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Impact of the equation-of-state -- gravity degeneracy on constraining the nuclear symmetry energy from astrophysical observables

Xiao-Tao He,1,2,∗ F. J. Fattoyev,1,† Bao-An Li,1,‡ and W. G. Newton1,§

1Department of Physics and Astronomy, Texas A&M University-Commerce, Commerce, TX 75429, USA
2College of Material Science and Technology, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, China

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There is a degeneracy between the equation of state (EOS) of super-dense neutron-rich nuclear matter and the strong-field gravity in understanding properties of neutron stars. While the EOS is still poorly known, there are also longstanding ambiguities in choosing Einstein’s General Relativity (GR) or alternative gravity theories in the not-so-well tested strong-field regime. Besides possible appearance of hyperons and new phases, the most uncertain part of the nucleonic EOS is currently the density dependence of nuclear symmetry energy especially at supra-saturation densities. At the same time, the EOS of symmetric nuclear matter (SNM) has been significantly constrained at saturation and supra-saturation densities. To provide information that may help break the EOS-gravity degeneracy, we investigate effects of nuclear symmetry energy within its uncertain range determined by recent terrestrial nuclear laboratory experiments on the gravitational binding energy and space-time curvature of neutron stars within GR and the scalar-tensor subset of alternative gravity models, constrained by recent measurements of the relativistic binary pulsars J1738+0333 and J0348+0432. In particular, we focus on effects of the following three parameters characterizing the EOS of super-dense neutron-rich nucleonic matter: (1) the incompressibility $K_0$ of symmetric nuclear matter (SNM), (2) the slope $L$ of nuclear symmetry energy at saturation density and (3) the high-density behavior of nuclear symmetry energy. We find that the variation of either the density slope $L$ or the high-density behavior of nuclear symmetry energy leads to large changes in both the binding energy and curvature of neutron stars while effects of varying the more constrained $K_0$ are negligibly small. The difference in predictions using the GR and the scalar-tensor theory appears only for massive neutron stars, and even then is significantly smaller than the differences resulting from variations in the symmetry energy. We conclude that within the scalar-tensor subset of gravity models, the EOS-gravity degeneracy has been broken by the recent relativistic pulsar measurements, and that measurements of neutron star properties sensitive to the compactness constrain mainly the density dependence of the symmetry energy at saturation and supra-saturation densities.

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∗ hext@nuaa.edu.cn
† farrooh.fattoyev@tamuc.edu
‡ bao-an.li@tamuc.edu
§ william.newton@tamuc.edu
I. INTRODUCTION

Neutron stars (NSs) are among the most dense and neutron-rich objects in the Universe. They are thus a natural testing ground for fundamental theories of strong-field gravity and the nature of matter under extreme conditions. Our knowledge about super-dense matter in NSs remains quite poor, see, e.g., [1–3], and may be compounded by the possible break-down of Einstein’s General Relativity (GR) in the strong-field regime [4, 5]. There is a possible degeneracy between the nuclear equation of state (EOS) for matter in NSs and the theory of gravity applied to describe their structure. How to break this degeneracy is a longstanding problem to which many recent studies have been devoted: see, e.g., [4–19]. A better understanding of the EOS will certainly help constrain the strong-field gravity theory, and vice versa. Besides the possible appearance of hyperons and quarks in super-dense matter in NSs, the most uncertain part of the nucleonic EOS is currently the nuclear symmetry energy at supra-saturation densities [20]. Ideally, effects of the uncertain nuclear EOS should be examined simultaneously within GR and a large sample of alternative gravity theories. This however requires much collaborative work of the whole community. In this work, we examine the extent to which a gravity-EOS degeneracy exists when models of gravity are restricted to GR and Scalar-Tensor (ST) theories, and when variations of the EOS are the result of variation of (1) the incompressibility $K_0$ of symmetric nuclear matter (SNM), (2) the slope $L$ of nuclear symmetry energy at saturation density and (3) the high-density behavior of nuclear symmetry energy, within ranges constrained by terrestrial nuclear data. We find that the variation of either the density slope $L$ or the high-density behavior of nuclear symmetry energy within their uncertainty ranges lead to significant changes in both the binding energy and curvature of NSs, while effects of varying $K_0$ are negligibly small. In particular, the variations are significantly greater than those that result from ST theories of gravity, leading to the conclusion that within those subset of gravity models, measurements of neutron star properties constrain mainly the EOS.

Neutron stars are very compact: the typical surface compactness parameter of NSs is of the order of $\eta \equiv 2GM/c^2R \approx 0.4$. While it is common to measure the strength of the gravitational field with the compactness parameter, this leaves out most of the useful information related to its strength, in particular, when one would like to test the deviations of GR from alternative theories of gravity in the strong-field regime. It was shown in Ref. [21] that for the gravitational field to be considered as strong it is more natural to choose a parameter $\xi \equiv 2GM/c^2r^3$, which is related to the non-vanishing components of the Riemann tensor in vacuum, $R_{001}^3 = -\xi$. Notice that the curvature is the lowest order quantity of the gravitational field that cannot be set to zero by a coordinate transformation [21]. Recently, by using the venerable Akmal-Pandharipande EOS [22] the authors of Ref. [5] have studied the radial profile of compactness and curvature in neutron stars and have concluded that the GR is in a less tested regime than the EOS over the entire region of neutron star. In other work, by using a very stiff EOS with nuclear incompressibility of $K_0 = 546$ MeV [23], it was shown that the fractional gravitational binding energy of the PSR J0348+0432 with mass $2.01 \pm 0.04M_\odot$ [24] is twice as large as that of both NSs in the celebrated double pulsar system PSR J0737-3039, whose masses have been measured with a very high precision. Given that the observation of these pulsars resulted in some of the most precise tests of GR to date, it was concluded that the great difference in their fractional gravitational binding energies keeps PSR J0348+0432 in a special place far outside the presently tested binding energy range, and therefore this NS could serve as a tested for theories of strong-field gravity. Particularly interesting is the study of the scalar-tensor theory of gravity, whose parameters have a well-defined constraint from a ten-year pulsar timing observation of PSR J1738+0333 [25].

