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Ordinary Muon Capture in Hydrogen Reexamined

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ABSTRACT — The rate of muon capture in a muonic hydrogen atom is calculated in heavy-nucleon chiral perturbation theory up to next-to-next-to leading order. To this order, we present the systematic evaluation of all the corrections due to the QED and electroweak radiative corrections and the proton-size effect. Since the low-energy constants involved can be determined from other independent sources of information, the theory has predictive power. For the hyperfine-singlet μp capture rate Γ_0, our calculation gives $\Gamma_0 = 712 \pm 5\, s^{-1}$, which is in excellent agreement with the experimental value obtained in a recent high-precision measurement by the MuCap Collaboration.

I. INTRODUCTION

In a recent MuCap Collaboration experiment [1], the rate Γ_0 of muon capture from the hyperfine-singlet state of a μp atom was measured to 1 % precision. The reported experimental value is

$$\Gamma_0^{\rm exp}(\mu^- p \to \nu_\mu n) \ = \ 714.9 \pm 5.4(stat) \pm 5.1(syst)\,{\rm sec}^{-1}\,. \tag{1}$$

As is well known [2], the μp capture process is the primary source of information on the pseudoscalar form factor, $G_P(q^2)$, which appears in the nucleon matrix element of the axial-vector weak current (see Eq.(6)); for recent reviews, see [3, 4]. To be more specific, μp capture is sensitive to the quantity $g_P \equiv G_P(q^2 = -0.88 m_\mu^2)$, where q^2 is the four-momentum transfer squared relevant to μp capture (m_μ is the muon mass). Bernard *et al.* [5] used heavy baryon chiral perturbation theory (HBχPT) to calculate $G_P(q^2)$; their results essentially reproduce those obtained earlier by Adler and Dothan [6] based on PCAC, and by Wolfenstein [7] with the use of the dispersion relations. It is to be emphasized, however, that the systematic expansion scheme of HBχPT allows us to conclude that the corrections to the expression for $G_P(q^2)$ obtained by Bernard *et al.* are very small [8]. The value of g_P derived from HBχPT is $g_P = 8.26 \pm 0.23$ [5]. Meanwhile, the empirical value of g_P extracted from $\Gamma_0^{\rm exp}$ with the use of the theoretical framework provided in Ref.[9] is $g_P^{exp} = 8.06 \pm 0.55$ [1], which is consistent with the theoretical value.

It is to be noted that a theoretical treatment of μp capture that matches the 1 % experimental accuracy requires a rigorous treatment of the radiative corrections (RCs) of order α. Czarnecki *et al.* [9] calculated the relevant RCs within the theoretical framework developed by Sirlin and Marciano [10, 11]. In this approach (to be referred to as the S-M approach), the RCs of order α are decomposed into so-called “outer” and “inner” corrections. The outer correction is a universal function of the lepton velocity and is model-independent, whereas the inner correction is affected by the short-distance physics and hadron structure. The inner corrections arising from photon and weak-boson loop diagrams are divided into high-momentum and low-momentum contributions. The former is evaluated in the current-quark picture, while the latter is estimated with the use of the phenomenological electroweak nucleon form factors. The expression for Γ_0 including the RCs of order α due to Czarnecki *et* *al.* [9] was used by the MuCap Group in deducing the above-mentioned value of g_P^{exp} from $\Gamma_0^{\rm exp}$. We remark that, although the estimates of inner corrections in the S-M approach are considered to be reliable to the level of accuracy quoted in the literature, the possibility that these estimates may contain some degree of model dependence is not totally excluded. This motivates us to present here a calculation of the RCs of order α based on model-independent effective field theory (EFT).

In this note we evaluate the RCs for μp capture based on HBχPT, which is an effective low-energy field theory of QCD, see e.g., Refs. [12–14]. In HBχPT, the short-distance hadronic and electroweak processes are subsumed into a well-defined set of low-energy constants (LECs), which means that the LECs should systematically parameterize the inner corrections. Therefore, provided that there are sufficient sources of information to fix these LECs, the HBχPT approach gives model-independent results with the possibility to estimate higher-order corrections. In two of the earlier publications we used the same EFT approach to evaluate RCs to order α for the neutron β-decay [15], and for the inverse β-decay reaction, $\bar{\nu}_e p \to e^+ n$, at low energies [16]. It is to be noted that the EFT treatments of the μp capture process, neutron β-decay and the $\bar{\nu}_e p \to e^+ n$ reaction involve the same LECs. Therefore, if we can determine these LECs with the use of experimental information for one process, we can make model-independent predictions for observables for the other reactions.

The remainder of this article is organized as follows. In Section II we explain the basic ingredients that enter into the HBχPT calculation of the μp capture rate. We describe in Section III the evaluation of the RCs to order α, and give in Section IV the numerical results for the μp capture rate including the RCs. Finally, Section V is dedicated to discussion and conclusions.

II. HBχPT CALCULATION OF THE μp CAPTURE RATE

The theoretical framework is essentially the same as the one employed in Ref. [15]. We therefore present here only a brief recapitulation of our formalism, relegating details to Ref. [15]. HBχPT assumes that the characteristic four-momentum for the process, $Q \ll \Lambda_\chi \simeq 1$ GeV, where Λ_χ is the chiral scale. This theory contains two perturbative expansions, one in terms of the expansion parameter $Q/\Lambda_\chi \ll 1$ and the other in terms of $Q/m_N \ll 1$, where m_N is the nucleon mass. Since $m_N \simeq \Lambda_\chi$, the two expansions are considered simultaneously. When we include RCs in our considerations, a third expansion parameter α enters the theory. Our concern here is to carry out a HBχPT calculation up to next-to-next-to-leading order (NNLO), i.e., to order $(Q/\Lambda_\chi)^2 \simeq \alpha \simeq 1/137$. In what follows, we first describe the contributions that arise from the expansions in Q/Λ_χ and Q/m_N. This part is based on the previous HBχPT results that can be found in Refs. [17–19]. (We follow the notations used in Ref. [17].) We then proceed to explain our calculation of RCs.

