{\Lambda_{NN}^*(l, k_p)}{\Lambda_{NN}} \right)^n . \tag{41}$$

In turn we could have used $\Lambda_{NN}^*(l, k_p)$ as the basis of an alternative definition of the peripheral demotion

$$\frac{\Lambda_{NN}^*(l, k_p)}{\Lambda_{NN}} = \left(\frac{Q}{M} \right)^{\nu'_{\text{OPE}}(l)} . \tag{42}$$

In terms of the scale Λ_{NN}^*, if we choose a bound state with $|k_p| = |i\gamma_B| = m_\pi$ we will get a smaller Λ_{NN}^* (i.e. more strength for the OPE potential) and more demotion ($\nu'_{\text{OPE}} > \nu_{\text{OPE}}$). On the contrary the virtual state / resonance condition – let's say at a momentum $|k_p| = |-i\gamma_V| = |k_R| = m_\pi$ – entails a bigger Λ_{NN}^*, i.e. a smaller $\Lambda_{NN}/\Lambda_{NN}^*$ ratio, and less peripheral demotion ($\nu'_{\text{OPE}} < \nu_{\text{OPE}}$). Actually we are more interested in the possibility that we might have been overestimating the demotion, which means that we only have to consider the virtual-state/resonance case. We have performed the calculations in Appendix B and checked that the effect of a change in conditions from a threshold bound state to a resonance is actually tiny for $\nu_{\text{OPE}}(l)$, usually of the order of $\Delta \nu_{\text{OPE}} \sim -(0.05 - 0.2)$. If we compare this change to the uncertainty related to Q/M, which lies in the range $|\Delta \nu_{\text{OPE}}| \sim 0.5 - 2$, we see that corrections to the threshold bound state condition can be safely ignored in most partial waves.

C. The Peripheral Perturbative Expansion Revisited

Now that we have a power-counting argument for the centrifugal suppression of the singlets, we want to check how it stands against concrete calculations. The approach we find most convenient is the comparison of multiple iterations of the OPE potential. We will see how at low energies the expansion parameter of perturbative OPE is indeed set by $Q/(b_l \Lambda_{NN})$. The ratio of iterated versus non-iterated diagrams has been already used in the past as a tool for determining the convergence of the EFT series [36], but calculations have been usually limited to just a few iterations of OPE. While this might
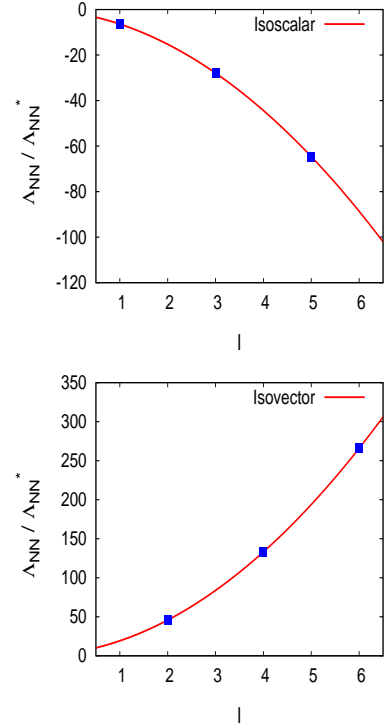

FIG. 4: Ratio between the actual value of Λ_{NN} appearing in the OPE potential and the critical Λ_{NN}^* generating a bound state at threshold for l-wave. The ratio is calculated for isoscalar and isovector channels. In the isoscalar case the ratio is negative as the OPE potential is actually repulsive in the isoscalar channels $^1P_1$, $^1F_3$, $^1H_5$ and so on.

not be a drawback in S-wave scattering, peripheral waves require the evaluation of higher orders of perturbation theory in order to be able to estimate the expansion parameter.