Disentangling the EOS-Gravity degeneracy is important if we are to make use of astrophysical observables to measure neutron star properties. Potential observables directly related to the neutron star compactness include the binding energy [26, 27], surface redshift [28] and maximum spin frequency of a neutron star [29], and it is these observables that we shall focus on in this paper.

The nuclear symmetry energy $S(\rho)$ represents the penalty imposed on the energy of the nuclear system as one departs from the symmetric limit of equal number of protons and neutrons at density $\rho$. Within the parabolic approximation of the nuclear EOS for neutron-rich matter, $S(\rho)$ is simply the difference in specific energy between pure neutron matter (PNM) and SNM. The bulk parameters characterizing the EOS of SNM near nuclear saturation density $\rho_0$ are well constrained from fitting models to the ground state properties of finite nuclei, and to the breathing-mode energies of giant resonances in medium-to-heavy nuclei [30–37]. The incompressibility of nuclear matter is constrained in the far tighter range of $K_0 = 240 \pm 20$ MeV, which is well below the value predicted three decades ago, for example, in Ref. [23]. Moreover, combining information from studying the collective flow and kaon production in relativistic heavy-ion collisions in several terrestrial nuclear physics laboratories the EOS of SNM at higher densities has also been limited in a relatively small range up to about $4.5\rho_0$ [38] consistent with the latest observations of two-solar mass neutron stars [24, 39]. However, despite intensive efforts devoted to constraining the density dependence of the nuclear symmetry energy, its knowledge still remains largely uncertain even around nuclear saturation density $\rho_0$ [40–52]. It plays a vital role not only in describing the structure of rare isotopes and their reaction mechanisms [20, 53–55], but also determines uniquely the proton fraction essential for understanding the cooling mechanism and appearance
of exotic species in neutron stars [2, 56]. Moreover, it affects significantly the structure, such as the radii, moment of inertia, tidal polarizability and the core-crust transition density, as well as the frequencies and damping times of various oscillation modes of neutron stars (For a full review please refer to the topical issue on this subject matter [57]). Furthermore, the uncertainty in the high-density behavior of the symmetry energy is quite large with predictions from all varieties of nuclear models diverging dramatically [58–60]. Besides the well known difficulties of treating accurately quantum many-body problems, our poor knowledge about the spin-isospin dependence of three-body and many-body forces, the short-range behavior of nuclear tensor force and the resulting isospin-dependence of short-range nucleon-nucleon correlations, coupled with the lack of sensitive experimental probes are mainly responsible for the current situation. Whereas several observables have been proposed in the past [20], and some hints of the high-density behavior of nuclear symmetry energy have been reported recently [61, 62], there is still no widely accepted conclusion regarding the density dependence of nuclear symmetry energy [63]. Therefore, the symmetry energy is still regarded as the most uncertain part of the EOS of neutron-rich nuclear matter.

The paper has been organized as follows. First, in Sec. II we will overview the underlying nuclear EOS models used in this study. Special attention is paid to the uncertainties of the EOS of both SNM at saturation and the nuclear symmetry energy at both saturation and supra-saturation densities. Second, in Sec. III we provide the necessary details required to calculate the gravitational binding energy and discuss its sensitivity to the nuclear symmetry energy, in particular. Next, in Sec. IV we discuss the role of the full contraction of Riemann curvature tensor as a suitable choice of curvature. In particular, we will show that while the relative strength of the gravitational field could be several times larger from the surface to the core of the neutron star, it strongly depends on the choice of the underlying EOS. Then, in Sec. V we will briefly overview the necessary formula to calculate the mass versus radius relation and the gravitational binding energy in the scalar-tensor models of gravity. We will show that current observational constraints on parameters of this theory puts them at a disadvantage in breaking the degeneracy between the uncertainties in the EOS and alternative models of gravity. Later, in Sec. VI we discuss the possibility of determination of the nuclear symmetry energy from two particular astrophysical observables sensitive to the compactness of the star: the surface gravitational redshift of non-rotating NSs and the maximum spin frequency of rapidly rotating NSs. And finally in Sec. VII we offer our concluding remarks.

II. THE EQUATION OF STATE OF STELLAR MATTER

The equation of state of neutron-rich nucleonic matter, in general, can be expressed as

\[ E(\rho, \alpha) = E_0(\rho) + S(\rho)\alpha^2 + \mathcal{O}(\alpha^4) , \]

(1)

where \( E(\rho, \alpha) \) and \( E_0(\rho) \) are the binding energy per nucleon in asymmetric nuclear matter of isospin asymmetry \( \alpha = (\rho_n - \rho_p)/\rho \) and in SNM \( (\alpha = 0) \), respectively. Here \( S(\rho) \) is referred to as the nuclear symmetry energy, which represents the energy cost of departing from symmetric limit of equal neutrons and protons, and \( \rho = \rho_n + \rho_p \) is the total baryon density with \( \rho_n \) (\( \rho_p \)) being the neutron (proton) density. It is customary to characterize the EOS further in terms of bulk nuclear parameters by expanding both \( E_0(\rho) \) and \( S(\rho) \) in Taylor series around nuclear saturation density

\[ E_0(\rho) = B_0 + \frac{1}{2} K_0 \chi^2 + \mathcal{O}(\chi^3) , \]

(2)

\[ S(\rho) = J + \chi \frac{1}{2} K_{\text{sym}} \chi^2 + \mathcal{O}(\chi^3) , \]

(3)

where \( \chi \equiv (\rho - \rho_0)/3\rho_0 \) quantifies the deviations of the density from its saturation value. As discussed in the Introduction the binding energy at saturation \( B_0 \) and the nuclear incompressibility coefficient \( K_0 \) are tightly constrained by terrestrial experimental data using ground state properties of finite nuclei and energies of giant resonances. On the other hand, while the symmetry energy at saturation \( J \) is more or less known, its density slope \( L \) is largely unconstrained. Moreover the value of the curvature of the symmetry energy at saturation \( K_{\text{sym}} \) still has a huge uncertain range.

