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Effects of initial flow velocity fluctuation in event-by-event (3+1)D hydrodynamics

Longgang Pang,^{1,2} Qun Wang,² and Xin-Nian Wang^{1,3}

¹*Nuclear Science Division, MS 70R0319, Lawrence Berkeley National Laboratory, Berkeley, CA 94720*

²*Interdisciplinary Center for Theoretical Study and Department of Modern Physics,
University of Science and Technology of China, Hefei 230026, China*

³*Institute of Particle Physics and Key Laboratory of Quarks and Lepton Physics,
Central China Normal University, Wuhan 430079, China*

Hadron spectra and elliptic flow in high-energy heavy-ion collisions are studied within a (3+1)D ideal hydrodynamic model with fluctuating initial conditions given by the AMPT Monte Carlo model. Results from event-by-event simulations are compared with experimental data at both RHIC and LHC energies. Fluctuations in the initial energy density come from not only the number of coherent soft interactions of overlapping nucleons but also incoherent semi-hard parton scatterings in each binary nucleon collision. Mini-jets from semi-hard parton scatterings are assumed to be locally thermalized through a Gaussian smearing and give rise to non-vanishing initial local flow velocities. Fluctuations in the initial flow velocities lead to harder transverse momentum spectra of final hadrons due to non-vanishing initial radial flow velocities. Initial fluctuations in rapidity distributions lead to expanding hot spots in the longitudinal direction and are shown to cause a sizable reduction of final hadron elliptic flow at large transverse momenta.

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I. INTRODUCTION

One of the striking phenomena in high-energy heavy-ion collisions at the Relativistic Heavy-ion Collider (RHIC) and the Large Hadron Collider (LHC) is the collective flow [1–7] generated by the tremendous pressure of the dense matter, from the quark-gluon plasma in the early time to the hadronic resonance gas in the late stage of the evolution. Such collective transverse flow influences not only the transverse momentum spectra but also the azimuthal anisotropy or anisotropic flow of the final hadrons [8–10]. Anisotropic flow arises from the collective expansion of dense matter with initial geometric anisotropy. The simplest example is the hydrodynamic expansion of dense matter with a smooth initial energy density distribution in non-central heavy-ion collisions. The initial radial pressure gradient in the almond-shaped dense matter is asymmetric in the azimuthal angle. Such an asymmetric pressure gradient drives the system into a transverse expansion and transforms the initial geometric asymmetry into momentum asymmetry in azimuthal angle. The second Fourier coefficients of hadron azimuthal distributions are known as the elliptic flows. The large values of elliptic flow as measured in semi-central heavy-ion collisions at RHIC and LHC suggest the formation of a strongly coupled quark-gluon plasma close to a perfect fluid [11–17]. Comparisons of experimental data on elliptic flow and viscous hydrodynamic simulations can now provide phenomenological constraints on the specific shear viscosity (the ratio of shear viscosity to entropy density) of the quark-gluon plasma [18–22].

One of the critical inputs for the hydrodynamic model of heavy-ion collisions is the initial condition. Smoothed distributions of the initial energy density from either a Glauber or Color Glass Condensate (CGC) model of parton production were used in some recent hydrodynamic

calculations [10, 19–23]. However, one has to resort to event-by-event hydrodynamic simulations [24–35] to take into account the fluctuation in the initial conditions. Such fluctuating initial conditions have been shown to be responsible for odd harmonic flows (harmonic coefficients of the azimuthal anisotropy), such as the triangular flow [36], as well as the double-peak structure of dihadron correlation [38–44] in the final hadron spectra. These odd harmonic flows and dihadron correlations persist even in the most central heavy-ion collisions due to fluctuation of the initial local parton density [45]. Because of approximate longitudinal boost invariance of the local parton density in the initial condition, anisotropic flows in transverse momentum spectra are also correlated in pseudo-rapidity leading to the observed “ridge” structure in dihadron correlation in azimuthal angle and pseudo-rapidity [44–51].

Most of recent event-by-event hydrodynamic studies [28–31, 33–35, 52] employ the Glauber [53] or CGC model [54, 55] of parton production for the initial transverse energy density distribution. The Monte Carlo (MC) Glauber model assumes an initial energy density that is proportional to the transverse density of the number of wounded nucleons or a linear combination of the number of wounded nucleons and binary nucleon-nucleon collisions. The Monte Carlo (MC) CGC inspired models use the Kharzeev-Levin-Nardi (KLN) description [56, 57] of initial gluon production per wounded nucleon pair whose produced gluon multiplicity also depends on the impact parameter. Such fluctuating or bumpy initial energy density distributions in the event-by-event hydrodynamic simulations affect both the transverse momentum spectra and the azimuthal anisotropic flow as compared to the event-averaged smooth initial conditions, due to the increased local pressure gradient around hot spots or cold valleys. The transverse expansion of these hot spots amid

the overall expanding medium also leads to large values of odd harmonic azimuthal anisotropy of the final hadron spectra which will give rise to a conic structure in the di-hadron azimuthal correlation [45, 49, 50].

The initial conditions used for recent event-by-event hydrodynamic simulations have mostly assumed zero local flow velocities. Some of the hydro simulations [24–26] have included local velocity and longitudinal fluctuation. But their effects have not been systematically studied before. Since initially produced partons are basically uncorrelated in different nucleon-nucleon collisions, such an assumption is approximately correct for initial conditions that are smoothed over a transverse area much larger than the nucleon size. However, when fluctuation of initial energy density over the range of a nucleon size or smaller is considered, the initial local flow velocities are nonzero and their effects are non-negligible. The nonzero local flow velocity can arise from multiple parton correlation in mini-jets production which becomes the dominant mechanism for initial parton production in high-energy heavy-ion collisions at RHIC and LHC [58–60]. Mini-jets are clusters of many partons collimated in phase space. After initial thermalization, correlations of these partons associated with a pair of mini-jets are not necessarily destroyed and thus lead to non-vanishing local flow velocities. Such local flow velocities are expected to increase the final hadron multiplicity and the inverse slope of hadron transverse momentum spectra. They should also lead to small initial collective radial flow velocity at the outer region of the dense matter, which was found empirically important to explain the experimental data on HBT correlations [61]. The initial local transverse flow velocities due to mini-jets can also lead to intrinsic same-side and away-side dihadron correlations which are not induced by the collective expansion of the dense matter. Similarly, fluctuations in the local longitudinal flow velocity are also important and should be included in (3+1)D hydrodynamics [62].

In this paper, we will study the effects of fluctuating initial flow velocities within an ideal (3+1)D hydrodynamic model using the initial conditions as provided by the AMPT (A Multi-phase Transport) Monte Carlo model [63]. Such initial conditions contain mini-jets given by the HIJING Monte Carlo model [58, 59] and initial thermalization via the parton cascade within the AMPT model. We will study the effects of initial local flow velocity fluctuations in both transverse and longitudinal directions on the final hadron multiplicity distribution, transverse momentum spectra, elliptic flow as well as dihadron correlations. We will show the importance of the initial local flow velocity fluctuation in the study of hadron spectra and elliptic flow and therefore the current effect of extracting shear viscosity from comparisons between hydro calculations and experimental data.

The rest of this paper is organized as follows. In Sec. II, we will give a brief description of the (3+1)D ideal hydrodynamic model that we have developed, including a new projection method for calculation of the freeze-

out hyper surface and determination of initial conditions from the AMPT model. We will compare the calculated hadron spectra and elliptic flow with experimental data from RHIC and LHC in Sec. III. We will investigate in detail the effects of initial local flow velocities on hadron spectra and elliptic flow by comparing hydrodynamic simulations with and without initial local flow velocities in Sec. IV. We study the sensitivity of final hadron spectra and elliptic flow in Sec. V with a comparison between hydro results with a Chemical Equilibrated (CE) Equation Of State (EoS) and Partial Chemical Equilibrium (PCE) EoS. We conclude in Sec. VI with a discussion on the dihadron correlations from event-by-event hydrodynamics. We will give a brief description of the SHASTA algorithm that we use to solve the (3+1)D hydrodynamic equations in the Appendix.

II. IDEAL (3+1)D HYDRODYNAMICS

A. Conservation equations

The ideal hydrodynamic model of high-energy heavy-ion collisions is based on the assumption that local thermal equilibrium is achieved at some initial time τ_0 and the evolution of the system afterwards can be described by conservation equations for energy-momentum tensor and net baryon current,

$$\partial_\mu T^{\mu\nu} = 0, \quad (1)$$

$$\partial_\mu J^\mu = 0, \quad (2)$$

where the energy-momentum tensor and net baryon current can be expressed as

$$\begin{aligned} T^{\mu\nu} &= (\varepsilon + P)u^\mu u^\nu - P g^{\mu\nu}, \\ J^\mu &= n u^\mu, \end{aligned} \quad (3)$$

in terms of the local energy density ε , pressure P , the metric tensor $g^{\mu\nu}$, net baryon density n (or any conserved charges) and time-like 4-velocity u^μ with $u^2 = 1$. The above 5 equations contain 6 variables and can be closed by the equation of state (EoS) $P = P(\varepsilon, n)$. For systems with zero net baryon density, the net baryon current is always zero and the above equations contain only 5 variables. This will be the case that we consider in this paper. We will use the parameterized EoS $s95p-v1$ by Huovinen and Petreczky [64] which has a cross-over between the lattice QCD results at high temperature at $T = 250$ MeV and hadron resonance gas below $T = 180$ MeV. Such an EoS is valid only for zero baryon number density or chemical potential. For simplicity, however, we will assume the same EoS for all region of dense matter in our calculation, including large rapidity region where the net baryon density is nonzero.

The velocity of a fluid element in Cartesian coordinates $x^\mu = (t, x, y, z)$ is defined as [65]

$$u^\mu \equiv \frac{dx^\mu}{d\sigma} \equiv u_0(1, \vec{v}_\perp, \tilde{v}_z) \quad (4)$$

where $d\sigma^2 = dt^2 - dx^2 - dy^2 - dz^2$ and spatial components of the flow velocity are defined as $\tilde{v}_i = u^i/u_0$ ($i = x, y, z$). The time-component is $u_0 = 1/\sqrt{1 - \tilde{v}^2}$.