We will begin with the standard ratio of once iterated OPE versus tree level for the l-wave, which we define as

$$R_l(k) = \frac{\langle k, l|VG_0V|k, l\rangle}{\langle k, l|V|k, l\rangle} , \tag{43}$$

where we define the matrix elements above as

$$\langle k, l|V|k, l\rangle \;=\; \int_{r_c}^{\infty} dr\, u_k^{[0]}(r)\, V(r)\, u_k^{[0]}(r) , \tag{44}$$

$$\langle k, l|VG_0V|k, l\rangle \;=\; \int_{r_c}^{\infty} dr\, u_k^{[0]}(r)\, V(r)\, u_k^{[1]}(r) , \tag{45}$$

which roughly corresponds to the diagram ratio already evaluated in Ref. [36] for the S waves. The ratio R_l also coincides with the ratio of the one-loop over the tree level phase shifts

$$R_l(k) = \frac{\delta_l^{[2]}(k)}{\delta_l^{[1]}(k)} . \tag{46}$$

Once we consider the peripheral demotion factor b_l, we expect

TABLE I: Power counting of OPE for peripheral singlet partial waves. We show the critical value of Λ_{NN} that renders the central potential non-perturbative in each of the singlets. The actual power counting assignment for OPE in each of the partial waves depends on the expansion parameter of nuclear EFT, which is not known exactly. However assuming the baseline values of $Q = m_\pi$ and $M \sim 0.5 - 1.0\,\mathrm{GeV}$, we expect the expansion parameter to be between $1/7$ and $1/3$. Using this range of values, we calculate the OPE demotion in each partial wave.

$^S L_J$	$\Lambda_{NN}/\Lambda_{NN}^*(l)$	N$^\nu$LO
$^1P_1$	-6.40	N$^{1.0-1.7}$LO
$^1F_3$	-27.9	N$^{1.7-3.0}$LO
$^1H_5$	-64.6	N$^{2.1-3.8}$LO
$^1J_7$	-116.4	N$^{2.4-4.3}$LO
$^1L_9$	-183.3	N$^{2.7-4.7}$LO
$^1N_{11}$	-265.4	N$^{2.9-5.1}$LO
$^1D_2$	45.8	N$^{2.0-3.5}$LO
$^1G_4$	133.1	N$^{2.5-4.5}$LO
$^1I_6$	265.9	N$^{2.9-5.1}$LO
$^1K_8$	444.0	N$^{3.1-5.5}$LO
$^1M_{10}$	667.4	N$^{3.3-5.9}$LO

R_l to behave as

$$R_l(k) \sim \frac{Q}{b_l \Lambda_{NN}}\,. \tag{47}$$

However there is a catch: the ratio R_l can actually be computed analytically for the OPE potential at low energies and it turns out to be much smaller than expected from the previous argument. For showing this we write the perturbative expansion of the reduced wave functions in terms of a Green's function

$$u_k^{[n+1]}(r) = M_N \int_0^\infty dr' G_l(r,r')V(r')\,u^{[n]}_l(r') \tag{48}$$

where the Green's function is defined by

$$
\begin{aligned}
k\,G_l(r,r') &= \hat{j}_l(kr)\,\hat{y}_l(kr')\theta(r-r') \\
&+ \hat{j}_l(kr')\,\hat{y}_l(kr)\theta(r'-r)\,,
\end{aligned}
\tag{49}
$$

with $\hat{j}_l$ and $\hat{y}_l$ the reduced spherical Bessel functions we already introduced in Section I. If we take into account that $u_k^{[0]}(r) = \hat{j}_l(kr)$ and expand the reduced Bessel functions in powers of kr, we can evaluate R_l yielding

$$
\begin{aligned}
R_l &= -\frac{\sigma\tau}{2l+1}\left(\frac{m_\pi}{2^{2l+1}\Lambda_{NN}}\right) \\
&\times \left[1 + \frac{8 + (l+1)(2l+1)(6l+5)}{2(2l-1)(2l+3)}\frac{k^2}{m_\pi^2}\right. \\
&+ \left.\frac{(l+1)(8+(l+1)(2l+1)(6l+5))}{(2l-1)(2l+3)}\frac{k^4}{m_\pi^4} + \dots\right] \\
&\sim \frac{Q}{2^{2l+1}\Lambda_{NN}}\,,
\end{aligned}
\tag{50}
$$

that is, much more suppressed than expected (owing to $|\Lambda_{NN}| < 2^{2l+1}|\Lambda_{NN}^*|$).

