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Phys. Rev. C **91**, 044901 — Published 6 April 2015

DOI: [10.1103/PhysRevC.91.044901](https://doi.org/10.1103/PhysRevC.91.044901)

Thermally Fluctuating Second-Order Viscous Hydrodynamics and Heavy-Ion Collisions

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Abstract

The fluctuation-dissipation theorem requires the presence of thermal noise in viscous fluids. The time and length scales of heavy ion collisions are small enough so that the thermal noise can have a measurable effect on observables. Thermal noise is included in numerical simulations of high energy lead-lead collisions, increasing average values of the momentum eccentricity and contributing to its event by event fluctuations.

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I. INTRODUCTION

Observables in high energy heavy ion collisions are consistent with the formation of a very hot, nearly ideal fluid [1, 2]. It remains difficult to understand the physical reasons why the fluid thermalizes in about 1 fm/c, has shear viscosity to entropy density ratio η/s close to its proposed lower bound of $1/4\pi$, and why such a system, about 10 fm across, can be described with hydrodynamics. Because of the small size of the fluid, thermal fluctuations in the fluid should contribute significantly to two-particle correlations and event-by-event fluctuations. This requires extending the well-known results of [3] to relativistic hydrodynamics, preferably in the Landau frame [4]. Measuring the effect of thermal fluctuations can provide an important independent measurement of transport coefficients.

Thermal noise in the energy-momentum tensor has an autocorrelation function proportional to $\delta^4(x-x')$, which makes the variance of the cell-averaged energy and momentum densities proportional to $1/(\Delta V \Delta t)$. As a result, no matter how small the viscosity, there exists a minimum spatial grid size below which the results of non-perturbative, thermally fluctuating hydrodynamic simulations using white noise are unreliable and plagued by pathologies such as negative energy densities and gradients so large as to negate the application of hydrodynamics. This limit is not just a numerical artifact but is related to the coarse graining implicit in hydrodynamics. Examination of thermal noise in [5] showed how the transport coefficients themselves encode the limit of resolution of hydrodynamics; including thermal noise in the most straightforward way has a limiting resolution built into it, unlike the algorithms without noise. Viscous hydrodynamical codes for heavy-ion collisions even without thermal noise will have, for very limited times and spatial extents, viscous corrections that lead to unphysical pressures in the ideal part of the energy-momentum tensor $T^{\mu\nu}$. The regions with large viscous corrections are outside of the freeze-out surface; hydrodynamical codes have methods which either turn off or tame the large viscous corrections in this region as a practical matter of keeping these terms from disturbing the simulation where hydrodynamics is valid [6]. With the introduction of the noise term, these unphysical pressures occur more often, and these *ad hoc* methods are called more frequently to the point where one should be more skeptical of the accuracy of the results of these codes.

There appear to be two options to address the aforementioned difficulty. If thermal fluctuations are to be included self-consistently and non-perturbatively in the space-time

evolution of the system, it is probably necessary to use colored rather than white noise. White noise correlators are proportional to Dirac delta-functions in space and time. Their use is justified if the grid cells are larger than or at least comparable to the correlation length. At sufficiently fine resolutions, the noise should actually be colored and correlated across cells. Colored noise for the energy-momentum tensor appropriate for the matter created in heavy ion collisions has not been worked out, although for the baryon current it has been [7]. The other option is to treat the noise as a perturbation on a noise-less background. That is the approach first implemented in the Bjorken 1+1 dimensional fluid model in [4]. It ought to be an accurate description of thermal fluctuations unless the equation of state has some critical behavior where fluctuations would be greatly amplified and carry the system to states far away from the average one; see, for example, [8].

In this paper we will use the 3+1 dimensional second-order viscous code MUSIC [9, 14] and treat thermal fluctuations perturbatively. A similar effort is discussed in [10]. We will apply it to Pb-Pb collisions at energies available at the LHC (Large Hadron Collider). We verify that single-particle distributions are not affected by the implementation of noise. Multi-particle distributions are affected measurably and this is demonstrated by a shift and increase in width of the momentum eccentricity distribution.

II. LINEARIZED RELATIVISTIC HYDRODYNAMICS

A. Israel-Stewart hydrodynamics and thermal noise

A robust method for simulating thermal noise must be both conservative, keeping

$$\partial_\mu T_{\text{tot}}^{\mu\nu} = 0 \tag{1}$$

where $T_{\text{tot}}^{\mu\nu}$ is the full energy momentum tensor, and be able to handle discontinuities in the thermodynamic variables.

The energy-momentum tensor is separated into an averaged part $T_0^{\mu\nu}$, the fluctuating part of the ideal energy-momentum tensor $\delta T_{\text{id}}^{\mu\nu}$ which arises from fluctuations in the local flow velocity and energy density, a similar fluctuating part of the tensor $\delta W^{\mu\nu}$, and the noise part $\Xi^{\mu\nu}$:

$$T_{\text{tot}}^{\mu\nu} = T_0^{\mu\nu} + \delta T_{\text{id}}^{\mu\nu} + \delta W^{\mu\nu} + \Xi^{\mu\nu}. \tag{2}$$

The subscript “0” signifies that the quantity is the average value and does not include thermal noise, while the prefix “ δ ” signifies that this is the contribution that fluctuates event by event with thermal expectation values. The fluctuations $\delta T_{\text{id}}^{\mu\nu}$ are determined by fluctuations in the thermodynamical variables δu^μ and δP ; the equal-time autocorrelation for these fluctuations are local and determined purely by equipartition of energy:

$$\langle \delta u^\mu(\mathbf{x}, t) \delta u^\nu(\mathbf{x}', t) \rangle = \frac{T_0}{e_0 + P_0} (u_0^\mu u_0^\nu - g^{\mu\nu}) \delta^3(\mathbf{x} - \mathbf{x}'), \quad (3)$$

$$\langle \delta P(\mathbf{x}, t) \delta P(\mathbf{x}', t) \rangle = (e_0 + P_0) T_0 \frac{\partial P}{\partial e} \delta^3(\mathbf{x} - \mathbf{x}'). \quad (4)$$

In the linearized limit, $\delta T_{\text{id}}^{\mu\nu} = -\delta P g^{\mu\nu} + \delta P (1 + (\frac{\partial e}{\partial p})) u_0^\mu u_0^\nu + (e_0 + P_0) (\delta u^\mu u_0^\nu + u_0^\mu \delta u^\nu)$, which determines its equal-time autocorrelation function.