We will work in this paper with the invariant-time coordinate $X^\mu = (\tau, x, y, \eta_s)$ where $\tau = \sqrt{t^2 - z^2}$ and the spatial rapidity η_s are defined as,

$$\begin{aligned} t &= \tau \cosh \eta_s, \\ z &= \tau \sinh \eta_s. \end{aligned} \quad (5)$$

The metric tensor $g^{\mu\nu} = \text{diag}(1, -1, -1, -1/\tau^2)$ and correspondingly $g_{\mu\nu} = \text{diag}(1, -1, -1, -\tau^2)$ are given by the invariant line element $ds^2 = g_{\mu\nu}dX^\mu dX^\nu = d\tau^2 - dx^2 - dy^2 - \tau^2 d\eta_s^2$ in the invariant-time coordinate. The velocity 4-vector in this coordinate is,

$$\begin{aligned} U^\mu &\equiv \frac{dX^\mu}{ds} = \frac{dX^\mu}{dx^\nu} \frac{dx^\nu}{ds} = \frac{dX^\mu}{dx^\nu} u^\nu \\ &= \begin{pmatrix} u^0 \cosh \eta_s - u^z \sinh \eta_s \\ \tilde{u}_\perp \\ \frac{1}{\tau}(-u^0 \sinh \eta_s + u^z \cosh \eta_s) \end{pmatrix} \equiv U^\tau \begin{pmatrix} 1 \\ \tilde{v}_\perp \\ \frac{v_\eta}{\tau} \end{pmatrix} \end{aligned} \quad (6)$$

where v_z , \tilde{v}_\perp and v_η are defined as [66],

$$\begin{aligned} \tilde{v}_\perp &= \tilde{v}_\perp \cosh(y_v)/\cosh(y_v - \eta_s), \\ v_\eta &= \tanh(y_v - \eta_s), \end{aligned}$$

y_v denotes the rapidity of the longitudinal flow velocity as given by $\tilde{v}_z = \tanh y_v$ and $U^\tau = 1/\sqrt{1 - v_\perp^2 - v_\eta^2}$.

Since we assume an EoS that is independent of the local baryon number density or chemical potential, the baryon current conservation equation decouples from the energy-momentum conservation. We will regard $T^{\tau\tau}$, $T^{\tau x}$, $T^{\tau y}$, $T^{\tau\eta}$ and J^τ as independent variables in the conservation equations. Other components of the energy-momentum tensor and the baryon current can be expressed in terms of these 5 variables from the definitions in Eq. (3). For example, from definition $T^{\tau x} = (\varepsilon + P)U^\tau U^x$, one can express $T^{xx} = (\varepsilon + P)U^x U^x + P = v_x T^{\tau x} + P$. With these independent variables, we can use the SHASTA (SHarp And Smooth Transport Algorithm) algorithm, which is designed to solve partial differential equations with the form $\partial_t(T) + \partial_i(v_i T) = S$, to solve the hydrodynamic equations. These conservation equations can be cast in the following form by variable substitutions,

$$\begin{aligned} \partial_\tau(\tau T^{\tau\tau}) - \tau \nabla \cdot (\mathbf{v} T^{\tau\tau}) &= S^\tau, \\ \partial_\tau(\tau \tilde{T}^{\tau\perp}) - \tau \nabla \cdot (\mathbf{v} \tilde{T}^{\tau\perp}) &= \tilde{S}^\perp, \\ \partial_\tau(\tau T^{\tau\eta}) - \tau \nabla \cdot (\mathbf{v} T^{\tau\eta}) &= S^\eta, \\ \partial_\tau(\tau J^\tau) - \tau \nabla \cdot (\mathbf{v} J^\tau) &= 0, \end{aligned} \quad (7)$$

with the source terms,

$$\begin{pmatrix} S^\tau \\ \tilde{S}^\perp \\ S^\eta \end{pmatrix} = \begin{pmatrix} \tau \nabla \cdot (\mathbf{v} P) - v_\eta^2 (T^{\tau\tau} + P) - P \\ -\tau \tilde{\partial}_\perp P \\ -(1/\tau) \partial_\eta P - 2T^{\tau\eta} \end{pmatrix}, \quad (8)$$

where $\nabla \cdot (\mathbf{v} R) = \tilde{\partial}_\perp \cdot (\tilde{v}_\perp R) + \partial_{\eta_s}(v_\eta R)/\tau$ for any variable R . The energy density is determined from $T^{\tau\nu}$ through a root finding method by iterating the following equation,

$$\varepsilon = T^{\tau\tau} - \frac{M^2}{T^{\tau\tau} + P(\varepsilon)}, \quad (9)$$

to an accuracy $|\delta\varepsilon| < 10^{-15}$, where $M^2 = (T^{\tau\perp})^2 + (\tau T^{\tau\eta})^2$. The initial value of ε for the iteration is approximated by $\varepsilon = T^{\tau\tau}$. The flow velocity is given by

$$\tilde{v}_\perp = \tilde{T}^{\tau\perp}/[T^{\tau\tau} + P(\varepsilon)], \quad (10)$$

$$v_\eta = \tau T^{\tau\eta}/[T^{\tau\tau} + P(\varepsilon)]. \quad (11)$$

B. FCT-SHASTA Algorithm

The conservation equations in Eq. (7) have the general form of coupled convective diffusion equations which can be solved using extended FCT (Flux-Corrected Transport)-SHASTA algorithm [67–69]. Here we give a brief overview of the algorithm using the following 1D partial differential equation as an example:

$$\partial_t \rho + \partial_x(v\rho) = 0. \quad (12)$$

The FCT-SHASTA algorithm first evolves this equation by a transport and diffusion stage which ensures the solution's monotonicity and positivity,

$$\begin{aligned} \rho_j^{td} &= \frac{1}{2} Q_-^2 (\rho_{j-1}^n - \rho_j^n) + \frac{1}{2} Q_+^2 (\rho_{j+1}^n - \rho_j^n) \\ &\quad + (Q_+ + Q_-) \rho_j^n, \end{aligned} \quad (13)$$

$$Q_+ = \left(\frac{1}{2} - v_j^{1/2} \frac{\delta t}{\delta x}\right) / \left[1 + (v_{j+1}^{1/2} - v_j^{1/2}) \frac{\delta t}{\delta x}\right], \quad (14)$$

$$Q_- = \left(\frac{1}{2} + v_j^{1/2} \frac{\delta t}{\delta x}\right) / \left[1 - (v_{j-1}^{1/2} - v_j^{1/2}) \frac{\delta t}{\delta x}\right], \quad (15)$$

where td stands for “transport and diffusion”, j denotes the discretized space index in δx and n the time step in δt , $v_j^{1/2}$ denotes the value of v_j at half time-step $n + 1/2$ which is evaluated with a 2-step Runge-Kutta method. The derivation of Q_- and Q_+ can be found in the Appendix. In the zero-velocity limit (where $Q_+ = Q_- = 1/2$) the above solution becomes,

$$\begin{aligned} \rho_j^{td} &= \rho_j^n - \frac{1}{8} (\rho_{j+1}^n - 2\rho_j^n + \rho_{j-1}^n) \\ &= \rho_j^n - f_{j+1/2} + f_{j-1/2}, \end{aligned} \quad (16)$$

from which one can clearly identify the diffusion term $\frac{1}{8}(\rho_{j+1}^n - 2\rho_j^n + \rho_{j-1}^n)$. The general form of the diffusion can be expressed by the flux $f_{j\pm 1/2} = \pm \frac{1}{8}(\rho_{j\pm 1} - \rho_j)$. In the second stage of the FCT-SHASTA algorithm, an anti-diffusion term is added to the transported and diffused result. Usually the anti-diffusion term are calculated in the FCT by subtracting low order from high order transport results to make sure the anti-diffusion is accurate

enough. Since ripples arise in high order transport algorithm, the flux in the anti-diffusion term must be corrected to make sure no new maximum or minimum are produced. In our calculation, the flux limiter developed by Zalesak [68] in the anti-diffusion stage is extended to a full multi-dimensional FCT algorithm. In this multi-dimensional algorithm the 1D FCT-SHASTA algorithm with time-splitting [69] is used along one direction at a split-time step, while a $x \rightarrow y \rightarrow \eta_s \rightarrow y \rightarrow x$ rotation is used to extend the FCT-SHASTA algorithm to multi-dimensions and to suppress the numerical eccentricity produced in transverse direction during the hydrodynamic evolution.

We will use the second order midpoint Runge-Kutta method to include the source term S in a differential equation,

$$\frac{d\rho}{dt} = S, \quad (17)$$

which involves two steps,

1. $\rho^{n+1/2} = \rho^n + \frac{1}{2}\delta t S(t, \rho^n)$
2. $\rho^{n+1} = \rho^n + \delta t S(t + \delta t/2, \rho^{n+1/2})$

The energy density and velocity calculated in half time step $\delta t/2$ can be used in the FCT-SHASTA algorithm to improve the numerical precision as described in the Appendix. The 2nd order Runge Kutta method can significantly improve the numerical accuracy and remove the numerical diffusion for much larger time steps.

We use the following simplified conservation equation to describe the numerical method and procedures of a combined FCT-SHASTA and the 2nd order Runge-Kutta algorithm in solving hydrodynamic equations in our study,

$$\partial_\tau \mathcal{T} + \partial_i(v_i \mathcal{T}) = S, \quad (18)$$

where we use $\mathcal{T} \equiv \tau T^{\tau\nu}$ to denote one component of the energy-momentum tensor.

1. Calculate source term at time step n : $S = S(\tau^n, \varepsilon^n, v_i^n, \mathcal{T}^n)$.
2. Evolve $\partial_\tau \mathcal{T} + \partial_i(v_i \mathcal{T}) = 0$ to time step $n + 1/2$ by using SHASTA algorithm to get $\mathcal{T}^{n+1/2}$.
3. Update to $\mathcal{T}^{n+1/2} = \mathcal{T}^{n+1/2} + 0.5\Delta\tau S(\tau^n, \varepsilon^n, v_i^n, \mathcal{T}^n)$ and use the root-finding method to calculate the energy density and velocity $\varepsilon^{n+1/2}, v_i^{n+1/2}$.
4. Calculate source term at half-time step $n + 1/2$: $S = S(\tau^{n+1/2}, \varepsilon^{n+1/2}, v_i^{n+1/2}, \mathcal{T}^{n+1/2})$.
5. Evolve $\partial_\tau \mathcal{T} + \partial_i(v_i \mathcal{T}) = 0$ to time step $n + 1$ by using SHASTA algorithm with half time step velocity $v_i^{n+1/2}$ to obtain \mathcal{T}^{n+1} .

6. Update to $\mathcal{T}^{n+1} = \mathcal{T}^{n+1} + \Delta\tau S(\tau^{n+1/2}, \varepsilon^{n+1/2}, v_i^{n+1/2}, \mathcal{T}^{n+1/2})$ and calculate the energy density and velocity for the next time step $\varepsilon^{n+1}, v_i^{n+1}$ via root-finding method.

We refer readers to the Appendix for more details about the SHASTA algorithm. We have used 1D SHASTA algorithm with time-splitting [69] to solve the hydrodynamic equations and it is easy to implement parallel computing in the future. In the current event-by-event simulations, only high level parallel computing is used where events run on separate CPU's. The most time consuming part in hydrodynamic simulations is to calculate spectra of direct thermal hadrons and decay products from hundreds of resonances for comparison with experimental data. The CPU hours used in our simulations are significantly reduced by our improved algorithm to calculate the freeze-out hyper-surface.

