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# Meaning of the nuclear wave function

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## Abstract

**Background** The intense current experimental interest in studying the structure of the deuteron and using it to enable accurate studies of neutron structure motivate us to examine the four-dimensional space-time nature of the nuclear wave function, and the various approximations used to reduce it to an object that depends only on three spatial variables.

**Purpose** The aim is to determine if the ability to understand and analyze measured experimental cross sections is compromised by making the reduction from four to three dimensions.

**Method** Simple, exactly-calculable, covariant models of a bound-state wave state wave function (a scalar boson made of two constituent-scalar bosons) with parameters chosen to represent a deuteron are used to investigate the accuracy of using different approximations to the nuclear wave function to compute the quasi-elastic scattering cross section. Four different versions of the wave function are defined (light-front spectator, light-front, light-front with scaling and non-relativistic) and used to compute the cross sections as a function of how far off the mass-shell (how virtual) is the struck constituent.

**Results** We show that making an exact calculation of the quasi-elastic scattering cross section, involves using the light-front spectator wave function. All of the other approaches fail to reproduce the model exact calculation if the value of Bjorken  $x$  differs from unity. The model is extended to consider an essential effect of spin to show that constituent nucleons cannot be treated as being on their mass shell even when taking the matrix element of a ‘good’ current.

**Conclusions** Developing realistic light-front spectator wave functions to meet the needs of current and planned experiments is a worthwhile activity.

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Keywords:

## I. INTRODUCTION

Nuclear theorists have made tremendous progress during the last decade in computing nuclear spectra from first principles [1–3]. Two and three- nucleon interactions, with a traceable connection to QCD [4–7], have been used in exact calculations of the energy levels of nuclei with  $A \leq 12$ . Furthermore, a variety of new techniques to treat heavy nuclei have been developed. Nevertheless, some fundamental questions regarding the nature of the nuclear wave function remain.

The nuclear wave function depends on only three of the four available space-time variables. The usual derivation of three-dimensional physics starts with the four-dimensional Bethe-Salpeter equation for two nucleons, which in principle makes a non-perturbative sum of the effects of all interactions, and reduces it to a three-dimensional equation without changing the unitarity properties. The result of the procedure is that the square of the four-momentum of the nucleons is equal to the mass squared; the nucleons are placed on their mass-shell [8]. If the relativistic phase-space factor is replaced by the non-relativistic version, the resulting equation is the Lippmann-Schwinger LS equation, equivalent to the Schroedinger equation. The two-nucleon potential, as constrained by phase shifts computed within the LS equation, is then used to compute nuclear properties, by solving or approximating the many-body Schroedinger equation. Three-nucleon forces are also included. Within this procedure the constituent nucleons of nuclear wave functions are on their mass-shell. However, the sum of their basis-dependent single-particle energies is not the energy of the nucleus; the nucleons are therefore termed as being off the energy shell.

Carrying out the reduction from four to three dimensions can be effected using either the standard equal time formulation in which the relative time is set to 0, or the light-front procedure in which the relative value of  $z + ct$  is set to 0 [9–12]. An exception to this procedure is the use of the Gross equations [13–16], rooted in atomic physics [17], that places only one nucleon (the ‘spectator’) on its mass shell. No applications of this procedure to nuclei with  $A > 3$  exist at this time.

The purpose of the present manuscript is to examine and determine the limitations of the three-dimensional approach to the nuclear wave function through exact and approximate evaluations of quasi-elastic scattering on a two-body system which is a semi-realistic, but completely Lorentz-invariant version of the deuteron. We concentrate on the deuteron because it is the simplest nucleus, and because there is now intense experimental interest in a variety of measurements that focus on its wave function. For example, there is much attention on studying the wave function at high momentum transfer [18–20]. Another experiment of high interest is the proposed measurement of  $A_{zz}$  (JLab LOI12-14-002), available by using a tensor polarized deuteron target, aimed specifically at studying the deuteron wave function [21]. In the quasielastic region,  $A_{zz}$  can be used to compare light cone calculations with calculations that incorporate the virtual nature of the struck nucleon, and is an important quantity to determine for understanding tensor effects. Such effects are related to the dominance of  $pn$  correlations in nuclei [22–26]. The measurements are planned to occur at values of Bjorken  $x = \frac{Q^2}{2m_N\nu}$  significantly greater than unity. Furthermore, the light-front deuteron wave function is needed to interpret existing and planned spectator-tagging experiments [27, 28] aimed at determining neutron structure.

The outline of the remainder of this paper follows. Our simple model is defined in Sec. II. The deuteron is treated as a scalar boson that is a bound state of two different scalar bosons. The vertex function is taken as a constant. Such models have long been used [29, 30] to illustrate relativistic aspects of complicated dynamical situations. The definition of constituent virtuality is presented and its importance is illustrated in Sec. III. The model exact calculation of the analog of quasi-elastic electron-deuteron scattering cross section is presented in Sec. IV. We neglect the influence of final state interactions throughout this paper. This simplification allows us to focus on the influence of virtuality. Furthermore, the effects of final state interactions can be minimized through the appropriate choice of kinematics [20]. A discussion of four different wave functions: light-front-spectator, light front, light front with Bjorken scaling, and non-relativistic is presented in Sec. V. Cross sections obtained using these different models are compared with the model exact cross sections in Sec. VI. All of the models, except the light-front-spectator, fail badly if the value of  $x$  differs significantly from unity. The vertex function of the simple model of Sect. II is generalized in Sect. VII, where it is shown that the qualitative conclusion just stated does not depend on using a constant vertex function. One aspect of spin is considered in Sec. VIII where it is shown that it is necessary to consider the virtual nature of constituent fermions, even if computing the matrix element of a ‘good’ current. The final section presents a summary and discussion of the possible implications of the work presented here.

