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Spontaneous topological transitions of electromagnetic fields in spatially inhomogeneous CP-odd domains

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Metastable CP-odd domains of the hot QCD matter are coupled to QED via the chiral anomaly. The topology of electromagnetic field in these domains is characterized by magnetic helicity. It is argued, using the Maxwell-Chern-Simons model, that spatial inhomogeneity of the domains induces spontaneous transitions of electromagnetic field between the opposite magnetic helicity states.

I. INTRODUCTION

A possible existence of the metastable CP-odd domains in hot QCD matter has been actively discussed, especially in the context of the relativistic heavy-ion collisions [1]. These domains are described by a scalar field θ whose interaction with the electromagnetic field $F^{\mu\nu}$ is given by the anomalous term in the QED Lagrangian [2–6]

$$\mathcal{L}_A = -\frac{c_A}{4}\theta\tilde{F}_{\mu\nu}F^{\mu\nu}, \quad (1)$$

where $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}F^{\lambda\rho}$ is the dual field tensor and

$$c_A = \frac{N_c e^2}{2\pi} \sum_f q_f^2 \quad (2)$$

is a constant. Together with the Maxwell's term $-(1/4)F_{\mu\nu}^2$, Eqs. (1),(2) constitute the Maxwell-Chern-Simons (MCS) model [1, 4–6], which is a useful tool for systematic study of the CP-odd effects in QED. The anomalous term (1) induces a number of remarkable effects, some of which may have already been experimentally observed, see reviews [7, 8].

As has been recently pointed out in [9, 11] the electromagnetic field inside the CP-odd domains is described by Chandrasekhar-Kendall (CK) states [12, 13] which are spherical waves with definite magnetic helicity. Magnetic helicity determines the topology of the CK state and is a topological invariant proportional to the number of twisted and linked flux tubes.

The θ -field is usually modeled by a spatially homogenous time-dependent function $\theta(t)$, which is suitable to study the temporal evolution of topological configurations of electromagnetic field in matter with chiral asymmetry [9, 10, 14, 15]. Indeed, in the presence of the chiral imbalance, magnetic helicity is not conserved, hence an initial CK state, as well as θ , undergo non-trivial topological evolution [9, 10, 14, 15] that manifests itself in transitions of electromagnetic field between the states with different magnetic helicity. A significant change of the θ -field typically occurs over the time τ which is of the order of the inverse electrical conductivity of the QCD matter. The lattice calculations indicate that near the critical point $\tau \sim 36$ fm. [16–18]. At significantly shorter time intervals the time-dependence of θ can be neglected. Thus, θ can be approximated by a constant for processes that occur at distances much shorter than the domain size R , [19, 20].

The main subject of this paper is interaction of electromagnetic fields with a CP-odd domain at time intervals shorter than τ and distances of order R . Since the CK states are spatially extended configurations, they are sensitive to the spatial variations of θ , in particular to the finite size of the domain. Interaction of the electromagnetic field with the spatial gradient of θ induces transitions between different CK states. Calculation of the corresponding transition rates is the main subject of this paper. The main result is given by (33). It shows that the only possible transitions are $\{l \rightarrow l \pm 1, h \rightarrow h\}$ and $\{l \rightarrow l, h \rightarrow -h\}$ with $m \rightarrow m$ in both cases. In particular, the transition rate between the states of opposite magnetic helicity is described by (36). It indicates a possibility of spontaneous generation of magnetic helicity independently of the presence of the chiral imbalance.*

* Another mechanism that does not require the initial chirality imbalance for the magnetic helicity generation was recently discussed in [21].

II. TRANSITIONS BETWEEN THE CK STATES

A. Maxwell-Chern-Simons model

Integrating by parts and dropping the full derivative, the anomalous term (1) can be cast in a more convenient form

$$\mathcal{L}_A = \frac{c_A}{2} \epsilon_{\mu\nu\lambda\rho} \partial^\mu \theta A^\nu \partial^\lambda A^\rho. \quad (3)$$

Under the gauge $A^0 = 0$ (3) can be written as

$$\mathcal{L}_A = \frac{c_A}{2} \left[\dot{\theta} \mathbf{A} \cdot \mathbf{B} - \nabla \theta \cdot (\mathbf{A} \times \mathbf{E}) \right]. \quad (4)$$

As the precise spatial distribution of θ is not known, it makes sense to reduce the complexity of the problem, while keeping its essential features, by assuming that θ is time-independent and its gradient given by

$$\nabla \theta = \mathbf{P} f(r), \quad (5)$$

where “the chiral polarization” \mathbf{P} is a constant, and f is a smooth dimensionless function of the radial coordinate r obeying the boundary conditions $f(0) = 1$ and $f(\infty) = 0$.