C. Freeze Out and Hadronization

We will use the Cooper-Frye formula [70] to calculate the momentum distribution for particle i with degeneracy g_i :

$$E \frac{dN_i}{d^3P} = \frac{dN_i}{dY p_T dp_T d\phi} = g_i \int_\Sigma p^\mu d\Sigma_\mu f(p \cdot u), \quad (19)$$

where $d\Sigma_\mu$ is the normal vector of a small piece of freeze-out hyper-surface beyond which the temperature falls below the freeze-out temperature T_f or energy density ε falls below the freeze-out density ε_f . Hadrons pass through the freeze-out surface element is assumed to obey thermal distribution at temperature T_f ,

$$f(p \cdot u) = \frac{1}{(2\pi)^3} \frac{1}{e^{((p \cdot u - \mu_i)/T_f)} \pm 1}, \quad (20)$$

where \pm stands for fermions and bosons respectively, u is the flow velocity. All resonances are assumed to freeze out from the same hyper surface and decay into stable particles. The invariant energy of particle in comoving frame is,

$$E = p \cdot u \quad (21)$$

$$= u^\tau [m_T \cosh(Y - \eta_s) - \vec{p}_\perp \cdot \vec{v}_\perp - m_T \sinh(Y - \eta_s) v_\eta]$$

where $p^\mu = (m_T \cosh(Y - \eta_s), \vec{p}_\perp, m_T \sinh(Y - \eta_s)/\tau)$. In order to calculate the spectra, we need to know the freeze-out hyper surface $\Sigma = (\tau_f, x, y, \eta_s)$ at freeze-out time τ_f . The normal vector for one piece of freeze-out hyper surface in invariant-time coordinate is,

$$d\Sigma_\mu = (\tau_f dx dy d\eta_s, -\tau_f d\tau dy d\eta_s, -\tau_f d\tau dx d\eta_s, -d\tau dx dy) \quad (22)$$

In a simple cuboidal method by Hirano [66], finite grid sizes $\Delta\tau, \Delta x, \Delta y$ and $\Delta\eta_s$ are used to calculate $d\Sigma_\mu$.

The surface elements are calculated independently for each direction. In the τ -direction, a cuboidal volume $d\Sigma_\tau = \tau_f \Delta x \Delta y \Delta \eta_s$ is recorded when the freeze-out temperature falls between $T(\tau_n, x, y, \eta_s)$ at time step n and $T(\tau_{n+1}, x, y, \eta_s)$ at time step $n+1$. The norm vector points to the low energy density direction along the τ axis. The freeze-out time τ_f and 4-velocity u are calculated from interpolation between time step τ_n and τ_{n+1} . In the x -direction, $d\Sigma_x = -\tau \Delta \tau \Delta y \Delta \eta_s$ is recorded at time step n where T_f falls between $T(\tau_n, x_i, y, \eta_s)$ and $T(\tau_n, x_{i+1}, y, \eta_s)$. Hirano's method seems to overestimate the freeze-out hyper surface in one single cell where the total cuboidal volume is always added without considering the cut-through position. After the decomposition of $p^\mu d\Sigma_\mu$ to 4 directions across several hydrodynamic cells, one finds that the overestimated part in one cell actually fills up the underestimated part in another. It is quite a good approximation as long as the velocity at the freeze-out hyper surface does not change too much and its 3 components can be treated as the flow velocity on the cube edge. The requirement can be easily fulfilled by using one single hydro cell cube. The problem is that the size of the data file for the whole freeze-out hyper surface strongly depends on the time and space grid size used in solving the hydrodynamic equations. If a smaller grid size is used (for example $\Delta\tau = 0.01$ fm and $\Delta x = 0.1$ fm) to improve the numerical accuracy in the transport stage, the data file becomes huge and the calculation of hadron spectra for hundreds of resonances will be very time consuming.

To improve the computation efficiency for finer grids, one can divide the whole hyper surface into smaller pieces inside interpolation cubes each extending to several hydrodynamic grid cells along 4 directions [10, 71–73]. Each piece of surface elements is presented by the intersections s_i on edges of an interpolation cube where the hyper surface cuts through, and its area is approximated by a group of triangles (in the (2+1)D case) or a group of tetrahedra (in the (3+1)D case) formed by these intersections. The main task is to triangulate these intersections in (2+1)D or (3+1)D hydro, and calculate the areas of the triangles or volumes of the tetrahedra piece by piece.

In the algorithm developed by Kataja, Ruuskanen (KR) and collaborators [71, 72] and later used in the Azhydro code by Kolb [10] for the (2+1)D case, these intersections on the edges of an interpolation cube are ordered into a circular sequence. The area S of one piece of hyper surface inside an interpolation cube is approximated by the summation of the areas of a group of triangles with each triangle constructed by connecting two nearby intersections with the center point O of all the intersections (as shown in the left panel of Fig. 1),

$$S = \sum_{i=1}^N \Delta O s_i s_{i+1}, \quad (23)$$

where N is the number of intersections and $s_{N+1} = s_1$. The flow velocity and energy density on this piece of sur-

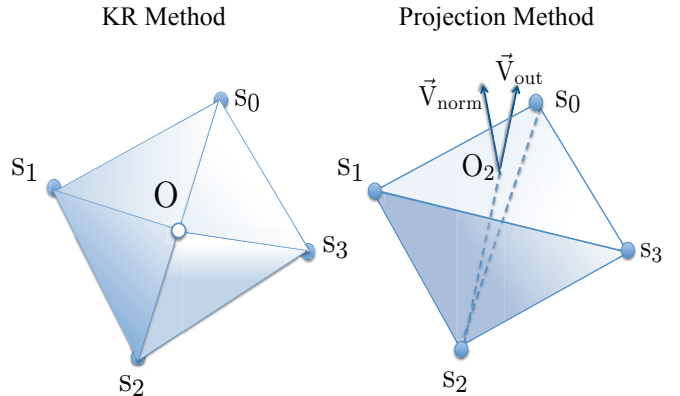


FIG. 1: (Color online) An example of (2+1)D freeze-out hyper surface. In the projection method, there are 4 triangles formed on the convex hull $\diamond s_0 s_1 s_2 s_3$. Only two are chosen by the criterion $\vec{V}_{norm} \cdot \vec{V}_{LE} > 0$, where \vec{V}_{LE} is the low energy density direction and \vec{V}_{norm} is the outward normal vector of each surface triangle.

face element used in the Cooper-Frye formula is approximated by interpolation of their values at the 8 corners of the cube. In the KR method for the (2+1)D case, the most difficult part is to order these intersections in a circular sequence and this is achieved by using a bit-chart of the distances between the mid-points of any 2 of the 12 edges of the interpolation cube. Once the intersections are ordered, the area of one triangle formed by the center point O and two neighboring intersections can be calculated from:

$$\vec{V}_{norm}^{2+1D} = \frac{1}{2} \begin{vmatrix} n & i & j \\ A_0 & A_1 & A_2 \\ B_0 & B_1 & B_2 \end{vmatrix} = n d\Sigma_0 + i d\Sigma_1 + j d\Sigma_2, \quad (24)$$

where A and B are the two vectors that span the triangle in (2+1)D hydro. This method can be considered as one variant of the Marching Cubes algorithm [74].

The above KR method has been extended to (3+1)D hydro recently by the McGill group [73]. A bit-chart for 32 edges on the $d\tau dx dy d\eta_s$ super cube was constructed. The triangles are replaced by tetrahedra. The volume of one tetrahedron is then,

$$\vec{V}_{norm}^{3+1D} = \frac{1}{6} \begin{vmatrix} n & i & j & k \\ A_0 & A_1 & A_2 & A_3 \\ B_0 & B_1 & B_2 & B_3 \\ C_0 & C_1 & C_2 & C_3 \end{vmatrix} = n d\Sigma_0 + i d\Sigma_1 + j d\Sigma_2 + k d\Sigma_3, \quad (25)$$

where A , B and C are the three vectors that span the tetrahedron in the (3+1)D case.

In this paper, we develop a projection method to calculate the freeze-out hyper surface $d\Sigma_\mu$ in our (3+1)D

hydrodynamic simulation. We illustrate the method here for the (2+1)D case as shown in the right panel of Fig. 1. The extension to (3+1)D or (n+1)D is straightforward. Here we list a step-by-step procedure for calculating the freeze-out hyper surface in our projection method:

- Identify interaction points whenever energy density difference $\epsilon - \epsilon_{fr}$ changes sign between two grid points, where ϵ_{fr} is the freeze-out energy density.
- An “n-simplex” is an n-dimensional polytope which is the convex hull of its n+1 vertices. A 2-simplex is a triangle, a 3-simplex is a tetrahedron, a 4-simplex is a pentachoron [75].
- One can calculate the area of one piece of hyper surface directly if there are only 3 intersections on the edges of the interpolation cube. For 4 or more intersections, select any 4 intersections $s_{i=0,1,2,3}$ to construct one 3-simplex $\diamond_{s_0s_1s_2s_3}$ that has 4 triangles on its surface as shown in the right panel of Fig. 1.
- Consider one triangle $\Delta_{s_0s_1s_3}$ whose center is denoted by O_2 . The vector between the fourth intersection point and the center of the chosen triangle, $\vec{V}_{out} = s_2\vec{O}_2$ is defined as the outward direction of the tetrahedron on $\Delta_{s_0s_1s_3}$ side. In the (3+1)D case, \vec{V}_{out} is defined as $s_n\vec{O}_n$ where s_n is the intersection point opposite to center of the tetrahedron O_n .
- The normal vector of $\Delta_{s_0s_1s_3}$ has two directions (this is also true for 3+1D case). Choose the outward normal vector \vec{V}_{norm} that satisfies $\vec{V}_{norm} \cdot \vec{V}_{out} > 0$.
- For an interpolation cube with more than 4 intersections, select another intersection s_i outside the first constructed 3-simplex or tetrahedron $\diamond_{s_0s_1s_2s_3}$. This intersection s_i is considered out of the tetrahedron on a triangle $\Delta_{s_0s_1s_3}$ side when $\vec{V}_{norm} \cdot O_2\vec{s}_i > 0$. Find all possible triangles of the tetrahedron that the intersection s_i is out of. Form a new tetrahedron between s_i and each of these triangles. Remove those triangles that are shared between the first and the new tetrahedra to form a new convex hull.
- Repeat the above step for the rest of intersections and the previous formed convex hull until a closed convex hull is formed with all the outer triangles that one has constructed.
- Consider \vec{v}_{iLE} as the low energy density flow vector for each intersection point on one triangle $\Delta_{s_0s_1s_3}$. Then define $\vec{V}_{LE} = \sum_{i=0,1,3} \vec{v}_{iLE}$ as the low energy density flow vector of the triangle $\Delta_{s_0s_1s_3}$. This triangle will be considered as a piece of the freeze-out hyper surface only when $\vec{V}_{norm} \cdot \vec{V}_{LE} > 0$.

