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The ground state phase transition in the Nilsson mean-field plus standard pairing model

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The ground state phase transition in Nd, Sm and Gd isotopes is investigated by using the Nilsson mean-field plus standard pairing model based on the exact solutions obtained from the extended Heine-Stieltjes correspondence. The results of the model calculations successfully reproduce the critical phenomena observed experimentally in the odd-even mass differences, odd-even differences of two-neutron separation energy, α-decay and double β^--decay energy of these isotopes. Since the odd-even effects are the most important signatures of pairing interactions in nuclei, the model calculations gain microscopic insight into the nature of the ground state phase transition manifested by the standard pairing interaction.

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I. INTRODUCTION

Quantum phase transitions (QPTs) in nuclei have been analyzed extensively in both experiment and theory [1–6]. These studies have provided new insights and understanding of the evolution of nuclear shapes and energy level structures in transitional regions [7]. Theoretical studies of the QPTs in nuclei are typically based on the collective model and the interacting boson model [1, 4, 8–11]. Recently, the new microscopic insight for the traditional collective states in nuclei has been provided by using the projected shell model (PSM) [12]. The analysis of the resulting wave functions for the excited 0^+ states indicates clear features of quantum oscillations, with large fluctuations in deformation for soft nuclei and strong anharmonicities in the oscillations for rigidly deformed nuclei [13]. The relativistic density-functional theory has also been exploited in determining parameters in the collective Hamiltonian [14, 15], which is thus provided a microscopic method to study the low-lying spectra of nuclei in the transitional region. Generally, the QPT in nuclei is referred to as the ground state phase transition, though the concept can also be applied to excited states. Classically, the QPT in nuclei occurs at zero temperature, which can be related to different geometrical shapes of the systems. The critical phenomena of the QPT may be manifested by a sudden change in many quantities of the ground or lower excited states, which are called effective order parameters of the QPT, such as excitation energy ratios, ground state quadrupole moment, the isomer shifts, two-neutron separation energy, *etc.* Since a nuclear system is finite, only a crossover may be observed instead of a dramatic and sudden change or discontinuity in these quantities predicted from the classical limit of the theories. [7]

Nuclear pairing correlation, as an important part of the residual interactions necessary to augment any nuclear mean-field theory, represents one of the main and longstanding pillars of current understanding of nuclear structure [17]. Particularly, the pairing interaction in the nuclear shell model plays a key role to reproduce ground-state properties of nuclei, such as binding energies, two-neutron (proton) separation energies, odd-even effects, and excitation spectra, etc. [18–24]. Very recently, the ground state phase transition in some even-even, odd-A, and odd-odd nuclei in the $A \sim 150$ mass region with the analysis of some experimental observables (the effective order parameters) has been investigated [25]. The experimental evidence reveals that the odd-even mass difference may reach the maximal or minimal value around the critical point at the neutron number $N = 90$. The analysis of Sm isotopes based on the Nilsson mean-field plus the extended pairing model also indicates possible existence of critical phenomena which may microscopically be related to pairing interactions [25]. However, the nature of the critical behavior, including the role of pairing interaction, is still far from being clear. In particular, it is known that the odd-even effect (i.e., the fact that the mass of an odd-even nucleus is larger than the mean of the two adjacent even-even nuclear masses [19]) provides the most significant evidence of pairing interactions, but how these interactions relate to the phase transition is yet to be resolved. Therefore, it is important to explore possible microscopic mechanism with an appropriate pairing model that can account for the ground state phase transition reflected by the related odd-even effects in nuclei.

The purpose of this work is to systematically study the ground state phase transition in Nd, Sm, and Gd isotopes by using the Nilsson mean-field plus standard pairing model. It has been observed recently [26] that solutions of the model can be obtained from zeros of the associated extended Heine-Stieltjes polynomials, which makes it feasible to apply the model with many valence nucleon pairs over a large number of single-particle levels. In addition, a recent study [27] provides a refined method to solve the nonlinear Richardson equation for both deformed and nearly spherical nuclei based on the polynomial approaches shown in Refs. [26, 28]. By using the extended Heine-Stieltjes polynomials, the effect of the pairing interaction as well as the spherical mean-field on the spectral statistical properties have been studied [29]. It is shown that there are many exceptional values of the pairing strength G within the critical region from the normal (localized) to the nonlocal (pair condensate) phase transition in the model, and at these exceptional values the level statistics is ascribed to quantum chaos. The goal of this work is to gain some insight into the possible correlation between the pairing interaction and the related ground state phase transition described by the Nilsson mean-field plus standard pairing model.

It is known that Nd, Sm, and Gd isotopic chains display a transition from spherical to axially deformed shape, which is the typical first-order ground state phase transition described in the collective model or the interacting boson model in nuclei [8]. In this paper, the odd-even mass differences, odd-even differences of two-neutron separation energy, α-decay and double β^--decay energy of these nuclei will be calculated. The ground-state phase transition of these isotopes will be analyzed systemically based on the model results. The role of the pairing interaction in the ground state phase transition will also be addressed.