## II. MODEL DYNAMICS AND MODEL SCATTERING PROCESS

The nuclear dynamics are modeled by a version of the  $\phi^3$  model, generalized to  $D\phi\chi$  so that one scalar particle a ‘deuteron’  $D$  of mass  $M$  is a bound system of two different scalar particles  $\phi, \chi$  of mass  $m$ , only one of which interacts with a scalar probe of four momentum  $q$ . The interaction between the probe and the struck nucleon is taken to be a constant,  $g$ . The deuteron vertex function  $\Gamma(k, P)$  is also taken as a constant,  $G$ . This set of dynamics corresponds to the 0’th order chiral perturbation theory version of the deuteron. The model allows for all matrix elements to be computed in covariant fashion and there is no need to limit the kinematics. The values of  $m, M$  are those of the average nucleon mass and mass of the physical deuteron. Thus  $M = 2m - B$ , with  $B = 0.0022$  GeV. These scales  $B$  and  $M$  span the range of mass scales that would enter into a more realistic model.

The quasi-elastic scattering reaction of interest is shown in Fig. 1. A scalar ‘deuteron’ of 4-momentum  $P$  encounters a virtual space-like scalar photon of four-momentum  $q$ , leading to a final state in which the struck nucleon has momentum  $k + q$ , and is a real particle of positive energy. The spectator  $s$  has four-momentum  $p_s$  with  $p_s^2 = m^2$  and its energy is greater than 0. We consider only this diagram here so as to concentrate on the fundamental aspects. Therefore the usually important effects of final state interactions are neglected throughout this paper. We use the convention that in the deuteron rest-frame the four-momentum  $q$  is given by  $q = (\nu, 0, 0, -\sqrt{Q^2 + \nu^2})$ , with  $Q^2 = -q^2 > 0$ . It is useful to define two kinematic variables

$$x \equiv \frac{Q^2}{2m\nu}, \quad (1)$$

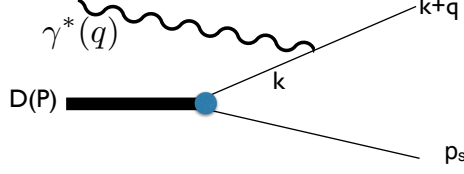


FIG. 1: (Color online) Scattering process of interest

which ranges between 0 and about  $M/m \approx 2$ , and

$$\xi \equiv \frac{Q^2}{Mq^-} = -\frac{q^+}{M} = \frac{m}{M} \frac{2x}{1 + \sqrt{1 + \frac{4m^2x^2}{Q^2}}} = \frac{m}{M} \frac{2xQ}{Q + \sqrt{Q^2 + 4m^2x^2}}, \quad (2)$$

$\xi$  is the Nachtmann variable for the given target, and it is limited by momentum conservation to be less than unity. The  $\pm$  components of the four-momentum  $V$  of any particle is defined here as  $V^\pm \equiv V^0 + V^3$ . Another useful variable is the light-front variable  $\alpha$ , defined via

$$p_s^+ \equiv (1 - \alpha)P^+ = E_s + p_{sz}. \quad (3)$$

In the deuteron rest frame  $P^\pm = M$ .

### III. NUCLEON VIRTUALITY

The virtual nucleon in Fig. 1 has four momentum  $k$  given by  $k = P - p_s$ . The quantity

$$V \equiv m^2 - (P - p_s)^2 = m^2 - k^2 \equiv -X \quad (4)$$

measures the deviation of the about to be struck nucleon from its mass shell. The founding assumption of nearly all nuclear wave functions is that the virtuality vanishes.

Proceed by examining the quantity  $X = -V$ . Given that  $k^- = P^- - p_s^-$ , the use of Eq. (3) gives  $k^+ = \alpha M$ , in the deuteron rest frame, and also  $k^- = P^- - p_s^- = M - \frac{k_\perp^2 + m^2}{(1-\alpha)M}$ , where  $\mathbf{k}_\perp$  is the momentum of the spectator nucleon. Furthermore

$$X(\alpha, k_\perp) = (P^+ - p_s^+)(P^- - p_s^-) - k_\perp^2 - m^2 = \alpha(M^2 - \frac{k_\perp^2 + m^2}{\alpha(1-\alpha)}). \quad (5)$$

We examine the delta function  $\delta(X + 2k \cdot q - Q^2)$  to determine the relevant value of  $\alpha$  as a function of  $q$  and  $k_\perp$ :

$$X + q^+(M - \frac{k_\perp^2 + m^2}{M(1-\alpha)}) + q^-\alpha M - Q^2 = 0. \quad (6)$$

The vanishing of the effect of the virtuality at high momentum transfer and energy can most readily be observed by using light-front variables. The argument of the delta function shown in Eq. (6) can be rewritten as

$$(\alpha - \xi)(q^- M + M^2) - (1 - \xi)\frac{k_\perp^2 + m^2}{(1-\alpha)} = 0. \quad (7)$$

It is worthwhile to point out that in the Bjorken limit of  $Q^2/\nu^2 \ll 1$ ,  $q^+ \ll q^-$ , the last two terms of Eq. (6) are much larger than the first two terms. In that case, one may ignore the the first two terms, so that in this scaling limit  $\alpha = \xi$ .

We need to examine the effects of the first two terms of Eq. (7). This is a quadratic equation in  $\alpha$ , which can be solved, yielding the result

$$\alpha = \frac{1}{2} \left( 1 + \xi - \sqrt{(1 - \xi)^2 - \epsilon 4C(1 - \xi)} \right), \quad (8)$$

$$\epsilon \equiv \frac{M}{M + q^-} < 1, \quad C \equiv \frac{(k_\perp^2 + m^2)}{M^2} \quad (9)$$

obtained using the condition that when  $\epsilon C = 0$ ,  $\alpha = \xi$ , and  $k_\perp$  is the perp component momentum of the spectator. The relations Eq. (8) and Eq. (2) determine the spectator momentum for each value of  $k_\perp$ .