Muon capture being a low-energy process, the relevant weak interaction can be expressed as the local current-current interaction, and the transition amplitude for the ordinary muon capture (OMC) process in hydrogen, $\mu^- p \to \nu_\mu n$, is given by

$$\begin{aligned}\mathcal{M}_{fi} &= \frac{G_\beta}{\sqrt{2}}\langle n\nu_\mu|\hat{l}_\alpha \hat{j}^\alpha|(\mu^- p)_{\rm atom}\rangle \approx \frac{G_\beta}{\sqrt{2}}\sqrt{\frac{m_\mu + m_N}{2m_\mu m_N}}\,\Psi_{\mu p}(\mathbf{0})\langle n\nu_\mu|\hat{l}_0\hat{j}^0 - \hat{\mathbf{l}}\cdot\hat{\mathbf{j}}|\mu p\rangle \\ &\equiv \frac{\mathcal{N}_{\rm rel}G_\beta}{2}\sqrt{\frac{m_\mu + m_N}{2m_\mu m_N}}\,\Psi_{\mu p}(\mathbf{0})\mathcal{T}_{\rm NR}\,. \end{aligned} \tag{2}$$

In the above, $G_\beta \equiv G_F V_{ud}$, where G_F is the Fermi coupling constant determined from the muon decay rate, and the V_{ud} is the CKM matrix element given in Ref. [23]. $\Psi_{\mu p}(\mathbf{0})$ is the μp-atomic wave function at $\mathbf{r} = 0$. The normalization factor $\mathcal{N}_{\rm rel}$, which arises from "matching" between the standard relativistic normalization of spinors and the corresponding non-relativistic normalizations is given by $\mathcal{N}_{\rm rel} = 4m_N\sqrt{m_\mu E_\nu}$, where

$$E_\nu = \frac{(m_\mu + m_p)^2 - m_n^2}{2(m_\mu + m_p)} = 99.149\,\mathrm{MeV}\,. \tag{3}$$

The non-relativistic transition amplitude, $\mathcal{T}_{\rm NR}$, in Eq.(2) is written as

$$\mathcal{T}_{\rm NR} = \chi_n^\dagger \chi_\nu^\dagger\,\widehat{\mathcal{M}}\,\chi_\mu \chi_p\,, \tag{4}$$

where $\chi_{p,n}$ and $\chi_{\mu,\nu}$ are the nucleon and lepton two-spinors, respectively; the explicit expression for the operator $\widehat{\mathcal{M}}$ will be given in what follows.

The matrix element of the leptonic weak current operator, $\hat{l}_\alpha$ in Eq.(2) is given by $l_\alpha \equiv \langle\nu|\hat{l}_\alpha|\mu\rangle = \overline{u}_\nu\gamma_\alpha(1-\gamma_5)u_\mu$, which in the present case takes the form

$$l_0 = \frac{1}{\sqrt{2}}\,\chi_\nu^\dagger(1 - \vec{\sigma}\cdot\hat{\nu})\chi_\mu\,, \quad \mathbf{l} = \frac{-1}{\sqrt{2}}\,\chi_\nu^\dagger(1 - \vec{\sigma}\cdot\hat{\nu})\vec{\sigma}\chi_\mu\,, \tag{5}$$

where $\hat{\nu}$ is the unit vector in the direction of the neutrino momentum. The matrix elements of the nucleon weak current operator, $\hat{j}^\alpha = \hat{j}^\alpha_v - \hat{j}^\alpha_a$, are given by[30]

$$\begin{aligned}\langle n(p')|\hat{j}^\alpha_v|p(p)\rangle &\equiv j^\alpha_v = \overline{u}_n(p')\left[F_1^v(q^2)\gamma^\alpha + F_2^v(q^2)\frac{i\sigma^{\alpha\beta}q_\beta}{2m_N}\right]u_p(p) \\ \langle n(p')|\hat{j}^\alpha_a|p(p)\rangle &\equiv j^\alpha_a = \overline{u}_n(p')\left[G_A(q^2)\gamma^\alpha\gamma_5 + G_P(q^2)\frac{q_\beta}{m_\mu}\gamma_5\right]u_p(p)\,, \end{aligned} \qquad (6)$$

where $F_1^v(q^2)$, $F_2^v(q^2)$, $G_A(q^2)$ and $G_P(q^2)$ are the vector, weak-magnetism, axial-vector and pseudoscalar form factors, respectively, and where the m_N is the average nucleon mass, $m_N = \frac{1}{2}(m_p + m_n)$. In the *rest frame of the initial proton*, the non-relativistic nucleon currents in HBχPT are given by[31]

$$\begin{aligned} j^\alpha_v &= \mathcal{N}_n\,\bar{n}_v(p')\Bigg\{\left[\frac{2m_N}{E'+m_N}F_1^v(q^2) - \frac{E'-m_N}{E'+m_N}F_2^v(q^2)\right]v_\alpha \\ &\quad\times\left[\frac{1}{E'+m_N}(F_1^v(q^2)+F_2^v(q^2)) - \frac{1}{2m_N}F_2^v(q^2)\right]q_\alpha \\ &\quad+\frac{2}{E'+m_N}[S_\alpha, S\cdot q\,](F_1^v(q^2)+F_2^v(q^2))\Bigg\}\,p_v(0)\,, \\ j^\alpha_a &= \mathcal{N}_n\,\bar{n}_v(p')\Bigg\{G_A(q^2)\left[2S_\alpha - \frac{2(S\cdot q)\,v_\alpha}{E'+m_N}\right] \\ &\qquad\qquad + G_P(q^2)\frac{2(S\cdot q)\,q_\alpha}{m_\mu(E'+m_N)}\Bigg\}\,p_v(0)\,, \end{aligned} \qquad (7)$$

with the heavy nucleon spinors, $n_v(r')$ and $p_v(0)$ defined as [12]

$$n_v(p') = \sqrt{\frac{2m_N}{E'+m_N}}\,\frac{1}{2}(1+\not{v})u_n(p')\,, \quad p_v(0) = \frac{1}{2}(1+\not{v})u_p(p)\,. \qquad (8)$$

