Screening of nucleon electric dipole moments in nuclei

Satoru Inoue, Vladimir Gudkov, Matthias R. Schindler, and Young-Ho Song

Phys. Rev. C **93**, 055501 — Published 11 May 2016

DOI: 10.1103/PhysRevC.93.055501

Screening of Nucleon Electric Dipole Moments in Nuclei

Satoru Inoue,[1,*] Vladimir Gudkov,[1,†] Matthias R. Schindler,[1,‡] and Young-Ho Song[2,§]

[1]*Department of Physics and Astronomy, University of South Carolina, Columbia, SC 29208*

[2]*Rare Isotope Science Project, Institute for Basic Science, Daejeon 305-811, Korea*

Abstract

A partial screening of nucleon electric dipole moments (EDMs) in nuclear systems, which is related to the Schiff mechanism known for neutral atomic systems, is discussed. It is shown that the direct contribution from the neutron EDM to the deuteron EDM is partially screened by about 1% in a zero-range approximation calculation.

PACS numbers: 24.80.+y, 11.30.Er, 21.10.Ky

*Electronic address: `inoue@mailbox.sc.edu`

†Electronic address: `gudkov@sc.edu`

‡Electronic address: `mschindl@mailbox.sc.edu`

§Electronic address: `yhsong@ibs.re.kr`

I. INTRODUCTION

The possible observation of permanent electric dipole moments (EDMs), which violate both time-reversal invariance and parity [1], can be important evidence of physics beyond the Standard Model, and EDMs have been the subject of intense experimental and theoretical investigations for more than 50 years (see, for example, recent reviews [2–5] and references therein).

The Schiff theorem (see [6] and its extensions [7–11]) states that in a neutral system of non-relativistic point-like particles with Coulomb interactions, the particles' intrinsic electric dipole moments are completely screened. As a consequence, in a neutral atomic system the nuclear EDM is screened and only the residual EDM of the nucleus after screening, known as the nuclear Schiff moment, can be observed. Though the measurement of the nuclear EDM without electrons would have less uncertainty from a theoretical point of view, the acceleration of the charged nucleus in an electric field made this approach impractical and most EDM searches have focused on neutral systems. However, with the recent advance in using the fine-tuned momentum technique in storage rings, direct measurements of nuclear EDMs will be feasible in the near future [12–15].

The value of the nuclear EDM can be defined (see, for example [16, 17] and references therein) as

$$\vec{d} = \langle JJ|\hat{D}_{\not{T}\not{P}}|JJ\rangle, \tag{1}$$

where $|JJ\rangle$ is a nuclear state with total spin J and its projection also equal to J. The EDM operator $\hat{D}_{\not{T}\not{P}}$ contains direct contributions from the intrinsic nucleon EDMs,

$$\hat{D}^{nucleon}_{\not{T}\not{P}} = \sum_i \frac{1}{2}[(d_p + d_n) + (d_p - d_n)\tau_i^z]\boldsymbol{\sigma}_i \tag{2}$$

and contributions from the nuclear EDM polarization operator,

$$\hat{D}^{pol}_{\not{T}\not{P}} = \sum_i Q_i \boldsymbol{r}_i, \tag{3}$$

which describes the polarization of the nucleus due to time reversal invariance violating (TRIV) potentials. Here, d_n and d_p are the neutron and proton EDMs, and Q_i and r_i are the charge and position of the i-th nucleon in center-of-mass coordinates. A non-zero nuclear EDM results in an energy shift of the system in an external electric field. Usually, contributions from the polarization operator are larger then contributions from the intrinsic nucleon EDMs, however both contributions are important.

In this paper we show that a partial screening of intrinsic EDMs, similar to the one considered in the Schiff theorem, can occur in a charged system of particles which also interact by strong interactions. The origin of this effect lies in the interactions of the individual EDMs with the electric field created by the charged particles in the system. While there has been recent work to calculate the EDMs of light nuclei [16–22], this effect has not been considered. A one-photon exchange contribution to the nucleon-nucleon potential in which one of the vertices is the nucleon EDM was derived in an effective field theory framework in Ref. [23], but its effects have not been explicitly considered in subsequent calculations. We will estimate the magnitude of this screening in the simple case of deuterons, using first a zero-range approximation, and then a square well potential. Finally, we discuss the generalization to the case of larger nuclei, where this screening effect also exists.