Does R_l mean that we have underestimated the peripheral demotion? Not necessarily: the ratio R_l, and in general lower-order perturbation theory, exaggerates the effect of the centrifugal barrier. The reasons for this mismatch are not completely clear. It might be related to the interplay of the regular ($\sim r^{l+1}$) and irregular ($1/r^l$) components of the wave function u_k at low energies ($k \ll m_\pi$) and intermediate distances ($m_\pi r \sim 1$), with the irregular piece giving a larger contribution and only appearing at higher order perturbation theory. Be it as it may, the bottomline is that we have to go to higher-order perturbation theory to probe the expansion parameter.

For exploring higher-order perturbation theory, we define the matrix element of the n-iterated OPE potential as

$$
\begin{aligned}
\langle V_l^{[n]}\rangle &= \langle k,l|\,\underbrace{VG_0\dots G_0 V}_{n\ \text{insertions of OPE}}\,|k,l\rangle \\
&= \int_{r_c}^\infty dr\,u_k^{[0]}(r)\,V(r)\,u_k^{[n-1]}(r)\,.
\end{aligned}
\tag{51}
$$

We can now define a generalized $R_l^{[n]}$:

$$R_l^{[n]} = \frac{\langle V_l^{[n]}\rangle}{\langle V_l^{[n-1]}\rangle}\,. \tag{52}$$

We expect the scaling

$$R_l^{[n]} \sim \frac{Q}{b_l \Lambda_{NN}}\,, \tag{53}$$

for large n. This expectation is robust and well-grounded as a consequence of the potential only having a specific fraction of the strength necessary to bind.

We present the calculations of the ratios of the $(n+1)$-th versus the n-th order of perturbation theory in Fig. 5. For convenience we have chosen the inverse of the ratio $R_l^{[n]}$. The reason is the inverse of the expansion parameter is a more natural indication of how well the perturbative expansion converges: the bigger $|1/R_l^{[n]}|$ the more convergent. As a matter of fact, for $k = 0$ we have

$$\frac{1}{R_l^{[n]}} \sim \frac{\Lambda_{NN}}{\Lambda_{NN}^*(l)}\,, \tag{54}$$

which when evaluated yield the values of the second column of Table I, i.e. $1/R_1^{[n]} \sim -6.40$, $1/R_2^{[n]} \sim 45.8$, $1/R_3^{[n]} \sim -27.9$ and so on. As can be seen in Fig. 5, $1/R_l^{[n]}$ does converge to the predicted value at momenta below the pion mass ($k < m_\pi$), where we have extended the previous calculations to seventh order in perturbation theory to be able to appreciate the convergence pattern. For momenta of the order of the pion mass the expansion parameter is not so good as for $k < m_\pi$ but compatible with the EFT expectations, where we have to take into account that the $Q/b_l\Lambda_{NN}$ expansion contains powers of both $m_\pi/b_l\Lambda_{NN}$ and $k/b_l\Lambda_{NN}$. At larger momenta $k > m_\pi$ – and even as we approach $k \sim M \sim 0.5\,\mathrm{GeV}$, the expansion still works rather well. This might be puzzling from the EFT perspective but has a natural explanation in terms of the form of

the central OPE potential. For large momenta the exponential decay of central OPE is irrelevant and we are left with a Coulomb-like potential ($V \sim 1/r$). Coulomb is perturbative at momenta above the inverse Bohr radius. In the case of central OPE the equivalent is $k_B \sim m_\pi^2/\Lambda_{NN}$, which is actually smaller than m_π. Yet this is a particular feature of central OPE that is not expected to happen for other contributions of the EFT nuclear potential.

In the calculations of Fig. 5 we have reached seventh-order perturbation theory for checking the expansion parameter. Yet this is numerically demanding – more so for low momenta and high partial waves – and we limit the analysis to $k_{cm} \geq 20\,\mathrm{MeV}$ for $l \leq 4$, to $k_{cm} \geq 60\,\mathrm{MeV}$ for $l = 5$ and to $k_{cm} \geq 140\,\mathrm{MeV}$ for $l = 6$. Though this is a limitation, the calculations are still very meaningful. For $l \leq 4$ the ratios do converge to the expansion parameter predicted in Table I with an unrelated method, providing a cross-check of the calculation. For $l = 5, 6$ the situation is less clear – we are limited to higher momenta and apparently seventh order perturbation theory is not enough to stabilize the ratios – but nonetheless the ratios still seem to converge to the predicted value. One interesting feature that is evident from the calculations of Fig. 5 is that the low orders of perturbation theory converges actually faster than the high orders. The practical implication of this phenomenon is that results at tree level are more accurate than what is to be expected from the expansion parameter of the series. This might in turn point out towards choosing the higher-order estimates for the demotion.