The value of  $\alpha$  must be such that the spectator energy  $E_s \geq m$ . This means that

$$\frac{1}{2} [M(1 - \alpha) + \frac{k_\perp^2 + m^2}{(1 - \alpha)M}] \geq m \quad (10)$$

Note that  $M = 2m - B$ ,  $B > 0$ . Eq. (10) holds for all values of  $\alpha$  such that  $\alpha < 1$ . Conservation of four-momentum leads to a limit on  $x \equiv \frac{Q^2}{2m\nu}$  :

$$x \leq \frac{M}{m} \frac{1}{1 + \frac{4m^2 - M^2 + 4k_\perp^2}{Q^2}}. \quad (11)$$

This equation can also be written as a limit on  $k_\perp^2$  :

$$Q^2 + 4m^2 - M^2 + 4k_\perp^2 < 2M\nu. \quad (12)$$

Armed with the value of  $\alpha$  which depends on  $x, Q^2$  and  $k_\perp$ , we may compute the value of  $V$  for different kinematic situations. The results are shown in Figs. 2, 3 and 4. We see that,

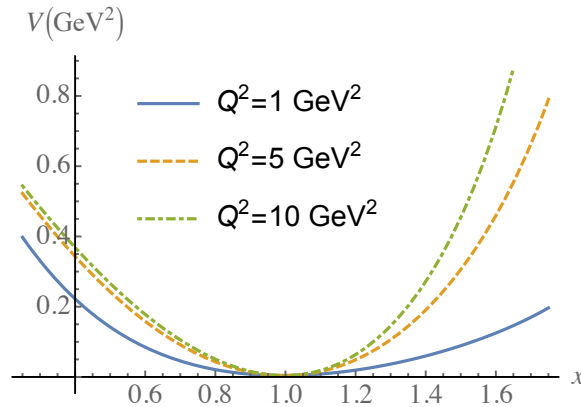


FIG. 2: (Color online) Virtuality as a function of  $x, Q^2$  for  $k_\perp = 0$ .

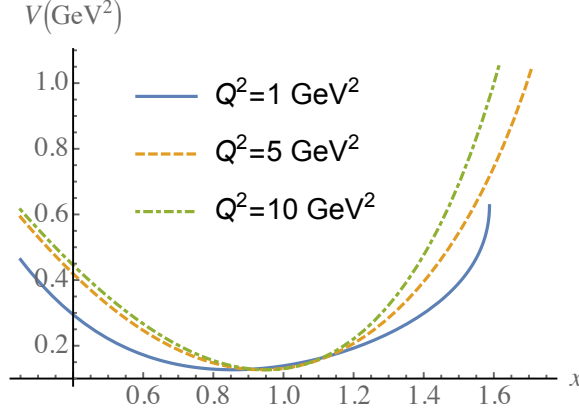


FIG. 3: (Color online) Virtuality as a function of  $x, Q^2$  for  $k_{\perp} = 0.25$  GeV.

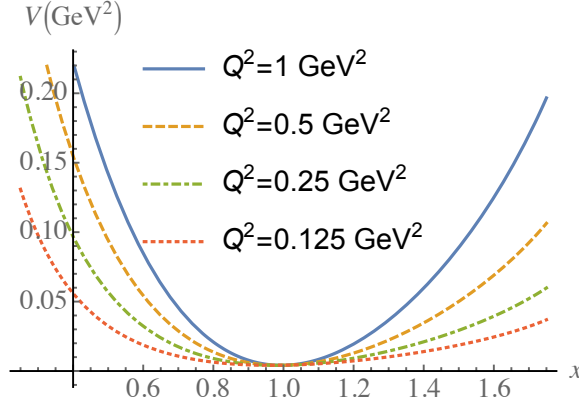


FIG. 4: (Color online) Virtuality as a function of  $x, Q^2$ , lower values of  $Q^2$  for  $k_{\perp} = 0$ .

except for small values of  $Q^2$  and  $x$  near unity, the value of  $V$  is generally larger than  $0.1$   $\text{GeV}^2$ . This corresponds to a momentum of  $300$   $\text{MeV}/c$ , which is not an ignorable scale in nuclear physics. Thus in general approximating  $V$  by  $0$  is expected to be a dangerous approximation. This means that the usual nuclear procedure of treating the nucleons as being on their mass shell is not valid, and that the connection between the scattering amplitude and the usual equal-time or light front three-dimensional wave functions is severed.

#### IV. EXACT MODEL QUASI-ELASTIC CROSS SECTION

The cross section is for the absorption of a space-like scalar “photon” of four-momentum  $q$  on a two body system of scalar mesons which is our toy model of the deuteron. In this case

$$d\sigma = \frac{(2\pi)^4}{j} \frac{d^4p}{(2\pi)^3} \delta_+(p^2 - m^2) \frac{d^3p_s}{2E_s(2\pi)^3} \delta^4(P + q - p - p_s) |\mathcal{M}|^2, \quad (13)$$

where  $\mathcal{M}$  is the invariant amplitude, and  $j = 4M|\vec{q}|$  is the flux factor. Define  $k = P - p_s$ , the four-momentum of the struck particle. Anticipating the use of light-front variables, we state

$$\frac{d^3p_s}{2E_s} = \frac{d^2k_\perp dk^+}{2(P^+ - k^+)}, \quad (14)$$

so that integration over the four-momentum-conserving delta function yields

$$d\sigma = \delta_+((k + q)^2 - m^2) \frac{d^2k_\perp dk^+}{2(P^+ - k^+)(2\pi)^2 j} |\mathcal{M}|^2. \quad (15)$$