II. SCHIFF THEOREM

For completeness we present a short proof of the Schiff theorem that applies to neutral systems, and can be adapted to charged systems (see [11] for an example of the Schiff theorem applied to ions). The total Hamiltonian of non-relativistic particles in a constant electric $\vec{E}$ field, interacting through electrostatic forces can be written as:

$$H = T + V_{C-C} + V_{C-D} + V_C^{\rm ext} + V_D^{\rm ext}, \tag{4}$$

where

$$T = -\sum_{i=1}^{N} \frac{\vec{\nabla}_i^2}{2m_i}, \tag{5}$$

$$V_{C-C} = \frac{1}{2}\sum_{i\neq j} \frac{Q_i Q_j}{|\vec{x}_i - \vec{x}_j|}, \tag{6}$$

$$V_{C-D} = \sum_{i\neq j} Q_i \vec{d}_j \cdot \vec{\nabla}_j \frac{1}{|\vec{x}_i - \vec{x}_j|} = -\sum_{i\neq j} Q_i \vec{d}_j \cdot \vec{\nabla}_i \frac{1}{|\vec{x}_i - \vec{x}_j|}, \tag{7}$$

$$V_C^{\rm ext} = -\sum_i Q_i \vec{x}_i \cdot \vec{E}, \tag{8}$$

$$V_D^{\rm ext} = -\sum_i \vec{d}_i \cdot \vec{E}, \tag{9}$$

and $\vec{d}_i$ is the intrinsic EDM of the i-th particle.

We consider the unperturbed Hamiltonian (which is different from the Hamiltonian in Ref. [6], but analogous to the approach of Ref. [9])

$$H_0 = T + V_{C-C}, \tag{10}$$

assuming that the energy levels of the unperturbed system are known,

$$H_0|n\rangle = E_n|n\rangle. \tag{11}$$

To demonstrate the screening effect, we calculate the energy shift ΔE_n due to the potential $V = V_{C-D} + V_C^{\text{ext}} + V_D^{\text{ext}}$ that is linear in both the external field and the intrinsic EDMs,

$$\Delta E_2 = \langle n|V_D^{\text{ext}}|n\rangle + \left(\langle n|V_C^{\text{ext}} \sum_{m\neq n} \frac{|m\rangle\langle m|}{E_n - E_m} V_{C-D}|n\rangle + \langle n|V_{C-D} \sum_{m\neq n} \frac{|m\rangle\langle m|}{E_n - E_m} V_C^{\text{ext}}|n\rangle \right). \tag{12}$$

We refer to the first term on the right-hand side of Eq. (12) as the direct term, and to the remaining terms as the indirect one. Introducing the displacement operator (note that here all particles are charged)

$$A \equiv \sum_i \frac{\vec{d}_i \cdot \vec{\nabla}_i}{Q_i}, \tag{13}$$

which commutes with T and satisfies the following operator identities,

$$[A, V_{C-C}] = V_{C-D}, \tag{14}$$

$$[A, V_C^{\text{ext}}] = V_D^{\text{ext}}, \tag{15}$$

one can re-write the indirect term as

$$\begin{aligned}
&\langle n|V_C^{\text{ext}} \sum_{m\neq n} \frac{|m\rangle\langle m|}{E_n - E_m} V_{C-D}|n\rangle + \langle n|V_{C-D} \sum_{m\neq n} \frac{|m\rangle\langle m|}{E_n - E_m} V_C^{\text{ext}}|n\rangle \\
&= \langle n|V_C^{\text{ext}} \sum_{m\neq n} \frac{|m\rangle\langle m|}{E_n - E_m} [A, H_0]|n\rangle + \langle n|[A, H_0] \sum_{m\neq n} \frac{|m\rangle\langle m|}{E_n - E_m} V_C^{\text{ext}}|n\rangle \\
&= \sum_{m\neq n} \left(\langle n|V_C^{\text{ext}}|m\rangle\langle m|A|n\rangle - \langle n|A|m\rangle\langle m|V_C^{\text{ext}}|n\rangle \right) \\
&= \langle n| - [A, V_C^{\text{ext}}]|n\rangle \\
&= -\langle n|V_D^{\text{ext}}|n\rangle.
\end{aligned} \tag{16}$$

This term exactly cancels the direct contribution and thus proves Schiff's theorem. Note that the potential V_{C-D}, which describes the interactions of intrinsic particle EDMs with the electric fields from other particles, is essential for the proof of the cancellation.