As our base models we employ two recently established EOSs for neutron-rich nucleonic matter within the IU-FSU Relativistic Mean Field (RMF) model and the SkIU-FSU Skyrme-Hartree-Fork (SHF) approach [49, 64, 65]. The IU-FSU EOS was obtained by adjusting the parameters of the RMF to satisfy the latest constraints from terrestrial nuclear experiments and astrophysical observations [65]. Moreover, its predictions closely match the state-of-the-art nuclear many-body calculations for the EOS of low-density pure neutron matter (PNM). By construction, the SkIU-FSU model is a non-relativistic counterpart of the IU-FSU, and they have similar EOSs for both SNM with \( K_0 = 231.3 \text{ MeV} \), and PNM around and below \( \rho_0 \). In particular, the slope of the symmetry energy at saturation for both models is \( L = 47.2 \text{ MeV} \). At subsaturation densities, \( S(\rho) \) is almost identical for these models. However, their
Models have same $K_0 = 231.3$ MeV

To test the sensitivity of the gravitational binding energy and space-time curvature of neutron stars to the variations of properties of neutron-rich nuclear matter around saturation density, we have further introduced four additional EOSs using the IU-FSU as the base model. By tuning two purely isovector parameters of the IU-FSU model one can efficiently modify the density dependence of the symmetry energy [64]. This tuning is carried out under the constraint that the value of the symmetry energy at a subsaturation density of $\rho \approx 0.103\rho_0$ is fixed at 26.0 MeV. Such tuning leaves the EOS of SNM unchanged and guarantees that predictions for the ground-state properties of finite nuclei will not deviate from their experimental values [66]. In this way we created two RMF models with density slopes of the symmetry energy at saturation density of $L = 60$ MeV and $L = 100$ MeV. Finally, by properly adjusting all parameters of the RMF model, so that their bulk properties match those of the IU-FSU baseline model, we have created two additional RMF models with differing incompressibility coefficients of $K_0 = 220$ MeV and $K_0 = 260$ MeV. We have ensured that predictions of these two models reproduce the experimental data on charge radii and binding energies of some selected closed-shell nuclei by slightly adjusting the binding energy at saturation. Shown in Fig. 1 are the density dependence of the symmetry energy for four of these models. Notice we did not display models with $K_0 = 220$ MeV and $K_0 = 260$ MeV, because the density dependence of the symmetry energy in these models are almost indistinguishable from that of the original IU-FSU.

FIG. 1. (color online). Density dependence of the nuclear symmetry energy for the four models discussed in the text. The solid black line corresponds to the symmetry energy predictions in SkIU-FSU, the dashed black line corresponds to the original IU-FSU model, the dotted red and dash-dotted blue lines are symmetry energy predictions in the IU-FSU-like models with density slope $L = 60.0$ MeV (IU-FSU$^1$) and $L = 100.0$ MeV (IU-FSU$^2$), respectively. Notice that we did not display results for the IU-FSU-like models with incompressibility coefficients of $K_0 = 220$ MeV and $K_0 = 260$ MeV, since their predictions are almost identical to the one of the original IU-FSU model.

III. GRAVITATIONAL BINDING ENERGY OF NEUTRON STARS

We calculate the core EOS assuming a minimal model of neutrons, protons, electrons and muons in chemical equilibrium. We employ the EOS of Baym, Pethick, and Sutherland [67] for the outer crust. Because the quantities we will be calculating are not sensitive to the detailed composition of the inner crust, we simply use polytropic EOSs
that connect the uniform liquid core to the bottom layers of the outer crust defined by the neutron dripline. The stellar profiles are then determined by integrating the usual Oppenheimer-Volkoff (OV) equations [68]

\[
\frac{dP(r)}{dr} = -\frac{G}{c^2 r^2} \left[ \mathcal{E}(r) + P(r) \right] \left[ M(r) + 4\pi r^3 \frac{P(r)}{c^2} \right] \left[ 1 - \frac{2GM(r)}{c^2 r} \right]^{-1},
\]

(4)

\[
\frac{dM_G(r)}{dr} = 4\pi r^2 \mathcal{E}(r)
\]

(5)

where G is the gravitational constant, \( P(r), \mathcal{E}(r) \) and \( M(r) \) are the pressure, energy density and mass profiles as a function of radial distance \( r \). Given the boundary condition \( P(0) = P_c, M(0) = 0 \), supplemented by an EOS, the radius of the neutron star \( R \) will then be determined from the condition of \( P(R) = 0 \). The corresponding total mass enclosed is referred to as the gravitational mass of the neutron star, which is

\[
M_G = \frac{1}{c^2} \int_0^R 4\pi r^2 \mathcal{E}(r) \, dr,
\]

(6)

One may also define the so-called baryonic mass of the neutron star as the mass of the star that has been disassembled into its component baryons, which is given by \( M_B \equiv N m_B \). Here \( m_B \) is the baryon mass and \( N \) is the total number of baryons in the star that can be found by a volume integration of the baryon density \( \rho_B(r) \) in the Schwarzschild geometry,

\[
N = \int_0^R 4\pi r^2 \rho_B(r) \left[ 1 - \frac{2GM(r)}{c^2 r} \right]^{-1/2} \, dr,
\]

(7)

Then the gravitational binding energy in mass units is simply defined as \( \mathcal{B} \equiv M_G - M_R \) [69], which is always a negative quantity for a gravitationally bound system. It is also important to specify the baryon mass \( m_B \). If we set it to be the mass of a single nucleon, then \( \mathcal{B} \) would be the total binding energy that contains the contribution from the nuclear binding energy as well. In order to compare the predicted binding energy with estimates of the baryon mass of the pre-collapse cores, the nuclear binding energy must be taken into account. It is therefore appropriate to set the baryon mass to the atomic mass unit instead. Therefore following Refs. [1, 26, 27, 70] we set it to \( m_B \equiv 931.5 \) MeV/c². It has useful to express the fractional gravitational binding energy \( \frac{\mathcal{B}}{M_G} \) as a function of the compactness parameter \( \eta_R \) [27, 71, 72], which for neutron stars is several orders of magnitude larger than that of regular stars such as our Sun [5, 71]. As a result, the binding energy contribution to the total mass of a neutron star is significant—up to 25% of its rest mass [72–74] for stars near the maximum mass—although the exact size of the binding energy strongly depends on the underlying EOS [27].