II. THE NILSSON MEAN-FIELD PLUS STANDARD PAIRING MODEL AND ITS EXACT SOLUTION

The Hamiltonian of the Nilsson mean-field plus standard pairing model for either the proton or the neutron sector is given by

$$\hat{H} = \sum_{i=1}^{n} \epsilon_i \hat{n}_i - G \sum_{ii'} S_i^+ S_{i'}^-, \tag{1}$$

where the sums run over all given i-Nilsson levels of total number n, $G > 0$ is the overall pairing interaction strength, $\{\epsilon_i\}$ are the single-particle energies obtained from the Nilsson model, $n_i = a_{i\uparrow}^\dagger a_{i\uparrow} + a_{i\downarrow}^\dagger a_{i\downarrow}$ is the fermion number operator for the i-th Nilsson level, and $S_i^+ = a_{i\uparrow}^\dagger a_{i\downarrow}^\dagger$ $[S_i^- = (S_i^+)^\dagger = a_{i\uparrow} a_{i\downarrow}]$ is pair creation [annihilation] operator, The up and down arrows in these expressions refer to time-reversed states.

According to the Richardson-Gaudin method [30], the exact k-pair eigenstates of (1) with $\nu_{i'} = 0$ for even systems or $\nu_{i'} = 1$ for odd systems, in which i' is the label of the Nilsson level that is occupied by an unpaired single particle, can be written as

$$|k;\xi;\nu_{i'}\rangle = S^+(x_1^{(\xi)}) S^+(x_2^{(\xi)}) \cdots S^+(x_k^{(\xi)}) |\nu_{i'}\rangle, \tag{2}$$

where $|\nu_{i'}\rangle$ is the pairing vacuum state with the seniority $\nu_{i'}$ that satisfies $S_i^- |\nu_{i'}\rangle = 0$ and $\hat{n}_i |\nu_{i'}\rangle = \delta_{ii'} \nu_i |\nu_{i'}\rangle$ for all i. Here, ξ is an additional quantum number for distinguishing different eigenvectors with the same quantum number k and

$$S^+(x_\mu^{(\xi)}) = \sum_{i=1}^{n} \frac{1}{x_\mu^{(\xi)} - 2\epsilon_i} S_i^+. \tag{3}$$

in which $x_\mu^{(\xi)}$ ($\mu = 1, 2, \cdots, k$) satisfy the following set of Bethe ansatz equations (BAEs):

$$1 - 2G \sum_i \frac{\rho_i}{x_\mu^{(\xi)} - 2\epsilon_i} - 2G \sum_{\mu'=1(\neq\mu)}^{k} \frac{1}{x_\mu^{(\xi)} - x_{\mu'}^{(\xi)}} = 0, \tag{4}$$

where the first sum runs over all i-levels and $\rho_i = -\Omega_i/2$ with $\Omega_i = 1 - \delta_{ii'} \nu_{i'}$. For each solution, the corresponding eigenenergy is given by

$$E_k^{(\xi)} = \sum_{\mu=1}^{k} x_\mu^{(\xi)} + \nu_{i'} \epsilon_{i'}. \tag{5}$$

Through the Heine-Stieltjes correspondence, one can find solutions of (4) by solving the second-order Fuchsian equation [26]:

$$A(x) y''(x) + B(x) y'(x) - V(x) y(x) = 0, \tag{6}$$

where $A(x) = \prod_{i=1}^{n} (x - 2\epsilon_i)$ is an n-degree polynomial, the polynomial $B(x)$ is given as

$$B(x)/A(x) = \sum_{i=1}^{n} \frac{2\rho_i}{x - 2\epsilon_i} - \frac{1}{G}, \tag{7}$$

and $V(x)$ are called Van Vleck polynomials [31] of degree $n-1$, which are determined according to Eq. (6). In search for polynomial solutions of Eq. (6), we write

$$y(x) = \sum_{\mu=0}^{k} a_\mu x^\mu, \quad V(x) = \sum_{\mu=0}^{n-1} b_\mu x^\mu, \tag{8}$$

where $\{a_\mu\}$ and $\{b_\mu\}$ are the expansion coefficients to be determined. Substitution of (8) into Eq.(6) yields two matrix equations. By solving these two matrix equations, we can obtain the solutions [26] of $\{a_\mu\}$ and $\{b_\mu\}$.

Furthermore, if we set $a_k = 1$ in $y(x)$, the coefficient a_{k-1} becomes equal to the negative sum of the $y(x)$ zeros, $a_{k-1} = -\sum_{\mu=1}^{k} x_\mu$, and hence, yields the corresponding eigen-energy according to Eq. (5),

$$E_k = -a_{k-1} + \nu_{i'} \epsilon_{i'}. \tag{9}$$

III. THE GROUND-STATE PHASE TRANSITION IN Nd, Sm, AND Gd ISOTOPES

The main objective of this work is to reveal the ground-state phase transition and its relationship with the pairing interaction in Nd, Sm, and Gd isotopes from the Nilsson mean-field plus standard pairing model perspective. Model calculations for Nd, Sm, and Gd isotopes are performed for valence neutrons in the sixth HO shell with 22 Nilsson levels (orbits) for valence neutrons. The quadrupole deformation and hexadecapole deformation parameters in the Nilsson model are obtained from Ref. [32]. Hence, the single-particle energies ϵ_β in the model Hamiltonian (1) for these three isotopic chains are determined by the Nilsson model results. As an approximation, only valence neutron pair-excitations are considered, while proton pair-excitations, which only contribute to excited states of the model, are not included.