B. Chandrasekhar-Kendall states

The CK photons are elementary excitations of the electromagnetic field that have energy ω orbital angular momentum l , its projection m and magnetic helicity h . It satisfies the dispersion relation $\omega = k$, where k is the magnitude of the momentum that does not have a definite direction in the CK state. In the radiation gauge $A_0 = 0$, $\nabla \cdot \mathbf{A} = 0$ the CK photons are described in the coordinate representation by the wave functions

$$\mathbf{A}_{klm}^h(\mathbf{r}, t) = \frac{1}{\sqrt{2kR}} h k \mathbf{W}_{klm}^h(\mathbf{r}) e^{-ikt}, \quad (6)$$

where $\mathbf{W}_{klm}^h(\mathbf{r})$ are the eigenfunctions of the curl operator

$$\nabla \times \mathbf{W}_{klm}^h(\mathbf{r}) = h k \mathbf{W}_{klm}^h(\mathbf{r}) \quad (7)$$

obeying the orthogonality conditions

$$\int \mathbf{W}_{k'l'm'}^{h'*}(\mathbf{r}) \cdot \mathbf{W}_{klm}^h(\mathbf{r}) d^3r = \frac{\pi}{k^2} \delta(k - k') \delta_{ll'} \delta_{mm'} \delta_{hh'}. \quad (8)$$

For a typical photon energy, the domain radius R is so large, that $kR \gg 1$. This allows one to treat the CK photon energy spectrum as approximately continuous. The electromagnetic potential can be written as

$$\mathbf{A}(\mathbf{r}, t) = \sum_{lmh} \int_0^\infty \frac{R dk}{\pi \sqrt{2kR}} (h k a_{klm}^h \mathbf{W}_{klm}^h(\mathbf{r}) e^{-ikt} + h.c.) , \quad (9)$$

where a_{klm}^h is the operator obeying the usual bosonic commutation relations

$$\left[a_{k'l'm'}^{h'\dagger}, (a_{klm}^h)^\dagger \right] = \frac{\pi}{R} \delta(k' - k) \delta_{ll'} \delta_{mm'} \delta_{hh'}, \quad (10)$$

etc. It is convenient to choose the quantization axis z as the direction of vector \mathbf{P} and define the spherical coordinates θ and ϕ with respect to it. The functions $\mathbf{W}_{klm}^h(\mathbf{r})$ can be expressed in terms of the spherical harmonics and the orbital angular momentum operator $\mathbf{L} = -i(\mathbf{r} \times \nabla)$ as [22]

$$\mathbf{W}_{klm}^h(\mathbf{r}) = \mathbf{T}_{klm}^h(\mathbf{r}) - i h \mathbf{P}_{klm}^h(\mathbf{r}), \quad (11)$$

where

$$\mathbf{T}_{klm}^h(\mathbf{r}) = \frac{j_l(kr)}{\sqrt{l(l+1)}} \mathbf{L}[Y_{lm}(\theta, \phi)], \quad \mathbf{P}_{klm}^h(\mathbf{r}) = \frac{i}{k} \nabla \times \mathbf{T}_{klm}^h(\mathbf{r}), \quad l \geq 1. \quad (12)$$

Although functions \mathbf{T}_{klm} and \mathbf{P}_{klm} also form a complete set on a unit sphere (at fixed k), they do not have definite magnetic helicity.

C. Transition rate

The scattering matrix element describing the scattering of the CK photon off the θ -field is

$$\langle k'l'm'h'|S_A|klmh\rangle = \frac{c_A}{2} \int d^4x \langle k'l'm'h'|\dot{\theta} \mathbf{A} \cdot (\nabla \times \mathbf{A}) + \nabla \theta \cdot (\mathbf{A} \times \dot{\mathbf{A}})|klmh\rangle, \quad (13)$$

where $|klmh\rangle \neq |k'l'm'h'\rangle$. Substituting (9) into (13) and using (7)-(8) one derives