The viscous and noise contributions to the energy-momentum tensor are tied directly to the transport coefficients. This being the case, viscous hydrodynamics has to be considered here. Additionally, because of the relativistic velocities which are commonplace to heavy-ion collisions, superluminal propagation speeds must be removed from all modes of the theory. The Israel-Stewart equations are derived using some considerations of the divergence of the entropy current, and describe the evolution of the average terms in $T_{\text{tot}}^{\mu\nu}$ [11–13]:

$$\partial_\mu T_{0\text{id}}^{\mu\nu} = -\partial_\mu W_0^{\mu\nu}, \quad (5)$$

$$\Delta_\alpha^\mu \Delta_\beta^\nu (u_0 \cdot \partial) W_0^{\alpha\beta} = -\frac{1}{\tau_\pi} (W_0^{\mu\nu} - S_0^{\mu\nu}) - \frac{4}{3} (\partial \cdot u_0) W_0^{\mu\nu}, \quad (6)$$

which are closed by the equation of state and the condition in the Landau frame that $u_{0\mu} W_0^{\mu\nu} = 0$, where $\Delta^{\mu\nu} = g^{\mu\nu} - u_0^\mu u_0^\nu$, $\Delta^\mu = \Delta^{\mu\nu} \partial_\nu$, and where the first order viscous correction in the Landau-Lifshitz frame is $S_0^{\mu\nu} = \eta (\Delta^\mu u_0^\nu + \Delta^\nu u_0^\mu - \frac{2}{3} (\partial \cdot u_0) \Delta^{\mu\nu})$. $W_0^{\mu\nu}$ is symmetric in μ and ν and transverse in the Landau-Lifshitz frame; this tensor and its numerical solution for heavy ion collisions is discussed in [14]. The linearized response of fluctuations to thermal noise is now a separate equation, found by taking the difference between Eqs. 1 and 5:

$$\partial_\mu (\delta T_{\text{id}}^{\mu\nu} + \delta W^{\mu\nu} + \Xi^{\mu\nu}) = 0. \quad (7)$$

The equation of motion for $\delta W^{\mu\nu}$ are simple to find but tedious: first, in first-order hydro-

dynamics, the viscous correction is determined with a simple equation:

$$\begin{aligned}
\delta W_{(1)}^{\mu\nu} \equiv \delta S^{\mu\nu} &= \eta(\Delta^\mu \delta u^\nu + \Delta^\nu \delta u^\mu - \frac{2}{3}(\partial \cdot \delta u)\Delta^{\mu\nu}) \\
&+ \delta\eta(\Delta^\mu u_0^\nu + \Delta^\nu u_0^\mu - \frac{2}{3}(\partial \cdot u_0)\Delta^{\mu\nu}) \\
&+ \eta(\delta\Delta^\mu u_0^\nu + \delta\Delta^\nu u_0^\mu - \frac{2}{3}(\partial \cdot u_0)\delta\Delta^{\mu\nu}), \tag{8}
\end{aligned}$$

where the variations $\delta\Delta^{\mu\nu}$ and $\delta\Delta^\mu$ are up to first order in δu^μ . Note that $\delta\eta = \frac{\partial\eta}{\partial T}\delta T$ is necessary to include for any temperature-dependent shear viscosity; it tends to be very small. The variation of the relaxation time $\delta\tau_\pi$ is ignored and in the simulation of the thermal fluctuations, τ_π is held fixed at 0.2 fm/c. One can confirm that this simple variation of $S^{\mu\nu}$ satisfies the Landau-Lifshitz condition at linear order:

$$(u_0 + \delta u)_\mu (S_0 + \delta S)^{\mu\nu} = u_{0\mu} S_0^{\mu\nu} + u_{0\mu} \delta S^{\mu\nu} + \delta u_\mu S_0^{\mu\nu} = 0. \tag{9}$$

The equation for $\partial_t \delta W^{\mu\nu}$ at second order in derivatives can be found in two ways: one is to start with the equation for $W^{\mu\nu}$,

$$(u \cdot \partial)W^{\mu\nu} = -\frac{1}{\tau_\pi}(W^{\mu\nu} - S^{\mu\nu}) - \frac{4}{3}(\partial \cdot u)W^{\mu\nu} - u^\mu((u \cdot \partial)u_\alpha)W^{\alpha\nu} - u^\nu((u \cdot \partial)u_\alpha)W^{\mu\alpha},$$

examine its fluctuations up to linear order, and include them in one equation:

$$\begin{aligned}
(u_0 \cdot \partial)\delta W^{\mu\nu} &= -\frac{1}{\tau_\pi}(\delta W^{\mu\nu} - \delta S^{\mu\nu}) - \frac{4}{3}(\partial \cdot \delta u)W_0^{\mu\nu} - \frac{4}{3}(\partial \cdot u_0)\delta W^{\mu\nu} \\
&- \delta u^\mu((u_0 \cdot \partial)u_{0\alpha})W_0^{\alpha\nu} - u_0^\mu(((\delta u \cdot \partial)u_{0\alpha})W_0^{\alpha\nu} + ((u_0 \cdot \partial)\delta u_\alpha)W_0^{\alpha\nu} + (((u_0 \cdot \partial)u_{0\alpha})\delta W^{\alpha\nu})) \\
&- \delta u^\nu((u_0 \cdot \partial)u_{0\alpha})W_0^{\alpha\mu} - u_0^\nu(((\delta u \cdot \partial)u_{0\alpha})W_0^{\alpha\mu} + ((u_0 \cdot \partial)\delta u_\alpha)W_0^{\alpha\mu} + ((u_0 \cdot \partial)u_{0\alpha})\delta W^{\alpha\mu})) \\
&- (\delta u \cdot \partial)W_0^{\mu\nu}. \tag{10}
\end{aligned}$$

Alternatively, starting with

$$\delta [\Delta_\alpha^\mu \Delta_\beta^\nu (u \cdot \partial)W^{\alpha\beta}] = -\frac{1}{\tau_\pi}(\delta W^{\mu\nu} - \delta S^{\mu\nu}),$$

(not a closed equation for δW because Δ_α^μ is singular), a closed system of equations can be obtained by requiring the viscous part of $T^{\mu\nu}$ to be transverse:

$$(u + \delta u)_\mu (W_0 + \delta W)^{\mu\nu} = u_{0\mu} W_0^{\mu\nu} + \delta u_\mu W_0^{\mu\nu} + u_{0\mu} \delta W^{\mu\nu} = 0,$$

which also yields Eq. 10.