The total binding energy of a nuclear system in the model is given by

$$E_B = E_B^{(\text{core})} + E_B(\nu) + E_B(\pi), \tag{10}$$

where $E_B^{(\mathrm{core})}$ is the binding energy of the core, taken to be that of ^{132}Sn, which is reasonably approximated by a constant and given by the experimental binding energy of ^{132}Sn, and $E_B(\nu)$ and $E_B(\pi)$ are the ground state energy of the model of the neutron and the proton sector, respectively, calculated from (1). With the previous mentioned approximation, $E_B(\pi)$ is taken as a constant because the number of valence protons is fixed in a chain of isotopes, with which the ground state energy $E_B(\pi)$ calculated from (1) is almost a constant for a chain of isotopes. Furthermore, possible residual interactions between protons and neutrons are also neglected.

There are several quantities related to the binding energies of adjacent nuclei in a chain of isotopes. They are the odd-even mass difference defined by

$$P(Z,N) = E_B(Z,N+1) + E_B(Z,N-1) - 2E_B(Z,N), \tag{11}$$

where $E_B(Z,N)$ is the binding energy of a nucleus with proton number Z and neutron number N, the two-neutron separation energy S_{2n} defined by [33]

$$S_{2n}(Z,N) = E_B(Z,N) - E_B(Z,N-2), \tag{12}$$

the α-decay energy Q_α defined as [33]

$$Q_\alpha(Z,N) = E_B(Z-2,N-2) - E_B(Z,N) + E_B(2,2), \tag{13}$$

where $E_B(2,2)$ is the binding energy of ^{4}He, and the double β^--decay energy $Q_{2\beta^-}$ defined by [33]

$$Q_{2\beta^-}(Z,N) = E_B(Z+2,N-2) - E_B(Z,N) + 2M_n - 2M_p, \tag{14}$$

where M_n and M_p are neutron and proton mass, respectively. These quantities have been recognized as qualified effective order parameters to identify the phase transitions in both even-even and odd-A nuclei. Furthermore, these quantities only depend on the number of nucleons with abundant experimental data available [2, 4–6]. However, through the analysis of these quantities and their odd-even differences [25], it is observed that nearly all the odd-even differences reach their extreme values (maximum or minimum) around the critical point, which, therefore, are more sensitive and suitable to be used as effective order parameters to manifest the shape phase transition. Hence, in the following, besides the odd-even mass differences $P(Z,N)$, the odd-even differences of two-neutron separation energy, α-decay energy and double β^--decay energy will be calculated, where the odd-even difference of the two-neutron separation energy is given by

$$D(S_{2n}(Z,N)) = S_{2n}(Z,N-1) - S_{2n}(Z,N), \tag{15}$$

the odd-even difference of the α-decay energy is given by

$$\begin{aligned} &D(Q_\alpha(Z,N)) \\ &= Q_\alpha(Z,N-1) - Q_\alpha(Z,N) \\ &= (E_B(Z,N) - E_B(Z,N-1)) \\ &\quad -(E_B(Z-2,N-2) - E_B(Z-2,N-3)), \end{aligned} \tag{16}$$

and the odd-even difference of double β^--decay energy is given by

$$\begin{aligned} &D(Q_{2\beta^-}(Z,N)) \\ &= Q_{2\beta^-}(Z,N-1) - Q_{2\beta^-}(Z,N) \\ &= (E_B(Z,N) - E_B(Z,N-1)) \\ &\quad -(E_B(Z+2,N-2) - E_B(Z+2,N-3)). \end{aligned} \tag{17}$$

It is clearly shown in (15), (16), and (17) that the binding energies contributed from the valence protons in related nuclei to these odd-even differences are canceled out, which justifies the validity of the approximation with $E_B(\pi)$ taken as a constant in our model calculation.

For each isotopic chain, the neutron pairing interaction strength G used in the standard pairing model are adjusted by fitting the binding energies and the odd-even mass differences. The neutron pairing strengths G determined for $^{144-155}$Nd, $^{146-159}$Sm, and $^{148-161}$Gd are shown in Table I.

TABLE I: Pairing interaction strength G determined from the binding energies and the odd-even mass differences of $^{144-157}$Nd, $^{146-157}$Sm, and $^{148-159}$Gd.