The kinematics in the rest-frame of the proton is as follows. The four-momenta of the initial proton and the outgoing neutron are $p = (m_N, \mathbf{0})$ and $p' = (E', \mathbf{p}')$, respectively, where $E' = \sqrt{m_N^2 + \mathbf{p}'^2}$ and $\mathbf{p}' = -\mathbf{p}_\nu$. The four-momentum transfer in the OMC process is $q = p' - p = (q_0, \mathbf{q})$, with $q_0 = E' - m_N = \frac{\mathbf{p}_\nu^2}{2m_N} + \mathcal{O}(m_N^{-2})$ and $\mathbf{q} = -\mathbf{p}_\nu$. Expanding the proton and neutron spinors in Eq.(7) up to $\mathcal{O}(m_N^{-2})$ leads to

$$\begin{aligned} j_0(q) &= \chi_n^\dagger\,[f_1^v(q) + (\vec{\sigma}\cdot\hat{\nu})f_3^a(q)]\,\chi_p\,, \\ \mathbf{j}(q) &= -\chi_n^\dagger\,[i(\vec{\sigma}\times\hat{\nu})f_2^v(q) + \hat{\nu}f_3^v(q) + \vec{\sigma}f_1^a(q) + \hat{\nu}(\vec{\sigma}\cdot\hat{\nu})f_2^a(q)]\,\chi_p\,, \end{aligned} \qquad (9)$$

where the non-relativistic polar-vector form factors are related to the standard Lorentz covariant form factors in the proton rest frame via

$$f_1^v(q) = F_1^v(q^2)\left(1 - \frac{q^2}{8m_N^2}\right) + \frac{q^2}{4m_N^2}\,F_2^v(q^2)\,,$$

$$f_2^v(q) = \frac{|\mathbf{q}|}{2m_N}\left[F_1^v(q^2)+F_2^v(q^2)\right] , \quad f_3^v(q) = \frac{|\mathbf{q}|}{2m_N} F_1^v(q^2) , \tag{10}$$

while the non-relativistic axial-vector form factors are related to the covariant axial form factors via

$$f_1^a(q) \ = \ G_A(q^2)\left(1-\frac{q^2}{8m_N^2}\right) , \quad f_2^a(q) = -\frac{|\mathbf{q}|^2}{2m_\mu m_N}\left(1+\frac{q^2}{8m_N^2}\right)G_P(q^2) , \tag{11}$$

$$f_3^a(q) = \frac{|\mathbf{q}|}{2m_N}\left(G_A(q^2)+\frac{q^2}{2m_\mu m_N}\, G_P(q^2)\right) . \tag{12}$$

The non-relativistic form factors appearing in Eqs. (10), (11) and (12) have been calculated in Refs. [17, 18, 20, 21], up to next-to-next-to leading order (NNLO) or $\mathcal{O}((Q/\Lambda_\chi)^3)$, in HB$\chi$PT. In the proton rest-frame, they are given by

$$\begin{aligned}
f_1^v(q) =& \, 1+\kappa_V\frac{q^2}{4m_N^2} \\
& -\frac{1}{(4\pi f_\pi)^2}\left\{q^2\left(\frac{2}{3}\, g_A^2+2B_{10}^{(r)}\right)+q^2\left(\frac{5}{3}\, g_A^2+\frac{1}{3}\right)\ln\left[\frac{M_\pi}{\lambda}\right]\right. \\
& \left. -\int_0^1 dz\left[M_\pi^2(3g_A^2+1)-q^2z(1-z)(5g_A^2+1)\right]\ln\left[1-z(1-z)\frac{q^2}{M_\pi^2}\right]\right\} , \\
f_2^v(q) =& \, \frac{|\mathbf{q}|}{2m_N}\left\{1+\kappa_V+g_A^2\frac{4\pi m_N M_\pi}{(4\pi f_\pi)^2}\int_0^1 dz\left[1-\sqrt{1-z(1-z)\frac{q^2}{M_\pi^2}}\,\right]\right\} , \\
f_3^v(q) =& \, \frac{|\mathbf{q}|}{2m_N} , \\
f_1^a(q) =& \, g_A\left(1-\frac{q^2}{8m_N^2}\right)+\frac{q^2}{(4\pi f_\pi)^2}\tilde{B}_3 , \\
f_2^a(q) =& \, \frac{|\mathbf{q}|^2}{q^2-M_\pi^2}\left\{g_A\left(1+\frac{q^2}{8m_N^2}\right)-\frac{2M_\pi^2}{(4\pi f_\pi)^2}\tilde{B}_2\right\}+\frac{|\mathbf{q}|^2}{(4\pi f_\pi)^2}\tilde{B}_3 , \\
f_3^v(q) =& \, \frac{|\mathbf{q}|}{2m_N}g_A\left(1-\frac{q^2}{q^2-M_\pi^2}\right) .
\end{aligned} \tag{13}$$

In terms of the quantities derived above, the operator $\widehat{\mathcal{M}}$ [see Eq.(2)] is written as

$$\begin{aligned}
\widehat{\mathcal{M}} \ =& \ (1-\vec{\sigma}_l\cdot\hat{\nu})\left[\, f_1^v(q)-i\vec{\sigma}_l\cdot(\vec{\sigma}_N\times\hat{\nu})f_2^v(q)-(\vec{\sigma}_l\cdot\hat{\nu})f_3^v(q)\right. \\
& \left. -(\vec{\sigma}_l\cdot\vec{\sigma}_N)f_1^a(q)-(\vec{\sigma}_l\cdot\hat{\nu})(\vec{\sigma}_N\cdot\hat{\nu})f_2^a(q)+(\vec{\sigma}_N\cdot\hat{\nu})f_3^a(q)\,\right] ,
\end{aligned} \tag{14}$$

where $\vec{\sigma}_l$ and $\vec{\sigma}_N$ are the spin matrices acting on the lepton and nucleon spinors, respectively. We may choose the direction of the emitted neutrino as our z-axis, i.e.,

