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Plasma effects on resonant fusion

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I investigate the effects of plasma interactions on resonance-enhanced fusion rates in stars. Starting from basic principles we derive an expression for the fusion rate that can serve as a basis for discussion of approximation schemes. The present state-of-the-art correction algorithms, based on the classical correlation function for the fusing particles and the classical energy shift for the resonant state, do not follow from this result, even as an approximation. The results of expanding in a perturbation solution for the case of a weakly coupled plasma are somewhat enlightening. But at this point we are at a loss as to how to do meaningful calculations in systems with even moderate plasma coupling strength. Examples where this can matter are: the effect of a possible low energy $^{12}\text{C} + ^{12}\text{C}$ resonance on X-ray bursts from accreting neutron stars or on supernova 1A simulations; and the calculation of the triple α rate in some of the more strongly coupled regions in which the process enters, such as accretion onto a neutron star.

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I. 1. INTRODUCTION

Nuclear fusion rates play a central role in stellar evolution theory. Their calculation depends primarily on laboratory data, and the open questions that persist have mostly to do with these measurements and their extrapolations to very low energies. Another issue that presents itself is that of the influence of the surrounding plasma on the fusion rates. This is a mature subject, to say the least, for the case of non-resonant fusion, but there are some questions that still are not settled, and some of these pertain to the special case of resonance-enhanced fusion, which is the subject of the body of this paper.

To set the context, I begin with a review of the way the plasma corrections work for the ordinary case in which the energy variation of the vacuum value of [cross-section \times velocity] is mainly from the Coulomb wave-functions, rather than from a resonant peak. A lore has developed, the “basically classical” approach [1]- [6], which follows the following path ¹:

1.) calculation of an effective two particle correlation function (down to a small distance) in a classical simulation;

2.) defining an effective two body potential proportional to the logarithm of this function, written in the form $V_{\text{eff}} = Z^2 e^2 / r + V_{\text{sc}}(r)$ where V_{sc} is referred to as the “screening potential”;

3.) in the potential V_{eff} calculating the tunneling through the barrier to determine the correlation function at zero separation $K(r = 0)$. This correlator then is assumed to embody all of the effects of the plasma on the fusion rate.

Implementing the above, I consider fusing particles A and B, with respective number densities, n_a and n_b . I write the rate, w , at which one specified nucleus of type

B undergoes fusion as,

$$w = n_a \langle \sigma v \rangle \frac{K(0)}{K_0(0)}. \quad (1)$$

and I consider all of the energy dependence of the vacuum cross-section σ to come from the Coulomb potential. The factor $K_0(0)$ is the correlator in the absence of the plasma, and serves to cancel the Coulomb factor that comes in σ .

Both in this usual case, where the input from experiment is the low energy Coulomb-corrected cross-section, and in the resonance case, where the input is the resonance energy and partial width, I use the word “experimental”, slightly inaccurately, to mean what the data would be when the effects of the atomic electrons in the real experiment have been removed. This is not a trivial matter and in some cases there are considerable uncertainties as to the results. However these uncertainties are small on the scale of corrections that I shall discuss.

Turning to the resonant case, in the absence of plasma the contribution of a narrow resonance to the fusion rate is,

$$w_0 = n_a \zeta e^{-E_r/T} \exp\left(-\pi e^2 Z^2 \sqrt{\frac{\mu}{E_r}}\right), \quad (2)$$

where E_r is the resonance energy and ζ is given by the product of $\Gamma_r^{(0)}$, defined as what the partial width in the elastic channel would be in the absence of the Coulomb force between these nuclei, and irrelevant kinematic factors. The final factor in (2) is the square of the exponential factor in the Coulomb wave-function; and μ is the reduced mass of the fusing particles.

This paper discusses approaches to incorporating plasma effects into the rates for the resonant case. These plasma effects can be important in applications. Refs. [7] and [8] suggest that the existence a resonant state in the $^{12}\text{C} + ^{12}\text{C}$ system in the vicinity of $E_r = 1.5$ MeV in energy might explain features of X ray bursts from accreting neutron stars. Refs. [7] and [9] discuss the screening

¹ Everything in the present paper pertains to cases in the thermonuclear regime, rather than in the pycnonuclear regime.

corrections to resonant rates, motivated by this possibility. Following these references, I choose as an example a density of $\rho = 5 \times 10^9 \text{gc}^{-3}$ for a pure ^{12}C plasma, and a temperature region $4 < T_8 < 9$.

For purposes of exposition I assume that the screening potential V_{sc} is nearly constant in the classically inaccessible region for the two particle system with resonant energy, and make the replacement $V_{sc}(r) \approx V_{sc}(0)$ in this region. In the above application this is a good approximation². I write $V_{sc}(0) = -\gamma/T$ and take the dimensionless parameter γ from ref. [9],

$$\gamma = 2.293\Gamma^{0.25} + 1.053\Gamma - 0.5551 \ln(\Gamma) - 2.35, \quad (3)$$

where Γ is the usual plasma coupling. Under the sample conditions of composition and density it is given by, $\Gamma = 6.2/T_9$. A first guess as to the form for the screened rate formula could be,

$$w_{scr} = n_a \zeta e^{-E_r/T} \exp\left(\frac{-\pi e^2 Z^2 \sqrt{\mu}}{\sqrt{E_r + \gamma T}}\right), \quad (4)$$

where the penetration factor has been altered by subtracting the screening potential at the origin, $-\gamma T$, from the energy in the incoming state. In normal fusion, where there is an integral over energy and the other energy dependent factors are the phase space and the statistical factor $\exp[-E/T]$, a simple shift of integration variable by $-\gamma T$ gives the usual enhancement factor $\exp(\beta\gamma)$ in rate, but this does not apply to the resonance case, it appears.