Nucleus	G (MeV)	Nucleus	G (MeV)
^{144}Nd	0.520	^{145}Nd	0.527
^{146}Nd	0.415	^{147}Nd	0.437
^{148}Nd	0.385	^{149}Nd	0.423
^{150}Nd	0.374	^{151}Nd	0.418
^{152}Nd	0.392	^{153}Nd	0.454
^{154}Nd	0.414	^{155}Nd	0.465
^{146}Sm	0.470	^{147}Sm	0.420
^{148}Sm	0.340	^{149}Sm	0.320
^{150}Sm	0.295	^{151}Sm	0.300
^{152}Sm	0.275	^{153}Sm	0.240
^{154}Sm	0.246	^{155}Sm	0.300
^{156}Sm	0.290	^{157}Sm	0.336
^{148}Gd	0.415	^{149}Gd	0.360
^{150}Gd	0.265	^{151}Gd	0.130
^{152}Gd	0.200	^{153}Gd	0.075
^{154}Gd	0.170	^{155}Gd	0.060
^{156}Gd	0.210	^{157}Gd	0.050
^{158}Gd	0.233	^{159}Gd	0.150

A. Odd-even mass differences

In recent work [25], it is observed that the odd-even mass difference may serve as one of the effective order parameters to identify the ground state phase transition. The odd-even mass difference $P(Z,N)$ of the three chains of isotopes are calculated according to Eq. (11). This quantity is more sensitive to pairing correlations compared to the binding energies. As shown in Fig. 1, the odd-even mass differences of the three chains of isotopes calculated from the present model are very close to the corresponding experimental values.

Moreover, both the theoretical $P(Z,N)$ values as functions of N and the corresponding experimental values for even-even

Nd, Sm, and Gd shown in panel (a) of Fig. 1 have a peak or a valley at $N = 90$. The sudden increase or decrease in $P(Z, N)$ for these nuclei at $N = 90$ is obvious, which may be regarded as one of the effective order parameters to identify the QPT in these nuclei. Similar transitional behavior can also be observed in $P(Z, N)$ values for odd-A Nd, Sm, and Gd as shown in panel (b), in which $P(Z, N)$ drops to its minimum at $N = 89$. It is known that $N \approx 90$ nuclei, such as ^{150}Nd, ^{152}Sm, ^{154}Gd with mass number $A \approx 150$ are near the critical point of the first-order quantum phase transition, namely, the critical point of spherical (vibrational) to the axially deformed (rotational) shape phase transition [8]. The above conclusion, however, is made based on the analysis within the collective model or the interacting boson model. In the Nilsson mean-field plus standard pairing model, the quantum phase transition in these nuclei is interpreted as the competition between the local single-particle energies provided from the Nilsson mean-field, in which the quadrupole-quadrupole interaction is replaced by the deformation, and the pairing interaction. Therefore, the critical behavior of $P(Z, N)$ described in this model is mainly due to the fact that the pairing interaction strength in these nuclei reaches a critical point. The same conclusion also applies to other odd-even differences shown below.

B. Odd-even differences of two-neutron separation energy

The two-neutron separation energy has also been taken as an effective order parameter to identify the phase transition, which was investigated extensively in both experimental and theoretical approaches for even-even and odd-A systems [7, 8].

As shown in Fig. 2, the odd-even differences of two-neutron separation energy $D(S_{2n})$ also has a valley near $N \approx 90$ for even-even Nd, Sm, and Gd (Fig. 2(a)) and odd-A nuclei (Fig. 2(b)), which may also be regarded as a signature of the ground state phase transition. The odd-even differences of two-neutron separation energy $D(S_{2n})$ obtained from this model reproduce the experimental data remarkably well. Namely, both the theoretical and experimental value of $D(S_{2n})$ for Nd, Sm, and Gd reach the minimum value at $N = 90$ for the even-even cases or at $N = 91$ for the odd-A cases as shown in Fig. 2. Our results indicate that the odd-even differences of two-neutron separation energy may be can serve as an effective order parameter to identify the phase transition, at least for Nd, Sm, and Gd isotopes in the present model.

C. Odd-even differences of α-decay and double β^--decay energy

The odd-even differences of α-decay energy $D(Q_\alpha)$ and double β^--decay energy $D(Q_{2\beta^-})$ in Sm and Eu isotopic chains were investigated previously [25], and the results suggested show that $D(Q_\alpha)$ and $D(Q_{2\beta^-})$ all reach their maximal values around $N \approx 90$ based on experimental data but without theoretical model calculations. The odd-even differences of α-decay energy $D(Q_\alpha)$ and double β^--decay energy $D(Q_{2\beta^-})$ for Sm, Gd, and Nd are also calculated in the present model