These equations now describe linearized perturbations in Israel-Stewart hydrodynamics, and from the response function $G_R(\omega)$ in Fourier space for any mode of the theory, the autocorrelation at finite temperature

$$G_S(\omega) = \frac{2T}{\omega} \text{Im}\{G_R(\omega)\}, \quad (11)$$

which is the statement of the fluctuation-dissipation theorem. The autocorrelations $\langle \delta u^\mu \delta u^\nu(\omega, \mathbf{k}) \rangle$, $\langle \delta W^{\mu\nu} \delta W^{\alpha\beta}(\omega, \mathbf{k}) \rangle$, and even the correlation functions for products of δu^μ and $\delta W^{\mu\nu}$ can now be determined. The autocorrelation for any of these quantities can be found with simple, yet tedious, algebra in Fourier space, as was demonstrated in Appendix A of [5], while their expressions in real space are more involved. We only note here that, thanks to the transversality of $\delta W^{\mu\nu}$ in this frame, the autocorrelations of δu^i and δe remain the same as above (this is shown for a fluid at rest in [5]).

As discussed in [5], the autocorrelation of $\Xi^{\mu\nu}$ can be found using

$$\langle \partial_\mu \Xi^{\mu\nu}(\mathbf{x}, t) \partial_\alpha \Xi^{\alpha\beta}(\mathbf{x}', t') \rangle = \left\langle \partial_\mu (-\delta T_{\text{id}}^{\mu\nu} - \delta W^{\mu\nu})(\mathbf{x}, t) \times \partial_\alpha (-\delta T_{\text{id}}^{\alpha\beta} - \delta W^{\alpha\beta})(\mathbf{x}', t') \right\rangle$$

and the response functions for modes in Israel-Stewart hydrodynamics. Both [10] and [5] discuss the shape of $\Xi^{\mu\nu}$ in time: it is colored noise that does not instantly decorrelate in time, much like the momentum p in the Langevin equation. Also like p , it can be expressed as the solution to a stochastic equation itself, except that this equation now is driven by a source term of white noise, much like the noise term ξ in the Langevin equation. One can define $\delta W' \equiv \delta W + \Xi$ to make one single stochastic partial differential equation to describe the sum of these two terms:

$$\begin{aligned} (u_0 \cdot \partial) \delta W'^{\mu\nu} &= -\frac{1}{\tau_\pi} (\delta W'^{\mu\nu} - \delta S^{\mu\nu} - \xi^{\mu\nu}) - \frac{4}{3} (\partial \cdot \delta u) W_0^{\mu\nu} - \frac{4}{3} (\partial \cdot u_0) \delta W'^{\mu\nu} \\ &\quad - \delta u^\mu ((u_0 \cdot \partial) u_{0\alpha}) W_0^{\alpha\nu} - u_0^\mu (((\delta u \cdot \partial) u_{0\alpha}) W_0^{\alpha\nu} + ((u_0 \cdot \partial) \delta u_\alpha) W_0^{\alpha\nu} + (((u_0 \cdot \partial) u_{0\alpha}) \delta W'^{\alpha\nu})) \\ &\quad - \delta u^\nu ((u_0 \cdot \partial) u_{0\alpha}) W_0^{\alpha\mu} - u_0^\nu (((\delta u \cdot \partial) u_{0\alpha}) W_0^{\alpha\mu} + ((u_0 \cdot \partial) \delta u_\alpha) W_0^{\alpha\mu} + ((u_0 \cdot \partial) u_{0\alpha}) \delta W'^{\alpha\mu})) \\ &\quad - (\delta u \cdot \partial) W_0^{\mu\nu}. \end{aligned} \quad (12)$$

Note the noise term $\xi^{\mu\nu}$: it has the autocorrelation

$$\begin{aligned} \langle \xi^{\mu\nu}(x) \xi^{\alpha\beta}(x') \rangle &= \left[2\eta T_0 (\Delta^{\mu\alpha} \Delta^{\nu\beta} + \Delta^{\mu\beta} \Delta^{\nu\alpha}) \right. \\ &\quad \left. + 2(\zeta - 2\eta/3) T_0 \Delta^{\mu\nu} \Delta^{\alpha\beta} \right] \delta^4(x - x'), \end{aligned} \quad (13)$$

as discussed in [5].

This is now a closed set of equations, however some intuition for the differences between δT_{id} , $\delta W^{\mu\nu}$, and $\Xi^{\mu\nu}$ should be developed. We conclude this subsection with a discussion of fluctuations in a fluid at rest: when $u^\mu = (1, \mathbf{0})$, the averaged energy-momentum tensor

$$T^{\mu\nu} = \begin{pmatrix} e_0 & 0 & 0 & 0 \\ 0 & P_0 & 0 & 0 \\ 0 & 0 & P_0 & 0 \\ 0 & 0 & 0 & P_0 \end{pmatrix}$$

the fluctuations of the energy-momentum tensor $\delta T^{\mu\nu}$ are non-zero only when $\mu = \nu$ or when either μ or ν equals zero, and represents the coarse-graining error in the total enthalpy and flow velocity of a fluid element:

$$\delta T_{id.}^{\mu\nu} = \begin{pmatrix} \delta e & (e_0 + P_0)\delta u^1 & (e_0 + P_0)\delta u^2 & (e_0 + P_0)\delta u^3 \\ (e_0 + P_0)\delta u^1 & \delta P & 0 & 0 \\ (e_0 + P_0)\delta u^2 & 0 & \delta P & 0 \\ (e_0 + P_0)\delta u^3 & 0 & 0 & \delta P \end{pmatrix}$$

These fluctuations are not entirely random: they must obey energy and momentum conservation and their evolution in time is determined by the equation of state and transport coefficients. In a viscous fluid, there also exist fluctuations in δW^{ij} . At first order, setting the bulk viscosity $\zeta = 0$,

$$\delta W^{ij} = \eta \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2\partial_x \delta u^x - \frac{2}{3}\nabla \cdot \delta u & \partial_x \delta u^y + \partial_y \delta u^x & \partial_x \delta u^z + \partial_z \delta u^x \\ 0 & \partial_x \delta u^y + \partial_y \delta u^x & 2\partial_y \delta u^y - \frac{2}{3}\nabla \cdot \delta u & \partial_y \delta u^z + \partial_z \delta u^y \\ 0 & \partial_x \delta u^z + \partial_z \delta u^x & \partial_y \delta u^z + \partial_z \delta u^y & 2\partial_z \delta u^z - \frac{2}{3}\nabla \cdot \delta u \end{pmatrix}.$$