$\hat{\nu} \equiv \hat{z}$. In the helicity basis, the amplitude $\mathcal{T}_{\rm NR}$ appearing in Eqs.(2) and (4) is given as

$$\mathcal{T}_{\rm NR} \equiv \widetilde{M}(h;S,S_z) = \sum_{S_z^p,S_z^\mu} \langle \frac{1}{2}S_z^p;\frac{1}{2}S_z^\mu \mid \frac{1}{2}\frac{1}{2};SS_z\rangle \, \langle \frac{1}{2}S_z^n;\frac{1}{2},\frac{-1}{2}|\widehat{\mathcal{M}}|\frac{1}{2}S_z^p;\frac{1}{2}S_z^\mu\rangle \quad (15)$$

where $h\!=\!L$ ($h\!=\!R$) corresponds to the positive (negative) helicity state of the final-state neutron, and $S\!=\!0$ ($S\!=\!1$) represents the hyperfine-singlet (triplet) state of the muonic hydrogen atom [32]. The constraint, $S_z = S_z^n - \frac{1}{2}$, reduces the eight possible helicity amplitudes in Eq.(15) to the following three:

$$\begin{aligned}
\widetilde{M}(L;0,0) &= \sqrt{2}\,(f_1^v + 2f_2^v + f_3^v + 3f_1^a + f_2^a + f_3^a)\ , \\
\widetilde{M}(L;1,0) &= \sqrt{2}\,(f_1^v - 2f_2^v + f_3^v - f_1^a + f_2^a + f_3^a)\ , \\
\widetilde{M}(R;1,-1) &= 2\,(f_1^v + f_3^v - f_1^a - f_2^a - f_3^a)\ .
\end{aligned} \quad (16)$$

Finally, since the binding energy of the muonic hydrogen atom can safely be ignored, the total OMC rate in a hyperfine state S is given as

$$\begin{aligned}
\Gamma_S &= \frac{1}{2(m_\mu\!+\!m_N)}\cdot\frac{1}{2S\!+\!1}\int\frac{d^3\mathbf{p}'}{(2\pi)^3 2E'}\frac{d^3\mathbf{p}_\nu}{(2\pi)^3 2E_\nu}\,(2\pi)^4\delta^4(P_I - p' - p_\nu)\,|\mathcal{M}_{fi}|^2 \\
&= \frac{G_\beta^2\mathcal{N}_{\rm rel}^2}{16 m_\mu m_N}\cdot\frac{1}{2S\!+\!1}\,|\Psi_{\mu p}(\mathbf{0})|^2\!\int\!\frac{d^3\mathbf{p}'}{(2\pi)^3 2E'}\frac{d^3\mathbf{p}_\nu}{(2\pi)^3 2E_\nu}\,(2\pi)^4\delta^4(P_I\!-\!p'\!-\!p_\nu)\sum_{S_z,h}\left|\widetilde{M}(h;S,S_z)\right|^2 \\
&= \frac{G_\beta^2\mathcal{N}_{\rm rel}^2}{2S\!+\!1}\cdot\frac{|\Psi_{\mu p}(\mathbf{0})|^2}{64\pi\, m_\mu m_N}\left(\frac{E_\nu}{E_\nu\!+\!\sqrt{m_N^2\!+\!E_\nu^2}}\right)\sum_{S_z,h}\left|\widetilde{M}(h;S,S_z)\right|^2\ ,
\end{aligned} \quad (17)$$

where P_I is the initial total four-momentum. If we ignore radiative corrections and identify $\Psi_{\mu p}(\mathbf{0})$ with the lowest-order $1s$-state Coulomb wave function, $\Phi_{1s}(\mathbf{0}) = (\alpha^3\mu_{\mu p}^3/\pi)^{1/2}$ with $\mu_{\mu p} \equiv m_\mu m_p/(m_\mu\!+\!m_p)$, then the last line in Eq.(17) agrees with Eq.(26) in Ref. [17]. In the present work, however, we do include radiative corrections, and it turns out that, at $\mathcal{O}(\alpha)$ under consideration, there appear two types of significant radiative corrections to $\Phi_{1s}(\mathbf{0})$, and these corrections will be discussed in the following section.

We now evaluate the form factors in Eq.(13), which determine the helicity amplitudes in Eq.(16). Table I shows the numerical values of the nucleon weak form factors and helicity amplitudes calculated for the four-momentum transfer, $q^2 = q_*^2 \equiv -0.88m_\mu^2$, relevant to OMC. These numerical values were obtained with the use of the following input parameters: $g_A = 1.266$, $\kappa_V = 3.706$, $f_\pi = 92.42$ MeV, $M_\pi = 139.57$ MeV, and $m_N = 938.919$ MeV. The LECs appearing in Eq.(13) are determined following Refs. [12, 20]. First, $\tilde{B}_2$ is fixed from the Goldberger-Treiman (G-T) discrepancy relation,

$$\frac{2M_\pi^2}{(4\pi f_\pi)^2\, g_A}\,\tilde{B}_2 = \frac{g_A\, m_N}{g_{\pi NN} f_\pi} - 1\,. \tag{18}$$

For $g_{\pi NN} = 13.40$ and $g_A = 1.266$ (see, e.g., PDG2002 [22]), this relation leads to $\tilde{B}_2 = -0.176$. The values of g_A and $g_{\pi NN}$ have been slightly changing over the years; if we use the latest values $g_{\pi NN} = 13.05$ and $g_A = 1.270$ (taken from PDG2012 [23]), we obtain $\tilde{B}_2 = -0.498$. To what extent the existing uncertainties in g_A and $g_{\pi NN}$ affect the calculated μp capture rate will be discussed in the last section. The LEC, $\tilde{B}_3$, is fixed from the nucleon axial radius,

$$\tilde{B}_3 = \frac{g_A}{2}\,(4\pi f_\pi)^2\,\frac{\langle r_A^2\rangle}{3}\,.$$