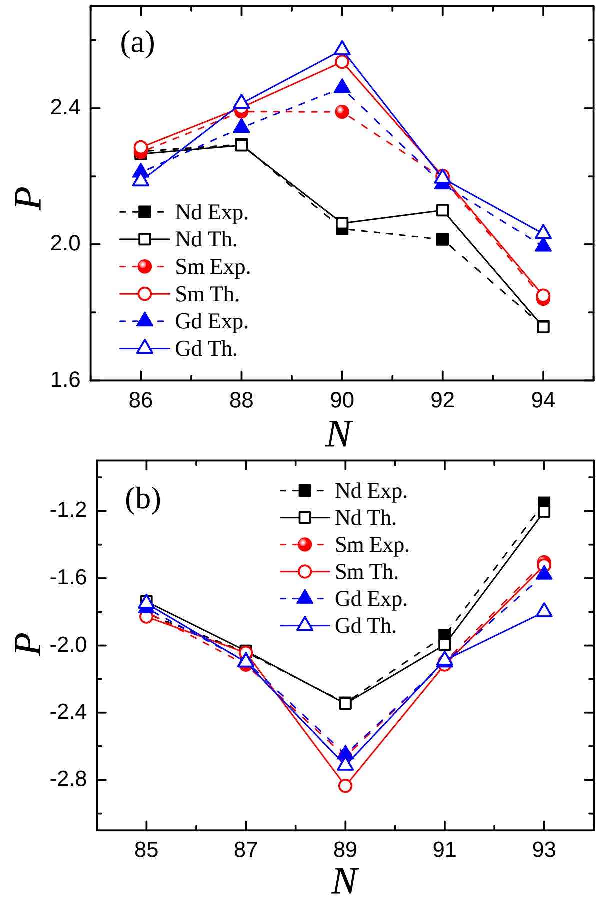

FIG. 1: (Color online) The odd-even mass differences (in MeV) for $^{145-155}$Nd, $^{147-157}$Sm, and $^{149-159}$Gd. Experimental values are denoted as "Exp.", which are taken from [33], the theoretical values calculated in the the Nilsson mean-field plus standard pairing model are denoted as "Th." for the even-even cases (upper panel) and the odd-A cases (lower panel).

according to (16) and (17). As shown in Fig. 3, the present model results of $D(Q_\alpha)$ closely follow the experimental trends for Sm and Gd isotopes. For even-even cases, there is a peak in $D(Q_\alpha)$ around the critical point at $N = 90$ for Sm and Gd as shown in Fig. 3(a). Similarly, $D(Q_\alpha)$ for odd-A nuclei exhibits the same behavior at the critical point. $D(Q_\alpha)$ reaches its maximum around $N = 91$ for Sm and Gd as shown in Fig. 3(b). The emergence of the apparent peak in $D(Q_\alpha)$ with the variation of N may also provide a sign of the ground state phase transition. Similar peak also emerges in the odd-even differences of double β^--decay energy $D(Q_{2\beta^-})$. As shown in Fig. 4, in comparison with the corresponding experimental data, the calculated results of $D(Q_{2\beta^-})$ from the present model for Nd and Sm reproduce the critical phenomenon observed experimentally, in which the dramatic change occurs at $N = 90$ in these even-even nuclei and at $N = 91$ in these odd-A nuclei.

The consistence between the experimental and the theoretical results shown in Figs. 1-4 indicates that the present model describes the ground state quantities of these nuclei rather

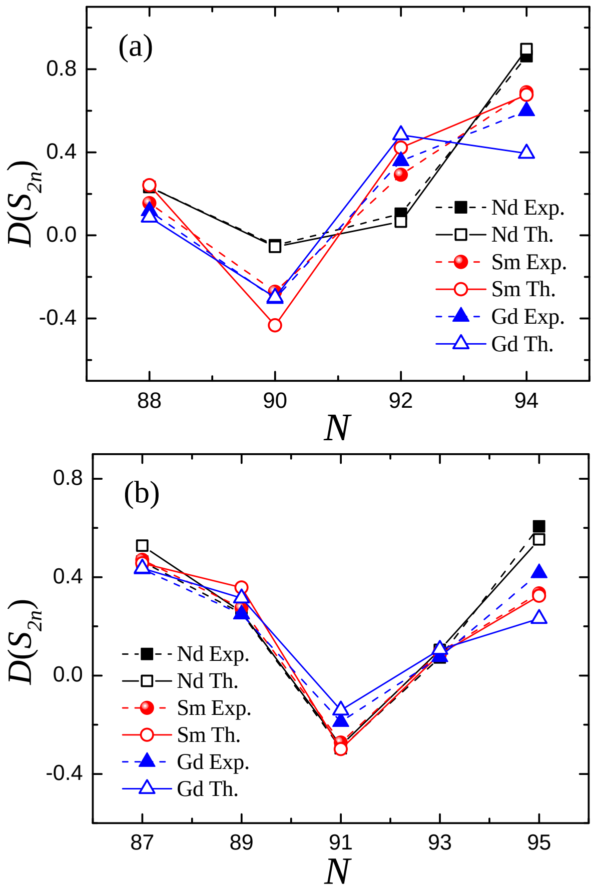

FIG. 2: (Color online) The odd-even differences of the two-neutron separation energy (in MeV) for $^{147-155}$Nd, $^{149-157}$Sm, and $^{151-159}$Gd. Experimental values are denoted as "Exp.", which are taken from [33], the theoretical values calculated from the present model are denoted as "Th." for the even-even cases (upper panel) and the odd-A cases (lower panel).