In refs. [7] and [9], the result (4) is extended by including a plasma-induced resonance-energy shift Δ_{E_r} given by making the replacement, $E_r \rightarrow E_r + \Delta_{E_r}$, and leading to,

$$w_{scr+shft} = n_a \zeta e^{-(E_r + \Delta_{E_r})/T} \exp\left(\frac{-\pi e^2 Z^2 \sqrt{\mu}}{\sqrt{E_r + \gamma T + \Delta_{E_r}}}\right). \quad (5)$$

The question now is, “what is Δ_{E_r} ?”. Both ref. [7] and ref. [9] choose it to be given by $E_{12} - 2E_6$, where E_Z is the free energy shift of an ion with charge eZ due to interaction with the plasma. In this case, in the “basically classical” approach it is a theorem that $\Delta_{E_r} = V_{sc}(0)$ [5]. But in the next section I shall argue that the energies of the incoming ^{12}C ions probably should not have been subtracted, in which case we would have had $\Delta_{E_r} \approx 2V_{sc}(0)$. What do I mean by “probably”? Actually, only that if one were to start from the correct *ab initio* approach, and ask how, making reasonable assumptions, anything like the above structure can result, then one would come

to the conclusion that these energies should not be subtracted. When one looks more deeply into the problem, however, it looks more and more as though nothing about this whole general approach can be trusted. This conclusion will be explained below.

For now, just in order to show the importance of the issues, I compare the two suggestions made above for the parameter range specified above. Fig.1 shows the ratios of screened to unscreened rates for my two different assumptions as to the value of Δ_{E_r} , plotted over a range of temperatures.

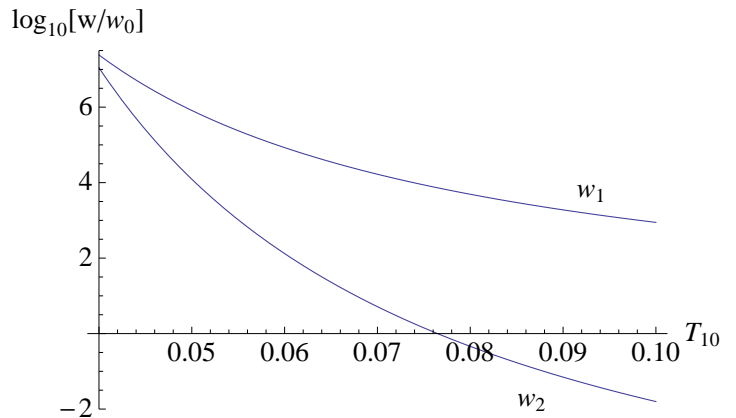


FIG. 1: The logarithms of the ratios of the screened rates, w_1 and w_2 to unscreened rates w_0 from (5). The rate w_2 has been calculated taking Δ_{E_r} as just the resonance energy shift, while the rate w_1 is the result if, in calculating Δ_{E_r} , the energy shifts of the incoming nuclei are subtracted.

Looking at fig. 1, it can be stipulated that the subject of this paper can be important, even in a region of moderate values of the plasma coupling. But I am not here addressing data, nor the likelihood of existence of the possible resonance. I am only examining the underlying theory, and I must characterize (5) as an *ansatz*, rather than as a derived result. This is in contrast to what I would argue *vis-a-vis* the case of most applications of (1) to non-resonant fusion, but even the latter is subject to the following caveat: eq. (1), giving the rate correction in terms of K , is true only in the case that the energy release Q in the fusion is much greater than the Gamow energy, E_G [10] [11]. For the case of resonant fusion the analogue to the value of Q is, in effect, not only not large compared to E_G , it is negative. It is my opinion that one can only begin serious consideration of plasma effects in the resonant case only after developing a correct formal framework for the discussion. This is taken up in the next section.

II. 2. FORMAL FRAMEWORK

I quote some general relations that are derived in the appendix. In these derivations for the fusion reaction, $A+B \rightarrow R$, I take the resonance, R , as a particle, with a

² It is easy to go beyond this simplification, using the mechanics presented in ref. [9].

point coupling to the fusing nuclei. Described in terms of field operators $\psi_a(\mathbf{r})$ and $\psi_b(\mathbf{r})$ for the fusing particles, and $\psi_r(\mathbf{r})$ for the resonance, a zero-range fusion interaction is,

$$H_{\text{fus}} = g \int d\mathbf{r} \psi_r^\dagger \psi_a \psi_b + H.C., \quad (6)$$

where calculation of the decay rate for $R \rightarrow A+B$ gives g^2 in terms of $\Gamma_r^{(0)}$,

$$g^2 = 2^{-1/2} \pi \mu^{-3/2} \Gamma_r^{(0)} E_r^{1/2}. \quad (7)$$

Since the above model gives back exactly the Breit-Wigner formula when one sums the chain of repeated interactions to find the scattering amplitude, I judge it to be adequate for my purposes, as long as the resonance width in the fusion channel is small. The total width will not enter the explicit results, but for the whole application I assume that the decay is mostly into other channels. This simulation of the resonance should be good as long as the barrier penetration calculation doesn't care about the difference between $r = 0$ and the nuclear radius.

For purposes of exposition I think of singled-out reacting particles, A and B. The “plasma” is composed of all other particles. The rate will then have a factor V^{-2} where V is the volume; when translational invariance is restored and there are finite densities of the species A and B then V^{-2} is to be replaced by the product of number densities $n_a n_b$ to get rate per volume, or just by n_a to get rate per nucleus B as in the convention stated above. I make this replacement *ab initio* in what follows. It will also eliminate superfluous complication in what follows to take both the nucleus B (described by ψ_b) and the resonance R to have infinite mass, and to position them at the origin. The corrections, expressed as a multiplicative factor, will be very nearly the same as in the equal mass case.

With this simplification, in the appendix I derive expressions (A.8) and (A.11), which together give the resonant fusion rate,

$$\begin{aligned} w = & g^2 Z_P^{-1} n_a \int_{-\infty}^{\infty} dt \\ & \times \sum_{\text{Pl}, \text{Pl}'} \left\langle \left[\mathbf{r}_a = \mathbf{0}, \text{B}, \text{Pl} \left| e^{-(\beta - it)H} \right| \mathbf{r}_a = \mathbf{0}, \text{B}, \text{Pl}' \right] \right. \\ & \left. \times \left\langle R, \text{Pl}' \left| e^{-itH} \right| R, \text{Pl} \right\rangle \right], \end{aligned} \quad (8)$$

where $\beta = 1/T$ and Z_P is the partition function for the plasma. The notation, Pl, stands for all of the coordinates of the plasma particles collectively. The symbol B within the first bracket indicates that nucleus B is present (at $r = 0$) in the space of states in which this calculation is to be carried out. In the second bracket there is only the plasma and the single nuclear state R, the latter fixed at the origin.