The value of the iso-vector axial radius $\langle r_A^2\rangle^{1/2}$ has large uncertainty, see e.g., Ref. [12]. From the empirical axial form factor $G_A(t) = g_A/(1-t/m_A^2)^2$, we find $\langle r_A^2\rangle^{1/2} = \sqrt{12}/m_A = 0.62$ fm (0.57 fm) for $m_A = 1100$ MeV (1200 MeV). We adopt the value $\langle r_A^2\rangle^{1/2} = 0.65$ fm cited in Ref. [20] to find $\tilde{B}_3 = 3.08$. The last of the LECs in Eq.(13), $\tilde{B}_{10}^{(r)}$, is related to the nucleon iso-vector form factor [20]

$$\frac{1}{6}\,\langle r_V^2\rangle = -\frac{2\tilde{B}_{10}^{(r)}(\Lambda_\chi)}{(4\pi f_\pi)^2} - \frac{1+7g_A^2}{6(4\pi f_\pi)^2} - \frac{5g_A^2+1}{3(4\pi f_\pi)^2}\ln\!\left(\frac{M_\pi}{\Lambda_\chi}\right)\,.$$

From the measured value of $\langle r_V^2\rangle$ we obtain $\tilde{B}_{10}^{(r)} = 0.63$ for $\Lambda_\chi = 1$ GeV. These values of the LECs were used in obtaining the numerical results given in Table I.

TABLE I: The OMC form factors Eq.(13) and helicity amplitudes Eq.(16) at $q_*^2 = -0.88 m_\mu^2$, obtained for $g_A = 1.266$ and $g_{\pi NN} = 13.40$.

$f_1^v(q_*)$	$f_2^v(q_*)$	$f_3^v(q_*)$	$f_1^a(q_*)$	$f_2^a(q_*)$	$f_3^a(q_*)$	$\widetilde{M}(L;0,0)$	$\widetilde{M}(L;1,0)$	$\widetilde{M}(L;1,-1)$
0.971	0.244	0.053	1.245	-0.419	0.044	3.447	-0.766	0.153

III. RADIATIVE CORRECTIONS

In this section we consider radiative corrections to OMC, which consist of the usual QED loop corrections and loop corrections involving a weak-interaction vertex. Relegating the discussion of the latter to the end of the section, we first discuss the QED loop corrections.

The initial state in $\mu^- p$ capture is a charge-neutral $\mu^- p$-atom, and the final state involves only electrically neutral particles. Therefore, to the order in HBχPT under consideration, the bremsstrahlung process does not contribute to the "standard" radiative corrections (we ignore the higher order, i.e., $\mathcal{O}(\alpha/m_N)$ corrections which are negligible.) There are, however, two QED loop corrections to the initial state wave function which must be considered: the vacuum polarization correction, $\delta\psi_{1s}^{\rm VP}(\mathbf{0})$, and the correction due to the finite proton size, $\delta\psi_{1s}^{\rm FS}(\mathbf{0})$. Inclusion of these corrections changes the lowest-order muonic atomic wave function, $\Phi_{1s}(\mathbf{0})$, into $\Psi_{1s}(\mathbf{0})$:

$$\Psi_{1s}(\mathbf{0}) = \Phi_{1s}(\mathbf{0})\left[1 + \delta\psi_{1s}^{\rm VP}(\mathbf{0}) + \delta\psi_{1s}^{\rm FS}(\mathbf{0})\right] . \tag{19}$$

Eiras and Soto [24] calculated $\delta\psi_{1s}^{\rm VP}(\mathbf{0})$ to order $\mathcal{O}(\alpha)$, while Friar [25] discussed $\mathcal{O}(\alpha)$ contributions to $\delta\psi_{1s}^{\rm FS}(\mathbf{0})$.

The analytic expression for $\delta\psi_{1s}^{\rm VP}(\mathbf{0})$ derived by Eiras and Soto [24] reads

$$\begin{aligned}
\delta\psi_{1s}^{\rm VP}(\mathbf{0}) &= -\frac{\alpha}{\pi}\left[\left\{\frac{5}{9} - \frac{\pi}{4}\xi + \frac{1}{3}\xi^2 - \frac{\pi}{6}\xi^3 + \frac{1}{3}(\xi^4+\xi^2-2)F_1(\xi)\right\}_{\not\gamma}\right.\\
&\quad + \left\{\frac{11}{18} - \frac{2}{3}\xi^2 + \frac{2\pi}{3}\xi^3 - \frac{12\xi^4+\xi^2+2}{6}F_1(\xi) - \frac{4\xi^4+\xi^2-2}{6(\xi^2-1)}\left[1-\xi^2F_1(\xi)\right]\right\}_{\rm pole}\\
&\quad + \left\{\frac{2}{3} + \frac{\pi}{4}\xi - \frac{1}{9}\xi^2 + \frac{13\pi}{18}\xi^3 - \frac{1}{9}(13\xi^4 - 11\xi^2 - 11)F_1(\xi)\right.\\
&\quad \left.\left. -\frac{1}{3}(4\xi^3+3\xi)F_2(\xi) + \frac{1}{3}(4\xi^4+\xi^2-2)F_3(\xi) + \frac{1}{3}\left(4\xi^2+\frac{11}{3}\right)\ln\frac{\xi}{2}\right\}_{\rm multi-\gamma}\right]
\end{aligned} \tag{20}$$

where $\xi \equiv m_e/(\alpha\mu_{\mu p}) \sim \mathcal{O}(1)$; the expressions for the functions $F_i(\xi)$ $(i = 1, 2, 3)$ in Eq.(20) can be found in Ref. [24]. As explained in Ref. [24], the first curly bracket in Eq. (20) corresponds to zero photon exchange contributions, the second bracket corresponds to Coulomb pole subtraction terms, and the third bracket represents the multi-photon exchange contributions. Thus $\delta\psi_{1s}^{\rm VP}(\mathbf{0})$ consists of three parts:

$$\delta\psi_{1s}^{\rm VP}(\mathbf{0}) = [\delta\psi_{1s}^{\rm VP}(\mathbf{0})]_{\not\gamma} + [\delta\psi_{1s}^{\rm VP}(\mathbf{0})]_{\rm pole} + [\delta\psi_{1s}^{\rm VP}(\mathbf{0})]_{\rm multi-\gamma}. \tag{21}$$