To estimate the possible cancellation of nucleon EDMs in nuclei we apply a similar formalism. However, in this case one has to take into account some additional features: (a) the neutrality of some of the constituents (neutrons), (b) the acceleration of the whole charged system (nucleus) in the external electric field; and (c) the presence of strong nucleon-nucleon interactions. Unfortunately, strong interactions cannot be treated analytically by introducing an explicit strong interaction potential into Eqs. (4) and (11). Therefore, the energy shift of Eq. (12) has to be calculated numerically for each particular nucleus. In this paper we consider the deuteron, which is the simplest system proposed for nuclear EDM measurements in storage rings [12–15]. In future work, we plan to further investigate the screening of EDMs in heavier nuclei, which would be required for precision calculations of nuclear EDMs.

III. DEUTERON

The deuteron Hamiltonian in an external constant electric field can be written as

$$H = \frac{p_p^2}{2m_p} + \frac{p_n^2}{2m_n} + V(\vec{x}_p, \vec{\sigma}_p, \vec{x}_n, \vec{\sigma}_n) + e\vec{d}_n \cdot \vec{\nabla}_n \left(\frac{1}{|\vec{x}_p - \vec{x}_n|} \right) - e\vec{x}_p \cdot \vec{E} - (\vec{d}_p + \vec{d}_n) \cdot \vec{E}, \quad (17)$$

where $\vec{d}_p$ and $\vec{d}_n$ are the proton and neutron EDMs. $V(\vec{x}_p, \vec{\sigma}_p, \vec{x}_n, \vec{\sigma}_n)$ is the strong two-body potential that binds the nucleons. Here we do not consider TRIV nucleon-nucleon interactions because their contribution to the screening effect is of higher order in perturbation theory, and we find screening already without these interactions. After separating the motion of the center-of-mass $\vec{X}$ by changing coordinate variables,

$$\vec{X} \equiv \frac{m_p\vec{x}_p + m_n\vec{x}_n}{m_p + m_n}, \quad \vec{r} \equiv \vec{x}_p - \vec{x}_n, \quad (18)$$

we obtain the Hamiltonian

$$H = \frac{p_X^2}{2M} + \frac{p_r^2}{2\mu} + V(r, \vec{\sigma}_p, \vec{\sigma}_n) - e\vec{d}_n \cdot \vec{\nabla}_r \left(\frac{1}{r} \right) - e\vec{X} \cdot \vec{E} - \frac{em_n}{M}\vec{r} \cdot \vec{E} - (\vec{d}_p + \vec{d}_n) \cdot \vec{E}, \quad (19)$$

where $M = m_p + m_n = 2m_N$ is the total mass and $\mu = m_p m_n/(m_p + m_n) \approx m_N/2$ is the reduced mass, while m_N denotes the average nucleon mass. The center-of-mass motion is described by

$$H_X = \frac{p_X^2}{2M} - e\vec{X}\cdot\vec{E}, \tag{20}$$

and the remainder describes the relevant physics,

$$H_r = \frac{p_r^2}{2\mu} + V(r,\vec{\sigma}_p,\vec{\sigma}_n) - e\vec{d}_n\cdot\vec{\nabla}_r\left(\frac{1}{r}\right) - \frac{em_n}{M}\vec{r}\cdot\vec{E} - (\vec{d}_p+\vec{d}_n)\cdot\vec{E}. \tag{21}$$

To estimate the magnitude of the screening we consider the deuteron in the zero-range approximation. This model was first applied to deuteron EDM calculations in Ref. [24]. In this simplistic model the deuteron is described by the S-wave component of its wave function, which is taken as

$$\begin{aligned}\psi_d(\vec{r}) &= R_0(r)Y_0^0(\hat{r})\\ &= \sqrt{\frac{\kappa}{2\pi}}\frac{e^{-\kappa r}}{r}\ .\end{aligned} \tag{22}$$