$$\begin{aligned} \langle k'l'm'h'|S_A|klmh\rangle = \frac{c_A}{2} \frac{1}{2R} \int d^4x e^{i(k'-k)t} \frac{1}{\sqrt{kk'}} \Big\{ \dot{\theta} (hkk'^2 + h'k'k^2) \mathbf{W} \cdot \mathbf{W}'^* \\ + \nabla \theta \cdot (\mathbf{W} \times \mathbf{W}'^*) (ikk'^2 + ik'k^2) hh' \Big\}, \end{aligned} \quad (14)$$

where a shorthand notation is used: $\mathbf{W}_{klm}^h = \mathbf{W}$, $\mathbf{W}_{k'l'm'}^{h'} = \mathbf{W}'$. In the case of the time-independent domain of radius R described by (5), (14) simplifies

$$\langle k'l'm'h'|S_A|klmh\rangle = \pi c_A \delta(k' - k) \frac{k^2}{2R} \int_0^\infty dr r^2 f(r) \int d\Omega 2i \mathbf{P} \cdot (\mathbf{W} \times \mathbf{W}'^*) hh' \quad (15)$$

$$= \pi c_A \delta(k' - k) \frac{k^2}{2R} \int_0^\infty dr r^2 f(r) 2i hh' \mathbf{P} \cdot \mathbf{C}, \quad (16)$$

where

$$\mathbf{C} = \int \mathbf{W} \times \mathbf{W}'^* d\Omega = \int (\mathbf{T} \times \mathbf{T}'^* + hh' \mathbf{P} \times \mathbf{P}'^* - ih \mathbf{P} \times \mathbf{T}'^* + ih' \mathbf{T} \times \mathbf{P}'^*) d\Omega. \quad (17)$$

The eigenfunctions \mathbf{W}' in (16) and (17) are evaluated at $k' = k$. The first integral in (17) is proportional to

$$\begin{aligned} \int \epsilon_{ijk} (L_j Y_{lm}) (L_k^* Y_{l'm'}^*) d\Omega = \int \epsilon_{ijk} Y_{l'm'}^* L_k L_j Y_{lm} d\Omega = \frac{1}{2} \int \epsilon_{ijk} Y_{l'm'}^* [L_k, L_j] Y_{lm} d\Omega \\ = -i \int Y_{l'm'}^* L_i Y_{lm} d\Omega \equiv -i \langle l'm'|L_i|lm\rangle, \end{aligned} \quad (18)$$

where the commutator $[L_k, L_j] = i\epsilon_{kjs} L_s$ was used. The explicit expression for the matrix element of the angular momentum is

$$\begin{aligned} \langle l'm'|L_i|lm\rangle = \delta_{ll'} (m\delta_{mm'} e_z + \sqrt{l(l+1)-m(m-1)} \delta_{m',m-1} e_+ \\ + \sqrt{l(l+1)-m(m+1)} \delta_{m',m+1} e_-), \end{aligned} \quad (19)$$

where $\mathbf{e}_\pm = \frac{1}{2}(\mathbf{e}_x \pm i\mathbf{e}_y)$. Eq. (18) implies that

$$\int \mathbf{T} \times \mathbf{T}'^* d\Omega = -i \frac{j_l^2(kr)}{l(l+1)} \langle l'm'|L|lm\rangle. \quad (20)$$

The second integral in (17) reads

$$\int \epsilon_{ijk} P_j P_k'^* d\Omega = \frac{1}{k^2} \int \epsilon_{ijk} (\nabla \times \mathbf{T})_j (\nabla \times \mathbf{T}'^*)_k d\Omega = \frac{1}{k^2} \int \epsilon_{kjs} T_s'^* \nabla_j \nabla_i T_k d\Omega, \quad (21)$$

$$= -\frac{j_l(kr)j_{l'}(kr)}{\sqrt{l(l+1)}\sqrt{l'(l'+1)}} \frac{1}{k^2} \int \epsilon_{kjs} (L_s^* Y_{l'm'}^*) (p_j p_i L_k Y_{lm}) d\Omega, \quad (22)$$

where $\mathbf{p} = -i\nabla$. Integrating by parts and using $\mathbf{p} \cdot \mathbf{L} = 0$ and $[L_i, p_j] = i\epsilon_{ijk} p_k$, the integral in (22) can be rendered as

$$-\int \epsilon_{kjs} (L_s^* Y_{l'm'}^*) (p_j p_i L_k Y_{lm}) d\Omega = -i \int Y_{l'm'}^* p^2 L_i Y_{lm} d\Omega = -\frac{il(l+1)}{r^2} \langle l'm'|L_i|lm\rangle, \quad (23)$$

where I used $p^2 Y_{lm} = l(l+1)Y_{lm}/r^2$. Thus,

$$\int \mathbf{P} \times \mathbf{P}'^* d\Omega = -\frac{i}{k^2 r^2} [j_l(kr)]^2 \langle l'm'|L|lm\rangle. \quad (24)$$

Actually, integration by parts in (22) yields another term proportional to

$$\int_0^\infty \nabla [r^2 f(r) j_l(kr) j_{l'}(kr)] dr.$$

However, it vanishes due to the boundary conditions imposed on $f(r)$, see (5).