Now I ask: “What is the relation between (8) and the *ansatz* (5) ?” I begin by ignoring the coupling of the resonance state R to the plasma. The the final bracket in (8) is simply,

$$\left\langle R, \text{Pl}' \left| e^{-itH} \right| R, \text{Pl} \right\rangle = e^{-iE_r t} \delta(\text{Pl} - \text{Pl}'), \quad (9)$$

where the delta function is the product of delta functions for the individual plasma particles. Then the sum over plasma states in (8) gives the dependence $\exp(-iE_r t)$ times $K(\mathbf{0}, t)$, where K is the correlator as used in (1), but now extended to finite time by the replacement $\beta \rightarrow \beta - it$ and in the form of a trace over the plasma coordinates Pl. In the “basically classical” approach it is just,

$$K(\mathbf{0}, t) = \int \frac{d^3 p}{(2\pi)^3} e^{-E_p(\beta - it)} |\Psi_p(\mathbf{0})|^2, \quad (10)$$

where the Ψ_p is the wave-function at the origin in the screened potential. For the case considered in the introduction in which the screening part changes little within the classically inaccessible region I obtain,

$$|\Psi_p(\mathbf{0})|^2 \approx 2\pi e^2 Z^2 |\mathbf{p}|^{-1} \exp \left(-\pi e^2 Z^2 \sqrt{\frac{2\mu}{E_p - V_{sc}(0)}} \right). \quad (11)$$

Putting (10), (9) into (8) and using (11) one obtains (4), later to be modified by the plasma induced energy shift. Next I ask, “Does the governing formula (8) tell us how to do this modification”? Note that once I put in interactions with the plasma the final factor in (8), $\langle R, \text{Pl}' | \dots | R, \text{Pl} \rangle$ will not be a delta function in the plasma coordinates, nor diagonal in these coordinates. Thus I cannot eliminate the sum over states Pl' . Moreover the first bracket is no longer a trace and is then no longer given in terms of the correlation function.

However, in the spirit of “basically classical”, I can say “I determined the effective potential between ions A and B as the logarithm of the classical correlation function, in a calculation that traced over the plasma states. So I now imagine that when A and B are close to each other the plasma is in a single configuration, Pl_0 , a configuration that produces an electrical potential ϕ_{sc} at the origin. The sum of the interaction energies of these two particles with this potential, less the plasma-induced energy shifts of the individual ions when widely separated, gives the screening potential energy $V_{sc}(0)$. This is the energy shift that is to be used in calculating the first bracket, $\langle \dots \rangle$, in (8). Turning to the calculation of the plasma corrections to the second bracket, one sees that with the state Pl_0 replacing both Pl and Pl' in (8), thus eliminating the sum over Pl' , it becomes,

$$\left\langle R, \text{Pl}_0 \left| e^{-itH} \right| R, \text{Pl}_0 \right\rangle = e^{-i(E_r + \Delta E_r)t}, \quad (12)$$

which leads back to (5) with the second interpretation of Δ_{E_r} as given in the discussion following (5).

With this pseudo-derivation in hand, I feel more confident in declaring my belief that if any approach that leads to the form (5) makes sense, for strongly coupled systems, then Δ_{E_r} must be simply the energy shift of the resonance R in the plasma, as given by $e^2(Z_1 + Z_2)\phi_{sc}$, with no subtractions for the energy shifts of the isolated A and B states in the same plasma. Indeed, the object estimated in (12) has no way of knowing how the total charge $Z_1 + Z_2$ was split between the incoming nuclei. The above conclusion is at variance with the assumptions of refs. [7] and [9], and I have already illustrated in fig. 1 the very big difference that the choice makes in the rate formula.

That said, I believe that it is much more likely that the basically classical approach, which appears to have served well in ordinary fusion reactions, when the energy release Q is much greater than the Gamow energy, simply cannot be used to get meaningful results in this resonance fusion problem, with any choice of Δ_{E_r} .

I best can explain the essential difference between the resonance case and the normal fusion case by recalling some of the features of the latter. Again, strictly for economy of exposition, in the reaction $A + B \rightarrow C + D$, I take the two ions B and D to have infinite mass. As in the resonance case I take a zero range fusion interaction. Instead of a resonance energy parameter one now has an ordinary energy release parameter, Q . The analogue to (8) is now, as given in ref. [11],

$$w = g^2 Z_P^{-1} n_a \int_{-\infty}^{\infty} dt e^{iQt} \times \sum_{\text{Pl}, \text{Pl}'} \left\langle \left[\left| \mathbf{r}_a = \mathbf{0}, \text{B}, \text{Pl} \right| e^{-(\beta - it)H} \left| \mathbf{r}_a = \mathbf{0}, \text{B}, \text{Pl}' \right\rangle \times \left\langle \mathbf{r}_c = \mathbf{0}, \text{D}, \text{Pl}' \right| e^{-itH} \left| \mathbf{r}_c = \mathbf{0}, \text{D}, \text{Pl} \right\rangle \right] \right\rangle. \quad (13)$$

Now one can see how in the ordinary fusion case, when $Q \gg E_G$ the “basically classical” approach may suffice. In the $A+B \rightarrow C+D$ fusion process of (13) when the energy release Q is large compared with the Gamow energy, it is a reasonable approximation to set $t = 0$ in the exponent within the first bracket $\langle \rangle$ (pertaining to the initial $A + B$ system). If one further omits all plasma coupling of the fusion products to the plasma in calculation of the second bracket (pertaining to the $C+D$ system), and keeps just the kinetic energy term, H_0 , then the time integral just sets the energy of the fusion products to the value Q . Since I have taken the operator in the second bracket to be independent of the plasma coordinates, one can integrate $\delta(\text{Pl} - \text{Pl}')$ over the intermediate plasma configurations Pl' , and the first $\langle \rangle$, integrated over the space Pl , becomes the trace defining the static correlator, bringing one back to the standard result (1).