For our model

$$|\mathcal{M}|^2 = \frac{g^2 G^2}{X(\alpha, k_\perp)^2}, \quad (16)$$

where  $X(\alpha, k_\perp)$  is the absolute value of the inverse propagator. Then we write

$$d\sigma = \frac{d^2k_\perp}{8\pi^2 j} \int \frac{dk^+}{p_i^+ - k^+} \delta_+((k + q)^2 - m^2) \frac{g^2 G^2}{X(\alpha, k_\perp)^2} \quad (17)$$

The value of  $X(\alpha, k_\perp)$  is given by Eq. (5). We do the integral over  $dk^+$  to obtain

$$j \frac{8\pi^2}{g^2 G^2} \frac{d\sigma}{d^2k_\perp} = \frac{1}{X(\alpha, k_\perp)^2} \frac{1}{q^- M + M^2} \frac{1}{1 - 2\alpha + \xi}, \quad (18)$$

where  $E = \nu + M - E_s$  and  $\alpha$  as given by Eq. (8) and  $X$  of Eq. (5) are functions of  $\nu, Q^2, k_\perp^2$ . Note that the limit  $\alpha \leq 1$  is enforced by Eq. (8), and the limit  $\xi \leq 1$ . The use of light-front variables is not necessary, but their use does simplify the evaluation. We have obtained equivalent results using the standard energy-momentum variables.

It is convenient to define the quantity

$$\frac{d\Sigma}{d^2k} \equiv j \frac{8\pi^2}{g^2 G^2} \frac{\nu d\sigma}{d^2k_\perp} \quad (19)$$

The factor  $\nu$  is inserted because the cross section for a single free nucleon can be interpreted to have this factor. Thus  $\frac{d\Sigma}{d^2k}$  represents a cross section per nucleon. Results for  $\frac{d\Sigma}{d^2k_\perp}$  are shown for two ranges of  $Q^2$  in Fig. 5. We begin by noting that the cross sections look qualitatively similar to measured experimental data (see *e.g.* Fig. 6 of Ref. [20]) giving some credence to the simple model we use. Note also that the scaling limit (in which the cross sections depend on  $x$  but not on  $Q^2$ ) is obtained for  $Q^2$  of order 10's of GeV<sup>2</sup>.

## V. WAVE FUNCTIONS

We next relate the exact model calculations of the previous Section with various ideas about wave functions that are in the literature.



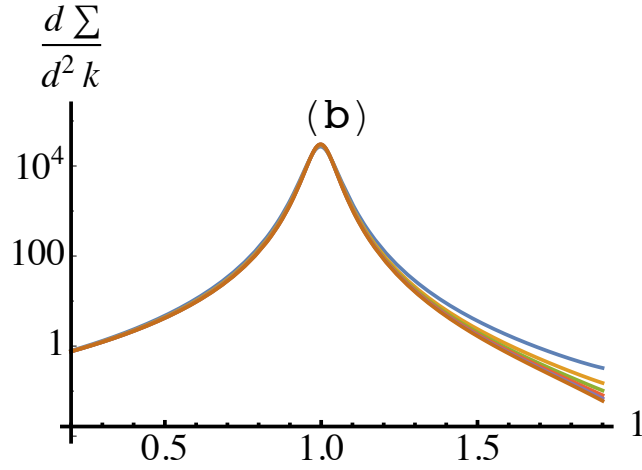
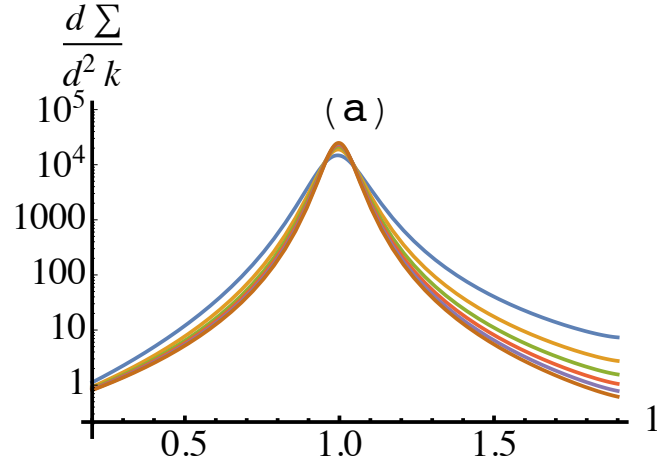


FIG. 5: (Color online)  $\frac{d\Sigma}{d^2k_\perp}$  as a function of  $x$  for two different ranges of  $Q^2$  with  $k_\perp = 0$ . (a)  $Q^2$  from 1 to 6 GeV<sup>2</sup>. (b)  $Q^2$  from 10 to 60 GeV<sup>2</sup>. For each case, the lower the value of  $Q^2$ , the higher the cross section. For larger values of  $Q^2$  the curves tend to coalesce.

### A. Exact model calculation uses the light-front spectator wave function

The Bethe-Salpeter [31] wave function for this model is given by

$$\Psi(k, P) = \frac{-iG}{(k^2 - m^2 + i\epsilon)((P - k)^2 - m^2 + i\epsilon)}. \quad (20)$$

This quantity does not enter in the calculation of the invariant amplitude, which involves only  $\frac{-iG}{(k^2 - m^2 + i\epsilon)} = -iG/X$ . However, the factor  $1/X$  can be obtained by doing an integration that places the spectator particle on the mass shell, so that  $(P - k)^2 = m^2$ , with  $(P^+ - k^+) > 0$ . The result of this integral is the spectator wave function of the Gross equation [13] times a kinematic factor, so that the object  $1/X$  of Eq. (5) corresponds to using the spectator wave function. Note that the integration used here involves light front coordinates to take advantage of the high energy of the incident virtual photon. Thus, computation of the exact cross section makes explicit use of a light-front version of the Gross equation wave function. We may even say that the light-front spectator wave function is designed to give the correct quasi-elastic scattering cross section. This wave function has the odd, but useful, feature that one constituent is virtual and the other spectator constituent, a spectator, is on its mass shell.