Turning to the third and fourth terms in (17) one obtains after integrating by parts and using the gauge condition $\nabla \cdot \mathbf{T} = 0$

$$\begin{aligned} \int (-ih\mathbf{P} \times \mathbf{T}'^* + ih'\mathbf{T} \times \mathbf{P}'^*) d\Omega &= -\frac{h+h'}{k} \int T_i'^* \nabla T_i d\Omega \\ &= -i \frac{j_l(kr) j_{l'}(kr)}{\sqrt{l(l+1)} \sqrt{l'(l'+1)}} \frac{h+h'}{k} \langle l'm' | L_i \mathbf{p} L_i | lm \rangle. \end{aligned} \quad (25)$$

Collecting (17),(20),(24) and (25) yields

$$\begin{aligned} \mathbf{C} &= -i \left[j_l^2(kr) \langle l'm' | \mathbf{L} | lm \rangle \left(\frac{1}{l(l+1)} + \frac{hh'}{k^2 r^2} \right) \right. \\ &\quad \left. + \frac{j_l(kr) j_{l'}(kr)}{\sqrt{l(l+1)} \sqrt{l'(l'+1)}} \frac{h+h'}{k} \langle l'm' | L_i \mathbf{p} L_i | lm \rangle \right]. \end{aligned} \quad (26)$$

The symmetry properties of the matrix elements imply that the first term in (26) describes transitions between the states with the same angular momentum $l \rightarrow l$, while the second one between the states with angular momentum different by one unit $l \rightarrow l \pm 1$.

Plugging (26) into (16) and bearing in mind that $h^2 = h'^2 = 1$ one derives

$$\begin{aligned} \langle k'l'm'h' | S_A | klmh \rangle &= \pi c_A \delta(k' - k) \frac{k^2}{R} \int_0^\infty dr r^2 f(r) \\ &\quad \times \left\{ j_l^2(kr) \left[\mathbf{P} \cdot \langle l'm' | \mathbf{L} | lm \rangle \left(\frac{hh'}{l(l+1)} + \frac{1}{k^2 r^2} \right) \right] \delta_{ll'} \right. \\ &\quad \left. + \frac{h+h'}{k} \mathbf{P} \cdot \langle l'm' | L_i \mathbf{p} L_i | lm \rangle \frac{j_l(kr) j_{l'}(kr)}{\sqrt{l(l+1)} \sqrt{l'(l'+1)}} \right\}. \end{aligned} \quad (27)$$

The explicit expression for the matrix element $\langle l'm' | L_i p_z L_i | lm \rangle$ is given by (A8) in the Appendix (recall that $\mathbf{P} = P \mathbf{e}_z$). It is helpful to define the auxiliary functions

$$\mathcal{R}_{ll'}(k) = \int_0^\infty dr r^2 f(r) j_l(kr) j_{l'}(kr), \quad (28)$$

$$\mathcal{R}'_{ll'}(k) = \frac{l(l+1)}{k^2} \int_0^\infty dr f(r) j_l(kr) j_{l'}(kr), \quad (29)$$

$$\mathcal{R}''_{ll'}(k) = \frac{1}{k} \int_0^\infty dr r f(r) j_l(kr) j_{l'}(kr). \quad (30)$$

Using all these equation and (19) in (27) one gets

$$\begin{aligned} \langle k'l'm'h' | S_A | klmh \rangle &= \pi c_A \delta(k' - k) \frac{k^2}{R} \delta_{m'm} \left\{ \frac{Pm}{l(l+1)} [hh' \mathcal{R}_{ll'}(k) + \mathcal{R}'_{ll'}(k)] \delta_{ll'} \right. \\ &\quad \left. + i(h+h')P [a_{lm} \delta_{l',l-1} + b_{lm} \delta_{l',l+1}] \mathcal{R}''_{ll'}(k) \right\}, \end{aligned} \quad (31)$$

where the coefficients a_{lm} and b_{lm} are given by (A9) and (A10). It is convenient to separate the part of the matrix element that describes the magnetic helicity flip $h' = -h$. To this end one can multiply the first term in the curly brackets by the identity $1 = \delta_{hh'} + (1 - \delta_{hh'})$. The term diagonal in all quantum numbers only contributes to the wave function renormalization and can be dropped at the leading order.