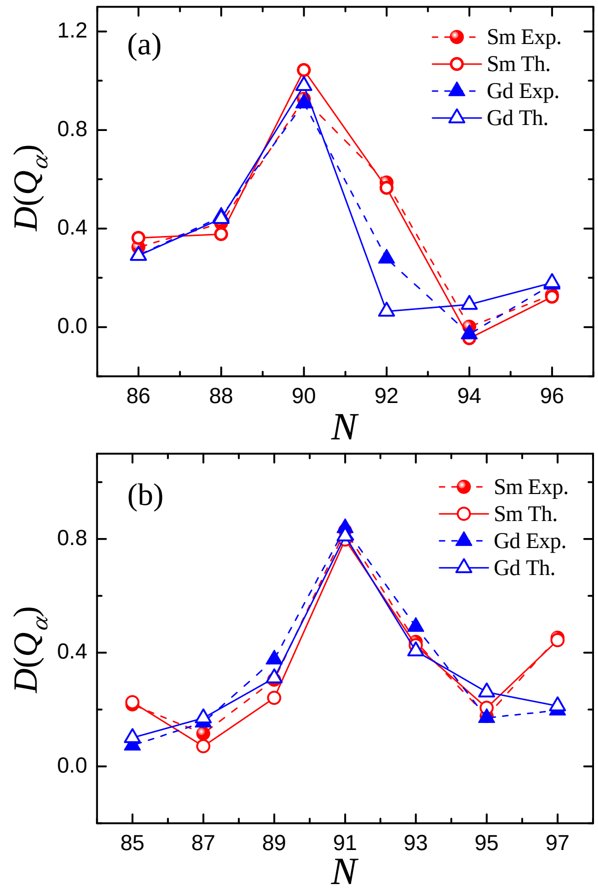

FIG. 3: (Color online) The odd-even differences of the α-decay energy (in MeV) for $^{147-157}$Sm and $^{149-159}$Gd. Experimental values are denoted as "Exp.", which are taken from [33], theoretical values calculated from the present model are denoted as "Th." for (Upper panel) even-even nuclei and (Lower panel) odd-A nuclei.

well. Since there are obvious changes in these odd-even differences around $N = 90$, which coincides with the critical point of spherical (vibrational) to the axially deformed (rotational) shape phase transition [8], these quantities may be taken as the effective order parameters to manifest the ground state phase transition in these nuclei.

D. Pairing strength

To gain the insight into the possible microscopic origin of the ground state phase transition described in the present model, the variation of the neutron pairing interaction strength G as a function of the neutron number N is studied. Figure 5 displays the pairing interaction strength G obtained from fitting the odd-even mass differences and the absolute value of the pairing interaction strength difference $|\Delta G|$, which is defined as $|\Delta G(Z,N)| = |G(Z,N+2) - G(Z,N)|$, as functions of N. It is clearly shown that $|\Delta G|$ and G vary non-monotonically with the increasing of the neutron number N for both the even-even and the odd-A cases. Particularly, $|\Delta G|$ reaches its minimum value or becomes flattening at the critical point N values, where the odd-even differences all reach the extreme values as shown in Fig. 1-4. Although the behaviors of G and $|\Delta G|$ of the even-even cases shown in Fig. 5(a) are a little different from those of the odd-A cases shown in Fig. 5(b) simply due to the effect of an extra unpaired neutron in the odd-A cases, an intimate link between the critical behaviors of the odd-even differences shown in Fig. 1-4 and the variation of G and $|\Delta G|$ at the critical points is obvious. Generally, the empirical formula of G is taken as $(G_1 \pm G_2\frac{N-Z}{A})\frac{1}{A}$(MeV) with $-(+)$ for neutron (proton), where G_1 and G_2 are adjusted to yield the odd-even mass differences [12]. In the present model, we fitted the neutron pairing interaction strength G, of which the results are shown in Table I, and found that, when N values are smaller than the critical point value N_c, the neutron pairing interaction strength G can indeed be empirically expressed as $G = G_1/A - G_2/A^2$ (MeV), while G follows the same empirical formula $G = G_1'/A - G_2'/A^2$ (MeV) with different fitting parameters $G_1' \neq G_1$ and $G_2' \neq G_2$ when $N \geq N_c$. Hence, the empirical formula of G for $N \leq N_c$ and that for $N \geq N_c$