D. Beyond One-Pion Exchange

At this point a question arises: how do we extend these ideas to TPE? NDA predicts leading TPE to be less important than OPE by Q^2. That is, if OPE is LO leading TPE is N^{2}LO [6]. But we have seen here that OPE is probably demoted by a larger factor for D and higher waves. It is natural to expect that a similar demotion will happen for TPE. The size of this effect will naturally depend on the partial wave and it might very well happen that in exceptional cases TPE might enter before OPE. This might explain why there are a few peripheral waves for which TPE is required to explain the phase shifts deduced from experimental data, such as $^1D_2$, $^3P_2$ or $^3D_3$ [14].

For OPE we have developed a power counting argument for the peripheral demotion. But this idea depends on an especial feature of the OPE potential in singlet waves: this is a regular potential that does not require regularization. For TPE the situation is different, it is a badly divergent potential at short distances and requires regularization. Exactly the same happens with OPE in the triplets. As a consequence

the argument we have developed here cannot be applied either for TPE or for OPE in the triplet channels. There are strategies to cope with this, though they will require serious scrutiny to check whether they work. The most obvious one is to renormalize these partial waves: if we add a contact-range interaction we might be able to apply the previous argument. The drawback of this idea is that it mixes short- and long-range physics. The factor by which we have to rescale TPE for having a bound state at threshold depends on the scattering volume of the channel (before rescaling TPE), which fixes the contact-range coupling. The meaning of this is that the rescaling factor is contaminated by the physical scattering length, which is undesirable. It might happen though that the effect is negligible, as happened with the bound-state versus resonance condition in the previous discussion.

If we strive for a solution that is manifestly independent of the existence of short-range physics two possibilities come to mind. The first is Birse's approach to tensor OPE [23], which adapts a series of techniques from molecular physics to study when tensor OPE is perturbative. A limitation of this approach is that it is formulated in the chiral limit, where the range of the OPE potential becomes infinite and it is thus similar to the potentials found in atomic physics. A possibility is to study the cut-off at which deeply bound states happen with TPE. Deeply bound states are non-physical bound states that occur when we have attractive singular interactions, such as TPE. They are of no consequence because their binding momenta are beyond the range of applicability of EFT. In addition techniques have been developed to get rid of them [12]. The point is that the more peripheral the wave the harder the cut-off for which deeply bound states appear. This might in turn gives us information on the quantitative nature of the partial wave suppression.

III. CONCLUSION

In this work we have analyzed from the EFT perspective – and in the particular case of the spin-singlet channels – the common wisdom observation that pion exchanges are perturbative in peripheral waves. For this we have studied the convergence of the perturbative expansion of the phase shifts numerically up to fourth order in perturbation theory. This calculation – which has been done for the first time up to such an order in this work – indicates that pions are perturbative in the singlets. In fact, the multiple iterations of one-pion-exchange potential turn out to be much more suppressed than expected even in a power counting (Kaplan-Savage-Wise) in which one-pion-exchange potential is treated as subleading.

To understand this pattern we have made use of a power counting argument to determine the actual demotion of one-pion-exchange potential with respect to leading order. The idea is to rescale the strength of the one-pion-exchange potential up to the point in which a bound state is generated at threshold. This critical strength can be translated into a critical Λ_{NN}^* – which does not correspond with the physical Λ_{NN} but will in general be softer – for which the perturbative expansion does not converge. The ratio $\Lambda_{NN}^*/\Lambda_{NN}$ corresponds