But for the resonance case I find no similar argument that can defend (12) or to lead back to (5).

III. 3. PERTURBATIVE RESULTS

As an alternative to the machinations of the last section, one can, beginning from (8), turn the perturbation crank to generate results for the weak coupling case. Though the results would of course not be applicable to the strongly coupled systems discussed earlier, one could hope that they would cleanly resolve the question of what to use for the resonance energy shift in (5), assuming for the moment that the calculation leads to the general structure of (5). But it will not, as will be shown. Nonetheless, in the appendix I work out these corrections in detail. For one thing, they comprise the only certain result as to the plasma corrections in the resonant case. For another, they highlight problems that were swept under the rug in the discussion that led to (12), and that raise doubts about any efforts to make a connection to the “basically classical” lore, as used in the speculations above.

First we give the result. The expansion is in powers of the couplings of the plasma particles to each other and to the fusing particles. The e 's in the Coulomb wave-functions of the fusing particles are not part of the expansion, of course. Because of the long range of the Coulomb potential, the leading corrections are of order $e^2 \kappa_D$, where κ_D is the Debye momentum.

After a long calculation, given in the appendix, involving the cancellation of many terms of this order, I find simply,

$$w = [1 + e^2 Z_1 Z_2 \kappa_D \beta] w_0, \quad (14)$$

which is just the original Salpeter correction, as used in standard weak-coupling applications, most particularly in the sun since that is the only venue in which relatively small corrections are of much interest. Is this what required so much work to learn, that the effect of the plasma is still just in the initial state equilibrium contact probability? That is, whether the final system is resonant or not, one has the same modification factor? Indeed if one looks back at the *ansatz*, (4) and chooses $\Delta_{E_r} = -\gamma T$, as in the case in which the screening potential does not change appreciably within the classically inaccessible region, one recovers the equivalent result, namely just the same screening factor $\exp(\gamma)$ that would modify a non-resonant fusion rate.³

However, in the perturbative calculation in the appendix one sees that the simple result depends completely on including the terms which, in the notation of (8), the intermediate plasma states Pl' are different from the initial plasma states in the trace, Pl . Described in terms of fundamental processes one can say that the incoming nuclei scatter a plasma particle and then this plasma particle is put back into its initial state through an interaction with the final resonance; or in an equivalent

³ This was pointed out in ref. [7].

description in terms of scattering amplitudes rather than correlation functions, that one must calculate interference terms between amplitudes in which a plasma particle scatters from a fusing nucleus and amplitudes in which it scatters from the resonance.

If one left out terms like this and calculated only the separate contributions from the interaction of the plasma with the initial state and from the interaction with the resonance, then the sum of these terms would not be given by (14). Depending on the parameters, it can be an order of magnitude greater than shown in (14). Yet going back to the strong coupling case, combining the two effects, classical screening and energy shift, is exactly what was prescribed. So it appears to me that it is not meaningful to calculate this way, and that, for example, neither of the “energy shift” models that were used in the comparison of fig. 1 are viable.

For comparison, if one did the same kind of perturbation calculation for the normal fusion case one would again obtain the Salpeter result with additional terms that cancel when one adds them up. In contrast to the resonance case, however, these individual terms are all reduced by at least one order of the factors $(Z_1 Z_2 e^2 \kappa_D)/Q$ or E_{Gamow}/Q , as long as these ratios are small. Thus in this case the perturbation analysis gives no cause to question the qualitative argument for the “basically classical” approach given at the end of the last section. In ref. [11] I argued the need for rethinking the plasma corrections for cases in which Q is so small that the condition is not fulfilled.

IV. 4. DISCUSSION

The strongest conclusion in this paper, which lacks definitive computational results for the systems of most interest, is that one should begin with a complete and correct formulation. It appears to me that the literature on this topic has depended on using physical intuition to graft together some pieces of lore that have served well in other contexts (but with some limitations that are usually unacknowledged). I do not believe that this provides a coherent approach starting from basic physics.

I list the two new results of this paper that have been proven, then turn to speculation:

1). The formula for reaction rate, (8). This was stated for the case in which one of the fusing particles and the final resonance both have very large mass, so that they each would stay at the origin. But it provides an adequate testing ground for the general questions that were raised in the present paper in relation, e.g., to $^{12}\text{C} + ^{12}\text{C}$ fusion.⁴

⁴ The corresponding formula for the case of general masses is easy to derive, but more complex. In the bracket for the fusing particles, the two particles that meet at 0 in the left hand state, $\langle |$,

2). The summation of the leading correction terms in the perturbative development. The outcome argued against my favorite guess as to how to do the energy shift calculation, but the details did show the necessity of following the plasma in a way in which one does not in the “basically classical” approach. I characterize the procedure of the latter as, “Do one big classical simulation, for the effective potential, and thereafter do your fusion calculation with the individual plasma states out of the picture.” There is evidence [12] that this procedure is adequate for determining the contact probability in the initial state, for an ordinary fusion reaction, as long as $E_{\text{Gamow}} \ll Q$. But when Q is small, or one is dealing instead with resonance fusion, I believe that it is insupportable.

If we had to choose between the two alternatives offered in the first section, we would follow the discussion just before and after (12) and use the energy shift of the resonance particle itself for Δ_{E_r} , with no subtractions for the energy shifts of the incoming particles. This gives the large relative suppressions seen in the $\log[w_2/w_0]$ curve in fig. 1.

Although in the above opinion I have reverted to the kind of guesswork that was deplored in the introduction, I add that the perturbative results do not argue as strongly as may have appeared for using the other option for Δ_{E_r} , as adopted by refs.[7] and [9]. First, suppose that one separates out the the normal Salpeter factor, $(1 + e^2 Z^2 \kappa_D \beta)$, from all other terms. Now the sum of these other terms is convergent even if one calculates them with unscreened Coulomb interactions, i.e., the infrared divergences in the individual terms have cancelled against each other.