The transition rate from the initial CK state with quantum numbers l, m, h to the final CK state with quantum numbers l', m' and h' is given by

$$w(lmh \rightarrow l'm'h') = \frac{1}{t} |\langle k'l'm'h' | S_A | klmh \rangle|^2 \frac{Rdk'}{\pi} \quad (32)$$

$$= \frac{c_A^2 k^4}{2R} \delta_{m'm} \left\{ 4P^2 (\mathcal{R}_{l'l'}''(k))^2 \delta_{h'h} [a_{lm}^2 \delta_{l',l-1} + b_{lm}^2 \delta_{l',l+1}] \right. \\ \left. + \frac{P^2 m^2}{l^2(l+1)^2} [\mathcal{R}_{l'l'}'(k) - \mathcal{R}_{ll'}(k)]^2 (1 - \delta_{hh'}) \delta_{ll'} \right\}. \quad (33)$$

where the identity $(h + h')^2 = 4\delta_{hh'}$ was used and one of the delta functions is replaced by $t/2\pi$. In particular, the rate of spontaneous magnetic helicity-flip $h \rightarrow -h$ is

$$w_{\text{flip}}(lm \rightarrow l'm') = \frac{c_A^2 k^4}{2R} \frac{P^2 m^2}{l^2(l+1)^2} [\mathcal{R}_{ll}'(k) - \mathcal{R}_{ll}(k)]^2 \delta_{l'l} \delta_{m'm}. \quad (34)$$

It is not difficult to verify that since $kR \gg 1$, $\mathcal{R}_{ll} \gg \mathcal{R}_{ll}'$ with the main contribution to the integral over r arising from the distances $1/k < r < R$. Since at large kr the spherical Bessel function can be approximated as $j_l(kr) \approx (kr)^{-1} \sin(kr - \pi l/2)$, one finds

$$\mathcal{R}_{ll}'(k) - \mathcal{R}_{ll}(k) \approx -\frac{1}{2k^2} \int_0^\infty f dr. \quad (35)$$

Thus, the helicity-flip transition rate is given by

$$w_{\text{flip}} = \frac{1}{8R} \left(c_A \frac{Pm}{l(l+1)} \int_0^\infty f dr \right)^2. \quad (36)$$

It is proportional to the domain radius and is independent of the CK state energy.

D. Estimates

To estimate the transition rate for the quark-gluon plasma produced in relativistic heavy-ion collisions, one needs to know the value of P . Alternatively, one can solve (5) to obtain

$$\theta(r) = P \int_0^r f(r') dr' + \theta_0, \quad (37)$$

where $\theta(0) = \theta_0$ is the value of θ in the domain's center. From the requirement that θ vanishes as $r \rightarrow \infty$ it follows that

$$\int_0^\infty f(r) dr = -\frac{\theta_0}{P}. \quad (38)$$

Eq. (36) can now be written as

$$w_{\text{flip}} = \frac{1}{8R} \left(c_A \frac{m}{l(l+1)} \theta_0 \right)^2. \quad (39)$$

In [25] the magnitude of the charge separation effect in a typical heavy-ion collision with $l = 4$ is reproduced with $\theta_0 \simeq \pi$ for $N_f = 2$. The domain size can be roughly approximated by the sphaleron size $R = 0.4$ fm [24]. Substituting these estimates into (39), yields for $l = m$ the magnetic helicity flip rate

$$w_{\text{flip}} \sim 0.7 \cdot 10^{-4} \text{ fm}^{-1}. \quad (40)$$

III. SUMMARY

In quark-gluon plasma, the electromagnetic field is initially in a state which is a superposition of states with definite angular momentum l , its projection m and magnetic helicity h (CK states). The main result of this paper are Eqs. (32)-(36) that give the transition rate between two CK states due to the spatial inhomogeneity of the CP-odd domain. Eq. (36) indicates that the spatially inhomogeneous CP-odd domains induce spontaneous flip of magnetic helicity. Thus, initially chirally symmetric electromagnetic field can spontaneously acquire magnetic helicity by means of interaction with hot QCD matter. Another possible way of magnetic helicity generation without any initial chirality imbalance was recently proposed in [21].