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[6] Here we follow consistency instead of convention: traditionally leading TPE is labeled as NLO because in Weinberg's counting there is no contribution to the nuclear potential that is Q-times smaller than OPE. If we are accounting for peripheral suppression, which in general are not integer powers of Q, this convention is no longer useful.
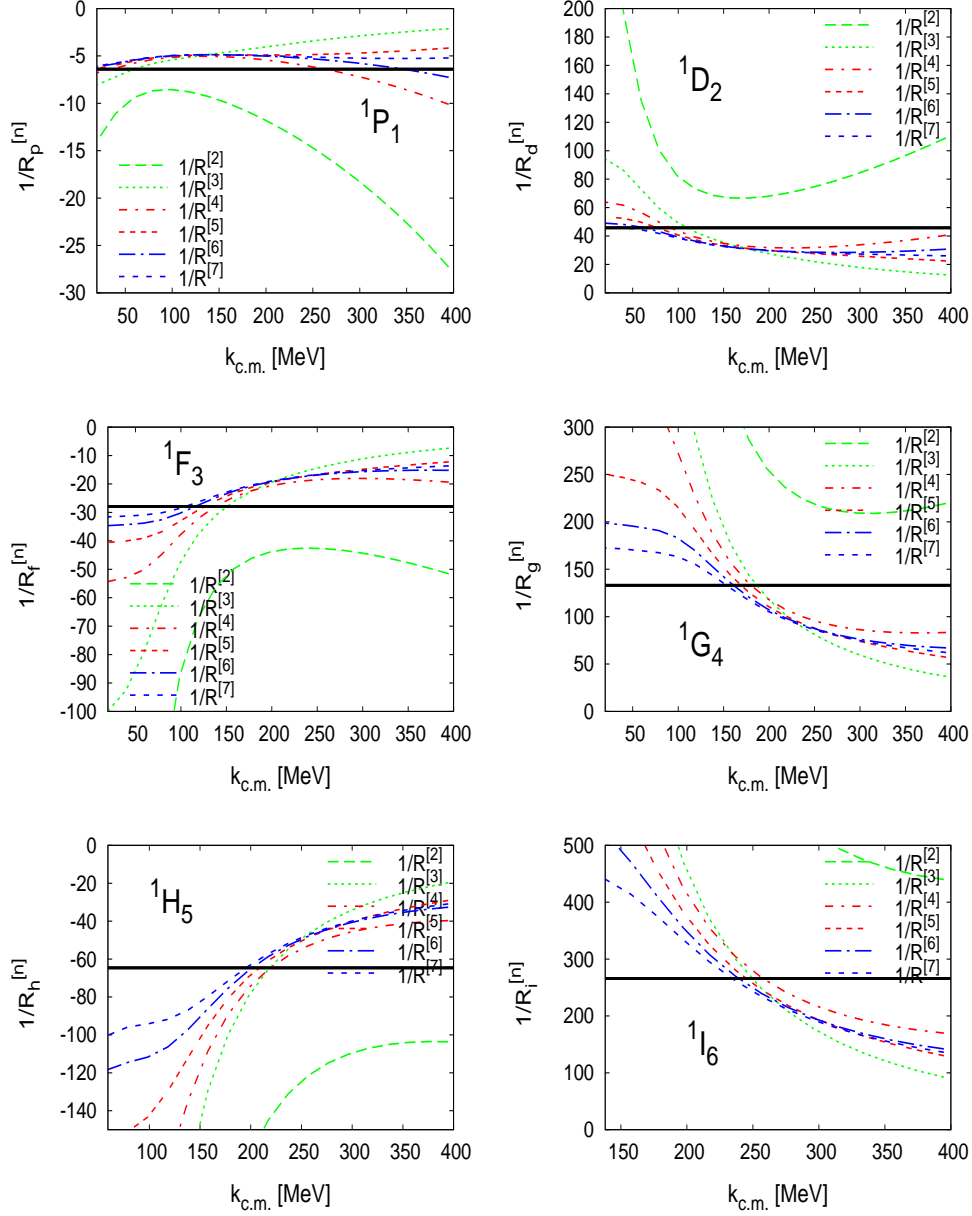
FIG. 5: Inverse ratios between the *n*-th order over the $(n-1)$-th order perturbation theory for OPE and the singlet partial waves with $l \leq 6$ (See the main text for a detailed explanation of the ratios.) The theoretical value to which the ratios should converge is shown in the black solid line.

to the expansion parameter of perturbative one-pion-exchange potential, which turns out to be quickly convergent. We have checked this prediction against concrete calculations, confirming the EFT argument. Actually even the $^1P_1$ partial wave is demoted beyond next-to-leading order and more peripheral waves can be demoted up to the point of being less important than subleading two-pion-exchange potential in naive dimensional analysis (it is yet to be seen how demoted will be leading and subleading two-pion-exchange potential in peripheral waves).