### B. The on-mass shell limit uses the light front wave function

The light front wave function is derived by taking the constituent particles to be on the mass-shell, denoted by OS. In this case  $k^2 - m^2 = 0$  and Eq. (6) becomes

$$q^+(M - \frac{k_\perp^2 + m^2}{M(1 - \alpha)}) + q^- \alpha M - Q^2 = 0. \quad (21)$$

The solution is given by

$$\alpha = \alpha_{\text{OS}} = \frac{1}{2} \left( -\sqrt{\epsilon_0 (4C\xi + 2\xi^2 - 2\xi) + (1 - \xi)^2 + \xi^2 \epsilon_0^2} + \xi(\epsilon_0 + 1) + 1 \right), \quad (22)$$

with  $\epsilon_0 \equiv \frac{M}{q^-}$ . The cross section is given by

$$j8\pi^2 d\sigma_{\text{OS}} = d^2 k_\perp \int \frac{dk^+}{P^+ - k^+} \delta_+(-\xi(M^2 - \frac{k_\perp^2 + m^2}{1 - \alpha}) + k^+ q^- - Q^2) \frac{g^2 G^2}{X^2(\alpha_{\text{OS}}, k_\perp)}. \quad (23)$$

The notation  $X(\alpha_{\text{OS}}, k_\perp)$  refers to using  $\alpha \rightarrow \alpha_{\text{OS}}$  in the defining equation Eq. (22). For calculations of elastic scattering the use of the light front wave function gives the exact result [30].

### C. Light front wave function-with scaling

For large value of  $Q^2$  and  $\nu$ , when  $Q^2/\nu$  is constant and  $x \equiv Q^2/2m\nu$ , (the scaling limit) one may ignore the  $k^2 - m^2$  and  $q^+ k^-$  appearing in the argument of the delta function of Eq. (6). In this case

$$\delta(X + 2k \cdot q - Q^2) = \delta(k^2 - m^2 + 2k \cdot q - Q^2) \rightarrow \delta(k^+ q^- - Q^2), \quad (24)$$

so that

$$\alpha = \xi, \quad (25)$$

and

$$j8\pi^2 d\sigma_{sc} = d^2k_{\perp} \int \frac{dk^+}{P^+ - k^+} \delta_+(k^+q^- - Q^2) \frac{g^2 G^2}{X(\alpha, k_{\perp})^2}, \quad (26)$$

yielding

$$j8\pi^2 \frac{d\sigma_{sc}}{g^2 G^2 d^2k_{\perp}} = \frac{1}{(1 - \xi)q^- M} \frac{1}{X^2(\xi, k_{\perp})}. \quad (27)$$

In the scaling limit, the relevant wave function is the light front wave function evaluated at a momentum fraction  $\xi$  that is determined only by  $x$  and  $Q^2$ .

The net result, so far, is that the exact calculation is handled by the spectator wave function. If one neglects the virtuality of the struck nucleon, one may use the light front wave function, but it is evaluated at a momentum fraction that depends upon  $k_{\perp}$  as well as on  $(x, Q^2)$ . Only in the scaling limit can one use the light front wave function, evaluated at the Nachtmann variable  $\xi$ .

#### D. Non-relativistic limit

We define the non-relativistic limit as using the non-relativistic approximation to the inverse propagator  $X$  of Eq. (5) in the expression for the cross section Eq. (18). Thus

$$X = M(M - 2E_s) \rightarrow -M(B + \frac{\vec{k}^2}{m}), \quad (28)$$

where in the non-relativistic approximation  $E_s = \sqrt{\vec{k}^2 + m^2} \approx m + \frac{\vec{k}^2}{2m}$ . Thus we use the non-relativistic limit and obtain

$$\frac{1}{X_{NR}} = \frac{-m}{M} \frac{1}{mB + k_z^2 + k_{\perp}^2}. \quad (29)$$

This is essentially the non-relativistic wave function for a delta function potential, and is also the zero range wave function of Bethe [32].

To evaluate the cross section in Eq. (18) we use  $X_{NR}$  of Eq. (29). It is necessary to determine  $k_z$  in terms of  $\alpha$ . In the non-relativistic theory all constituent particles are on their mass-shell, so  $k_z$  is determined from  $\alpha_{OS}$  via

$$\alpha_{OS} = \frac{E(k) + k_z}{2m}, \quad E(k) = m + \frac{\vec{k}^2}{2m}. \quad (30)$$

Solving this equation for  $k_z$  gives the result:

$$\frac{k_z}{m} = -1 + \sqrt{4\alpha_{OS} - 1 - \frac{k_{\perp}^2}{m^2}}, \quad (31)$$

with  $\alpha_{OS}$  given by Eq. (22). This is the root that has  $k_z = 0$  if  $\alpha_{OS} = 1/2$  and  $k_{\perp} = 0$ .