This behavior is similar in form to the cancellation of (photon) infrared divergences in the inclusive cross-sections for charged particle scattering, where individual contributions, like photon bremsstrahlung, are divergent in an order by order expansion. The point is that these cancellations do not carry forward in any fashion to the non-infrared-divergent parts. Returning back to the perturbative calculation, I believe that the cancellations of the infrared terms that restored the Salpeter form do not extend in any sense to the non-infrared divergent terms (which begin with terms of order e^4 but also include higher corrections of order Γ^3). Therefore, for applications in which the coupling is even of moderate strength, the only thing that really is learned from the elaborate calculations in the appendix is that it is wrong

meet again in the right hand state, $| \rangle$, but now at position \mathbf{r} . Correspondingly, in the bracket for the fusion product \mathbf{R} one goes from position \mathbf{r} to position $\mathbf{0}$. The whole expression has an additional integral over \mathbf{r} ; all of this serves to transfer total momentum from the initial to the final state. Center of mass motion is no longer totally trivial when we are in a plasma and Q is insufficiently large. The derivation of the above follows easily from the methods used in the appendix. For the ordinary fusion case the analogue is shown in detail in ref. [10], with a somewhat different representation of the operators involved, however.

to disregard the changes in the plasma, i. e. the sum over the states $|\text{PI}'\rangle$ in (8).

How one might imagine doing a real calculation, assuming unlimited computing resources? The guiding example could be the calculations of Militzer and Pollack [12] or Ogata [13], who do real quantum path integral evaluations for the static correlator at small distances, without the assumptions of the “mostly classical” approach. In ref [12] the authors write $e^{-\beta H} = [e^{-\beta H/N}]^N$ and choose N sufficiently large so that each factor $\langle \text{PI}' | e^{-\beta H/N} | \text{PI}'' \rangle$, effectively at high temperature, can be expanded in perturbation theory. The product of the N factors, involving N integrals over intermediate plasma configurations, is very expensive to compute, needless to say, but because of the negative real exponentials the calculation is feasible. In contrast, the result (13), with its factor $e^{-(\beta - it)H}$ and extra time integral, cannot be calculated directly by such a procedure, because of the oscillations. Clearly some technical progress is greatly needed.

An important fusion chain that requires resonance enhancement is the triple α process. There are suggestions for treating the associated plasma effects in ref. [14]. In red giant cores where the process makes its first appearance, the plasma coupling is too weak to make very much difference. But when, e.g., it takes place in surface layers of accreting neutron stars, all of the caveats of the present paper apply.

I also note recent considerations of the role of a possible $^{12}\text{C} + ^{12}\text{C}$ resonance on the ignition of a type 1a supernova [15] [16]. Though screening issues were not addressed in either of these works, the plasma coupling is strong enough in this application that they need to be.

V. ACKNOWLEDGMENT

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Appendix: Appendix

For completeness I begin by recapitulating some development from ref [10]. I calculate the time rate of change of the species A induced by the fusion interaction (6). Since the system is translationally invariant I can choose to evaluate this time derivative at at point $\mathbf{r} = \mathbf{0}$ and choose time to be zero as well. Directly from the Heisenberg equations and the commutation rules, writing the transform of (6) as,

$$H_{\text{fus}}(t) = \int d\mathbf{r} [h(\mathbf{r}, t) + h(\mathbf{r}, t)^\dagger], \quad (\text{A.1})$$

where $h = g\psi_a\psi_b\psi_r^\dagger$ I obtain the rate of change of n_a , the density of species A,

$$\left\langle \dot{n}_a(\mathbf{0}, \mathbf{0}) \right\rangle_\beta = -i \left\langle [n_a(\mathbf{0}, 0), H_{\text{fus}}(0)] \right\rangle_\beta = i \left\langle [h(\mathbf{0}, 0) - h^\dagger(\mathbf{0}, 0)] \right\rangle_\beta. \quad (\text{A.2})$$

The notation $\langle \dots \rangle_\beta$ indicates the thermal average in the medium, such that for an operator, O , one has,

$$\langle O \rangle_\beta \equiv Z_P^{-1} \text{Tr} \left[O \exp(-\beta[H + H_{\text{fus}} - \mu_a N_a - \mu_b N_b]) \right], \quad (\text{A.3})$$

where Z_P is the partition function, and where H is the Hamiltonian in the absence of the fusion term. I wish to calculate the rate to lowest non-vanishing order; i.e. to second order in h . Thus I now must consider the linear response of the average of the operator $[h(\mathbf{0}, 0) - h^\dagger(\mathbf{0}, 0)]$, as it appears in (A.2), to the perturbation H_{fus} , giving an expression for the rate,

$$w = -i \int_{-\infty}^0 dt \int (d\mathbf{r}) \left\langle [\dot{n}_e(\mathbf{0}, 0), H_{\text{fus}}(t)] \right\rangle_\beta = \int_{-\infty}^0 dt \int d\mathbf{r} \left\langle [h(\mathbf{0}, 0) - h^\dagger(\mathbf{0}, 0)], [h(\mathbf{r}, t) + h^\dagger(\mathbf{r}, t)] \right\rangle_\beta, \quad (\text{A.4})$$

where now the thermal average in the medium is to be calculated under the action of H alone. Note that,

$$\left\langle [h(\mathbf{0}, 0), h(\mathbf{r}, t)] \right\rangle_\beta = \left\langle [h^\dagger(\mathbf{0}, 0), h^\dagger(\mathbf{r}, t)] \right\rangle_\beta = 0. \quad (\text{A.5})$$

Using in addition the space-time translational invariance of the medium and the antisymmetry of the commutator,

$$\left\langle [h^\dagger(\mathbf{0}, 0), h(\mathbf{r}, t)] \right\rangle_\beta = - \left\langle [h(\mathbf{0}, 0), h^\dagger(-\mathbf{r}, -t)] \right\rangle_\beta, \quad (\text{A.6})$$

I write the rate as

$$w = - \int_{-\infty}^{\infty} dt \int d\mathbf{r} \left\langle [h(\mathbf{0}, 0), h^\dagger(\mathbf{r}, t)] \right\rangle_\beta. \quad (\text{A.7})$$