At first order in gradients, δW^{ij} is determined at each instant in time by derivatives of δu^i . In second order hydrodynamics, δW^{ij} evolves non-trivially in time, and it is helpful to imagine it to be an independent fluctuating variable as is suggested by the Israel-Stewart equations. Even in first-order hydrodynamics, δW^{ij} has a non-trivial autocorrelation func-

tion; in momentum space

$$\begin{aligned}
\langle \delta W^{ij} \delta W^{kl}(\mathbf{k}, \omega) \rangle &= \frac{2T}{w_0} [2\eta k^i k^j + (\zeta - 2\eta/3)\delta^{ij} k^2] [2\eta k^k k^l + (\zeta - 2\eta/3)\delta^{kl} k^2] \\
&\quad \times \text{Im} \left\{ 1/(\omega - c_s^2 |\mathbf{k}|^2 / \omega + i \frac{\zeta + 4\eta/3}{w_0} |\mathbf{k}|^2) \right\} \\
&+ \frac{2T\eta^2}{w_0} [k^i k^k \delta^{jl} + k^j k^k \delta^{il} + k^i k^l \delta^{jk} + k^j k^l \delta^{ik} - 4k^i k^j k^k k^l / |\mathbf{k}|^2] \\
&\quad \times \text{Im} \left\{ 1/(\omega + i \frac{\eta}{w_0} |\mathbf{k}|^2) \right\}.
\end{aligned}$$

In the fluid rest frame, $\Xi^{0\nu} = 0$. The physical significance of this is that $\Xi^{\mu\nu}$ does not represent the fluctuations of the energy density or momentum of the fluid elements, but instead the stochastic flux of energies and momenta between these elements. Without $\Xi^{\mu\nu}$, any $\delta T_{id}^{\mu\nu}$ should dissipate to its average value of zero in a viscous fluid, thanks to the dissipative term $\delta W^{\mu\nu}$; $\Xi^{\mu\nu}$ drives $\delta T^{\mu\nu}$ to its thermal expectation values.

In second-order hydrodynamics, $\Xi^{\mu\nu}$ has a non-trivial autocorrelation in time:

$$\langle \Xi^{\mu\nu}(\mathbf{x}, t) \Xi^{\alpha\beta}(\mathbf{x}', t') \rangle \propto \exp(-|t - t'|/\tau_\pi).$$

To achieve this autocorrelation in time, we make the $\Xi^{\mu\nu}$ itself a solution to a stochastic equation, this time driven by the otherwise unphysical $\xi^{\mu\nu}$.

B. Numerical solution of thermal noise in relativistic hydrodynamics

When using Bjorken coordinates to describe ultrarelativistic heavy-ion collisions, the derivatives ∂_μ above must be replaced with their covariant counterparts D_μ . The averaged part will be solved with the usual methods while the fluctuating and noise parts will be solved together, with the noise acting as a source term.

The steps for finding solutions for $\delta T^{\mu\nu}$ and $\delta W^{\mu\nu}$ are similar to those used for $T_0^{\mu\nu}$ and $W_0^{\mu\nu}$ in [9]:

- Determine $\delta W^{\mu\nu}$ at the next time step using the stochastic advective equation in Eq. 12. The first-order viscous term δS is calculated using δu^μ and δP from the current step.
- Next, determine $\delta T^{0\nu}$ at the next time-step using $D_0 \delta T^{0\nu} = -D_i \delta T^{i\nu} - D_\mu \delta W^{\mu\nu}$. The numerical method should be conservative; the change in $\delta W^{\mu\nu}$ during this timestep should be used to calculate $\partial_0 \delta W^{\mu\nu}$.

- Finally, determine δT^{ij} , as well as δp and δu^i , using

$$\delta T_{\text{id}}^{\mu\nu} = -\delta P g^{\mu\nu} + \delta P \left(1 + \left(\frac{\partial e}{\partial p} \right)_{n/s} \right) u_0^\mu u_0^\nu + (e_0 + P_0)(u_0^\mu \delta u^\nu + u_0^\nu \delta u^\mu) \quad (14)$$

and a root-finding algorithm using the values of $\delta T^{0\nu}$. The left hand side, $\delta T^{0\nu}$, was determined in the previous step, while δu^μ and δP are determined with this equality. Once δu^μ and δP are known, the remaining unknown terms in the energy-momentum tensor δT^{ij} can be determined for the next step.

For the first step in the list above, the MacCormack method is a predictor-corrector method which alternates between upstream and downstream differencing:

$$\begin{aligned} \bar{T}_i^{t\mu l} &= T_i^{t\mu l} - \frac{T_{i+1}^{x\mu l} - T_i^{x\mu l}}{\Delta x} \Delta t, \\ T_i^{t\mu l+1} &= \frac{T_i^{t\mu l} + \bar{T}_i^{t\mu l}}{2} - \frac{\bar{T}_i^{x\mu l} - \bar{T}_{i-1}^{x\mu l}}{2\Delta x} \Delta t, \end{aligned}$$

where $T_i^{t\mu l}$ is $T^{t\mu}$ averaged in the i -th cell at the beginning of the l -th timestep. This is written in one dimension; for three dimensions, this is iterated for each direction.

III. THERMAL FLUCTUATIONS IN HEAVY-ION COLLISIONS

Heavy-ion collisions are approximated fairly well as *boost-invariant* (only a function of τ and not η) when $\sqrt{s_{NN}}$ exceeds a few GeV. For this reason, it is advantageous to use τ - η coordinates instead of t, z , even if the hydrodynamic model is not boost-invariant. The space-time of the heavy-ion collision is still flat, but the Christoffel symbols for covariant derivatives are now non-zero:

$$\begin{aligned} D_\eta u^\tau &= \partial_\eta u^\tau + u^\eta, \\ D_\eta u^\eta &= \partial_\eta u^\eta + \frac{1}{\tau} u^\tau. \end{aligned} \quad (15)$$

The derivatives of any tensor can be determined by examining the derivatives of products of vectors. The derivatives must be modified for Bjorken coordinates.