- Use the criteria $\vec{V}_{norm} \cdot \vec{V}_{LE} > 0$ to select all the triangles of the closed convex hull that will form the freeze-out hyper surface of the interpolation cube, with contribution from each triangle given by \vec{V}_{norm} .

The above algorithm can be easily extended to the (3+1)D case with 3-simplex replaced by 4-simplex and triangles replaced by tetrahedra whose \vec{V}_{out} , \vec{V}_{norm} and \vec{V}_{LE} can be similarly defined.

If 4 intersections are coplanar in (2+1)D or 5 intersections are co-tetrahedron in (3+1)D hydro, random tiny placements with an amplitude of 10^{-9} will be applied to the intersections before the above procedure is applied in our algorithm. A numerical error of the order of 10^{-9} for the hyper surface calculation is negligible but this will keep the algorithm continue to run for all possible hyper surfaces. This has been checked for calculating the volume of a 3D cube with the same time coordinate τ in 4 dimensional space.

The hyper surface can cut through the same edge of the interpolation cube more than once which are recorded on the hydro cubes several of which make up the interpolation cube. In all above methods, if the number of cuts is odd, the intersections are approximated by a single point. An even number of intersections on a single edge of the interpolation cube are neglected. This approximation can be improved in the future by refining our projection method at the expense of minimal increase of computer time. However, one still has to neglect multiple intersections of the hyper surface with the basic hydro cubes.

We have compared our projection method with Hirono’s cuboidal freeze-out method, and find very tiny discrepancies in multiplicity distribution and p_T spectra. However, for small grid size, our projection method is much faster than the cuboidal freeze-out method in calculating the final particles’ spectra, especially in event-by-event simulations of higher order harmonic flows.

D. Resonance decays

To calculate the final hadron spectra from hydrodynamic simulations, one needs to include both direct thermal hadrons that go through the freeze-out surface and decay products from resonances which are assumed to freeze out at the same temperature. Resonances with mass up to 1.68 GeV are considered in the current calculation.

We have extended the numerical procedure for resonance decay from Kolb’s (2+1)D Azhydro code to our (3+1)D hydrodynamic simulations in this study. In the original Azhydro code, Bjorken scaling is assumed for the rapidity distributions of thermal hadrons and resonances. In this case, final hadron spectra at any given rapidity will contain decay products from resonances in all rapidities with a uniform and infinitely long distribution. In

our (3+1)D hydro calculation, this uniform and infinite rapidity distribution of resonances is replaced with the more realistic ones that are not smooth in each event as determined at the freeze-out in our hydro evolution with the fluctuating AMPT initial conditions. In our numerical calculations, we still have to limit the rapidity range to a finite value $[-8, 8]$ and use a linear interpolation outside this range assuming yields for all resonances to be zero at $Y = \pm 20$.

E. Initial Conditions

To incorporate fluctuations and correlations in both transverse and longitudinal flow velocities in event-by-event (3+1)D hydrodynamic simulations, we will use the AMPT model [63] to provide the local initial energy-momentum tensor in each hydrodynamic cell. The AMPT model uses the HIJING model [58–60] to generate initial partons from hard and semi-hard scatterings and excited strings from soft interactions. The number of excited strings in each event is equal to that of participant nucleons. The number of mini-jets per binary nucleon-nucleon collision follows a Poisson distribution with the average number given by the mini-jet cross section, which depends both on the colliding energy and the impact parameter through an impact-parameter dependent parton shadowing [58] in a nucleus. In this model, the total local energy-momentum density of partons and its fluctuations will be determined by the number of participants, binary nucleon-nucleon collisions, number of mini-jets per nucleon-nucleon collision and the fragmentation of excited strings. HIJING uses the Glauber model to determine the number of participants and binary nucleon-nucleon collisions with the Wood-Saxon nuclear distribution.

The formation times for partons from mini-jets produced via semi-hard scatterings are short. Their energy-momentum density can be used as part of the initial conditions for the hydrodynamic evolution. However, strong color fields in the soft strings take time to materialize and their contribution to the initial energy-momentum density at earlier times is hard to estimate. In the option that we use in AMPT, strings are melt via conversion of hadrons into quarks and anti-quarks after the string fragmentation which will participate in the parton cascade together with hard and semi-hard partons. The formation time of these soft partons are estimated according to their transverse momentum and energy ($t_f \sim 2p_0/p_T^2$). In our study we allow AMPT model to run through the parton cascade for an initial period of time. We record the space-time points of the last scattering or formation time for all the partons. Most of the partons are found to concentrate along the hyperbola of an initial proper time τ_0 . As an approximation, we simply assign the proper time τ_0 to all partons and use their 4-momenta to calculate the local energy-momentum tensor as the initial condition for our ideal hydro evolution. In this approxi-

mation for the initial conditions at a given proper time, parton interaction at large spatial rapidity at very late Cartesian time in the AMPT model is neglected. These are questionable approximations that one has to keep in mind when one considers theoretical uncertainties and future improvements. In principle, one should run the AMPT model to the end (no further interactions). The recorded particles' (both partons and hadrons) space-time positions when they cross the hyperbola with fixed τ_0 will provide the initial condition for the hydrodynamical evolution. This is, however, too demanding in computer time.

The 4-momenta and space coordinates of partons from the AMPT model according to the above description will be used to calculate the local energy-momentum tensor as the initial conditions for our event-by-event (3+1)D hydrodynamic simulations. Its value in each grid cell is approximated by a gaussian distribution in invariant-time coordinates,

$$T^{\mu\nu}(\tau_0, x, y, \eta_s) = K \sum_i \frac{p_i^\mu p_i^\nu}{p_i^\tau} \frac{1}{\tau_0 \sqrt{2\pi\sigma_{\eta_s}^2}} \frac{1}{2\pi\sigma_r^2} \times \exp \left[-\frac{(x-x_i)^2 + (y-y_i)^2}{2\sigma_r^2} - \frac{(\eta_s - \eta_{is})^2}{2\sigma_{\eta_s}^2} \right], \quad (26)$$

where $p_i^\tau = m_{iT} \cosh(Y_i - \eta_{is})$, $p_i^x = p_{ix}$, $p_i^y = p_{iy}$ and $p_i^\eta = m_{iT} \sinh(Y_i - \eta_{is})/\tau_0$ for parton i , which runs over all partons produced in the AMPT model simulations. We have chosen $\sigma_r = 0.6$ fm, $\sigma_{\eta_s} = 0.6$ in our calculations. The transverse mass m_T , rapidity Y and spatial rapidity η_s are calculated from the parton's 4-momenta and spatial coordinates. Note here that the Bjorken scaling assumption $Y = \eta_s$ is not used here because of early parton cascade before the initial time and the uncertainty principle applied to the initial formation time in AMPT. The scale factor K and the initial time τ_0 are the only two parameters that one can adjust to fit the experimental data on central rapidity density of produced hadrons.

Note that the Gaussian smearing in Eq. (26) smoothes out the energy-momentum tensor within several hydro grid cells. Such a smearing acts like an initial thermalization process similar to the parton cascade in the AMPT model within the initial time τ_0 . The initial matter in each grid cell is then assumed to reach a local thermal equilibrium and one can obtain the initial local energy density and flow velocity from Eq. (26) using the root finding method in Eqs. (9-11). This is equivalent to the prescription in Ref. [76] for a constant proper time surface.

III. HADRON SPECTRA FROM EVENT-BY-EVENT HYDRODYNAMICS

In this section, we will compare hadron spectra from our event-by-event (3+1)D ideal hydrodynamic simulations to experimental data at RHIC and LHC energies.

For $Au + Au$ collisions at the RHIC energy $\sqrt{s} = 200$ GeV, we use a scale factor $K = 1.45$ and an initial time $\tau_0 = 0.4$ fm/c in the initial conditions from the AMPT model. We have used grid spacings $\delta x = \delta y = 0.3$ fm, $\delta\eta_s = 0.2$ and $\delta\tau = 0.04$ fm/c with grid size $L_x \times L_y \times L_\eta = (30 \text{ fm}) \times (30 \text{ fm}) \times 6$. For $Pb + Pb$ collisions at the LHC energy $\sqrt{s} = 2.76$ TeV, we use $K = 1.6$, $\tau_0 = 0.2$ fm/c, $\delta x = \delta y = 0.2$ fm, $\delta\eta_s = 0.3$ and $\delta\tau = 0.03$ fm/c and grid size $L_x \times L_y \times L_\eta = (30 \text{ fm}) \times (30 \text{ fm}) \times 12$. With these grid spacings and sizes, we have checked that the increase of the total entropy of the system due to the numerical viscosity over the entire evolution duration of about 20 fm/c is less than 1%.

Shown in Figs. 2, 3 and 4 are pseudo-rapidity distributions for charged hadrons, p_T spectra for charged pions and p_T spectra for identified charged hadrons, respectively, from our event-by-event ideal hydrodynamic calculations. Also shown are experimental data for $Au + Au$ collisions with different centralities at the RHIC energy.

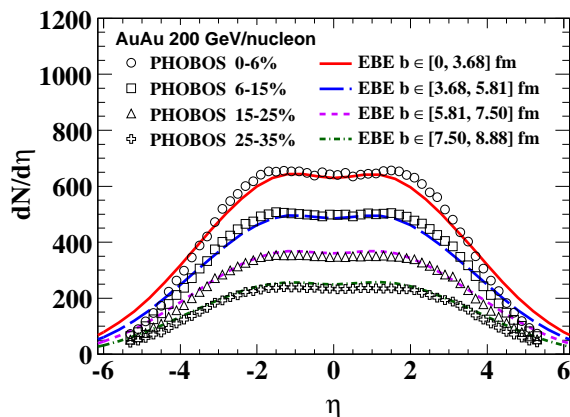


FIG. 2: (Color online) Charged hadron pseudo-rapidity distributions from event-by-event (3+1)D ideal hydro (lines) with AMPT fluctuating initial conditions compared to the PHOBOS experimental data [77] (symbols) for $Au + Au$ collisions at the RHIC energy $\sqrt{s}_{NN} = 200$ GeV. Centralities and the corresponding ranges of impact parameters are given by STAR's Glauber model results [78].