Here, $\kappa = \sqrt{m_N E_B}$ is the deuteron binding momentum, with the deuteron binding energy $E_B = 2.23\,\mathrm{MeV}$. Since V_C^{ext} and V_{C-D}^{ext} carry orbital angular momentum of 1, intermediate states in Eq. (12) must be $L=1$ components of scattering states. The zero-range potential does not affect $L=1$ states, and we can use the $L=1$ components of plane waves, which are

$$\begin{aligned}\Psi_{1,\vec{k}}(\vec{r}) &= R_{1,k}(r)\sum_{m=-1}^{1}Y_1^{m*}(\hat{k})Y_1^m(\hat{r})\\ &= \frac{4\pi i}{(2\pi)^{3/2}}\,j_1(kr)\sum_{m=-1}^{1}Y_1^{m*}(\hat{k})Y_1^m(\hat{r})\ .\end{aligned} \tag{23}$$

Thus, using these wave functions, the matrix elements $\langle 0|\vec{r}|1,\vec{k}\rangle$ and $\langle 0|\frac{\vec{r}}{r^3}|1,\vec{k}\rangle$ contributing to the second-order contribution in Eq. (12) can be calculated analytically.

The angular parts of the coordinate space integrals can be performed using the orthogonality of the spherical harmonics. The radial parts of the resulting matrix elements are given by

$$I_n(k) = \int dr\, r^n \frac{e^{-\kappa r}}{r}\,j_1(kr), \tag{24}$$

with $n=3$ and $n=0$, respectively, resulting in

$$I_3(k) = \frac{2k}{(k^2+\kappa^2)^2}, \tag{25}$$

$$I_0(k) = -\frac{\kappa}{2k} + \frac{k^2+\kappa^2}{2k^2}\arctan\left(\frac{k}{\kappa}\right). \tag{26}$$

Using these expressions as well as

$$E_0 = -E_B = -\frac{\kappa^2}{m_N} \ , \quad E_{1,k} = \frac{k^2}{m_N} \ , \tag{27}$$

the second-order energy shift ΔE_2 is given by[1]

$$\begin{aligned} \Delta E_2 &= \frac{4e^2\kappa m_N}{3\pi}\vec{d}_n \cdot \vec{E} \int_0^\infty dk \frac{k^2}{\kappa^2 + k^2} I_3(k) I_0(k) \\ &= \frac{\alpha}{12}\sqrt{\frac{m_N}{E_B}}\, \vec{d}_n \cdot \vec{E} \\ &\approx 0.013\, \vec{d}_n \cdot \vec{E} \ . \end{aligned} \tag{28}$$

Therefore, in the deuteron the direct contribution from the neutron EDM is partially screened by about 1%. This is consistent with the power counting estimate that can be made from Ref. [23], in which TV potentials based on one-photon exchange as well as on one-pion exchange are derived. Comparing the spin-isospin structure of the two potentials and using the result for the one-pion-exchange contribution to the deuteron EDM derived in Ref. [25], one can estimate the effect considered here to be about 1%. Because the neutron is not charged, there is no corresponding screening of the proton EDM. As seen from Eq. (28), the value of the screening would be reduced if the deuteron were more deeply bound. Strictly speaking, however, the zero-range approximation rests on the fact that the deuteron binding energy is small, and we cannot extrapolate to a deeply bound system. The power counting estimate would also not be appropriate for heavier systems in which the typical nucleon momenta are too large for the application of the EFT framework of Ref. [23].

To estimate the uncertainty of these calculations, we include effective range effects in the deuteron wave function of Eq. (22) by multiplying it by a factor of $(1 - \kappa\rho_t)^{-1/2}$, where $\rho_t \approx 1.76$ fm is the ${}^3S_1$ effective range. The second-order energy shift increases to

$$\Delta E_2 \approx 0.021\, \vec{d}_n \cdot \vec{E} \ , \tag{29}$$

which is comparable with the zero-range approximation result of Eq. (28).

To study how deviations from the zero-range potential can affect the size of the screening, we consider a square well potential as a model of the deuteron,

$$V(\vec{r}) = \begin{cases} -V_0, & r < r_0 \\ 0, & r \geq r_0 \ , \end{cases} \tag{30}$$

[1] We thank J. de Vries, C. Hanhart, A. Nogga, and A. Wirzba for pointing out a numerical error in our calculation.