We denote by $\Gamma_S^{(0)}$ the μp capture rate for the hyperfine-state S ($S = 0$ or 1), obtained by using $\Phi_{1s}(\mathbf{0})$ for $\Psi_{\mu p}(\mathbf{0})$ in Eq.(17). The use of $\Phi_{1s}(\mathbf{0})\big[1+\delta\psi_{1s}^{\rm VP}(\mathbf{0})\big]$ for $\Psi_{\mu p}(\mathbf{0})$ in Eq.(17) changes $\Gamma_S^{(0)}$ into

$$\Gamma_S^{(0)} + \delta\Gamma_S^{\rm VP} \equiv \Gamma_S^{(0)}[1+ 2\delta\psi_{1s}^{\rm VP}(\mathbf{0})]. \tag{22}$$

Table II shows the numerical results for $(\delta\Gamma_S)^{\rm VP}/\Gamma_S^{(0)} = 2\delta\psi_{1s}^{\rm VP}$. The first three columns show the individual contributions of the three terms in Eq.(21), while the fourth column gives $2\delta\psi_{1s}^{\rm VP}$, which is the sum of these three contributions. For comparison, in the fifth and sixth columns, we quote the values of $2\delta\psi_{1s}^{\rm VP}$ (in our notation) obtained in Refs. [9, 26]. Our result for $2\,\delta\psi_{1s}^{\rm VP}(\mathbf{0})$ agrees with the value given by Czarnecki

TABLE II: Corrections from vacuum polarization (VP) effects, $(\delta\Gamma_S)^{\rm VP}/\Gamma_S^{(0)} = 2\delta\psi_{1s}^{\rm VP}(\mathbf{0})$. The last two columns give the values of $2\delta\psi_{1s}^{\rm VP}(\mathbf{0})$ in Refs. [9, 26] for comparison.

Zero photon exchange $2[\delta\psi_{1s}^{\rm VP}(\mathbf{0})]_{\not\gamma}$	Coulomb pole subtraction $2[\delta\psi_{1s}^{\rm VP}(\mathbf{0})]_{\rm pole}$	Multi-photon exchange $2[\delta\psi_{1s}^{\rm VP}(\mathbf{0})]_{\rm multi-\gamma}$	Total VP contribution $2\,\delta\psi_{1s}^{\rm VP}(\mathbf{0})$	*Czarnecki, Marciano, & Sirlin* [9]	*Goldman* [26]
$1.045\frac{\alpha}{\pi}$	$0.358\frac{\alpha}{\pi}$	$0.250\frac{\alpha}{\pi}$	$1.654\frac{\alpha}{\pi}$	$1.73\frac{\alpha}{\pi}$	$2.95\frac{\alpha}{\pi}$

et al. [9] within $\sim$5%. Since the size of the $2\delta\psi_{1s}^{\rm VP}$ correction itself is about 0.4%, we can say this part of QED corrections is controlled with sufficient accuracy for our purpose.

The proton finite-size correction up to $\mathcal{O}(\alpha)$ is given as [25]

$$\delta\psi_{1s}^{\rm FS}(\mathbf{0}) = -\alpha\mu_{\mu p}\langle r\rangle_p\,. \tag{23}$$

where $\langle r\rangle_p$ is the first moment of the proton charge distribution, $\rho_p(\mathbf{r})$. Unfortunately, $\langle r\rangle_p$ cannot be measured directly, whereas the second moment, $\langle r^2\rangle_p$, can be extracted from experimental data. In order to evaluate $\langle r\rangle_p$, we assume a certain functional form of the proton charge distribution, $\rho_p(\mathbf{r})$, involving a single parameter, and after determining this parameter from the measured value of $\langle r^2\rangle_p$, we deduce $\langle r\rangle_p$ from the assumed $\rho_p(\mathbf{r})$. Table III gives $\langle r\rangle_p$ and $\langle r^2\rangle_p$ calculated for three different functional forms of $\rho_p(\mathbf{r})$. The results for the exponential form, $\rho_p(\mathbf{r})=1/(8\pi r_0^3)\mathrm{e}^{-(r/r_0)}$, are given in the fourth column; the exponential form corresponds to a dipole-type proton form factor (in momentum space), which reproduces very well the elastic electron-proton scattering data. We also present the results for two other commonly used forms for $\rho_p(\mathbf{r})$, the uniform distribution (second column), and the Gaussian form (third column); these results have been extracted from Ref. [25]. The last row in table III

TABLE III: First and second moments of the proton charge distribution calculated for various forms of $\rho_p(\mathbf{r})$ characterized by a single parameter.

	Uniform	Gaussian	Exponential
$\rho_p(\mathbf{r})$	$\frac{3}{4\pi R^3}\,\theta(R-r)$	$\left(\frac{1}{\sqrt{\pi}r_0}\right)^3 \mathrm{e}^{-(r/r_0)^2}$	$\frac{1}{8\pi r_0^3}\,\mathrm{e}^{-(r/r_0)}$
$\langle r\rangle_p$	$3R/4$	$2r_0/\sqrt{\pi}$	$3r_0$
$\langle r^2\rangle_p$	$3R^2/5$	$3r_0^2/2$	$12\,r_0^2$
$\langle r\rangle_p/\sqrt{\langle r^2\rangle}_p$	$\sqrt{15}/4=0.968$	$2\sqrt{2/3\pi}=0.931$	$\sqrt{3}/2=0.866$

shows the ratio $\langle r_p\rangle/\sqrt{\langle r^2\rangle}_p$ for each assumed form of $\rho_p(\mathbf{r})$. By taking the average of the results for these three cases, we deduce $\langle r\rangle_p=(0.916\pm0.051)\sqrt{\langle r^2\rangle}_p$; the "error estimate" here has been obtained by interpreting the scatter of the results in table III as a measure of uncertainty. Then, with the use of the experimental value of the proton r.m.s. radius, $\sqrt{\langle r^2\rangle}_p=0.862$ fm [27][33], we find $\langle r\rangle_p=0.790\pm0.044$ fm. Using this value in Eq.(23) leads to $\delta\psi_{1s}^{\mathrm{FS}}(\mathbf{0})\simeq-0.00275(1\pm0.056)$. Correspondingly, the finite-proton-size correction to the capture rate Γ_S in Eq.(17) is found to be $2\,\delta\psi_{1s}^{\mathrm{FS}}(\mathbf{0})=-0.0055(1\pm0.06)$. This result is essentially the same as that given in Eq.(8) of Ref. [9]. Thus, the finite-proton-size correction is of the same order as the vacuum polarization correction shown in Table II.