The importance of the peripheral demotion is not merely

academic. It has applications in few-body calculations, where the demotion can be used to improve and optimize the calculations. The way in which this is achieved is by only including the necessary number of iterations in the peripheral waves and by ignoring partial waves where tree-level one-pion-exchange potential is already higher order than the order of the calculation. Actually this is analogous to the common practice of ignoring the partial waves with angular momentum bigger than a certain critical value ($l \geq 5$ in most applications). The difference is that here we systematize this practice in a way that is compatible with the EFT expansion, providing guidelines for

future few-body calculations in nuclear EFT.

Yet this manuscript deals only with the one-pion-exchange potential in the peripheral singlets. For the peripheral demotion to be useful in few-body calculations, we have to extend the present analysis to peripheral triplets and also to the two-pion-exchange potential. This analysis is underway, though the tools that will be required are different that the ones we have used here. The fundamental difference of one-pion-exchange potential in peripheral triplets and of two-pion-exchange potential is that these potentials are singular, diverging faster than $1/r^2$ at distances below the pion Compton wavelength. Thus the calculation of their peripheral demotion will require the development of more sophisticated arguments that take into account the existence of a finite cut-off and how it relates to the other scales in the problem.

Appendix A: Integral Representation for the Phase Shifts

In this short appendix we derive Eq. (7), the integral representation of the phase shifts that we later use in perturbative calculations. The starting point is to consider the reduced Schrödinger equations for the free problem and for two particles interacting via a central potential V:

$$-v_l'' + \frac{l(l+1)}{r^2} v_l(r) = k^2 v_l, \qquad (A1)$$

$$-u_l'' + \left[2\mu V(r) + \frac{l(l+1)}{r^2} \right] u_l(r) = k^2 u_l. \qquad (A2)$$

Now we construct a Wronskian identity involving the free and non-free wave functions. For this, we first multiply the free equation above by the non-free wave function u_l and vice versa. Then, we calculate the difference between the two expressions we obtain. We end up with

$$(v_l u_l'' - v_l'' u_l) - 2\mu V(r) u_l v_l = 0. \qquad (A3)$$

The terms in the brackets are the derivative of $(v_l u_l' - v_l' u_l)$. We can thus integrate the expression above to reach:

$$(v_l u_l' - v_l' u_l)\Big|_{r_c}^{R} = 2\mu \int_{r_c}^{R} V(r) u_l(r;k) v_l(r;l) \, dr. \qquad (A4)$$

Now we simply take $v_l = \hat{j}_l(kr)$ and u_l as in Eq. (6). After removing the infrared and ultraviolet cut-offs ($R \to \infty$, $r_c \to 0$), we obtain

$$k \tan \delta_l = -2\mu \int_0^{\infty} V(r) u_k(r) v_k(r) \, dr, \qquad (A5)$$

that is, Eq. (7).

Appendix B: Peripheral Demotion and Resonances

In this appendix we discuss the different choices for the definition of Λ_{NN}^* and which effect do they have on the peripheral demotion of the OPE potential. As we will see the effect of changing the threshold bound state condition by a different condition is going to be rather small. This is so because the choice of rescaling up to the point of having a bound state at threshold, instead of at a finite binding energy, is not arbitrary. The EFT description of a bound state with the OPE potential will involve the scales m_π, Λ_{NN}^* and the wave number $\gamma = \sqrt{M_N B}$, with B the binding energy. By requiring $\gamma = 0$ we eliminate one of the variables in the problem: the only scales left are m_π and Λ_{NN}^*, which combine by means of a numerical factor to give an expansion parameter of one if there is a threshold bound state. That is, $\gamma = 0$ is the easiest and most convenient choice to isolate the scale Λ_{NN}^*.

On the contrary if $\gamma \neq 0$ the EFT expansion involves two ratios: m_π/Λ_{NN} and γ/Λ_{NN}. The failure of the expansion at a particular Λ_{NN} for a pole at $\gamma \neq 0$ does not necessarily mean that Λ_{NN} is the Λ_{NN}^* we are looking for. There might be a mismatch – probably a small one – between the two owing to the different numerical factors in each of the subexpansions. If we were able to take these details into account, it might very well happen that we end up with the original scale generating the threshold bound state.