## VI. MODEL RESULTS AND THE ACCURACY OF USING DIFFERENT WAVE FUNCTIONS

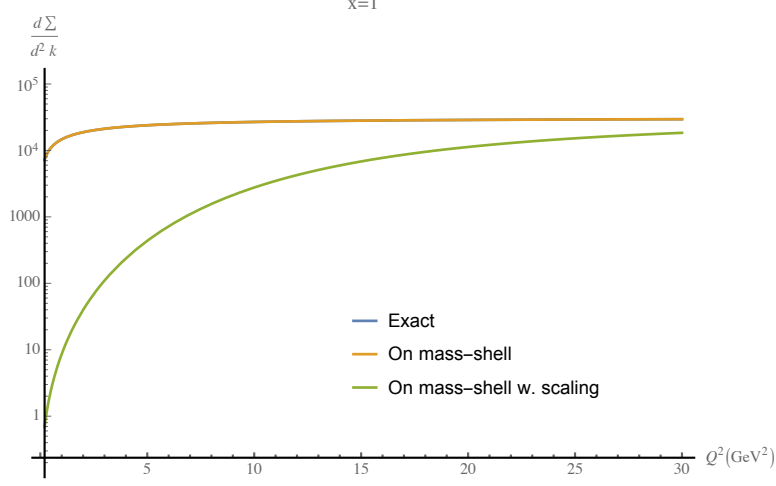


FIG. 6: (Color online)  $\frac{d\Sigma}{d^2k_\perp}$  for three models at  $x = 1$ ,  $k_\perp = 0$ . Only two curves are observable easily because of the confluence of the exact and on-mass shell light-front wave function approach.

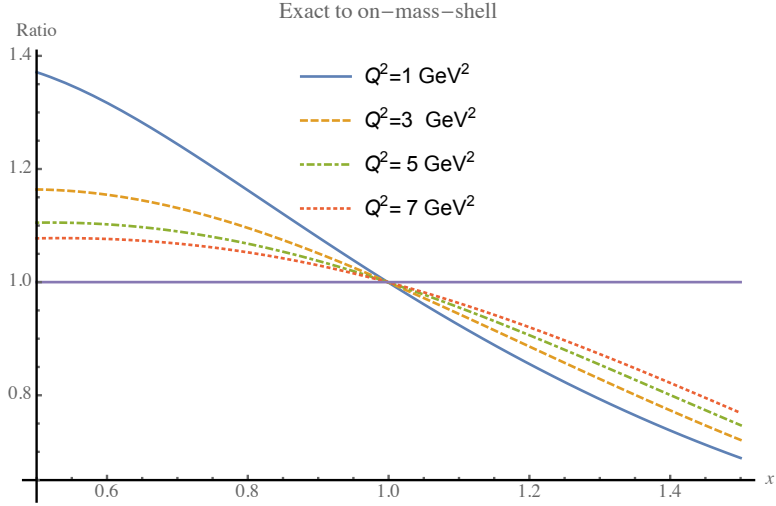


FIG. 7: (Color online) Exact to on-mass shell ratios as a function of  $x$ ,  $k_\perp = 0$  for different values of  $Q^2$ .

Fig. 6 shows cross sections at  $x = 1$ . We see that the exact and on-shell (0 virtuality) light-front approaches agree at all values of  $Q^2$  that are shown. The curves for these two methods are not distinguishable. In contrast, the use of Bjorken scaling is not valid unless the value of  $Q^2$  is very high.

That the accuracy of neglecting the virtuality holds only for values of  $x$  near unity is shown in Fig. 7. Significant errors are seen for values of  $x$  lower and higher than unity. The accuracy improves as the value of  $Q^2$  increases. However, these results show that the reliability of using light front wave functions is questionable if one is investigating high or low values of  $x$  for momentum transfers less than about 10 GeV<sup>2</sup>.

All of the previous cross sections are obtained using relativistic wave functions. The non-relativistic approximation is studied in Fig. 8. Using the non-relativistic approximation fails except for values of  $x$  near unity. The relative errors increase significantly with increasing  $Q^2$ . This indicates that using standard non-relativistic wave functions to analyze quasi-elastic scattering from deuteron targets may introduce an uncontrolled systematic error.

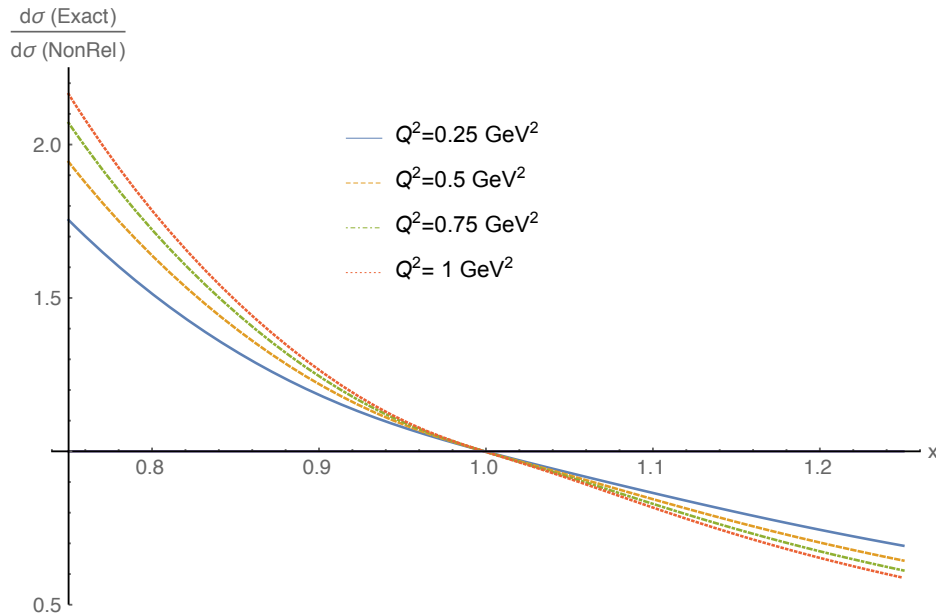


FIG. 8: (Color online) Non-relativistic approximation. Ratios of exact to non-relativistic cross sections are shown as a function of  $x$  for different values of  $Q^2$ .