The equation of state and transport coefficients for matter with temperatures above 120 MeV are highly non-trivial and are the focus of a continuing debate among lattice QCD practitioners and other nuclear physicists [15, 16]. These are some of the primary reasons

for studying heavy-ion collisions. Finding out how observables are sensitive to the equation of state and transport coefficients, and how to infer them from data, should be the goal of hydrodynamical simulations. For now, we use one equation of state, determined by lattice QCD calculations [15]. We also use temperature-independent values for η/s ; the relaxation time τ_π varies slowly in MUSIC as $1/T$ and for now, we approximate it as being constant at $0.2 \text{ fm}/c$ in the calculation of the fluctuations. Future work where other sources of event-by-event fluctuations are also included will use a variety of values for viscosity, relaxation time, and equation of state with the goal of using heavy-ion collisions and simulation together to determine the properties of hot nuclear matter.

The final hadronic observables are measured after the freeze-out of the flowing matter into freely streaming hadrons. The effect of non-equilibrium corrections to the energy-momentum tensor is the subject of research itself [17]. We extend the correction to freeze-out spectra discussed in [18] to include the contribution from thermal noise. The noiseless viscous correction to the thermal spectra in [18] is

$$f = f_0 + C f_0 (1 \pm f_0) W_{\alpha\beta} \frac{p^\alpha p^\beta}{2(e_0 + P_0)T^2},$$

where C is determined to make $T^{\mu\nu}$ for a fluid at freeze-out match $T^{\mu\nu}$ for the gas of particles described by f ; we will approximate it to be 1 but keep it in the equations for the sake of generality. The noise contribution δf connects both ideal and viscous fluctuations to particle spectra:

$$\begin{aligned} \delta f = & \delta f_0 + C \delta f_0 (1 \pm 2f_0) W_{\alpha\beta} \frac{p^\alpha p^\beta}{2(e_0 + P_0)T^2} + C f_0 (1 \pm f_0) \delta W'_{\alpha\beta} \frac{p^\alpha p^\beta}{2(e_0 + P_0)T^2} \\ & + C f_0 (1 \pm f_0) W_{\alpha\beta} \frac{p^\alpha p^\beta}{2(e_0 + P_0)T^2} \left(-\frac{\delta e + \delta P}{(e_0 + P_0)} - 2\frac{\delta T}{T_0} \right) \end{aligned} \quad (16)$$

where $\delta f_0 = -\frac{\exp(p \cdot u/T)}{(\exp(p \cdot u/T) \pm 1)^2} (\delta u \cdot p/T - (p \cdot u) \delta T/T_0^2)$ is the fluctuation of yields from ideal Cooper-Frye freeze-out.

Before moving on to observables at the LHC, some testing of the algorithms used for examining thermal noise must be performed. Figure 1 shows the results from simulations, in Cartesian coordinates, of thermal noise in a fluid at rest. The simulations track the cell averages of the thermodynamical variables, whose autocorrelations are proportional to $1/\Delta V$ and therefore approach the Dirac delta functions in Equations 3-4. All fluctuating variables

start at zero and are driven to non-zero values by $\xi^{\mu\nu}$. The plots show an agreement between the results of the numerical calculations and Equations 3-4 within about 5%. The cell sizes here are $0.352 \text{ fm} \times 0.352 \text{ fm} \times 0.352 \text{ fm}$; we will use the same grid sizes in the transverse directions in the following sections so that this test might estimate the error in our results.

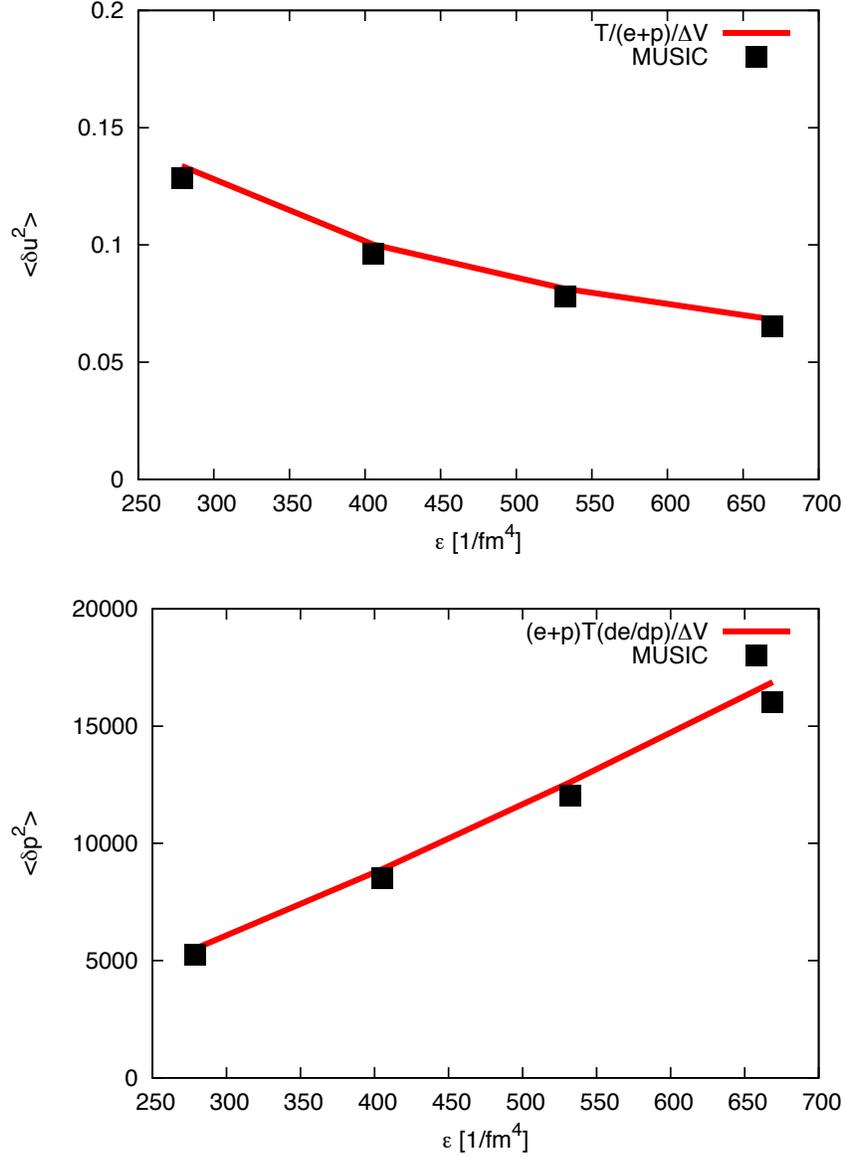


FIG. 1: The cell averages of autocorrelations $\langle \delta u^i \delta u^i / 3 \rangle$ and $\langle \delta p \delta p \rangle$ (in fm^{-8} as a function of energy density. (color online)