In Fig. 2 we display the fractional gravitational binding energy as a function of the total gravitational mass of a neutron star. Higher mass stars, and stars governed by softer EOSs (that is, softer symmetry energies or smaller values of incompressibility \( K_0 \)) are more compact and have higher absolute values of \( \frac{\mathcal{B}}{M_G} \) (see also [26, 75]). Now let us examine effects of the uncertain symmetry energy in more details. First, notice that with almost the same low-density symmetry energy up to about 1.5\( \rho_0 \), but different high-density behaviors, predictions for the \( \frac{\mathcal{B}}{M_G} \) in the SkIU-FSU and IU-FSU models are very different. This effect becomes more pronounced with the increase of the total gravitational mass. Indeed, larger neutron stars probe the symmetry energy at higher densities which is different in these two models due to their distinctively different high-density behaviors of the symmetry energy. Recall that by construction both the magnitude and the slope of the symmetry energy at saturation are identical in these models, i.e. \( J = 31.3 \) MeV and \( L = 47.2 \) MeV, respectively. Choosing the IU-FSU as a reference, then the relative changes in the fractional gravitational binding energies are 5.32%, 6.52% and 9.89% for 1.25\( M_\odot \), 1.4\( M_\odot \) and 1.9\( M_\odot \) neutron stars respectively (See Table. 1). Thus, the fractional gravitational binding energy becomes increasingly sensitive to the super-saturation behavior of the symmetry energy for increasing mass.

This is in contrast to the behavior of the binding energy in models with soft and stiff symmetry energies at saturation—e.g., the IU-FSU with \( L = 47.2 \) MeV and the IU-FSU² with \( L = 100 \) MeV. Again choosing the former as a reference model, then the relative changes in the fractional gravitational binding energies are −10.26%, −8.85% and −7.44% for 1.25\( M_\odot \), 1.4\( M_\odot \) and 1.9\( M_\odot \) neutron stars respectively. Thus the fractional binding energy is most sensitive to the saturation density slope of the symmetry energy for low mass neutron stars.

Finally, we note that, because the incompressibility coefficient of SNM, \( K_0 \), is presently well constrained by terrestrial nuclear physics data, the effect on the fractional binding energy \( \frac{\mathcal{B}}{M_G} \) over the constrained range of \( K_0 \) is almost negligible.

The discovery of the double pulsar system PSR J0737−3039 has enabled the accurate determination of many pulsar properties including both pulsar’s gravitational masses. In particular, the gravitational mass of the pulsar PSR J0737−3039B is determined to be \( M_G = 1.2489 \pm 0.0007 M_\odot \) [76], which is one of the lowest reliably measured mass for any
neutron star up to date. Its low mass has led to the suggestion that this NS might have been formed as a result of the collapse of an electron-capture supernova from a progenitor of a O-Ne-Mg white dwarf [26]. Under this assumption, it was estimated that the baryonic mass of the precollapse O-Ne-Mg core should lie between $1.366M_\odot < M_B < 1.375M_\odot$ [26] and $1.358M_\odot < M_B < 1.362M_\odot$ [77]. Using these two sets of constraints, it was shown in Ref. [27] that the upper limit of the density slope at saturation should be $L \approx 70$ MeV to be consistent with these observations. Notice that recently an even lower mass of $M_G = 1.230 \pm 0.007M_\odot$ has been reported through the long-term precision timing measurements of double neutron star system PSR J1756-2251 [78]. In Fig. 2 we confirm that only models with the soft symmetry energy are consistent with these set of constraints. Notably, models with even smaller values of $L$—such as IU-FSU with $L = 60$ MeV—could still be inconsistent, if the subsequent density dependence of the symmetry energy at higher densities is stiff. Such low mass stars, via their binding energy, probe the symmetry energy slope at saturation; and, once given a constrained range for $L$, they may also probe the symmetry energy at higher densities.

Finally, we remark that the fractional gravitational binding energy in the massive neutron star PSR J0348+0432 is twice larger than that of the PSR J0737-3039B independent of the EOS model used [24]. As noted in that paper, and shown in Fig. 2, whereas uncertainties of the EOS due to variations of the density dependence of the symmetry energy are of the order of 20% for a given star, the substantial differences in the $B/M_G$ of canonical and two-solar mass neutron stars makes the latter an ideal laboratory for testing GR in the strong field regime.

Thus we can conclude that, given that the EOS of SNM is well constrained by terrestrial data, the binding energy of low mass neutron stars probes the saturation density symmetry energy primarily, and then the high density behavior of the symmetry energy. Given constraints from low mass neutron stars, the resulting range for the symmetry energy as a function of density constrains $B/M_G$. For example, for the set of the EOSs used here we find that the fractional binding energy of a 1.9 solar-mass neutron star is between $-0.1380 \leq B/M_G \leq -0.1639$. There is no known mechanism of extracting baryonic mass in massive neutron stars however. In the next section we will analyze this further by contrasting the effect of the density dependence of the symmetry energy with measures of the strength of the gravitational field.
TABLE I. Predictions for the radius $R$, surface compactness parameter $\eta_R$, fractional gravitational energy $\frac{E}{M}$, full contraction of the Riemann curvature tensor at the surface $K_R$ of neutron stars with masses 1.25$M_\odot$, 1.4$M_\odot$ and 1.9$M_\odot$ as predicted by the various models used in the text. Also relative changes of the gravitational binding energy and surface full contraction of the Riemann curvature tensor with respect to the predictions of the original IU-FSU model are provided in percentage. Finally we give the radius, binding energy and curvature of a star calculated using the scalar-tensor model described in the text is given for the IU-FSU EOS (indicated by IU-FSU (ST)), together with the relative change in binding energy compared with the general relativistic result.