When one takes the medium to contain no nuclei of type R, so that there is no reverse reaction, one can omit the first term in the commutator in (A.7). Then inserting (6) one obtains,

$$w = g^2 \int_{-\infty}^{\infty} dt \int d\mathbf{r} e^{-iE_r t} \left\langle \psi_a^\dagger(\mathbf{r}, t) \psi_b^\dagger(\mathbf{r}, t) \psi_r(\mathbf{r}, t) \psi_r^\dagger(\mathbf{0}, 0) \psi_a(\mathbf{0}, 0) \psi_b(\mathbf{0}, 0) \right\rangle_\beta \equiv g^2 \int_{-\infty}^{\infty} dt y(t), \quad (\text{A.8})$$

where,

$$y(t) = Z_P^{-1} \int d\mathbf{r} \text{Tr} \left[\Lambda e^{-(\beta - it)H} \psi_a^\dagger(\mathbf{r}, 0) \psi_b^\dagger(\mathbf{r}, 0) \psi_r(\mathbf{r}, 0) e^{-itH} \psi_r^\dagger(\mathbf{0}, 0) \psi_a(\mathbf{0}, 0) \psi_b(\mathbf{0}, 0) \right], \quad (\text{A.9})$$

and where $\Lambda = \exp[\beta \sum \mu_i N_i]$.

The relation (A.8) is the fundamental formal result required to determine the fusion rate. I have taken the Schrodinger and Heisenberg representations to coincide at $t = 0$, and at this time the ψ operators can be expanded in annihilation operators for single particle states. The calculation that follows will be much simpler if we specialize to the case in which one of the fusing particles, say B, and the resonance, R, are much heavier than ion A; furthermore the plasma corrections to the order that I calculate, as a multiplicative factor, do not depend at all on the mass configuration. Therefore I simplify, *ab initio*, by taking the masses of B and R to be infinite, and situating a single B at $\mathbf{r} = \mathbf{0}$. I replace operators $\psi_b(\mathbf{r}, t)$, $\psi_r(\mathbf{r}, t)$ by operators $b(t)$, and $r(t)$ that respectively annihilate the indicated particles situated at the origin. Note that even though ions B and R no longer carry kinetic energy, these Heisenberg operators are still time dependent by virtue of the Coulomb interactions of the ions to which they refer. The result (A.9) is now replaced by ,

$$y(t) = Z_P^{-1} V^{-1} \text{Tr} \left[\Lambda e^{-(\beta - it)H} \psi_a^\dagger(\mathbf{0}, 0) b^\dagger(0) r(0) e^{-itH} r^\dagger(0) \psi_a(\mathbf{0}, 0) b(0) \right]. \quad (\text{A.10})$$

The rate that I calculate is now the rate of production of the resonance from the single ion B situated at $\mathbf{r} = \mathbf{0}$. When we recast the expression in a form in which we consider only one A particle, the factor V^{-1} in (A.10) gets replaced by a factor of n_a . Written a bit more explicitly to exhibit the plasma coordinates in an intermediate state (A.10) reads,

$$y(t) = Z_P^{-1} n_a \times \sum_{\text{Pl}, \text{Pl}'} \left[\left\langle \mathbf{r}_a = \mathbf{0}, \text{B}, \text{Pl} \right| e^{-(\beta - it)H} \right| \mathbf{r}_a = \mathbf{0}, \text{B}, \text{Pl}' \rangle \left\langle R, \text{Pl}' \right| e^{-itH} \left| R, \text{Pl} \right\rangle \right], \quad (\text{A.11})$$

which leads directly to (8).

To calculate the leading terms in a perturbation expansion I write the total Hamiltonian (not including H_{fus}) as $H = H_0 + H_I$, where H_0 includes all kinetic energies as well as the mutual Coulomb interactions of the plasma particles with each other and the Coulomb interaction between the distinguished nuclei A and B. Then H_I simply contains the Coulomb couplings between the nuclei, A, B, C and the plasma particles. Designating the total electrical potential of the plasma particles at position \mathbf{r} as $\phi(\mathbf{r})$ I have,

$$H_I = \int d\mathbf{r} e_a n_a(\mathbf{r}) \phi(\mathbf{r}) + [e_b b^\dagger b + e_r r^\dagger r] \phi(\mathbf{0}). \quad (\text{A.12})$$

Of course, if one were attempting to go beyond the weak coupling regime, one would have to face the fact that the potentials ϕ in (A.12) involve all of the coordinates of the individual plasma particles and that a numerical simulation, as will be required for a definitive answer in the strongly coupled case, will involve the explicit taking into account of the paths of all of these particles. In the discussion section I describe the difficulties in principle of carrying out such calculations. But in the present paper I calculate to second order in H_I , and in addition single out only the terms of order e^3 in which the superficial perturbative order e^4 has been lowered through the long range of the Coulomb force. For this purpose all I need to know about the plasma is the part of the field-field correlation function in momentum space that is singular for small k as κ_D goes to zero, namely [10],

$$\left\langle \phi(\mathbf{k}, t') \phi(-\mathbf{k}', t'') \right\rangle_{IR} \approx 4\pi \delta(\mathbf{k} - \mathbf{k}') \beta^{-1} \kappa_D^2 \frac{1}{(k^2 + \kappa_D^2) k^2}. \quad (\text{A.13})$$

I introduce the interaction picture through the identities,

$$e^{-H_a(\beta-it)} = e^{-H_0(\beta-it)} \Omega_+(-i\beta, t) \quad ; \quad e^{-iH_b t} = \Omega_-(0, t) e^{-iH_0 t}, \quad (\text{A.14})$$

where,

$$\Omega^{(+)}(-i\beta, t) = \exp\left[i \int_{-i\beta}^t dt' \hat{H}_I(t')\right]_+ \quad ; \quad \Omega^{(-)}(0, t) = \exp\left[-i \int_0^t dt' \hat{H}_I'(t')\right]_-, \quad (\text{A.15})$$

time-ordered and anti-time ordered, respectively, where,

$$\hat{H}_I(t) = e^{iH_0(t+i\beta)} H_I e^{-iH_0(t+i\beta)} \quad ; \quad \hat{H}_I'(t) = e^{-iH_0 t} H_I e^{iH_0 t}. \quad (\text{A.16})$$