The magnetic helicity evolution studied in [9, 10, 14, 15] requires finite electrical conductivity of the quark-gluon plasma. In the ideally conducting medium (with infinite electrical conductivity) the magnetic helicity evolution of [9, 10, 14, 15] is not possible. This is because the spatially homogeneous anomalous current is proportional to magnetic field and thus does not mix the CK states. What is argued in this paper is that even in the ideally conducting medium, spatial inhomogeneities in the anomalous current cause magnetic helicity evolution. Clearly, in a more comprehensive approach to magnetic helicity evolution both effects should be taken into account.

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Appendix A: The matrix element $\langle l'm'|L_i p_z L_i|lm\rangle$

It is advantageous to employ the raising and lowering operators $L_{\pm} = L_x \pm iL_y$ that act on the angular momentum eigenstates $|lm\rangle$ as

$$L_{\pm}|lm\rangle = \sqrt{(l \mp m)(l \pm m + 1)}|l, m \pm 1\rangle. \quad (A1)$$

Using (A1) one can reduce the matrix element of $L_i p_z L_i$ to the matrix elements of momentum:

$$\langle l'm'|L_i p_z L_i|lm\rangle = \langle l'm'|L_z p_z L_z|lm\rangle + \frac{1}{2}\langle l'm'|L_+ p_z L_-|lm\rangle + \frac{1}{2}\langle l'm'|L_- p_z L_+|lm\rangle \quad (A2)$$

$$= mm'\langle l'm'|p_z|lm\rangle + \frac{1}{2}\sqrt{(l'+m')(l'-m'+1)}\sqrt{(l+m)(l-m+1)}\langle l', m'-1|p_z|l, m-1\rangle \\ + \frac{1}{2}\sqrt{(l'-m')(l'+m'+1)}\sqrt{(l-m)(l+m+1)}\langle l', m'+1|p_z|l, m+1\rangle. \quad (A3)$$

The matrix elements of the momentum operator can be written as

$$\langle l'm'|p_z|lm\rangle = -i\langle l'm'|\frac{\partial}{\partial z}|lm\rangle = -\frac{i}{r}\langle l'm'|\sin^2\theta\frac{\partial}{\partial\cos\theta}|lm\rangle \quad (A4)$$

$$= \frac{i}{r}(l+1)\sqrt{\frac{(l+m)(l-m)}{(2l+1)(2l-1)}}\delta_{mm'}\delta_{l',l-1} - \frac{i}{r}l\sqrt{\frac{(l-m+1)(l+m+1)}{(2l+1)(2l+3)}}\delta_{mm'}\delta_{l',l+1}, \quad (A5)$$

where the following recursive relation for the associate Legendre polynomials was used, see **8.731** in [26]

$$\sin^2\theta\frac{d}{d\cos\theta}P_l^m(\cos\theta) = \frac{1}{2l+1}[(l+1)(l+m)P_{l-1}^m(\cos\theta) - l(l-m+1)P_{l+1}^m(\cos\theta)]. \quad (A6)$$

Substituting (A5) into (A3) yields

$$\langle l'm'|L_i p_z L_i|lm\rangle = \frac{i}{r}\delta_{m'm}\left\{(l-1)(l+1)^2\sqrt{\frac{(l-m)(l+m)}{(2l+1)(2l-1)}}\delta_{l',l-1} \right. \\ \left. + l^2(l+2)\sqrt{\frac{(l-m+1)(l+m+1)}{(2l+3)(2l+1)}}\delta_{l',l+1}\right\}. \quad (A7)$$

It is convenient to introduce a shorthand notation

$$\frac{\langle l'm'|L_i p_z L_i|lm\rangle}{\sqrt{l(l+1)}\sqrt{l'(l'+1)}} = \frac{i}{r}\delta_{m'm}[a_{lm}\delta_{l',l-1} + b_{lm}\delta_{l',l+1}] , \quad (\text{A8})$$

where

$$a_{lm} = \sqrt{\frac{(l-m)(l+m)(l-1)(l+1)^3}{l^2(2l+1)(2l-1)}} \quad (\text{A9})$$

$$b_{lm} = \sqrt{\frac{(l-m+1)(l+m+1)(l+2)l^3}{(2l+3)(2l+1)(l+1)^2}} . \quad (\text{A10})$$

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