IV. RESULTS AT THE LHC

With these modifications, a set of 200 thermally fluctuating events with impact parameter $b = 5.99$ fm at the LHC are calculated using MUSIC. The underlying event is initialized with smooth initial conditions and with all fluctuating parts of the energy momentum tensor set to 0. The impact parameter corresponds to a typical event in the 10-20% centrality class. The cells have transverse dimensions of $\Delta x = \Delta y = 45/128$ fm ≈ 0.35 fm. The cells begin evolving at $\tau_i = 0.4$ fm/c and then all cells are frozen out by about $\tau_f = 14.9$ fm/c. Figure 2 shows the evolution of $\delta e/e_0$, the ratio of the thermal fluctuation in local energy density to its average value. In some of these cells, $\delta e/e_0$ is greater than 1; this reflects the small cell sizes and the divergence of noise in the limit $\Delta V \rightarrow 0$. The central limit theorem keeps the integrals over this noise over the entire heavy-ion collision from leading to unphysical variances in observables.

Figure 3 shows the distributions of π^+ mesons for both the case without fluctuations and for the average of 30 thermally fluctuating events. Because $\langle \delta T^{\mu\nu} \rangle = \langle \delta W^{\mu\nu} \rangle = 0$, the average of a large ensemble of thermally fluctuating events should have no effect on $\langle dN/dp_T \rangle$, which Figure 3 demonstrates. This is one important test of the methods of Section II.

However, thermal noise affects not only the average value of the harmonic coefficient v_n but also leads to event-by-event variations in v_n at the same impact parameter. Thermal noise tends to increase average values of v_n . To see this, imagine a collision with zero impact parameter. In the averaged case, this leads to a cylindrically symmetrical expansion, and $v_n = 0$ for all $n \neq 0$. However in a single event with noise, $v_n \neq 0$ in general. Because each v_n is defined to be positive, $\langle v_n \rangle \neq 0$ when thermal noise is included.

For now, to avoid the extra complications of freeze-out to individual hadrons and viscous corrections to thermal distribution functions, we calculate the momentum eccentricity

$$\epsilon_p = \sqrt{\frac{\langle T^{xx} - T^{yy} \rangle^2 + \langle 2T^{xy} \rangle^2}{\langle T^{xx} + T^{yy} \rangle^2}}, \quad (17)$$

which is a generalization of the quantity discussed in [19]. It is a proxy for v_2 . Here $T^{\mu\nu}$ represents the total energy-momentum tensor, including viscous corrections as well as noise, which gives the best approximation to $\sum_i \frac{p_i^\mu p_i^\nu}{E_i} \delta^3(\mathbf{x} - \mathbf{x}_i(t))$.

Figure 4 shows ϵ_p versus proper time for a single noisy event, an averaged event, and

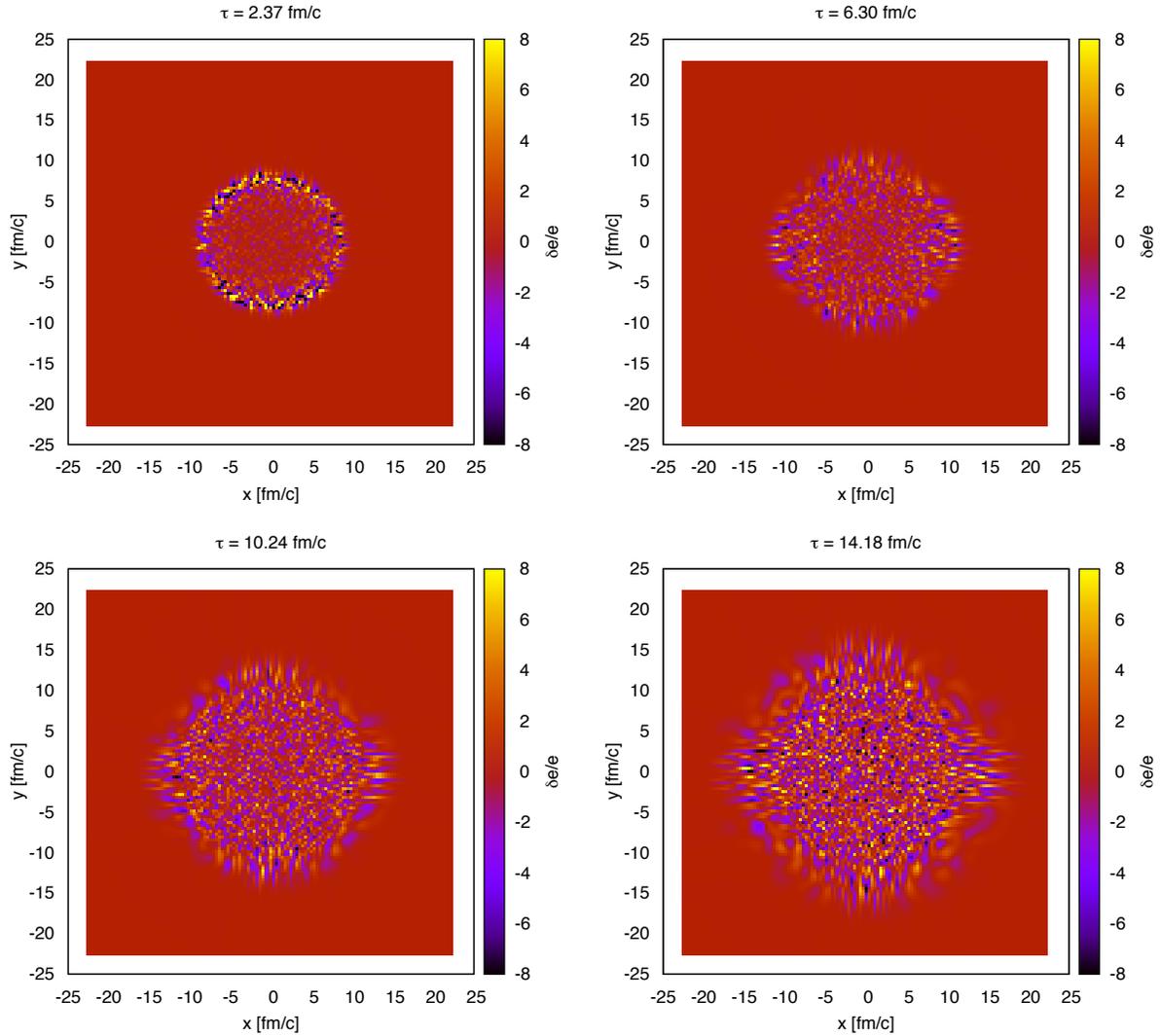


FIG. 2: Fluctuations in energy density in the transverse plane as a function of time in a typical Pb+Pb collision for $b = 5.99$ fm at LHC energies. (color online)

for the average of many noisy events. Thermal noise clearly leads to significant event-by-event variance of ϵ_p , but it also changes the average values. Interestingly, it leads to a very significant increase at early times.