Substituting in (A.9) I obtain,

$$y(t) = Z_P^{-1} V^{-1} \text{Tr} \left[\Lambda \psi_a(\mathbf{0}, 0) b(0) e^{-(\beta-it)H_0} \Omega^{(+)}(-i\beta, t) \psi_a^\dagger(\mathbf{0}, 0) b^\dagger(0) r(0) \Omega^{(-)}(0, t) e^{-itH_0} r^\dagger(\mathbf{0}, 0) \right]. \quad (\text{A.17})$$

In (A.17) I have also used the cyclic property of the trace to move the operators $\psi_a(\mathbf{0}, 0) b(0)$ to the front; in this form the needed diagonal matrix elements in the trace are between states with no B or R present. I now expand the Ω factors in powers of the coupling, retaining only terms of order e^2 , (or $e_a^2, e_a e_b$, etc.). Schematically, each will have powers of e coming from the thermal expectation of a product of two ion-electric-potential $\phi(\mathbf{r})$ operators; these are determined by the Hamiltonian H_0 . If this potential-potential correlator is itself expanded in powers of e , the expansion begins with terms of order e^2 , so that the rate corrections superficially would be of order e^4 but the small k singularity of (A.13) reduces the order to e^3 . In the present paper I pursue only these e^3 terms; the neglected terms will go as power e^4 and higher.

Expanding, I obtain,

$$\Omega^{(+)}(-i\beta, t) = 1 - i \int_{-i\beta}^t dt_1 \hat{H}_I(t_1) - \int_{-i\beta}^t dt_1 \hat{H}_I(t_1) \int_{-i\beta}^{t_1} dt_2 \hat{H}_I(t_2), \quad (\text{A.18})$$

and

$$\Omega^{(-)}(0, t) = 1 + i \int_0^t dt_1 \hat{H}_I(t_1) - \int_0^t dt_1 \hat{H}_I(t_1) \int_0^{t_1} dt_2 \hat{H}_I(t_2). \quad (\text{A.19})$$

The task is to put (A.18) and (A.19) into the rate equation (A.17) and select out the terms that are of order $e^2 \kappa_D$. We define $y(t) = y_1(t) + y_2(t) + y_3(t)$ where y_1, y_2, y_3 come respectively from the second order piece of $\Omega^{(+)}(-i\beta, t)$; the product of the first order parts of $\Omega^{(+)}(-i\beta, t)$ and $\Omega_b^{(-)}(0, t)$; and the second order part of $\Omega_b^{(-)}(0, t)$. The operator under the trace now factors into a part with a $\phi(\mathbf{k}, t') \phi(-\mathbf{k}', t'')$ and a part that depends only on the operators in the space of the nuclei A, B, R. To the order that I need to calculate, the partition function also factorizes into a product in each space. Therefore one can directly use (A.13) to evaluate the plasma part of the trace. Since the electric potential of the plasma, ϕ occurs in H_I in the term $\int d\mathbf{k} n_a(\mathbf{k}) \phi(\mathbf{k})$ the \mathbf{k} 's in the in (A.13) in the plasma factor still link to the \mathbf{k} 's in the fusing-particle factor. However since the leading (order e^3) behavior comes from very small k , I can evaluate at $k = 0$ in the fusing particle sector and use,

$$\int d\mathbf{k} d\mathbf{k}' \langle \phi(\mathbf{k}, t') \phi(-\mathbf{k}', t'') \rangle_{IR} = (2\pi)^3 \beta^{-1} \kappa_D. \quad (\text{A.20})$$

Furthermore one can make the replacement, for example,

$$\hat{H}_I = e_a e^{iH_0 t} \int \frac{d\mathbf{p} d\mathbf{k}}{(2\pi)^6} a_{\mathbf{p}}^\dagger a_{\mathbf{p}+\mathbf{k}} \phi(\mathbf{k}) e^{-iH_0 t} \rightarrow e_a \int \frac{d\mathbf{p}}{(2\pi)^6} a_{\mathbf{p}}^\dagger a_{\mathbf{p}+\mathbf{k}} \int d\mathbf{k} \phi(\mathbf{k}, \mathbf{t}), \quad (\text{A.21})$$

since in the limit of small \mathbf{k} , H_0 commutes with $\int d\mathbf{p} a_{\mathbf{p}}^\dagger a_{\mathbf{p}+\mathbf{k}}$. Then one sees that in the infrared limit there is effectively no t_1, t_2 dependence of the integrands in A.18) and (A.19); performing these integrals gives the respective factors,

$$\begin{aligned} f_1(t) &= \frac{\beta^2}{2} - \frac{t^2}{2} - i\beta t, \\ f_2(t) &= (t + i\beta)t, \end{aligned} \quad (\text{A.22})$$

and

$$f_3(t) = -\frac{t^2}{2}. \quad (\text{A.23})$$

With the t dependent factors separated from the matrix elements of operator products that I still have to evaluate I now write,

$$\begin{aligned} y_1(t) &= \beta\kappa_D f_1(t) Z_a^{-1} V^{-1} \left\langle 0_b, 0_r \left| \text{Tr}_a \Lambda_a \left\{ \left[\sum_{\mathbf{p}} a_{\mathbf{p}} \Psi_{\mathbf{p}}(\mathbf{0}) e^{-(\beta-it)E_{\mathbf{p}}} \right] b \right. \right. \right. \\ &\times \left[e_a \sum_{\mathbf{q}} a_{\mathbf{q}}^\dagger a_{\mathbf{q}} + e_b b^\dagger b \right] \left[e_a \sum_{\mathbf{q}'} a_{\mathbf{q}'}^\dagger a_{\mathbf{q}'} + e_b b^\dagger b \right] \left[\sum_{\mathbf{p}'} a_{\mathbf{p}'}^\dagger \Psi_{\mathbf{p}'}(\mathbf{0})^* \right] b^\dagger r r^\dagger \left. \right\rangle \left| 0_b, 0_r \right\rangle \\ &= (e_a + e_b)^2 f_1(t) W(t), \end{aligned} \quad (\text{A.24})$$

where $|0_b, 0_r\rangle$ stands for the state in the B,R part of the space that contains nothing; $\Lambda_a = \exp(\beta\mu_a N_a)$; the trace acts in the A+plasma space; and,

$$W(t) = 2^{-3/2} \pi^{-2} M_a^{3/2} \beta\kappa_D V^{-1} e^{\mu_a \beta} \int dE \sqrt{E} e^{-(\beta-it)E} \left| \Psi_E(\mathbf{0}) \right|^2. \quad (\text{A.25})$$