In addition to the two QED corrections discussed above, we need to consider the "standard" radiative corrections involving a weak-interaction vertex. It is to be noted that part of these corrections are already included in G_F, if one uses (as we do here) the value of G_F determined from the measured muon lifetime. In the following, what we simply call the "electroweak loop corrections" refer to those electroweak loop corrections that have not been accounted for by the use of the G_F derived from the muon lifetime. We remark that, to the order in HBχPT under consideration, the electroweak loop corrections are identical for μp capture and neutron beta-decay. We can therefore utilize the results obtained for neutron beta-decay in, e.g., Refs. [15, 29]. Since the muon velocity, β, in the initial μp-atomic state is essentially zero, we can take the limit of $\beta \to 0$ in the previous evaluations of the radiative corrections to the neutron beta-decay rate [15, 29], (In applying the results obtained for neutron β-decay to the μp capture case, we must drop the bremsstrahlung contributions, since both the initial and final states in μp capture contain only charge-neutral particles.) Then the electroweak radiative loop correction to the μp capture rate is obtained as

$$\Gamma_S^{(0)} \to \Gamma_S^{(0)}(1 + RC_{EW})\,, \tag{24}$$

with

$$RC_{\rm EW} = \frac{\alpha}{2\pi}\left\{\tilde{e}_V^R(m_N) + 3\ln\left[\frac{m_N}{m_\mu}\right] - \frac{27}{4}\right\} \tag{25}$$

In this expression the electroweak LEC, $\tilde{e}_V^R(m_N)$, subsumes short-distance physics not probed in the low-energy muon capture reaction. The value of this LEC at the scale, $\lambda = m_N$, has been determined in Refs. [15, 16] by comparing with the expressions for the short-distance radiative corrections derived by Sirlin and Marciano [10, 11] for the electroweak processes. The result is $\tilde{e}_V^R(m_N) = 19.5$. In the next section we discuss the numerical consequences of our evaluation of the $\mathcal{O}(\alpha)$ radiative and finite proton-size corrections discussed in this section.

IV. NUMERICAL RESULTS FOR THE CAPTURE RATES, Γ_0 AND Γ_1

As explained earlier, $\Gamma_0^{(0)}$ ($\Gamma_1^{(0)}$) denotes the hyperfine-singlet (hyperfine-triplet) capture rate calculated without including radiative corrections; *viz.,* $\Gamma_0^{(0)}$ and $\Gamma_1^{(0)}$ are obtained by identifying $\Psi_{\mu p}(\mathbf{0})$ in Eq.(17) with $\Phi_{1s}(\mathbf{0})$. Using the inputs listed in Table I, we obtain

$$\Gamma_0^{(0)} = 694\,\mathrm{s}^{-1} \quad \text{and} \quad \Gamma_1^{(0)} = 11.9\,\mathrm{s}^{-1}\,, \tag{26}$$

corresponding to the use of $g_A = 1.266$ and $g_{\pi NN} = 13.40$. The inclusion of the radiative corrections discussed in Section III modifies $\Gamma_S^{(0)}$ ($S = 0,\,1$) to Γ_S as

$$\Gamma_S = \Gamma_S^{(0)}(1 + RC_{\rm QED} + RC_{\rm EW})\ ;\ \ S = 0,\,1. \tag{27}$$

Here $RC_{\rm QED}$ represents the corrections arising from the change in the atomic μp wave function due to the vacuum-polarization and finite-proton-size effects, while, as explained earlier, $RC_{\rm EW}$ is the electroweak radiative correction:

$$RC_{\rm QED} = 2\delta\psi_{1s}^{\rm VP}(\mathbf{0}) + 2\delta\psi_{1s}^{\rm FS}(\mathbf{0}) \tag{28}$$

$$RC_{\rm EW} = \frac{\alpha}{2\pi}\left\{\tilde{e}_V^R(m_N) + 3\ln\left[\frac{m_N}{m_\mu}\right] - \frac{27}{4}\right\} . \tag{29}$$

We remark that, since the last two terms in Eq.(29) almost cancel each other, $RC_{\rm EW}$ has a pronounced dependence on the LEC, $\tilde{e}_V^R(m_N)$, which characterizes the short-distance processes.

The numerical consequences of including the radiative corrections are displayed in Table IV, where Γ_S ($S=0$, 1) are shown along with $\Gamma_S^{(0)}$ and the changes due to the individual contributions of $RC_{\rm QED}$ and $RC_{\rm EW}$. Again, these results have been obtained with the use of $g_A=1.266$ and $g_{\pi NN}=13.40$. Table IV demonstrates that the largest radiative correction to the OMC rate comes from $RC_{\rm EW}$, in conformity with the results reported in Ref. [9]. In particular, for the hyperfine-singlet OMC rate, which is of our main concern, $RC_{\rm EW}$ changes $\Gamma_0^{(0)}$ by $\sim$2 %.

TABLE IV: The hyperfine-singlet and -triplet OMC rates, Γ_0 and Γ_1 (in units of s^{-1}), calculated with and without radiative corrections (the proton-finite-size effect is included as part of $RC_{\rm QED}$) corresponding to $g_{\pi NN}=13.40$ and $g_A=1.266$.