## VII. OTHER DEUTERON WAVE FUNCTIONS

One might wonder if the qualitative results presented here are obtained only because of the simplicity of taking the deuteron vertex function to be a constant. Therefore we derive and use a more general wave function. Suppose instead of taking the vertex function to be constant  $\Gamma = G$ , we postulate that, for example,

$$\Gamma(k, P) = \frac{G \Lambda^2}{-k^2 + \Lambda^2 + m^2}, \quad (32)$$

where  $\Lambda$  is a parameter to be determined. This means that the factor  $1/X$  is replaced:

$$\frac{1}{X} = \frac{1}{k^2 - m^2} \rightarrow \frac{1}{k^2 - m^2} \frac{\Lambda^2}{-k^2 + \Lambda^2 + m^2} = \frac{1}{k^2 - m^2} - \frac{1}{k^2 - m^2 - \Lambda^2} \equiv \frac{1}{\tilde{X}} \quad (33)$$

The meaning of the second term may be identified by considering the non-relativistic limit

$$\frac{1}{\tilde{X}} = -\frac{m}{M} \left[ \frac{1}{mB + \vec{k}^2} - \frac{1}{mB + \Lambda^2 \frac{m}{M} + \vec{k}^2} \right], \quad (34)$$

and we observe that  $\frac{1}{\tilde{X}}$  is the Fourier transform of the Hulthén wave function [33]  $(e^{-ar} - e^{-br})/r$  with the parameters [34]

$$a = \sqrt{Bm} = 0.2316 \text{ fm}^{-1}, \quad b^2 - a^2 = \Lambda^2 m/M, \quad b = 1.3802 \text{ fm}^{-1}. \quad (35)$$

Evaluation yields  $\Lambda = 0.3795 \text{ GeV}$ .

The result Eq. (33) provides an alternate model wave function, which can be treated using the four different wave functions discussed above.

The use of this wave function is shown in Fig. 9. We see that the general shape and cross sections are about the same as obtained using the wave function of Sect. II. In particular, the requirement that  $Q^2$  values of 10's of  $\text{GeV}^2$  is reached to achieve scaling again occurs. Fig. 10 shows again that the differences between using the exact spectator wave function instead of the light-front wave function are very substantial. Similarly, the non-relativistic approximation fails, see Fig. 11. Thus the large effects of virtuality shown in the previous section seem to be general.

A final remark is that the model of Eq. (33) can be generalized to match to any s-wave function in the non-relativistic limit.

## VIII. VIRTUALITY OF SPIN 1/2 FERMIONS

Previous sections used a simple model involving spin-less particles. In this section, we consider what happens when the virtual particle is a fermion. There is a lore stating that when evaluating matrix elements of so-called “good currents”, with matrix elements that go to infinity in the infinite momentum frame, that the struck particles may be regarded as being on shell. This lore is not generally correct, as we shall show. The Feynman propagator of a virtual fermion of four momentum  $p$  can be written as

$$\frac{1}{\not{p} - m + i\epsilon} = \frac{(\not{p} + m)}{p^2 - m^2 + i\epsilon} = \frac{\sum_s u(p,s)\bar{u}(p,s)}{p^2 - m^2 + i\epsilon} + \frac{\gamma^+}{2p^+}, \quad p^+ > 0 \quad (36)$$

$$= -\frac{\sum_s v(p,s)\bar{v}(-p,s)}{p^2 - m^2 + i\epsilon} + \frac{\gamma^+}{2p^+}, \quad p^+ < 0. \quad (37)$$

In our notation the good current contains the operator  $\gamma^+$ , and  $\gamma^{+2} = 0$ , so the second terms of Eq. (36) and Eq. (37) proportional to  $\gamma^+$  do not contribute. Then one may use only on-shell spinors. This is the origin of the lore. However, there are two kinds of on-shell spinors, depending on whether the energy is positive or negative. For a virtual particle the value of  $p^+$  can be positive or negative, so that one may not neglect the possibility of intermediate negative energy states having a significant influence.

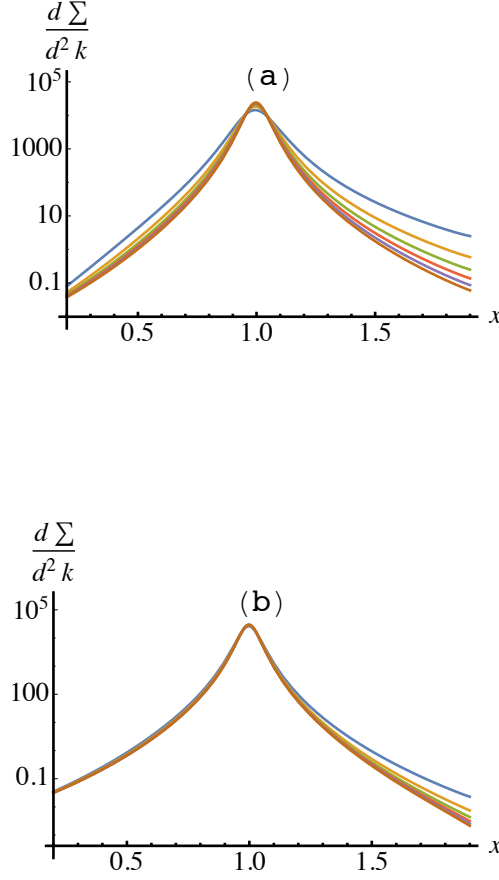


FIG. 9: (Color online)  $\frac{d\Sigma}{d^2k_\perp}$  as a function of  $x$  for two different ranges of  $Q^2$  with  $k_\perp = 0$ . (a)  $Q^2$  from 1 to 6 GeV<sup>2</sup>. (b)  $Q^2$  from 10 to 60 GeV<sup>2</sup>. For each case, the lower the value of  $Q^2$ , the higher the cross section. For larger values of  $Q^2$  the curves tend to coalesce. Deuteron wave function of Eq. (33)