<table>
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<th>Model</th>
<th>$L$(MeV)</th>
<th>$K_0$(MeV)</th>
<th>$M_C$(Me$\odot$)</th>
<th>$R$(km)</th>
<th>$\eta_R$</th>
<th>$\frac{E}{M}$</th>
<th>$K_R$(10$^{-2}$ km$^{-2}$)</th>
<th>$\frac{\Delta E}{E}$ (%)</th>
<th>$\frac{\Delta K_R}{K_R}$ (%)</th>
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IV. CURVATURE AS THE STRENGTH OF THE GRAVITATIONAL FIELD

In the absence of the cosmological constant the vacuum solution of the Einstein’s equations of GR, also known as the Schwarzschild exterior solution, suggests that it is natural to measure the strength of the gravitational field in terms of the compactness parameter, $\eta \equiv 2GM/c^2r$. It can take any value between $0 < \eta < 1$, with $\eta = 0$ corresponding to the flat Minkowski space in special relativity, while $\eta = 1$ corresponds to the strongest gravitational field that can be observed by an observer at infinity, the region which is known as the black hole event horizon. However, as it was shown in Ref. [21] the compactness parameter $\eta$, which is directly related to the Newtonian gravitational potential, could not serve as a fundamental parameter in describing the gravitational field in GR. Indeed the strength of the gravitational field is not characterized by a strong field potential, but by a large curvature [21]. There are several quantities that can be used as a measure of the curvature of gravitational field. These could, for example, be the Ricci scalar, the Ricci tensor, and the Riemann tensor. However, both the Ricci scalar and all components of the Ricci tensor vanish in the exterior region of compact objects, which makes them unsuitable choice for the measure of curvature. On the other hand, there is a non-vanishing component of the Riemann tensor even in the vacuum, $R_{\alpha \beta \gamma \delta} = -2GM/c^2r^3 \equiv -\xi$, which indicates that it should be a more relevant measure of curvature in neutron stars [5, 21]. Since there are totally 20 independent components of the Riemann tensor, it is best that one uses the full contraction of the Riemann tensor instead, which is a single scalar invariant. Therefore following the Ref. [5] we use the square root of the Kretschmann invariant, $K \equiv \sqrt{R_{\mu \nu \rho \sigma}R_{\mu \nu \rho \sigma}}$, which is a full contraction of the Riemann tensor. For a spherically symmetric mass configuration it can be expressed in terms of the pressure $P(r)$, energy density $\mathcal{E}(r)$, and mass $M(r)$ profiles as [5],

$$K^2 = \kappa^2 \left[ 3\left( \mathcal{E}(r) + P(r) \right)^2 - 4\mathcal{E}(r)P(r) \right] - \kappa \mathcal{E}(r) \frac{16GM(r)}{c^2r^3} + \frac{48G^2M^2(r)}{c^4r^6},$$

where $\kappa \equiv 8\pi G/c^4$. One can then readily evaluate the Kretschmann invariant while integrating the OV equations. Notice that outside the star this curvature has a desired form, $K \equiv 2\sqrt{3}\xi$.

It is common to measure the strength of gravity in neutron stars with respect to that of the solar system. At the surface of the Sun, the curvature is equal to $K_\odot \equiv 3.0 \times 10^{-17}$ km$^{-2}$, which may be regarded as a small quantity. On the other hand, neutron stars can have surface curvature of as large as 14 orders of magnitude greater. In Fig. 3 we display the ratio of the neutron-star surface curvature to the one of the Sun, $K_R/K_\odot$, as a function of the neutron star mass. Clearly, this ratio becomes even larger for more massive neutron stars. Notice that for less massive stars...
in the value of the surface curvature of a neutron star is significant to the symmetry energy at saturation density and at higher densities. There is a 25% reduction in the gravitational binding energy when the symmetry energy is modified, as can be observed in Table I.

Coupled with the sudden increase in the surface curvature of neutron stars close to their maximum mass, this emphasizes the fact that lower mass neutron stars are more natural laboratories for testing the EOS, and that only when the EOS is significantly constrained can high mass neutron stars come into their own as laboratories for testing various models of strong-field gravity. The effects of the density dependence of the symmetry energy are less pronounced on the curvature of massive neutron stars, but still significant. A curvature of $K_R = 4.0 \times 10^{14} \, K_0$ has been obtained for massive neutron stars in the range of $1.72 < M_\odot/M_\odot < 1.95$. Quantitatively, the relative variation in $K_R$ of a canonical 1.4 solar-mass NS predicted by the IU-FSU model and SkIU-FSU is 61%, and this change reduces to only 51% for a 1.9 solar-mass NS. This is comparable to the relative change between NSs with masses $1.4M_\odot$ and $1.9M_\odot$ within the two models: 83% and 71% for the IU-FSU and SkIU-FSU, respectively (See Table I).

Coupled with the sudden increase in the surface curvature of neutron stars close to their maximum mass, this emphasizes the fact that lower mass neutron stars are more natural laboratories for testing the EOS, and that only once the EOS is significantly constrained can high mass neutron stars come into their own as laboratories for testing alternative models for gravity.

In the same spirit as in the previous section, we now compare the effect of the uncertainties due to the variations of density dependence of the symmetry energy at saturation density and at high densities. There is a 25% reduction in the value of the surface curvature of a 1.4$M_\odot$ NS, when we compare models with the soft and stiff symmetry energies as controlled by their density slope at saturation of $L = 47.2$ MeV (IU-FSU) and $L = 100$ MeV (IU-FSU$^2$), correspondingly (See Table I). Obviously, the curvature is much larger when the soft symmetry energy is used.

This can be compared with a reduction in stiffness in the high-density EOS: comparing predictions of the SkIU-FSU and IU-FSU, we see that the surface curvature increases by about 21% for a 1.4$M_\odot$ NS and 26% for a 1.9$M_\odot$ NS. Thus the surface curvature is significantly more sensitive than the gravitational binding energy to the variation of the density dependence of the symmetry energy at both saturation densities and higher. Again, its variation with the incompressibility of SNM is negligible (See Table I and Fig. 3).