$\Gamma_0^{(0)}$	$\Gamma_0^{(0)}(1+RC_{\rm QED})$	$\Gamma_0^{(0)}(1+RC_{\rm EW})$	$\Gamma_0=\Gamma_0^{(0)}(1+RC_{\rm QED}+RC_{\rm EW})$
694.4	693.2	709.9	708.7
$\Gamma_1^{(0)}$	$\Gamma_1^{(0)}(1+RC_{\rm QED})$	$\Gamma_1^{(0)}(1+RC_{\rm EW})$	$\Gamma_1=\Gamma_1^{(0)}(1+RC_{\rm QED}+RC_{\rm EW})$
11.9	11.9	12.2	12.1

V. DISCUSSION AND CONCLUSIONS

In the previous section we have presented our numerical results obtained with the use of representative values for the relevant input parameters. We now discuss to what extent the uncertainties in these input parameters affect the calculated values of the μp capture rates, Γ_S $(S\!=\!0,1)$. We shall chiefly concentrate on the hyperfine-singlet rate Γ_0, a quantity of primary concern for most μp capture experiments.

As mentioned, the LEC, $\tilde{B}_2$, is determined from the G-T discrepancy [see Eq.(18)], and the fact that the current precision of the values of g_A and $g_{\pi NN}$ is somewhat limited leads to rather significant uncertainty in $\tilde{B}_2$. The results in Table IV were obtained for $\tilde{B}_2 = -0.176$, which corresponds to $g_{\pi NN}\!=\!13.40$ and $g_A\!=\!1.266$ taken from PDG2002 [22]. If we adopt $g_{\pi NN} = 13.05$ and $g_A = 1.270$ (values given in PDG2012 [23]), then we obtain $\tilde{B}_2 = -0.498$ and, correspondingly, $\Gamma_0^{(0)} = 701\, s^{-1}$ and $\Gamma_1^{(0)} = 11.7\ s^{-1}$. Thus, the uncertainty in $\tilde{B}_2$ changes $\Gamma_0^{(0)}$ by 7 s^{-1} (about 1% increase), and $\Gamma_1^{(0)}$ by 0.2 s^{-1} (about 2 % decrease). If we take into account (in the last column in Table IV) the mentioned variation in $\Gamma_0^{(0)}$, the corresponding change in Γ_0 ranges from 708.7 s^{-1} to 716.7 s^{-1}; thus

$$\Gamma_0 = 712.7\times(1\pm 0.005)\, s^{-1}, \tag{30}$$

where the relative error was deduced from the 1 % difference between the above-quoted two values of $\Gamma_0^{(0)}$.

We next consider the uncertainty in the proton axial radius, $(\langle r_A^2\rangle)^{1/2}$, discussed in Section II. The results shown in Table IV were obtained for $(\langle r_A^2\rangle)^{1/2} = 0.65$ fm. If we instead use $(\langle r_A^2\rangle)^{1/2} = 0.57$ fm, corresponding to $m_A = 1200$ MeV, we find $\Gamma_0^{(0)} = 697.3 s^{-1}$, an increase of 2.9 s^{-1} (or $\sim 0.5\%$). Again, if we consider (in the last column in Table IV) the scatter in the value of $\Gamma_0^{(0)}$, then the corresponding change in Γ_0 ranges from 708.7 s^{-1} to 712.9 s^{-1}, i.e.,

$$\Gamma_0 = 710.8\times(1\pm 0.0025)\, s^{-1} \tag{31}$$

where the relative error was deduced from the 0.5 % variation in $\Gamma_0^{(0)}$. Taking the average of the values in Eqs.(30) and (31), we arrive at

$$\Gamma_0 = 712\times(1\pm 0.006)\, s^{-1}\,, \tag{32}$$

where the error has been deduced from the quadratic sum of the errors in Eqs.(30) and (31).

In connection with Table IV we have pointed out that, of all the corrections of $\mathcal{O}(\alpha)$, the electroweak loop correction, $RC_{\rm EW}$, is largest; it increases $\Gamma_0^{(0)}$ by as much as $\sim$2 %. So, if $RC_{\rm EW}$ is not evaluated with sufficient accuracy, the theoretical error in Γ_0 can be larger than indicated by Eq.(32). As already mentioned, $RC_{\rm EW}$ is a sensitive function of the LEC, $\tilde{e}_V^R(M_N)$, as the last two terms in Eq.(29) nearly cancel each other. In the present work, following Ref.[15], we have determined $\tilde{e}_V^R(M_N)$ by comparing our HBχPT results with those obtained in the S-M method [10, 11]. Since this method is generally considered to be highly reliable, we believe that $\tilde{e}_V^R(M_N)$ is known with sufficient accuracy to make the uncertainty in Γ_0 related to $RC_{\rm EW}$ much smaller than 0.6 %, the error arising from the other sources [see Eq.(32)]. We remark that the same LEC, $\tilde{e}_V^R(m_N)$, also appears in neutron beta decay [15] and the inverse beta decay process, $\bar{\nu}_e+p \to e^+ + n$ [16]. It is therefore, in principle, possible to use either the neutron β-decay or the μp capture to control $\tilde{e}_V^R(m_N)$ and make predictions for the other processes involving the same LEC. This would allow us to deduce $\tilde{e}_V^R(m_N)$ without using the result of the S-M method.

In conclusion, the present HBχPT calculation of the hyperfine-singlet μp capture rate Γ_0, including radiative and proton finite-size corrections of $\mathcal{O}(\alpha)$, gives

$$\Gamma_0 = 712 \pm 5\, s^{-1}\,. \tag{33}$$

This is in excellent agreement with the experimental value quoted in Eq.(1). The 0.6 % theoretical error in Eq.(33) is dominated by the uncertainties in the input values of g_A and $g_{\pi NN}$ that enter into the G-T discrepancy.

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[33] It is to be noted that a recent muonic hydrogen atom experiment has questioned this value for the proton r.m.s. radius, see e.g., Ref. [28]