The experimental data on central pseudo-rapidity density of charged hadrons in the most central 0 – 5% collisions are used to calibrate the parameters, $K = 1.45$ and $\tau_0 = 0.4$ fm/c, in our model for the initial conditions. For other centralities, the range of impact parameters is varied according to the Glauber model [78]. We assumed the freeze-out temperature $T_f = 0.137$ GeV and the parameterized equation of state EoS s95p-v1 [64]. The spectra for charged hadrons include both direct thermal hadrons and decay products from resonances with masses up to 1.68 GeV. All hadron spectra agree with the experimental data well for all centralities and $p_T < 3$ GeV/c except for protons which are about a factor of 1.2 smaller than the data. The deficiency in proton spectra might be caused by finite chemical potential due to chemical freeze-

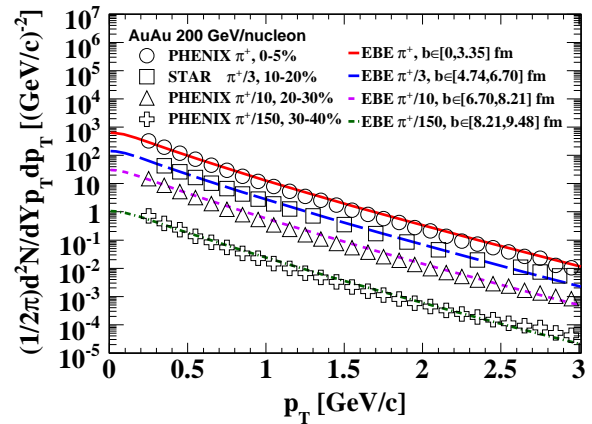


FIG. 3: (Color online) Transverse momentum spectra for π^+ from event-by-event (3+1)D ideal hydro (lines) with AMPT fluctuating initial conditions compared to the PHENIX [79] and STAR experimental data [80] (symbols) for $Au + Au$ collisions at different centralities at the RHIC energy $\sqrt{s}_{NN} = 200$ GeV

out time earlier than the pions and kaons [81, 82]. The charged hadron yields at large rapidity from our (3+1)D hydro calculation (Figs. 2) are somewhat larger than the experimental data. In these large rapidity regions, the net baryon density is quite large. One has to consider the evolution of the net baryon density coupled to the energy-momentum density and the EoS we used for zero baryon chemical potential is no longer valid. Inclusion of shear viscosity also slows down the longitudinal expansion and gives a narrower tail of the pseudo-rapidity distribution of charged hadrons [31, 84].

The overall geometric shape of the overlapped region and local density fluctuation of the initially produced dense matter will lead to azimuthal anisotropies in the final hadron spectra. The strong elliptic flow measured during the first years of RHIC experiments is considered as one of the evidences for a strongly coupled quark-gluon plasma (sQGP) formed in the central $Au + Au$ collisions [13]. Recent efforts have been focused on extracting the shear viscosity of the sQGP by comparing experimental data with viscous hydrodynamic calculations [19–22]. However, quantitative studies are complicated by uncertainties in the initial conditions [83]. Inclusion of fluctuations will add to the complexities of the problem. It is therefore useful to study the variation of anisotropic flows with more realistic initial conditions even in ideal hydrodynamics without viscous corrections. Most recent studies [27–31, 33–35] employed either Monte Carlo Glauber[53] or Monte Carlo KLN [54] initial conditions, both lack fluctuations in local flow velocity and longitudinal distribution in pseudo-rapidity.

The differential harmonic flow v_n of hadron spectra is

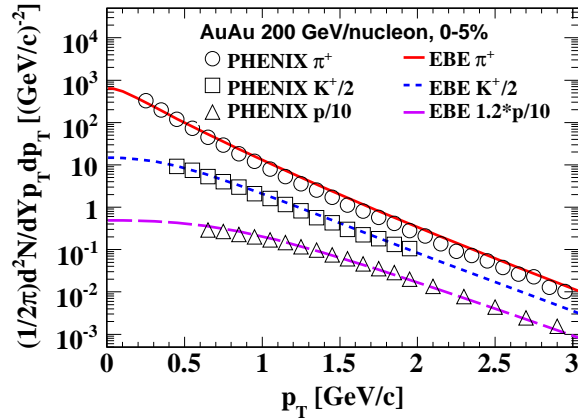


FIG. 4: (Color online) Transverse momentum spectra for identified particles from event-by-event (3+1)D ideal hydro (lines) with AMPT fluctuating initial conditions compared to the PHENIX experimental data (symbols) [79] for the most central (0-5%) $Au + Au$ collisions at the RHIC energy $\sqrt{s_{NN}} = 200$ GeV. A factor of 1.2 is multiplied to the hydro proton spectra due to possible early chemical freeze-out.

defined as:

$$v_n(p_T, \eta) = \frac{\int_0^{2\pi} d\phi \frac{dN}{d\eta p_T d p_T d\phi} \cos(n(\phi - \Psi_n))}{\int_0^{2\pi} d\phi \frac{dN}{d\eta p_T d p_T d\phi}}, \quad (27)$$

where Ψ_n can be the azimuthal angle for the participant-plane (PP) in coordinate space of initial partons in theoretical studies or event-plane (EP) in momentum space of the final hadrons in experimental analyses,

$$\Psi_n^{PP} = \frac{1}{n} \left(\arctan \frac{\langle r^2 \sin(n\phi_r) \rangle}{\langle r^2 \cos(n\phi_r) \rangle} + \pi \right), \quad (28a)$$

$$\Psi_n^{EP} = \frac{1}{n} \arctan \frac{\langle p_T \sin(n\phi_p) \rangle}{\langle p_T \cos(n\phi_p) \rangle}. \quad (28b)$$

The average in Eq. (28a) for Ψ_n^{PP} is over all initial partons weighted by their squared transverse coordinates $r^2 = x^2 + y^2$, while the average in Eq. (28b) for Ψ_n^{EP} is over final particles weighted by their transverse momenta. The corresponding hadronic flows will be denoted as v_n^{PP} and v_n^{EP} , respectively. Note that the final hadron spectrum from Cooper-Frye formula is a continuous distribution function. Therefore, integrations over the transverse momentum p_T , pseudo-rapidity η and azimuthal angle ϕ_p in calculating Ψ_n^{EP} will not introduce plane resolution due to finite number of particles per event which will have to be corrected for in experimental analyses.

In Fig. 5, we compare our ideal hydro calculation of v_2^{EP} (solid lines) with the PHENIX data [85] for charged hadrons within a pseudo-rapidity range $[-0.35, 0.35]$ in $Au + Au$ collisions at $\sqrt{s_{NN}} = 200$ GeV with different centralities. The event-planes in the PHENIX analysis were determined with 2 sub-events to correct for the

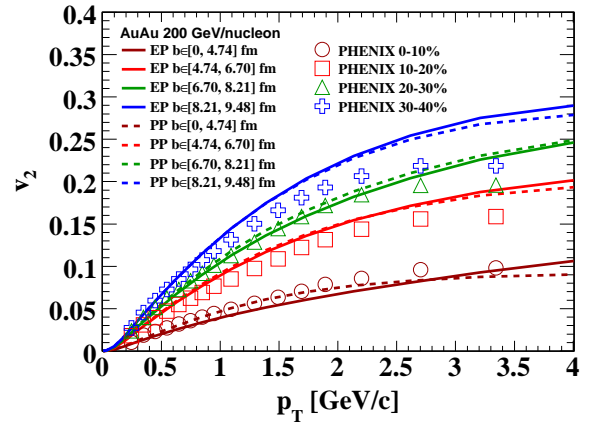


FIG. 5: (Color online) Elliptic flow for charged hadrons with respect to participant-planes (PP) (solid) and event-planes (EP) (dashed) from event-by-event (3+1)D hydrodynamic simulations with AMPT initial conditions compared to the PHENIX data [85] on v_2^{EP} for $Au + Au$ collisions at the RHIC energy $\sqrt{s_{NN}} = 200$ GeV.

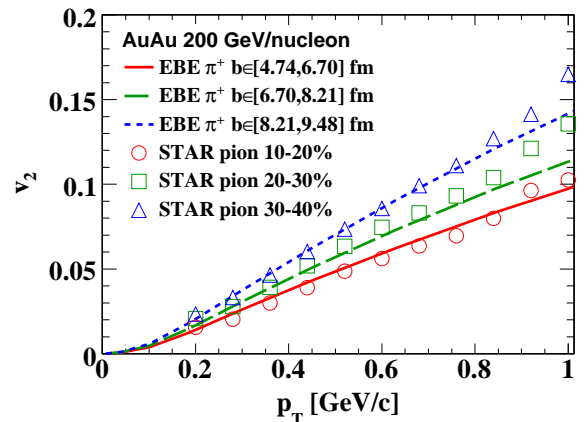


FIG. 6: (Color online) Elliptic flow for π^+ at different centralities from event-by-event ideal hydro simulations compared to the STAR data [2] for $Au + Au$ collisions at the RHIC energy $\sqrt{s_{NN}} = 200$ GeV.

event-plane resolution. Our hydro calculations fit the experimental results quite well at low p_T for all centralities. At higher p_T , viscous corrections and other non-equilibrium effects such as jet quenching [86–88] are expected to become important. Ideal hydrodynamics will fail, producing much larger elliptic flow than the experimental data. We also show v_n^{PP} (dashed lines) from our hydro calculations as determined by the participant-plane. It is a very good approximation of v_n^{EP} as determined by the event-plane, especially at low p_T . In the rest of this paper, we will focus on the elliptic flow v_2^{EP} with respect to event-planes.

To study the elliptic flow of identified hadrons, we

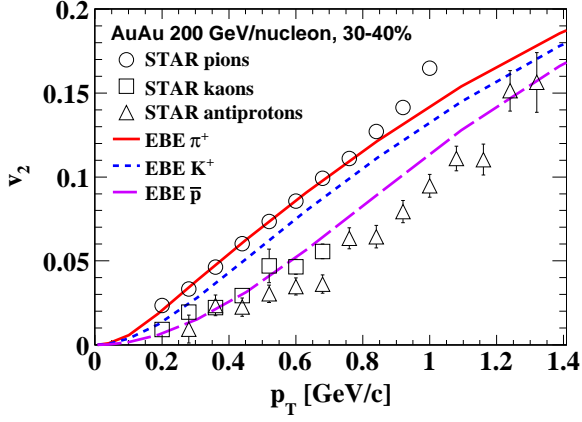


FIG. 7: (Color online) Elliptic flow for identified particles at centrality 30 – 40% from event-by-event ideal hydro simulations (lines) compared to the STAR data [2] (symbols).

show in Fig. 6 our ideal hydro results on v_2 for positively charged pions in $Au + Au$ collisions at the RHIC energy $\sqrt{s_{NN}} = 200$ GeV that fit the STAR experimental data [2] very well in 10–20%, 20–30% and 30–40% centrality bins for $p_T \leq 1$ GeV/c. Such an agreement with experimental data in all centralities cannot be achieved with smoothed initial conditions initial conditions according to Glauber model. One, however, can achieve the same agreement with smoothed initial condition but accounting for fluctuations in geometrical eccentricity [89]. In our hydro calculations, we used 100 events for each centrality bin. Much more events are needed for a true minimum bias (0–80%) calculation using the AMPT initial conditions. For one single centrality bin 30–40%, our hydro results on v_2 for identified charged pions and antiprotons fit the STAR data [2] reasonably well as shown in Fig. 7. But the hydro results for charged kaons deviates significantly from the STAR data.