Yet we will ignore these complications here. The point is to check the robustness of the estimations we have made. For that we will assume a different choice for the location of a pole in the amplitude and repeat the analysis of Section II B. Instead of $k_p = i\gamma_B = 0$ for a bound state at threshold, we will consider the case of a virtual state or resonance: $k_p = -i\gamma_V$ or $k_p = k_R$, where γ_V is real and k_R is complex. This will translate into a new value of the critical Λ_{NN}^* that depends on the position of the pole: $\Lambda_{NN}^* = \Lambda_{NN}^*(l, k_{\text{pole}})$ with $|k_{\text{pole}}| \sim Q$. Proceeding as in the $k_p = 0$ case, we define the peripheral demotion as

$$\frac{\Lambda_{NN}^*(l, k_{\text{pole}})}{\Lambda_{NN}} = \left(\frac{Q}{M} \right)^{\nu'_{\text{OPE}}(l, k_{\text{pole}})}, \qquad (B1)$$

where we have added the $'$ to distinguish the new estimations from the previous ones.

Virtual states and resonances are poles in the scattering amplitude in the second Riemann sheet of the complex energy plane. They are easy to locate in the case of contact-range potentials. For a finite-range potential, finding the virtual and resonant states is technically more challenging, due to the difficulty of choosing the second Riemann sheet in a numerical

TABLE II: Changes in the peripheral demotion of OPE if we change the definition of the critical $\Lambda_{NN}^*(l, k_p)$. While the original definition involved the critical value of Λ_{NN} that generates a bound state at threshold, here we use an alternative definition of Λ_{NN} that generates a phase shift of $\pi/2$ at $k = m_\pi$. This definition is in turn related to the existence of a nearby resonance at $k_R \sim m_\pi - i\gamma$ (where the approximation will be better the smaller γ). From the previous and new values of Λ_{NN}^* we can estimate the change in the peripheral demotion with respect to the shallow bound state condition, shown in the $\Delta\nu = \nu' - \nu$ column, where we have assumed Q/M to change between $1/7$ and $1/3$. As can be appreciated the changes are tiny and not worth considering. Owing to the resonances becoming narrower for higher partial waves we could only compute the shifts $\Delta\nu$ for $l \leq 6$. For higher partial waves we expect the shifts to be even tinier.

$^S L_J$	$\Lambda_{NN}/\Lambda_{NN}^*$	$\Lambda_{NN}/\Lambda_{NN}^*(m_\pi)$	$\Delta\nu = \nu' - \nu$
$^1P_1$	-6.40	-4.86	$-(0.14 - 0.25)$
$^1F_3$	-27.9	-24.9	$-(0.06 - 0.10)$
$^1H_5$	-64.6	-59.1	$-(0.05 - 0.08)$
$^1D_2$	45.8	33.31	$-(0.16 - 0.29)$
$^1G_4$	133.1	119.7	$-(0.05 - 0.10)$
$^1I_6$	265.9	250.2	$-(0.03 - 0.05)$

calculation. As we are interested in peripheral waves, the most natural outcome when we reduce the strength of the potential is that a bound state eventually becomes a resonance. This offers a simplification: resonances have a very clear signature in physical scattering. There is a jump of 180 degrees in the phase shifts near the location of the resonance, where the jump will be very steep if the resonance is narrow. In addition at a energy similar to the real part of the resonance pole the scattering amplitude saturates the unitarity bound and the phase shift reaches 90 degrees. Therefore the criterion we are going to use for $\Lambda_{NN}^*(l, k_R)$ will be

$$\cot \delta(k = m_\pi) = 0\,, \qquad (B2)$$

which in general will imply $|k_R| > m_\pi$, but only by a small amount if the resonance is narrow.

The change in $\nu_{\text{OPE}}(l)$ for the resonance condition are show in Table II. In general the new condition only entails a tiny change in $\nu_{\text{OPE}}(l)$ in the $-(0.05 - 0.2)$ range. This change is an order of magnitude smaller than the changes in $\nu_{\text{OPE}}(l)$ related to the uncertainty in the EFT expansion parameter Q/M and hence can be safely ignored.

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