In similar fashion one obtains,

$$\begin{aligned} y_2(t) &= \beta\kappa_D f_2(t) V^{-1} Z_a^{-1} \left\langle 0_b, 0_r \left| \text{Tr}_a \Lambda_a \left\{ \left[\sum_{\mathbf{p}} a_{\mathbf{p}} \Psi_{\mathbf{p}}(\mathbf{0}) e^{-(\beta-it)E_{\mathbf{p}}} \right] b \left[e_a \sum_{\mathbf{q}} a_{\mathbf{q}}^\dagger a_{\mathbf{q}} + e_b b^\dagger b \right] \right. \right. \right. \\ &\times r e_r r^\dagger c \left[\sum_{\mathbf{p}'} a_{\mathbf{p}'}^\dagger \Psi_{\mathbf{p}'}(\mathbf{0})^* \right] b^\dagger r^\dagger \left. \right\rangle \left| 0_b, 0_r \right\rangle \\ &= (e_a + e_b) e_c f_2(t) W(t), \end{aligned} \quad (\text{A.26})$$

and,

$$\begin{aligned} y_3(t) &= \beta\kappa_D f_2(t) V^{-1} Z_a^{-1} \left\langle 0_b, 0_r \left| \text{Tr}_a \Lambda_a \left\{ \left[\sum_{\mathbf{p}} a_{\mathbf{p}} \Psi_{\mathbf{p}}(\mathbf{0}) e^{-(\beta-it)E_{\mathbf{p}}} \right] b \right. \right. \right. \\ &\times r e_c^2 [r^\dagger r]^2 \left[\sum_{\mathbf{p}'} a_{\mathbf{p}'}^\dagger \Psi_{\mathbf{p}'}(\mathbf{0})^* \right] b^\dagger r^\dagger \left. \right\rangle \left| 0_b, 0_r \right\rangle \\ &= e_r^2 f_3(t) W(t). \end{aligned} \quad (\text{A.27})$$

In the above I did not impose charge conservation $e_r = e_a + e_b$, in order to best exhibit which interactions were contributing to which terms in the answer. Now eliminating e_r and summing the above three pieces, using $f_1(t) + f_2(t) + f_3(t) = \beta^2/2$, one obtains simply,

$$y(t) = 2^{-5/2} \pi^{-1} M_a^{3/2} \beta\kappa_D V^{-1} (e_a + e_b)^2 e^{\mu_a \beta} \int dE \sqrt{E} e^{-(\beta-it)E} a_{EaE}^\dagger \left| \Psi_E(\mathbf{0}) \right|^2. \quad (\text{A.28})$$

Substituting in (A.8), $w = g^2 \int dt y(t) \exp(-iE_r t)$, performing the time integration, then using,

$$g^2 = 2^{-1/2} \pi^{-1} M^{-3/2} \Gamma_r E_r^{1/2}, \quad (\text{A.29})$$

and,

$$e^{\beta\mu_a} = n_a (2\pi)^{3/2} (M_a T)^{-3/2}, \quad (\text{A.30})$$

and replacing the factor V^{-1} by n_b , I obtain for the change of the fusion rate due to the plasma interactions,

$$\delta w = \frac{1}{2} (e_a + e_b)^2 \beta\kappa_D w_0, \quad (\text{A.31})$$

where

$$w_0 = n_a n_b \left(\frac{2\pi}{M_a T} \right)^{3/2} \Gamma_r e^{-E_r/T} \exp \left(-\pi e^2 Z^2 \sqrt{\frac{M_a}{E_r}} \right). \quad (\text{A.32})$$

This gives, after the zero'th order term is added,

$$w = w_0 \left[1 + \frac{1}{2} \beta \kappa_D (e_a + e_b)^2 \right]. \quad (\text{A.33})$$

Before drawing conclusions from (A.33) I note that the factor n_a in the zero'th order term emerged computationally from,

$$n_a = (VZ)^{-1} \sum_{\mathbf{p}} \langle e^{-\beta(H - \mu_a N_a)} a_{\mathbf{p}}^\dagger a_{\mathbf{p}} \rangle = e^{\beta \mu_a} \int \frac{d^3 p}{(2\pi)^3} e^{-\beta p^2/2M}, \quad (\text{A.34})$$

But if one set out to calculate n_a in our basic formalism one would have had begun instead with,

$$\begin{aligned} n_a &= (VZ)^{-1} \sum_{\mathbf{p}} \langle e^{-\beta(H - \mu_a N_a)} a_{\mathbf{p}}^\dagger a_{\mathbf{p}} \rangle = \sum_{\mathbf{p}} \langle e^{-\beta H_0} a_{\mathbf{p}}^\dagger a_{\mathbf{p}} \Omega^{(+)}(-i\beta, 0) \rangle \\ &= \left(1 + \frac{e_a^2}{2} \beta \kappa_D \right) e^{\beta \mu_a} \int \frac{d^3 p}{(2\pi)^3} e^{-\beta p^2/2M}, \end{aligned} \quad (\text{A.35})$$

and similarly for n_b . That is to say, the plasma interactions have changed the relation between number density and chemical potential in just such a way that when the relation (A.33) is rewritten in terms of the actual density, the terms of order e_a^2 (and e_b^2) are removed [10], leaving us with the final answer ⁵,

$$w = w_0 [1 + e_a e_b \beta \kappa_D], \quad (\text{A.36})$$

the result (14) quoted in text.

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⁵ This last argument is given in more detail in ref. [10] , appendix B