Following the same procedure with the same parameters for initial conditions from the AMPT model, we can calculate hadron spectra in $Pb + Pb$ collisions at the LHC energy $\sqrt{s} = 2.76$ TeV. We have set the overall scale factor $K = 1.6$ and the initial time $\tau_0 = 0.2$ fm/c as constrained by the experimental data on the central rapidity density of charged hadrons. Shown in Fig. 8 are the charged hadron rapidity distributions in $Pb + Pb$ collisions at different centralities from our (3+1)D ideal hydro calculations. The ranges of impact parameters are selected according to the centralities in the ALICE data [91] for the central rapidity region.

The corresponding transverse momentum spectra of charge hadrons in the central rapidity region $|\eta| < 0.8$ for the most central $Pb + Pb$ collisions also agree with the experimental data well as shown in Fig. 9. The elliptic flow in $Pb + Pb$ collisions at the LHC energy $\sqrt{s} = 2.76$ TeV in Fig. 10 is very similar to that in $Au + Au$ collisions at RHIC (see Fig. 5). The ideal hydro results

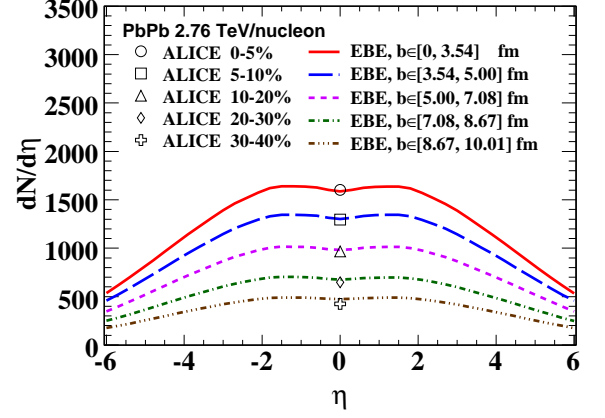


FIG. 8: (Color online) Charged hadron rapidity distributions from event-by-event ideal hydro calculations in $Pb + Pb$ collisions at $\sqrt{s} = 2.76$ TeV at different centralities as compared to the ALICE experimental data [91].

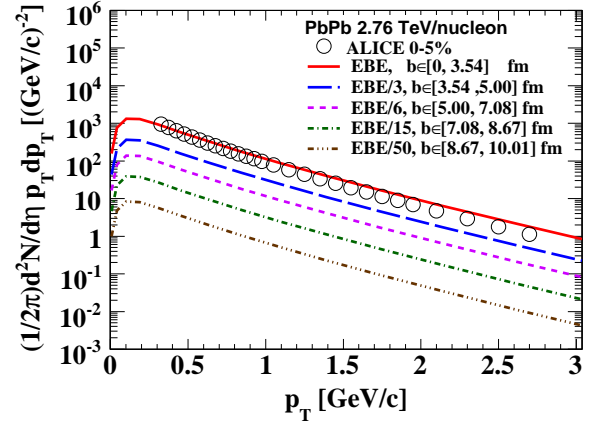


FIG. 9: (Color online) Transverse momentum spectra for charged hadrons in the central rapidity region $|\eta| < 0.8$ from event-by-event ideal hydro calculations in $Pb + Pb$ collisions at $\sqrt{s} = 2.76$ TeV as compared to the ALICE data [92].

agree with the ALICE experimental data well in central collisions but fail to describe the data at large p_T in peripheral collisions, indicating the importance of viscous or non-equilibrium corrections. Here the charged hadrons are restricted to $|\eta| < 2.5$ and event-planes are determined with charged hadrons in the pseudo-rapidity window $3.3 < |\eta| < 4.8$ as in the ATLAS data. Shown in Fig. 11 are v_2 for identified hadrons as compared to the preliminary ALICE data [90]. The (3+1)D hydro results describe the flavor dependence quite well, except anti-protons in the 10-20% centrality.

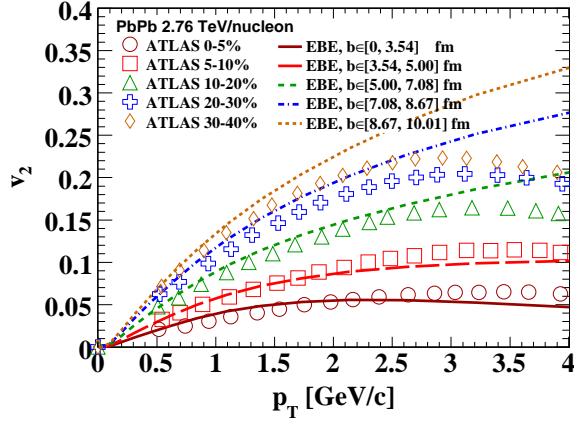


FIG. 10: (Color online) Elliptic flow of charged hadrons in the central rapidity region $|\eta| < 2.5$ from event-by-event ideal hydro calculations in $Pb + Pb$ collisions at $\sqrt{s} = 2.76$ TeV compared to the ATLAS data [44]. Event-planes are determined with charged hadrons in $3.3 < |\eta| < 4.8$.

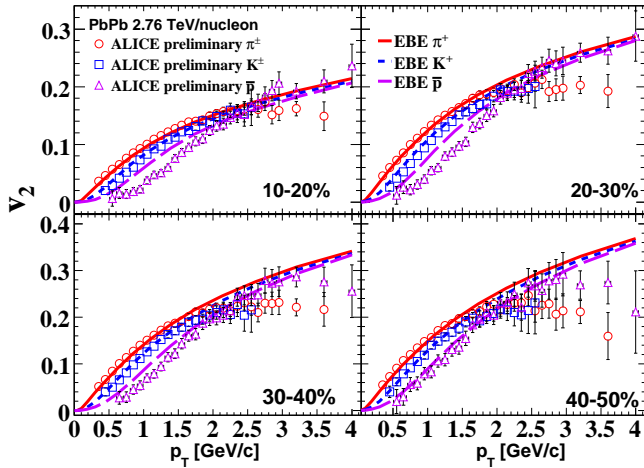


FIG. 11: (Color online) Elliptic flow from ideal hydro for identified particles in $Pb + Pb$ collisions with four different centralities at the LHC energy $\sqrt{s} = 2.76$ TeV compared to the preliminary ALICE data [90].

IV. EFFECTS OF LONGITUDINAL AND FLOW VELOCITY FLUCTUATIONS

Most of the fluctuating initial conditions such as MC Glauber or MC KLN model assume zero initial transverse flow velocity while the longitudinal flow velocity is assumed to be the same as the local spatial pseudorapidity η_s in the Bjorken scaling model. The latest (3+1)D viscous hydrodynamic model [29] also assumes zero initial transverse flow velocity and a Bjorken scaling scenario for the initial parton distribution in the longitudinal direction with an overall envelop function adjusted

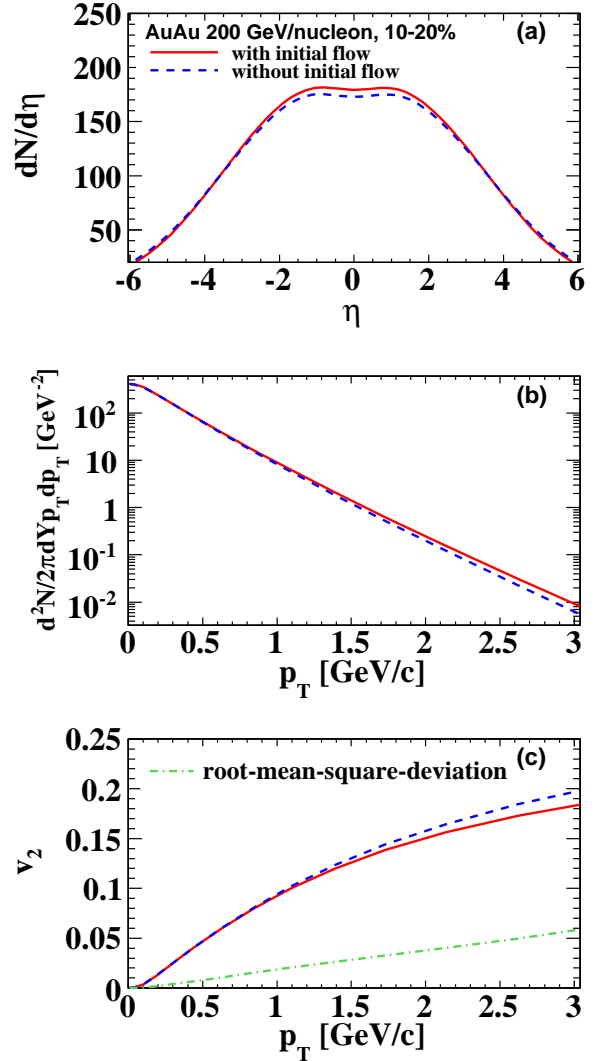


FIG. 12: (Color online) Event averaged multiplicity, p_T spectra and v_2 from ideal hydro with (solid) and without initial flow (dashed) velocity in the initial conditions for 10 – 20% $Au + Au$ collisions at $\sqrt{s} = 200$ GeV/nucleon.

to reproduce the final hadron rapidity distribution.

The initial condition in Eq. (26) from the AMPT model that we use in this study should contain non-vanishing local transverse and longitudinal flow velocities as well as fluctuations in the parton rapidity distribution. The AMPT model uses the HIJING model for initial parton production which contains many mini-jets as well as excited strings. After initial thermalization via parton cascade within the initial time τ_0 and the Gaussian smearing in Eq. (26), these mini-jets will lead to small but non-vanishing collective radial flow velocities as well as large local flow velocity fluctuations. The local fluctuating initial flow velocities due to mini-jets should also have strong back-to-back correlation in azimuthal angle. Such initial collective radial flow and local velocity fluctu-

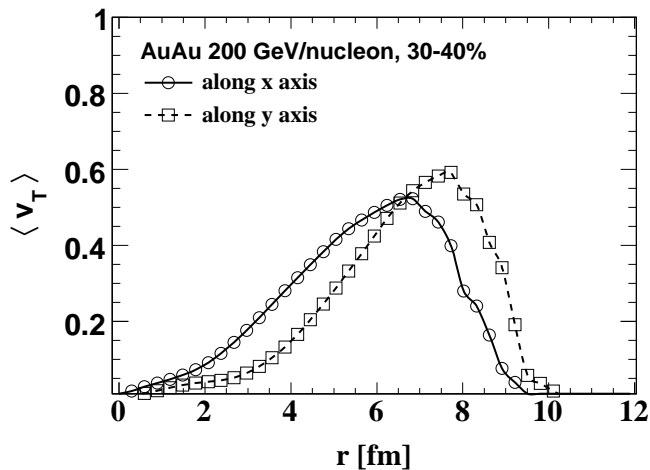


FIG. 13: The event-averaged initial radial flow velocity along the x and y -axis in the 30-40% semi-central $Au + Au$ collisions at $\sqrt{s} = 200$ GeV from the AMPT model at $\tau_0 = 0.4$ fm/c.

tuations should influence the final hadron spectra after hydrodynamic evolution.