with the parameters tuned to reproduce the deuteron binding energy $E_B = 2.23\,\text{MeV}$. In this model the deuteron radial wave function is

$$R_0(r) = \begin{cases} N_1 j_0(k'r)\ , & r < r_0 \\ N_2 \frac{e^{-\kappa r}}{\kappa r}, & r > r_0\ , \end{cases} \tag{31}$$

where

$$E_0 = -E_B = -\frac{\kappa^2}{m_N}\ , \quad V_0 - E_0 = \frac{k'^2}{m_N}\ . \tag{32}$$

Continuity of $R_0(r)$ and its derivative at $r = r_0$ requires the relation

$$k' \cot k' r_0 = -\kappa\ . \tag{33}$$

For a given r_0, we fix V_0 to be the smallest positive energy that satisfies Eq. (33).[2] The normalization coefficients $N_{1,2}$ are also fixed numerically so that the wave function is continuous and normalized to 1.

The $L = 1$ component of the scattering wave function is given by

$$\begin{aligned} \Psi_{1,\vec{k}}(\vec{r}) &= R_{1,k}(r) \sum_{m=-1}^{1} Y_1^{m*}(\hat{k}) Y_1^m(\hat{r}) \\ &= \frac{4\pi i}{(2\pi)^{3/2}} \sum_{m=-1}^{1} Y_1^{m*}(\hat{k}) Y_1^m(\hat{r}) \begin{cases} N_3 j_1(k''r), & r < r_0 \\ \left[j_1(kr) + ik f_1(k) h_1^{(1)}(kr) \right], & r > r_0, \end{cases} \end{aligned} \tag{34}$$

where $k'' = \sqrt{m_N(V_0 + E)}$. The scattering amplitude, f_1, and the normalization constant, N_3, are again found from continuity conditions. This solution is exact for a square well.

To determine the energy shift, we calculate the matrix elements I_3 and I_0 as defined in Eq. (24). The numerical results for the second-order energy shift ΔE_2 for different values of r_0 are given in Table I. As expected, the result converges to the zero-range value as $r_0 \to 0$. Despite the considerable r_0 dependence of ΔE_2 in the square well model, all values are within the expected range given by the estimates of Eqs. (28) and (29), and the screening is least in the simplistic zero-range approximation. Therefore, there is no indication that this effect can vanish in an improved treatment of deuteron and P-wave scattering wave functions.

[2] There are infinitely many solutions to V_0 in this equation, but other solutions correspond to deeper potentials with an excited state with binding energy E_B.

TABLE I: The second-order energy shift, ΔE_2, for selected values of r_0. The second column is the analytic zero-range result.

r_0 (fm)	0	0.1	0.5	1.0	1.5	2.0
ΔE_2 ($\vec{d}_n \cdot \vec{E}$)	0.013	0.013	0.014	0.015	0.017	0.018

IV. DISCUSSION

The above results show the existence of a Schiff-type screening of the neutron EDM contribution to the deuteron EDM. In the zero-range approximation, this contribution is reduced by roughly 1%. The size of the effect is at least in part due to the fact that the deuteron binding energy is very small. Within this approximation, the second-order energy shift ΔE_2 is proportional to $E_B^{-1/2}$; an increase in the binding energy may reduce the screening. However, the applicability of the zero-range approximation relies on the smallness of the binding energy.

For heavier nuclei, one must also consider the screening of proton EDM contributions due to electric fields from other protons. Our Eq. (12) applies for any nucleus, and the terms in parentheses give the size of the screening effect in general. V_{C-D} contains both proton and neutron EDMs, and in general they can both be screened; for the deuteron the proton EDM is not screened because there are no other charged particles. In the case of heavier nuclei, the strong NN potential cannot be modeled in a simple form and solved analytically. For an accurate answer, a numerical treatment for a given nucleus is required. However, we do find a scaling argument for the size of screening for heavier nuclei. The isovector nature of V_C^{ext} and V_{C-D} suggests that the intermediate states in Eq. (12) would be dominated by the giant dipole resonance [26, 27]. We considered a toy model for the dipole resonance, where proton and neutron distributions are each treated as uniform spheres, and the relative coordinate between the centers of the distributions undergoes harmonic oscillator motion. Scalings for the potentials in Eq. (12), the resonance frequency, and the nuclear radius are all known, and we find the second-order energy shift ΔE_2 to scale as $A^{2/3}$. This suggests that the screening effect would grow for larger nuclei, and that it should be considered in future calculations of nuclear EDMs.