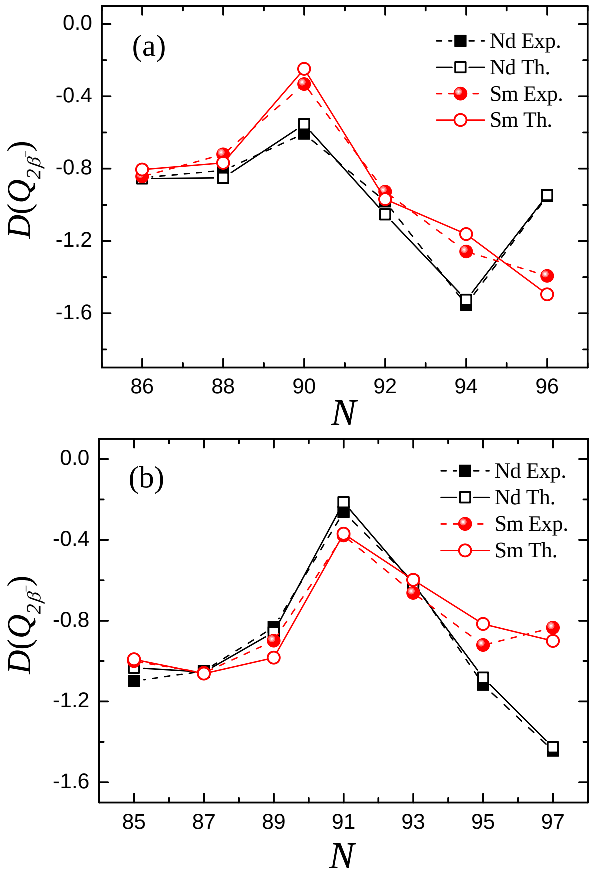

FIG. 4: (Color online) The odd-even differences of the double β^--decay energy (in MeV) for $^{145-155}$Nd and $^{147-157}$Sm. Experimental values are denoted as "Exp.", which are taken from [33], the theoretical values calculated from the present model are denoted as "Th." for the even-even cases (upper panel) and the odd-A cases (lower panel).

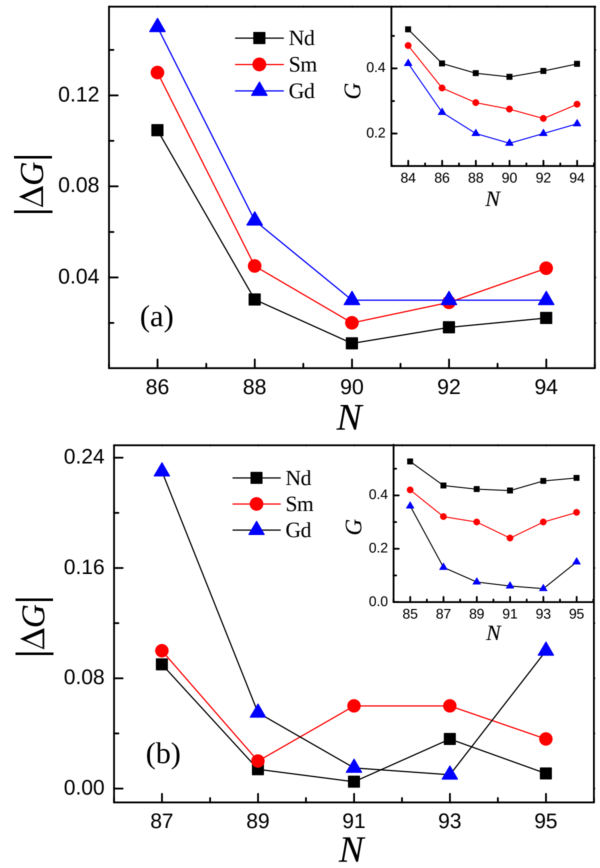

FIG. 5: (Color online) The absolute value of the pairing interaction strength difference $|\Delta G|$ (in MeV) determined under the present model for even-even nuclei (upper panel) and the odd-A nuclei (lower panel) for $^{153-163}$Nd, $^{155-165}$Sm, and $^{157-167}$Gd, in which the inset shows the pairing interaction strength G (in MeV) for these isotopes.

are different, which indicates that the derivative of the neutron pairing interaction strength G with respect to N for these isotope chains is discontinues at the critical point N_c in the model according to the present analysis. Since only a few values of N are used in the fitting, further analysis for more isotopes in a chain should be made in order to verify the present conclusion. The results shown in Figs. 1-4 seem to be directly correlated with the flattening of G and approach to the minimum value or flattering in $|\Delta G|$ near the critical point in the present model as shown in Fig. 5.

E. The information entropy

As suggested in the previous studies [29, 34], the information (Shannon) entropy seems suitable to reveal the (pairing) phase transition. To confirm that the noticeable changes in $P(Z,N)$, $D(S_{2n})$, $D(Q_\alpha)$, and $D(Q_{2\beta^-})$ around $N \approx 90$ in these isotopes are indeed related with the (pairing) phase transition in the ground state described in the present model, the information (Shannon) entropy under the preset model for the even-even Sm is also calculated. The information entropy measures the correlations among the mean-field single-pair product states with k pairs in the ground state $|\mathrm{g}\rangle \equiv |k; x; \nu_{j'}\rangle_{\mathrm{g}}$ of the model [29, 34], and is defined as

$$I_H(|\mathrm{g}\rangle) = -\sum_{i=1}^{d} |w_i|^2 \log_d(|w_i|^2), \tag{18}$$

where $\{w_i\}$ are the expansion coefficients of $|\mathrm{g}\rangle$ in terms of the mean-field single-pair product states, and d is the dimension of the space spanned by all possible single-pair product states, namely, k pairs distributed over the n levels of the Nilsson mean-field. The information entropy I_H varies within the closed interval $[0, 1]$. $I_H = 0$ corresponds to the ground state without the pairing interaction among valence nucleons. In this case, all valence nucleons are in the localized normal state. While $I_H = 1$ corresponds to the phase, in which the pairing interaction is extremely strong leading to the ground state with pair condensate, referred to as the delocalized superconducting phase. Obviously, the variation of I_H as a function of the pairing interaction strength G sketches the evolution from the localized normal phase towards the delo-