To check the influence of the initial flow velocity fluctuations on hadron spectra, we show in Fig. 12 the pseudo-rapidity distributions (top), transverse momentum spectra (middle) and elliptic flow of positively charged pions in 10-20% central $Au + Au$ collision from our hydro calculations with (solid) and without initial local flow velocities (dashed). These two initial conditions have the same initial local energy density distributions as determined by Eq. (9) with the initial energy-momentum tensor from AMPT simulations via Eq. (26). The local flow velocities as determined from Eqs. (10) and (11) are set to zero in the latter case. There is a slight increase in the hadron multiplicity or initial total entropy and the inverse slope of hadron spectra and a slight decrease of the elliptic flow at high p_T due to the fluctuating initial local flow velocities.

To understand the change in hadron transverse momentum spectra, we show in Fig. 13 the event-averaged initial radial flow velocities along the x and y -axis in 30-40% non-central $Au + Au$ collisions at $\sqrt{s_{NN}} = 200$ GeV at an initial time $\tau_0 = 0.4$ fm/c. One can see that parton interaction or thermalization during the initial time generates significant amount of radial flow that is anisotropic in azimuthal angle and is responsible for the harder transverse momentum spectra of final hadrons. Such radial flow velocities in the initial conditions for the hydrodynamic evolution is also shown to influence the HBT correlation of final hadrons [61] and other aspect of hydro evolution [93].

The fluctuation in the initial flow velocities seems to defuse the initial geometrical anisotropy a little and leads to slightly smaller elliptic flow for final hadrons as shown in the lower panel of Fig. 12. The calculated elliptic flow

is determined mainly by the geometric eccentricity and averaged over many events. We find, however, there are significant differences between the elliptic flows with and without the initial flow velocities on an event-by-event basis. We characterize these differences by the variance

$$\Delta v_2 = \sqrt{\sum_{i=1}^{N_{\text{event}}} (v_{2i}^{WIF} - v_{2i}^{WOF})^2 / N_{\text{event}}} \quad (29)$$

between the elliptic flow v_2^{WIF} with the initial flow velocity and v_2^{WOF} without, where N_{event} is the number of events. As shown by the dot-dashed line in the lower panel of Fig. 12, the variance is quite large reaching about 0.07 at $p_T = 3$ GeV/c. Note that we used hadrons in the rapidity range $3.1 < |\eta| < 3.9$ to determine the event-planes and calculate the elliptic flow for hadrons in the central rapidity region $|\eta| < 1.0$.

To investigate the effect of longitudinal fluctuation on hadron spectra we compare our event-by-event hydro calculations using the initial condition from the full AMPT results with that using a tube-like smooth initial longitudinal distribution. In the tube-like initial condition, we take the initial energy-density and transverse flow velocity from AMPT results in the central rapidity region and assume these transverse fluctuations to be the same along the longitudinal direction with an envelop function

$$H(\eta) = \exp[-\theta(|\eta| - \eta_0)(|\eta| - \eta_0)^2 / 2\sigma_w^2], \quad (30)$$

in rapidity. The length of the plateau η_0 in the central rapidity region and the width σ_w of the Gaussian fall-off at large rapidities are adjusted to fit the charged hadron rapidity distribution. As another comparison, we also calculate the elliptic flow from a one-shot AMPT initial condition, which is the average of many AMPT events each rotated by an angle to a common participant-plane. We prefer this as one-shot-tube initial condition since the initial parton density also has smooth tube-like distribution in the longitudinal direction. Shown in Fig. 14 are the transverse momentum spectra (upper panel) and elliptic flow (lower panel) of charged pions in semi-central (30-40%) $Au + Au$ collisions at the RHIC energy $\sqrt{s_{NN}} = 200$ GeV with the full AMPT initial conditions (solid lines) as compared to the initial conditions with a tube-like structure in the longitudinal direction (dot-dashed lines) and the one-shot AMPT with tube-like longitudinal distribution initial condition (dashed). The event-by-event fluctuations in the tube-like AMPT initial conditions significantly reduce elliptic flow of final hadrons with respect to the event-planes as compared to the one-shot-tube AMPT initial condition. The slope of the p_T spectra from the event-by-event tube-like initial conditions on the other hand is increased by the fluctuations (both the energy density and flow velocity) or hot spots in the transverse direction as compared to the spectra from one-shot-tube initial conditions. Similar results were found by both (2+1)D [32] and (3+1)D hydro [29] calculations. However, fluctuations in the longitudinal

direction in the full AMPT initial conditions have also hot spots in the longitudinal direction. The expansion of such longitudinal hot spots will dissipate more transverse energy into the longitudinal direction. This in turn decrease noticeably the value of the elliptic flow at large p_T compared to the results from tube-like event-by-event AMPT initial conditions. The slope of the p_T spectra is also significant smaller than that from event-by-event tube-like AMPT initial condition without fluctuation in the longitudinal direction.

Since anisotropic flow, at large p_T in particular, is used to extract transport coefficients (such as shear viscosity) from comparisons between experimental data and viscous hydrodynamics, the inclusion of fluctuation in initial rapidity distribution in the hydrodynamic calculations will be necessary for more qualitative studies.

In all the above calculations of elliptic flows for identified charged hadrons in the central rapidity region $|\eta| < 1.0$, the event-planes are determined using charged hadrons in $3.1 < |\eta| < 3.9$. To illustrate the sensitivity of the calculated elliptic flows to the rapidity selection for charged hadrons that determine the event-plane, we show in Fig. 15 elliptic flows for hadrons in the central rapidity region with the event-planes determined by hadrons in different rapidity windows. The dependence on the rapidity window is quite small. They are all slightly larger than the elliptic flow measured against the participant-planes (solid lines).

Since mini-jets in initial conditions from the AMPT model contain both near-side and away-side correlations, the fluctuations in the initial flow velocities that we use will also have important effects on final two-hadron correlations [94] in both azimuthal angle and rapidity. This is beyond the scope of our study in this paper and will be discussed in future studies.

V. PARTIAL CHEMICAL EQUILIBRIUM

Studies of hadron chemistry in high-energy heavy-ion collisions [95] indicate that chemical equilibrium is reached during the early stage of the hadronic phase of the dense matter. The flavor abundance of the final hadrons indicates a freeze-out temperature $T_{cf} \approx 158 - 164$ MeV at the RHIC and LHC energies [96]. This chemical freeze-out temperature is significantly higher than the kinetic freeze-out temperature $T_f = 137$ MeV that we used in our ideal hydro simulation. This is one of the reasons why the calculated proton spectra from our ideal hydro simulations are about a factor of 1.2 lower than the experimental measurements as shown in Fig. 4.

To take into account the earlier chemical freeze-out during the hadronic phase of the hydrodynamical evolution, one should use a Partial Chemical Equilibrated (PCE) EoS [97]. We compare the hadron spectra and differential elliptic flow from hydro calculations with Chemical Equilibrium (CE) version s95p-v1 (lines) and PCE version s95p-PCE165-v0 EoS (symbols) at both RHIC

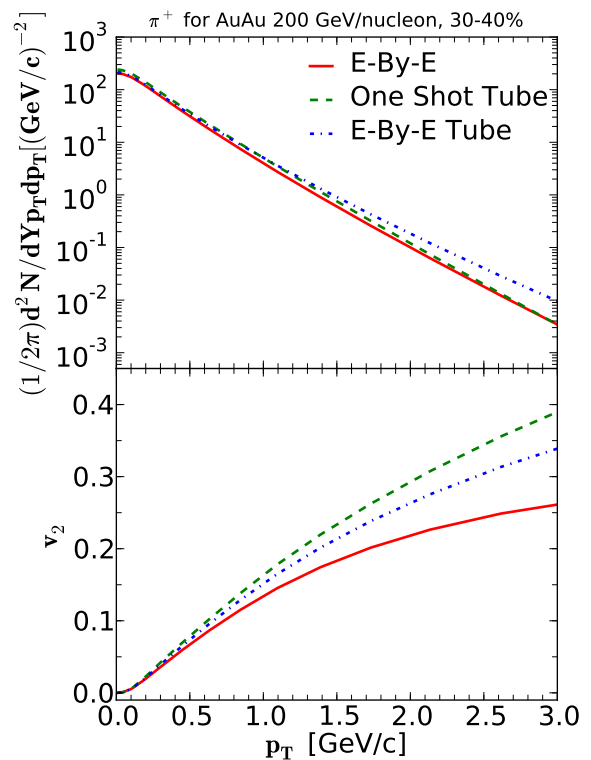


FIG. 14: Transverse momentum spectra (upper panel) and elliptic flow (lower panel) for charged pions from hydrodynamic simulations of 30-40% semi-central $Au + Au$ collisions at $\sqrt{s} = 200$ GeV with full fluctuating AMPT initial conditions (solid lines), tube-like AMPT initial conditions (dot-dashed lines) and one-shot AMPT tube-like initial conditions (dashed).

and LHC energies in Figs. 16 and 17. We have used a kinetic freeze-out temperature $T_{f,rz} = 137$ MeV in both calculations. Because of the finite chemical potential at the kinetic freeze-out in the hydro with a PCE EoS, the corresponding kaon and proton yields at low p_T is higher, improving agreement with the experimental data at RHIC. However, the slope of hadron spectra at high p_T is steeper than that from hydro with a CE EoS which is also below the experimental data. This can be improved a little by the viscous correction in the viscous hydrodynamics. As shown in Fig. 17, the elliptic flow from hydro with a PCE EoS hydro is also about 20% higher than that from hydro with a CE EoS, again leaving more rooms for improvement in the viscous hydrodynamics. One should note that the PCE EoS' are parameterized to take into account the higher chemical freeze-out temperature, which however is different in heavy-ion collisions at different colliding energy [96]. One therefore has to use different parameterization of PCE EoS at different colliding energies. The PCE EoS that was fitted to RHIC data, however, cannot describe the recent experimental data on identified hadron spectra from LHC using the

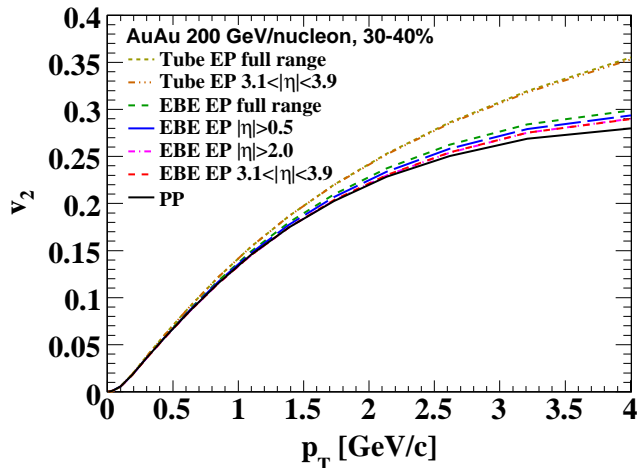


FIG. 15: Elliptic flows from (3+1)D hydro simulations of 30-40% semi-central $Au+Au$ collisions at $\sqrt{s} = 200$ GeV relative to the participant-planes (solid) and event-planes determined by charged hadrons in the full rapidity window (dashed), $|\eta| > 0.5$ (long-dashed), $|\eta| > 2.0$ (dot-dashed), and $3.1 < |\eta| < 3.9$ (dotted).

viscous hydrodynamics [98].