We next consider the strength of the gravitational field within the neutron star. Notice that towards the center of the star the compactness $\eta$ would approach zero. On the contrary, the curvature stays non-zero throughout the neutron star [5]. In order to study the strength of the gravitational field within the NS, we normalize the curvature profile $K(r)$ with respect to its value at the surface of the star $K_R$. The normalized profile tells us how many times stronger the curvature $K(r)$ gets as one approaches the central region. This also provides a convenient way to compare effects of the density dependence of the symmetry energy on the same basis. In Fig. 4 we plot the curvature profiles...
for a 1.4 and 1.9 solar-mass neutron stars using the EOSs considered in this work. For a canonical 1.4 $M_\odot$ NS the central curvature is 2.8 to 3.5 times stronger than the surface curvature depending on the EOS used. This ratio becomes larger for massive neutron stars, and in particular for a 1.9 solar mass NS it can be as large as 3.6 to 4.2 times stronger. Notice that the ratio becomes especially larger for models with the stiff saturation-density symmetry energy (See Fig. 4), whereas the high density behavior of the symmetry energy has a significantly smaller effect on the ratio of curvatures. Again, we observe a very small effect due to the variation of $K_0$.

Fig. 3 and Fig. 4 together give us a complete picture of the curvature $K(r)$ throughout the neutron star. First of all, the curvature of massive neutron stars at the surface is much larger than that of canonical NSs. Secondly, as one moves towards the center the strength of gravity becomes even stronger. Whereas this explicitly demonstrates the important role of massive neutron stars in studying the strong-field gravity, the uncertainties of the EOS resulting from our poor knowledge of the density dependence of the symmetry energy cannot be overlooked. It was put forward in Ref. [24] that the two solar mass neutron stars in PSR J0348+0432—which is the most massive neutron star observed up to date—can be used as a testbed for gravity in the strong field regime. This conclusion was based on the fact that its binding energy is far outside the presently tested binding energy range. While supporting this conclusion, we have also analyzed the effects of the density dependence of the symmetry energy on the strength of the gravitational field and have found that current uncertainties in the EOS are still very large to cleanly probe the fundamental physics in the extreme conditions. Nevertheless, massive NSs that are manifest as radio pulsars remain valuable probes of both models of gravity and theories of dense matter in the extreme conditions that are inaccessible to the terrestrial experiments. In the next section we will analyze our results for the case of a neutron star in the scalar-tensor theories.
V. NEUTRON STARS IN A SCALAR-TENSOR MODEL OF GRAVITY

Motivated by the fact that the properties of neutron stars are not only sensitive to the underlying EOS, but also to the strong-field behavior of gravity, in this section we will consider neutron stars in the simplest natural extension of the GR known as the scalar-tensor theories of gravity. According to these theories, in addition to the second-rank metric tensor, $g_{\mu\nu}$, the gravitational field is also mediated by a scalar field, $\phi$. The most general form of the action defining this theory is written as [79, 80]

$$ S = \frac{c^4}{16\pi G} \int d^4x \sqrt{-g^*} \left[ R^* - 2g^{*\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] + S_{\text{matter}} (\psi_{\text{matter}}, A^2(\phi)g^*_{\mu\nu}) , $$

where $R^*$ is the Ricci scalar curvature with respect to the so-called Einstein frame metric $g^*_{\mu\nu}$ and $V(\phi)$ is the scalar field potential. The physical metric, also known as Jordan frame metric, is defined as $g_{\mu\nu} = A^2(\phi)g^*_{\mu\nu}$. Notice that the GR is automatically recovered in the absence of the scalar field. The relativistic equations for stellar structure then can be written as

$$ \frac{dM_G(r)}{dr} = 4\pi r^2 E(r) A^4(\phi) + \frac{r^2}{2} \left( 1 - \frac{2GM(r)}{c^2 r} \right) \chi^2(r) + \frac{r^2}{4} V(\phi) , $$

$$ \frac{dM_B(r)}{dr} = 4\pi m_B r^2 \rho(r) A^3(\phi) , $$

$$ \frac{d\chi(r)}{dr} = \left[ 1 - \frac{2GM(r)}{c^2 r} \right]^{-1} \left\{ 4\pi \frac{G}{c^4} A^4(\phi) \left[ \alpha(\phi) \left( E(r) - 3P(r) \right) + r\chi(r) \left( E(r) - P(r) \right) \right] 
- \frac{2}{r} \left( 1 - \frac{GM(r)}{c^2 r} \right) \chi(r) + \frac{1}{2} r\chi(r) V(\phi) + \frac{1}{4} \frac{dV(\phi)}{d\phi} \right\} , $$

$$ \frac{dP(r)}{dr} = - \left( E(r) + P(r) \right) \left[ 1 - \frac{2GM(r)}{c^2 r} \right]^{-1} \left\{ 4\pi \frac{G}{c^4} r A^4(\phi) P(r) + \frac{G}{c^2 r} M(r) 
+ \left( 1 - \frac{2GM(r)}{c^2 r} \right) \left( \frac{1}{2} r^2 \chi^2(r) + \alpha(\phi) \chi(r) \right) - \frac{1}{4} V(\phi) \right\} , $$

where

$$ \alpha(\phi) \equiv \frac{\partial \ln A(\phi)}{\partial \phi} . $$

At the limiting case of $A(\phi) = 1$, and $V(\phi) = 0$ one recovers the general relativistic OV Eqs. (5). Following the Ref. [79] we set $V(\phi) = 0$ which can only appear in models of modified gravity, and consider a coupling function of the form

$$ A(\phi) = \exp \left( \alpha_0 \phi + \frac{1}{2} \beta_0 \phi^2 \right) . $$

Again once an EOS is supplemented, for a given central pressure $P(0) = P_c$ one can then integrate the equations above from the center of the star to $r \to \infty$. At the center of the star the Einstein frame boundary conditions are given as

$$ P(0) = P_c , \quad E(0) = E(c) , \quad \phi(0) = \varphi_c , \quad \chi(0) = 0 , $$

while at infinity we demand cosmologically flat solution to agree with the observation