The top panel of Figure 5 shows the probability distribution for events with impact parameter of $b = 5.99$ fm at the LHC. Without noise there is of course just one value. With noise there is a modest broadening of the distribution as well as an increase in the average value by about 5%. The bottom panel of figure 5 shows the distribution of ϵ_p for an “ultra-central” event with impact parameter $b = 0$ fm. Interestingly, $\langle \epsilon_p \rangle$ is non-vanishing thanks

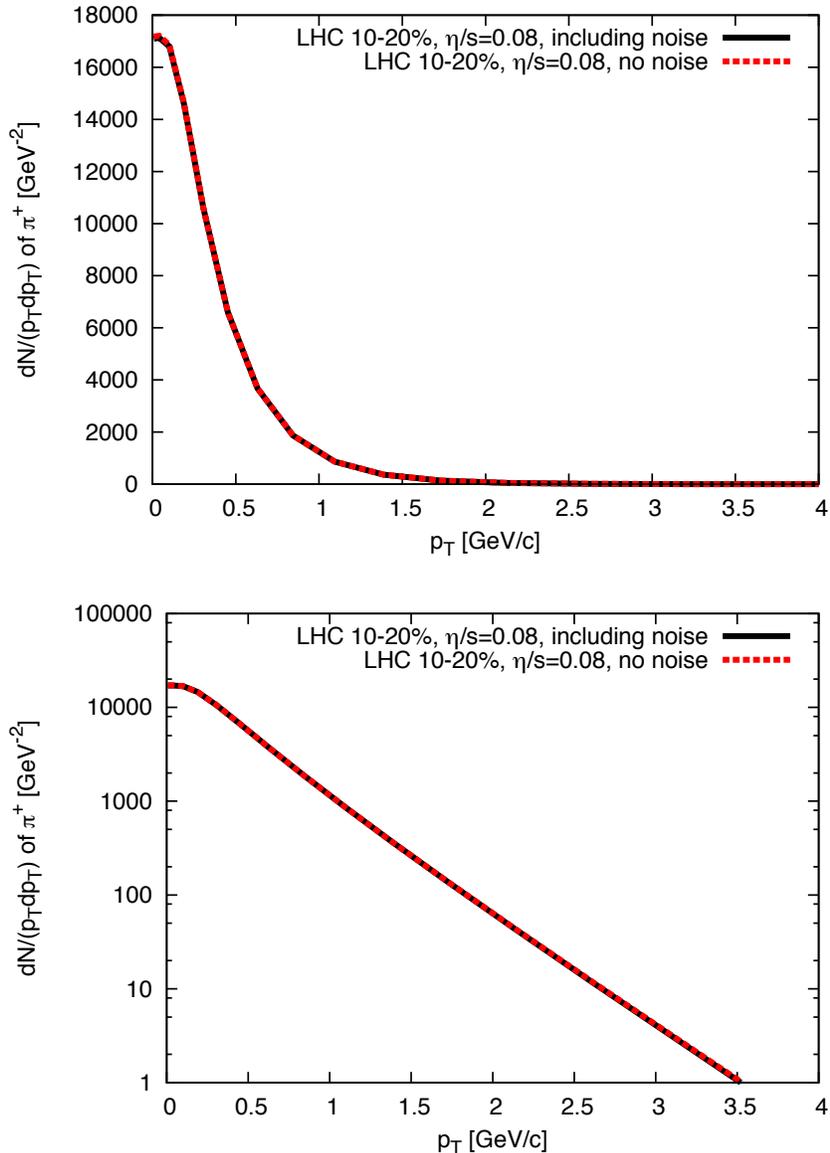


FIG. 3: Transverse momentum distribution for pions in linear and log scales showing that the single particle distribution is unaffected by noise. (color online)

to thermal noise driving all $\langle v_n \rangle$ to non-zero values, as argued above.

V. CONCLUSIONS

In this paper thermal noise has been included in second-order viscous hydrodynamics by an extension of the Israel-Stewart formalism. After some special considerations for the fluid produced in heavy ion collisions, the effect of these fluctuations was calculated for Pb-Pb

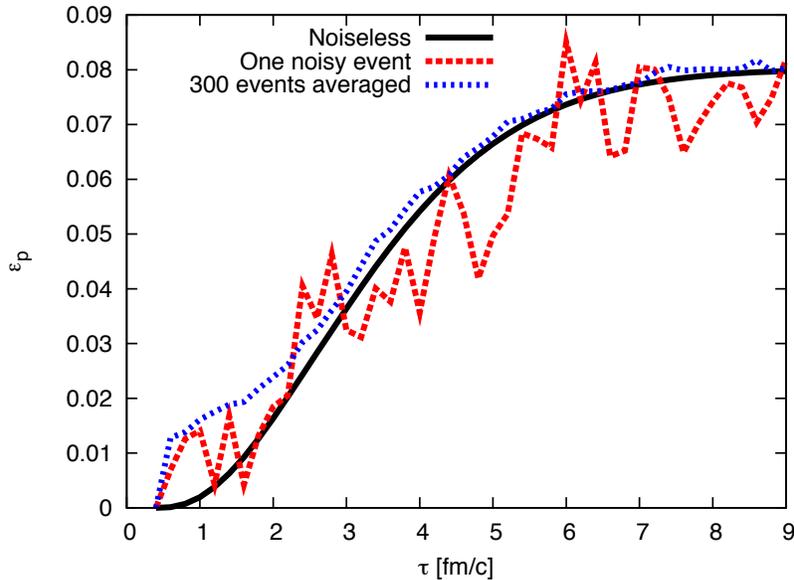


FIG. 4: The evolution in time of ϵ_p from the calculations of two $b = 5.99$ fm collisions with thermal noise as well as from a calculation without noise. Notice both the rapidly decorrelating variations in ϵ_p , coming from $\delta W'^{\mu\nu}$, as well as the variations over longer time-scales. (color online)

collisions at the LHC. A small variance was found in ϵ_p , the momentum eccentricity, for impact parameter $b = 5.99$ fm, while a significant variance was found for $b = 0$.