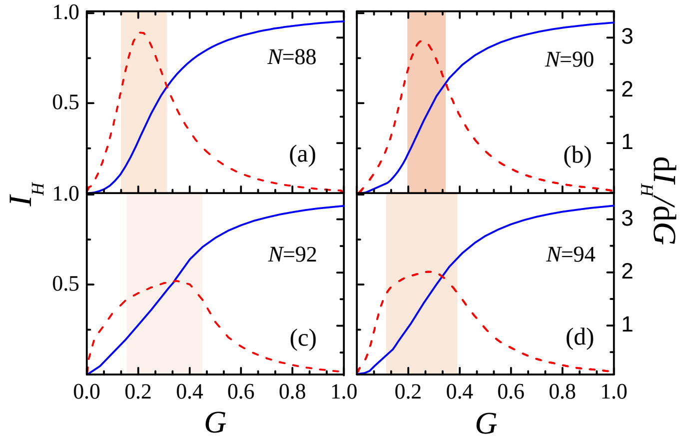

FIG. 6: (Color online) The information entropy I_H (solid line) and its derivative dI_H/dG (dashed line) of the ground state of even-even $^{150-156}$Sm as functions of G (in MeV). The shadowed area indicating the width of the peak in dI_H/dG is provided to guide to the eye.

calized superconducting phase. Since the behavior of I_H and its derivative dI_H/dG as a function of the pairing interaction strength G for these nuclei are quite similar, only those for even-even $^{150-156}$Sm are shown in Fig. 6. It is evident that I_H and dI_H/dG for even-even $^{150-156}$Sm calculated from the present model indicate that the system undergoes the phase transition from the localized normal phase with $G = 0$ and $I_H = 0$ to the delocalized superconducting (pair condensate) phase with sufficiently large G and $I_H \sim 1$ for a given valence neutron number N. Particularly, it is shown in Fig. 6 that the width of the peak in dI_H/dG is different from nucleus to nucleus. The width of the peak in dI_H/dG becomes the narrowest at $N = 90$ as shown in Fig. 6(b), which corresponds to the position, at which $P(Z, N)$, $D(S_{2n})$, $D(Q_\alpha)$, and $D(Q_{2\beta^-})$ all reach their extremal values shown in Figs. 1-4. The position of the extremal values in the odd-even differences is consistent to the position of the peak with the narrowest width in dI_H/dG, of which the critical point value is $G_c \simeq 0.27$ in the present model for even-even Sm. The results demonstrate that the ground state phase transition is indeed much more sensitive to the variation of the pairing interaction strength G around $N \approx 90$, which thus naturally explains the critical behavior in the present model, and confirms that the odd-even differences $P(Z, N)$, $D(S_{2n})$, $D(Q_\alpha)$ and $D(Q_{2\beta^-})$ are suitable to be taken as the order parameters of the ground state phase transition, at least in these isotopes.

IV. V. CONCLUSION

In summary, the Nilsson mean-field plus standard pairing model is applied to describe the ground state phase transition in Nd, Sm, and Gd isotopes. In comparison with the corresponding experimental data, the calculated results of the model for these isotopes, including the odd-even mass differences, the odd-even differences of two-neutron separation energy, α-decay and double β^--decay energy, reproduce the critical phenomena reasonably well. It is shown that both the theoretical and the experimental values of the odd-even differences for Nd, Sm, and Gd reach their extremal values at the critical point of the model around $N \approx 90$, which are consistent with the results obtained from the collective model and the interacting boson model studied previously [4, 8]. From the analysis of the information entropy and its derivative dI_H/dG in Sm isotopes, it is confirmed that the phase transition in the present model is much more sensitive to the variation of the pairing interaction strength G around $N \approx 90$, which thus provides the origin of the critical behaviors in the model, and confirms that these odd-even differences are indeed suitable to be taken as the order parameters of the ground state phase transition, at least in these isotopes. Moreover, it is shown that the variation of $|\Delta G|$ and G in the model is not monotonic with the increasing of the neutron number N. It seems that $|\Delta G|$ reaches the minimum or becomes flattening around $N \approx 90$, at which these odd-even differences all reach their extreme values. Therefore, our analysis provides a microscopic picture that the ground state phase transitional behaviors may be driven by the competition between the Nilsson mean-field and the pairing interaction based on the present model. It has been recognized that the emergence of the collective phenomena in these nuclei are related to the competition between the pairing interaction and the deformation-driven quadrupole-quadrupole interaction [4, 7]. However, the analysis shown in this work is based on the Nilsson mean-field plus standard pairing model, in which the quadrupole-quadrupole interaction is replaced by the deformation. Nevertheless, the phase transitional behaviors in these nuclei can still be clearly described in the present model.

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