Acknowledgments

We thank J. de Vries, C. Hanhart, A. Nogga, and A. Wirzba for pointing out the power-counting argument to estimate the expected size of the effect considered here. This material is based upon work supported by the U.S. Department of Energy Office of Science, Office of Nuclear Physics program under Award Number DE-FG02-09ER41621 (VG) and Award Number DE-SC0010300 (SI and MRS). The work of YS was supported by the Rare Isotope Science Project of the Institute for Basic Science funded by the Ministry of Science, ICT and the Future Planning and National Research Foundation of Korea (2013M7A1A1075764).

---

[1] L. Landau, Nucl.Phys. **3**, 127 (1957).

[2] V. Dmitriev and I. Khriplovich, Phys.Rept. **391**, 243 (2004).

[3] M. Pospelov and A. Ritz, Annals Phys. **318**, 119 (2005).

[4] J. Engel, M. J. Ramsey-Musolf, and U. van Kolck, Prog. Part. Nucl. Phys. **71**, 21 (2013), 1303.2371.

[5] T. Chupp and M. Ramsey-Musolf, Phys. Rev. **C91**, 035502 (2015), 1407.1064.

[6] L. I. Schiff, Phys. Rev. **132**, 2194 (1963).

[7] P. G. H. Sandars, Phys. Rev. Lett. **19**, 1396 (1967).

[8] V. V. Flambaum, I. B. Khriplovich, and O. P. Sushkov, Sov. Phys. JETP **60**, 873 (1984).

[9] C.-P. Liu, M. J. Ramsey-Musolf, W. C. Haxton, R. G. E. Timmermans, and A. E. L. Dieperink, Phys. Rev. C **76**, 035503 (2007).

[10] R. A. Sen'kov, N. Auerbach, V. V. Flambaum, and V. G. Zelevinsky, Phys. Rev. A **77**, 014101 (2008).

[11] V. V. Flambaum and A. Kozlov, Phys. Rev. **A85**, 022505 (2012), 1112.4595.

[12] I. Khriplovich, Phys.Lett. **B444**, 98 (1998).

[13] F. Farley, K. Jungmann, J. Miller, W. Morse, Y. Orlov, et al., Phys.Rev.Lett. **93**, 052001 (2004).

[14] Y. K. Semertzidis (Storage Ring EDM Collaboration), AIP Conf.Proc. **1182**, 730 (2009).

[15] A. Lehrach, B. Lorentz, W. Morse, N. Nikolaev, and F. Rathmann (2012), arXiv:1201.5773.

[16] C. P. Liu and R. G. E. Timmermans, Phys. Rev. **C70**, 055501 (2004).

[17] Y.-H. Song, R. Lazauskas, and V. Gudkov, Phys. Rev. **C87**, 015501 (2013), 1211.3762.

[18] J. de Vries, R. Higa, C.-P. Liu, E. Mereghetti, I. Stetcu, et al., Phys.Rev. **C84**, 065501 (2011).

[19] J. Bsaisou, C. Hanhart, S. Liebig, U. G. Meissner, A. Nogga, and A. Wirzba, Eur. Phys. J. **A49**, 31 (2013), 1209.6306.

[20] J. Bsaisou, J. de Vries, C. Hanhart, S. Liebig, U.-G. Meissner, D. Minossi, A. Nogga, and A. Wirzba, JHEP **03**, 104 (2015), [Erratum: JHEP05,083(2015)], 1411.5804.

[21] J. Bsaisou, U.-G. Meiner, A. Nogga, and A. Wirzba, Annals Phys. **359**, 317 (2015), 1412.5471.

[22] N. Yamanaka and E. Hiyama, Phys. Rev. **C91**, 054005 (2015), 1503.04446.

[23] C. M. Maekawa, E. Mereghetti, J. de Vries, and U. van Kolck, Nucl. Phys. **A872**, 117 (2011), 1106.6119.

[24] I. B. Khriplovich and R. A. Korkin, Nucl. Phys. **A665**, 365 (2000), nucl-th/9904081.

[25] J. de Vries, E. Mereghetti, R. G. E. Timmermans, and U. van Kolck, Phys. Rev. Lett. **107**, 091804 (2011), 1102.4068.

[26] M. Goldhaber and E. Teller, Phys. Rev. **74**, 1046 (1948).

[27] A. Bohr and B. Mottelson, *Nuclear Structure*, vol. 2 (Benjamin, New York, 1969).