$$ \lim_{r \to \infty} \varphi(r) = 0 . $$

The stellar coordinate radius is then determined by the condition of $P(r_s) = 0$. The physical radius of a neutron star is found in the Jordan frame as $R_{NS} = A^2(\varphi(r_s)) r_s$. Notice however that the physical stellar mass as measured by an observer at infinity, also known as the Arnowitt-Deser-Misner (ADM) mass $M_{\text{ADM}}$, matches with the coordinate
mass, since at infinity the coupling function approaches unity. It was shown in [81] that measurement of the surface atomic line redshifts from neutron stars could be used as a direct test of strong-field gravity theories. In particular, as a representative of the scalar-tensor theory a model with $\alpha_0 = 0$ and $\beta_0 = -8$ was used. Notice that however a significant improvement have been made since then in constraining the $\alpha_0$ and $\beta_0$ parameters. For example, very recently the authors of Ref. [25] put the most stringent constraint on these parameters from the observations of the 10-year timing campaign on PSR J1738+0333. According to these latest constraints the quadratic parameter should take values of $\beta_0 \gtrsim -5.0$. Moreover, as shown first in Ref. [79] predictions by models with $\beta_0 > -4.35$ can not in general be distinguished from the general relativistic results—i.e. from the model with $\{\alpha_0, \beta_0\} = \{0, 0\}$—due to the so-called “spontaneous scalarization” effect. Since we have already showed that the uncertainties in the SNM EOS due to the variation of $K_0$ has insignificant effects, in this section we will solely concentrate on the effects of the symmetry energy. Further, we will choose the upper bounds on the parameters of the scalar-tensor theory to be $\alpha_0^2 < 2.0 \times 10^{-2}$ and $\beta_0 > -5.0$ as constrained by the observation [25].

In Fig. 5 we present our results for the fractional gravitational binding energy as a function of the stellar mass (Fig. 5a) and the mass versus radius relation (Fig. 5b). The general relativistic (GR) predictions are then compared with the predictions of the scalar-tensor theories using different EOSs. We observe that the GR predictions give a lower absolute value of the fractional binding energy for a given stellar mass then the scalar-tensor theory, in general. However the difference is quite negligible as far as the uncertainties in the EOS is concerned. Moreover, low-mass neutron-stars are indistinguishable in these two models of gravity, because the critical value for the so-called “spontaneous scalarization” occurs only when NSs exceed masses of about 1.4 solar mass. To further illuminate the effects of compactness on the binding energy we can look at the mass versus radius relation. We observe that there is a slight change in this relation; notably, massive neutron stars have smaller radii in GR. If one uses a different value of $\beta_0 = -8.0$ as in the Ref. [81]—which has already been excluded by the observation—one can obtain very massive neutron stars. In particular, the IU-FSU EOS gives a maximum mass of $3.2M_\odot$ and a slightly larger radius of 14.5 km. However, under present observational constraints, differences in predictions using the GR and the scalar-tensor theories are much smaller than those due to the uncertainties in the EOS. Thus measurements of gravitational binding energy and neutron star radii will lead to significant constraints on the neutron star symmetry energy at saturation and high densities, rather than constraining gravity models between GR and scalar-tensor theories.

VI. FUTURE DETERMINATION OF THE NUCLEAR SYMMETRY ENERGY FROM THE SURFACE GRAVITATIONAL REDSHIFT AND THE MAXIMUM SPIN FREQUENCY

Having established that uncertainties in the EOS arising from the nuclear symmetry energy outweigh those within the set of Scalar-Tensor gravity theories, we now seek in this section for some possible astrophysical measurements (see Ref. [82] for review) that may help constrain the density dependence of nuclear symmetry energy. In Fig. 2 we showed the binding energy, which can be inferred from the masses of binary pulsars assuming a specific evolutionary pathway for which strong (albeit circumstantial) evidence exists for two systems in particular; currently such inferences favor a softer symmetry energy [26, 27]. It was shown by some of us previously that the pulsar moment of inertia is an observable sensitive to the density dependence of the nuclear symmetry energy [83, 84]. In particular, it was shown that knowledge of the moment of inertia for pulsar PSR J0737-3039A with a 10% accuracy could help discriminate among various equations of state. Moreover, it was recently shown by some of us that the observed symmetry energy effect on the tidal polarizability (also known as tidal deformability) parameter in binary neutron stars is very strong [49], and in particular it has been shown that future measurement of the tidal polarizability in binary mergers would stringently constrain the high-density behavior of the symmetry energy. Perhaps the simplest of all, the knowledge of stellar mass and radius even from a single neutron star would provide very useful information on the equation of state of neutron star matter as first pointed out by Ref. [85]. While the mass of the neutron star can be measured fairly accurately in binary systems, the measurement of its radius is usually very complicated. From the observational point of view, it is more relevant to describe stellar properties with the so-called radiation radius,

$$R_\infty = \frac{R}{\sqrt{1 - \eta_R}},$$

and the surface gravitational redshift,

$$z = \frac{1}{\sqrt{1 - \eta_R}} - 1.$$  \hspace{1cm} (20)

Notice that the expression for redshift is only valid in the slow-rotation approximation, where the Doppler effect due to the rotation of the star and the Lense-Thirring effect have been neglected. The radiation radius $R_\infty$ could be
FIG. 5. (color online). The fractional gravitational binding energy (a) and the mass versus radius relation (b) calculated using the EOSs considered in this work. The upper observational bound on the scalar-tensor theory parameters of \( \{\alpha_0, \beta_0\} = \{\sqrt{2.0 \times 10^{-5}}, -5.0\} \) have been used.

obtained from a combination of flux and temperature measurements from the neutron star’s surface and distance to the source. There are however some major uncertainties involved in the determination of \( R_\infty \) such as the distance, interstellar absorption, and details concerning the composition of the atmosphere and its magnetic field strength and structure [2]. Another possible combination to uniquely determine the stellar mass and radius could come from the inference of the surface gravitational acceleration and the gravitational redshift as pointed out by Ref. [28]. The gravitational redshift can be measured from the possible shift of atomic absorption lines in spectra of the stars. One of the main difficulties in measuring the gravitational redshift of signals emitted from the surface of neutron stars is due to the existence of strong surface magnetic field. Indeed, a value of \( z = 0.35 \) was reported [86] based on an
FIG. 6. (color online). The effect of the density dependence of the nuclear symmetry energy (a) and the incompressibility coefficient (b) on the surface gravitational redshift $z$ of neutron stars.

analysis of the stacked bursts in the low-mass X-ray binary EXO 0748-676. Unfortunately, subsequent observations could not confirm the existence of such spectral lines [87]. Nevertheless, future developments of large-area instruments in X-ray spectroscopy missions could make it possible to detect the spectral lines from the surface [88].