Thermal noise must contribute to event by event fluctuations thanks to the fluctuation-dissipation theorem; however, our results confirm that the contribution is often subleading when compared with results sampling initial-state fluctuations. Rather, the significance of thermal noise lies in its connection to transport coefficients: the variances caused by thermal noise are proportional to the transport coefficients, possibly allowing for a measurement of the shear viscosity in heavy ion collisions independent of the previous determinations based on elliptic flow. In [20], the ATLAS collaboration presented results for distributions of flow harmonics in several event classes; the variances of v_2 in the mid-central classes are approximately 0.05. The variances are explained well in many centrality classes with the methods employed in [21]. However, note the results from the “ultra-central” CMS 0-0.2% centrality class [22]. The average values and variances of v_n are on the order of our results from only thermally fluctuating hydrodynamics; the results from other calculations only explain some of the integrated v_n with unusually large values for η/s . A calculation combining initial state fluctuations and thermal fluctuations, aimed at describing the ultra-

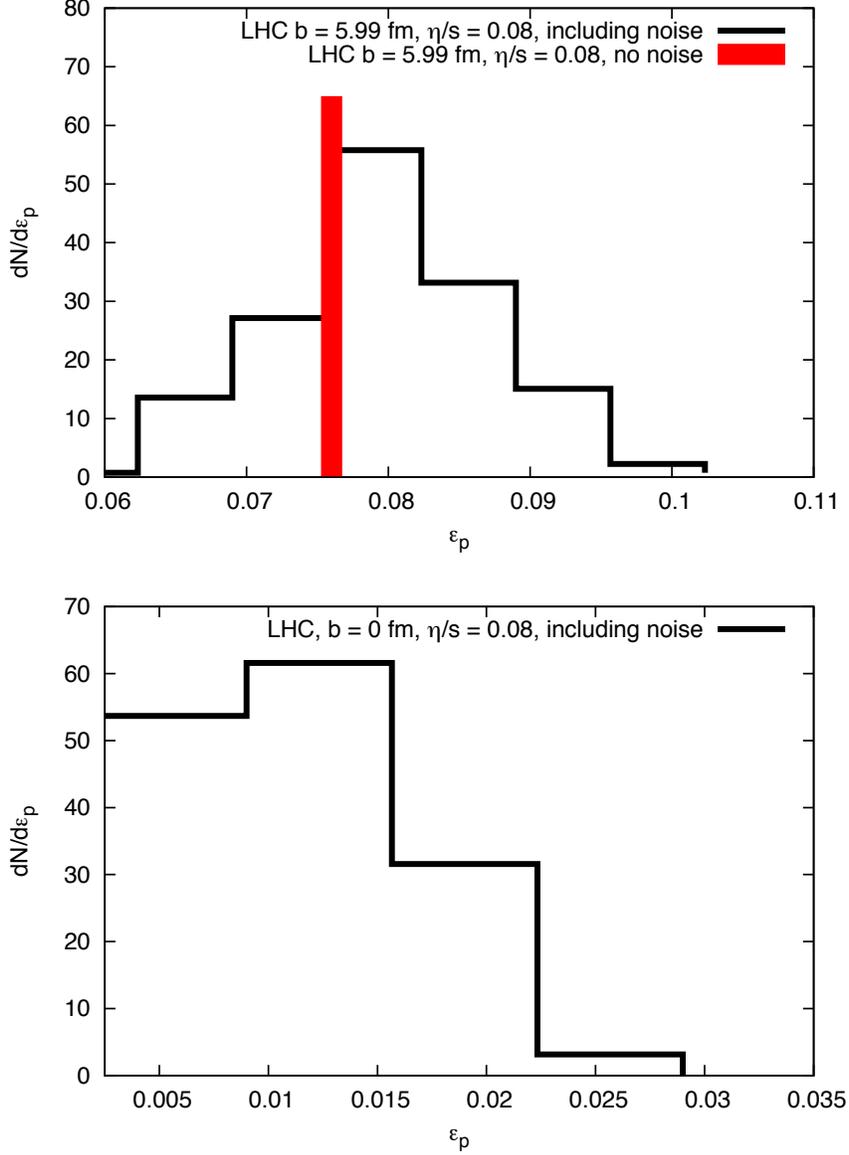


FIG. 5: Top panel ($b = 5.99$ fm): Inclusion of noise increases the average ϵ_p and broadens its distribution. Bottom panel ($b = 0$ fm): Even for exactly central collisions there is a nonzero average v_2 due to noise. (color online)

central event class, might not only explain this data but also provide another measurement of η/s .

Any attempt to disentangle the various sources of event-by-event fluctuations will necessarily utilize numerical simulations of both fluctuating initial conditions and thermal fluctuations. This is made necessary by the nonlinear response of the hydrodynamical variables

to large fluctuations in initial conditions, and by the dependence of the thermal noise on these same hydrodynamical variables. Looking at two-particle correlations in η , ϕ , and p_T both with and without thermal noise will settle once and for all where this effect can be measured.

Additional work in both the theory and simulation of thermal noise will improve the understanding of flow in heavy ion collisions. One important direction for theoretical improvements is to go beyond linear response: the quality of any truncated perturbative expansion is most reliably estimated with a full calculation at the next order. Recently, this work has become easier with the proposal of an effective action for hydrodynamics [23]. Finally, even linearized hydrodynamics poses problems for numerical simulation: the gradients become larger with decreasing cell size, seeming to rule out the possibility of using any higher-order method. In this paper we used the MacCormack method for its ability to perform well in the presence of shocks. However, a detailed investigation into the performance of numerical methods for linearized fluctuating hydrodynamics would be useful. Such work is underway.

VI. ACKNOWLEDGMENTS

JK and CY are supported by the U.S. DOE Grant No. DE-FG02-87ER40328. CG and SJ are supported by funding from the Natural Sciences and Engineering Research Council of Canada. BPS is supported under DOE Contract. No. DE- AC02-98CH10886. CY thanks E. Shuryak for very helpful suggestions. We are grateful for resources from the University of Minnesota Supercomputing Institute.

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