## IX. SUMMARY AND DISCUSSION

The need to understand the kinematic region in which  $x$  is not close to unity provides new challenges to relativistic treatments of the nuclear wave function. This need is driven by several experiments that aim at either determining deuteron structure or using known deuteron wave functions to determine neutron structure. In this paper, simple models are used to show that applying commonly used reductions of the Bethe-Salpeter equation from four dimensions to three dimensions severely compromises the ability to compute accurate cross sections for the interesting kinematic region in which the Bjorken  $x$  variable differs from unity. The only exact approach involves using the light-front-spectator wave function. In this

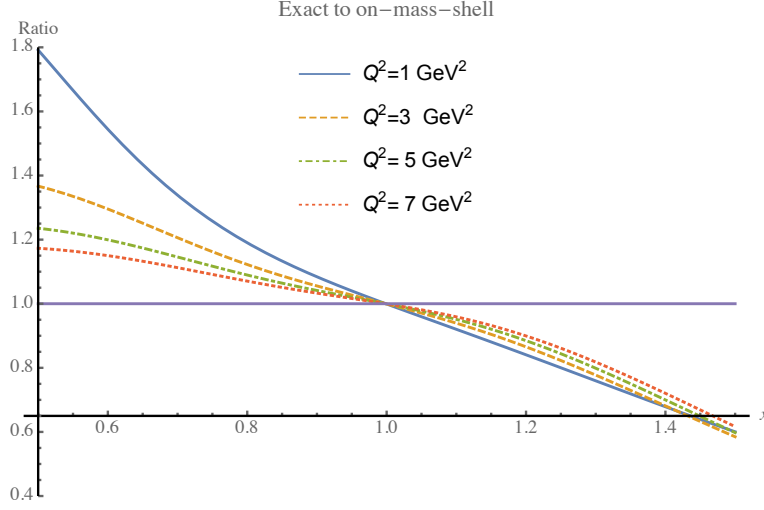


FIG. 10: (Color online) Exact to on-mass shell ratios as a function of  $x$ ,  $k_{\perp} = 0$  for different values of  $Q^2$ . Deuteron wave function of Eq. (33)

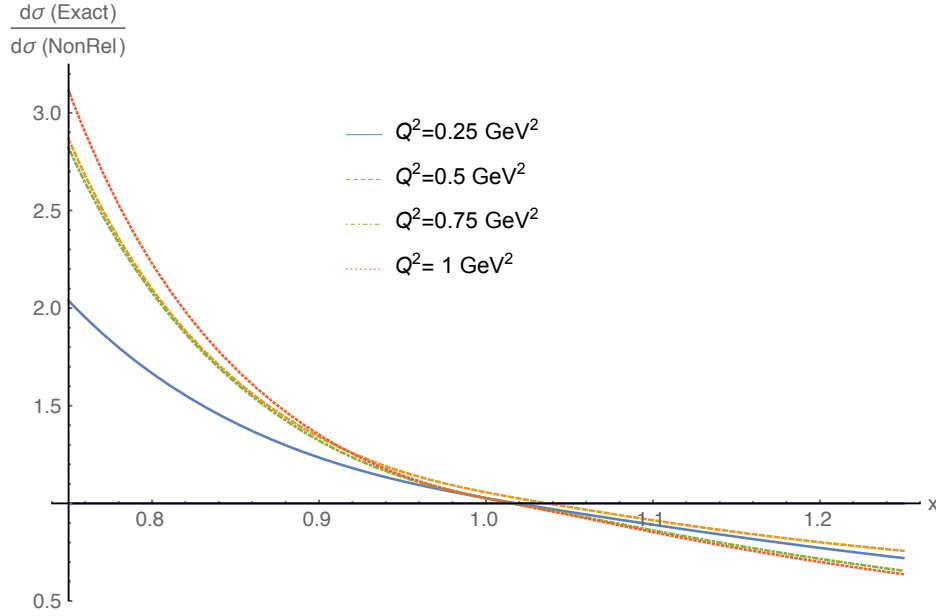


FIG. 11: (Color online) Non-relativistic approximation. Ratios of exact to non-relativistic cross sections are shown as a function of  $x$  for different values of  $Q^2$ . Deuteron wave function of Eq. (33)

case the wave function consists of one virtual constituent and one on-shell constituent, and the correct model scattering amplitude is obtained using the model's one-body current.

An alternative approach would be to use realistic nucleon-nucleon interactions. These can be interpreted relativistically [35], but the present results suggest that a complete treatment of the problem at hand would involve deriving the appropriate two-body current operators. Such



two-body currents are needed in the other quasipotential methods (involving only on-mass-shell constituents) discussed here to describe the physics that appears automatically in the impulse approximation to the spectator equation. The simple model used here demonstrates that the impulse approximation to the spectator equations already contain important terms that appear as two-body currents in other formalisms. The potential importance of these contributions for large  $x$  physics applications has implications for approved experiments at JLAB.

One current experiment [21] planned to specifically test the use of light-front spectator versus light-front wave functions is particularly relevant. The considerations presented here encourage us to predict that only the light-front spectator wave functions would reproduce the experimental results. The present results indicate that the accurate interpretation of future experiments would be simplified by developing realistic relativistic light-front spectator nuclear wave functions.

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