In Fig. 6, we plot the surface gravitational redshift as a function of the stellar mass for various equations of state discussed in previous sections. As it is evident from the figure, the effect of the density dependence of the nuclear symmetry energy is quite significant. For a canonical neutron star, uncertainties resulting from the density dependence of the symmetry energy would allow a range of predictions for the surface redshifts spanning from $0.197 < z < 0.244$.

Interestingly, had the $z = 0.35$ redshift in EXO 0748-676 been confirmed, its mass would then have to lie in the range of $1.71 M_\odot < M < 1.89 M_\odot$ similar to that found earlier in [89]. On the other hand, if future redshift measurements report $z > 0.6$ then all current models of the equation of state would be ruled out. Obviously, the effect of rotation would significantly alter the predictions for the redshift, and therefore should be taken into account. Such a study is in progress and results will be reported elsewhere.

Next, we plot in Fig. 7 the surface gravitational redshift $z$ as a function of the radiation radius $R_\infty$ by varying only the density dependence of the symmetry energy. The radiation radius of the low-mass X-ray source, EXO 1745-248 has been determined to be $R_\infty = 14.57 \pm 1.64 \text{ km (shaded band)}$ [90, 91]. It is seen that the IU-FSU model with $L = 100 \text{ MeV}$ is almost excluded by this observation regardless of the gravitational redshift for this source. Future precise determination of the radiation radius from the low-mass X-ray binaries would therefore more stringently constrain the density dependence of the symmetry energy.

A certain combination of the stellar mass and radii could alternatively be constrained by measuring the spin frequencies. The maximum spin frequency for a stellar structure is known as the Kepler frequency, which in Newtonian gravity can be calculated using $f_K = \frac{1}{2\pi} \sqrt{GM/R^3}$. The general relativistic expression for the maximum spin frequency is very complicated and can be found by solving an equation involving metric functions and their derivatives [95]. As first shown in Ref. [29], the maximum spin frequency for a maximum allowable mass configuration can be analytically written in terms of the stellar properties as $f_{K,\text{max}} = C \left( \frac{M_{\text{max}}}{M_\odot} \right)^{1/2} \left( \frac{R_{\text{max}}}{10 \text{ km}} \right)^{-3/2}$, where the $M_{\text{max}}$ and $R_{\text{max}}$ are the maximum allowable mass and the corresponding radius for static configurations. The most recent and updated analysis gives a fitting coefficient of $C = 1.08 \text{ kHz}$ [96]. While this expression is highly accurate in estimating the maximum spin frequency, it is limited to the maximum mass configuration only. Moreover, rotating stars assume an oblate shape thus allowing a larger maximum mass than their static counterparts for a given equation of state. Therefore, without relying on the approximate expression we have used the RNS code for rapidly rotating neutron stars [92–94]. In Fig. 8 we show the maximum spin rate as a function of the stellar mass.
FIG. 7. (color online). Surface gravitational redshift \( z \) as a function of stellar radiation radii \( R_\infty \) are constructed from the four EOSs discussed in the text.

J1748-2446ad—with a frequency of \( f = 716.356 \) Hz—is currently unknown [97]. Should it be a low-mass neutron star with \( M < 1.29 M_\odot \), models with larger density slope of the symmetry energy at saturation would be excluded. While not confirmed by subsequent observations, a weak evidence for an \( f = 1122 \) Hz signal has been reported during thermonuclear bursts from the XTE J1739285 neutron star [98]. Although this observation does not have strong statistical significance, our calculations indicate that its mass should be above \( 1.65 M_\odot < M < 2.12 M_\odot \) depending on the EOS model (see Fig. 8). Conversely, if such a fast-spinning neutron star had been found to have a mass of \( M = 1.7 M_\odot \) or less, then only models with the very soft nuclear symmetry energy such as the SkIU-FSU would have survived. Although at this stage there is no known pulsars with larger spin frequencies, future measurements of such frequencies would be decisive in ruling out models predicting the stiff nuclear symmetry energy.

VII. SUMMARY

There is a degeneracy between the EOS for super-dense matter and the strong-field gravity in understanding properties of neutron stars. Our goal is to provide information that may help break this degeneracy. Since the most uncertain part of the nucleonic EOS is currently the density dependence of nuclear symmetry energy especially at super-saturation densities, we have studied effects of the nuclear symmetry energy within its current uncertain range on the binding energy and curvature of neutron stars within GR and the scalar-tensor theory of gravity. We found that the gravitational binding energy is moderately sensitive to the underlying symmetry energy, with lower mass neutron stars \( \lesssim 1.4 M_\odot \) probing primarily the saturation-density stiffness of the symmetry energy, and higher mass stars being more sensitive to the high density behavior of the symmetry energy. For the most massive NSs discovered so far, we found an almost 20% change in their binding energies by varying the density dependence of the symmetry energy within its uncertainty range determined by recent terrestrial nuclear laboratory experiments. We also found that the binding energy is quite insensitive to the variation of the incompressibility \( K_0 \) of SNM EOS.

The curvature of neutron stars measured using the square root of the full contraction of the Riemann tensor was found to be significantly more sensitive to the variation of the density dependence of nuclear symmetry energy but not to that of the incompressibility \( K_0 \), with variations in surface curvature of order 50% over the whole mass range.

Within the scalar-tensor theory of gravity which is the simplest natural extension of the GR, using upper bounds on the parameters of the scalar-tensor theory from observation, we find negligible change in the binding energy of
neutron stars over the whole mass range, and significant changes in radii only for neutron stars well above $1.4M_\odot$. Even then, the changes in radii are smaller than those that result from variation of the stiffness of the symmetry energy.

Restricting ourselves to the scalar-tensor modifications to gravity, we conclude that astrophysical measurements of quantities related to the compactness and curvature of neutron stars such as the gravitational binding energy, atomic line redshifts, and the maximum spin frequency will help constrain the neutron star EOS further rather than the model of gravity. However, the large range of curvatures predicted by varying the symmetry energy may still mean that current uncertainties in the density dependence of nuclear symmetry energy are still too large to clearly break the EOS-gravity degeneracy in determining properties of neutron stars for a much wider class of gravity models.

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