VI. CONCLUSION

We have studied hadron spectra and elliptic flow in high-energy heavy-ion collisions within a (3+1)D ideal hydrodynamic model. The (3+1)D ideal hydrodynamic equations are solved numerically using an extended FTC-SHASTA algorithm. A projection method is developed to compute the freeze-out hyper surface for final hadrons. We carried out event-by-event hydrodynamic simulations with fluctuating initial conditions for the energy-momentum tensor from the AMPT model using a Gaussian smearing function for each initially produced parton. Such initial conditions provide both the local energy density as well as non-vanishing local flow velocities for the hydrodynamic evolution. With a set of parameters (widths of the Gaussian smearing, an overall scale factor and initial time), the hadron rapidity distributions, transverse momentum spectra as well as elliptic flows agree very well with experimental data, in particular at low p_T , at both the RHIC and LHC energies.

We also illustrated the effects of local flow velocity in initial conditions from the AMPT model on final hadron spectra. Due to rescattering during the initial time in the AMPT model, partons also develop some initial collective radial flow which leads to harder transverse momentum spectra as compared to hydro calculations without initial local flow velocities. The averaged elliptic flow however is not affected much. The fluctuation in the longitudinal distribution, however, is shown to reduce the elliptic flow at large p_T even in the central rapidity region. We

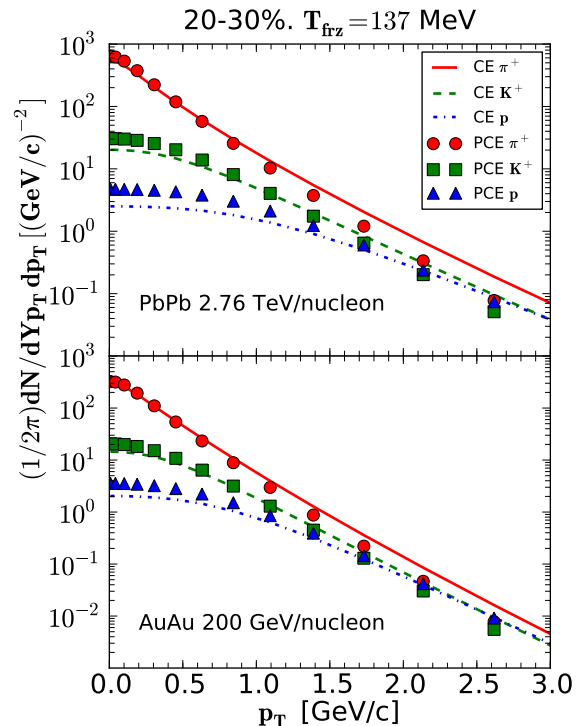


FIG. 16: Transverse momentum spectra of identified hadron spectra in semi-central (20-30%) heavy-ion collisions at RHIC (lower panel) and LHC (upper panel) energies from hydro simulations with CE (lines) and PCE (symbols) EoS.

also studied the sensitivities of the hydro results on the EoS and found influence of the early chemical freeze-out as parameterized in the PCE EoS to be important and should be considered in hydro calculations at different colliding energies separately.

Our (3+1)D ideal hydro calculations provide a good description of hadron spectra at low p_T in high-energy heavy-ion collisions. At larger p_T , viscous corrections are known to become important [19, 20], in particular for higher hadronic flows. An extension of the study to the viscous hydrodynamics will be necessary for hadron spectra at moderately large transverse momenta.

VII. APPENDIX: SHASTA ALGORITHM

There are high order and low order numerical algorithms to solve a partial differential equation. A low order algorithm keeps the solution monotonic but suffers from numerical diffusion, while a high order algorithm is more accurate but leads to dispersion (small ripples with new maximum and minimum during the evolution). The FCT (flux corrected transport) algorithm is developed to solve these problems in which the low order algorithm is used for the transport and diffusion, and a second step

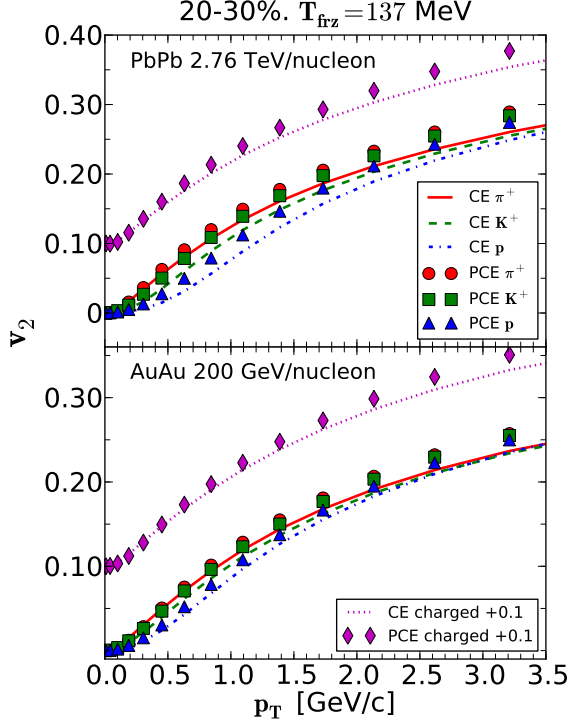


FIG. 17: Differential elliptic flow v_2 of identified and charged hadrons in semi-central (20-30%) heavy-ion collisions at RHIC (lower panel) and LHC (upper panel) energies from hydro simulations with CE (lines) and PCE (symbols) EoS.

high order algorithm is used to do anti-diffusion with a corrected flux (equals to the diffusion term with ripples corrected). The SHASTA (SHarp And Smooth Transport Algorithm) algorithm is one kind of FCT algorithms which is good at dealing with strong gradients and shocks [67–69]. We provide the basic steps of the FCT-SHASTA algorithm in this appendix.

Hydrodynamic equations contain mainly partial differential equations whose basic form in one dimension is:

$$\partial_t \rho + \partial_x(v\rho) = S(\rho, v), \quad (31)$$

where ρ can be mass density, energy density or momentum density and v is the velocity along x direction. In the following, we consider ρ as the mass density for simplicity. The source term S can be taken into account by a two-stage second-order mid-point Runge Kutta method:

$$\rho_j^{n+1} = \rho_j^n + \Delta t S_j^{1/2}(\rho_j^{n+1/2}, v_j^{n+1/2}), \quad (32)$$

where ρ_j^n denotes the mass density at grid point j and time step n , $S_j^{1/2}$ are calculated from $\rho_j^{n+1/2}$ and $v_j^{n+1/2}$ given by the first stage half time step calculation in the Runge-Kutta method. To explain the SHASTA algorithm, we only consider $\partial_t \rho + \partial_x(v\rho) = 0$ in this appendix.

A. The geometric interpretation of transport stage in SHASTA

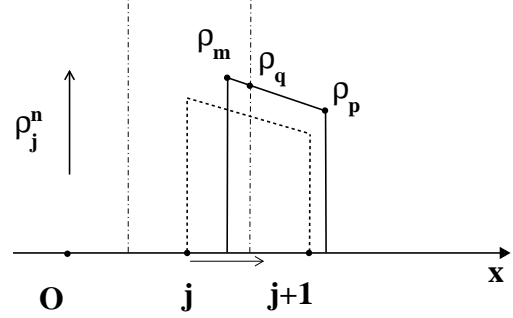


FIG. 18: The geometric explanation of SHASTA algorithm. The dashed trapezoid represents the total mass in one fluid element between grid point j and $j+1$ at time $t=0$. After one time step, the boundaries of this fluid element move to points m and p and the mass in this fluid element is conserved. One important principle of SHASTA algorithm is total mass conservation: $M = \sum_j \rho_j^n \delta x = \sum_j \rho_j^{n+1} \delta x$. The grid size δx and time step δt must satisfy $v\delta t < \delta x/2$ in the SHASTA algorithm to keep the positivity of the mass in each cell.

For the 1D FCT-SHASTA algorithm, we assume the velocity of the matter being transported at grid point j between time t and $t + \delta t$ can be approximated by $v_j^{1/2}$ at $t = t + \delta t/2$ which can be calculated from the 2-step Runge-Kutta method. The mass density ρ_j and ρ_{j+1} at the boundaries of one fluid element as illustrated in Fig. 18 (the dashed trapezoid) change to ρ_m and ρ_p after one time step. Since the mass in this fluid element is conserved,

$$\frac{1}{2}(\rho_j^0 + \rho_{j+1}^0)\delta x = \frac{1}{2}(\rho_m + \rho_p) \left[\delta x + (v_{j+1}^{1/2} - v_j^{1/2})\delta t \right]. \quad (33)$$

If we consider the two sides of the solid trapezoid vary in the same rate, we get

$$\rho_p = \rho_{j+1}^0 \delta x / \left[\delta x + (v_{j+1}^{1/2} - v_j^{1/2})\delta t \right], \quad (34)$$

$$\rho_m = \rho_j^0 \delta x / \left[\delta x + (v_{j+1}^{1/2} - v_j^{1/2})\delta t \right]. \quad (35)$$

Consider a cell centered at grid point j (between the dot-dashed lines in Fig. 18). The total mass in cell j at time $t + \delta t$ comes from the mass moved in from cell $j-1$ to j and the residual mass after some moved out from j to $j+1$. The mass density at point q where the right boundary of cell j intersects with the solid trapezoid is calculated from interpolation:

$$\rho_q = \rho_p + (\rho_m - \rho_p)(\delta x/2 + v_{j+1}^{1/2}\delta t) / \left[\delta x + \delta t(v_{j+1}^{1/2} - v_j^{1/2}) \right] \quad (36)$$

The residual mass is given by the area of the residual trapezoid:

$$\begin{aligned}\Delta m_{\text{re}} &= (\rho_m + \rho_q)l_{\text{re}}/2 \\ &= \delta x \left[\frac{1}{2}Q_+^2(\rho_{j+1}^n - \rho_j^n) + Q_+\rho_j^n \right],\end{aligned}\quad (37)$$

where $l_{\text{re}} = \delta x/2 - v_j^{1/2}\delta t$ and

$$Q_{\pm} = (1/2 \mp v_j^{1/2}\delta t/\delta x) / \left[1 \pm (v_{j\pm 1}^{1/2} - v_j^{1/2})\delta t/\delta x \right]. \quad (38)$$

Similarly, the mass transported from cell $j-1$ to cell j can be calculated as,

$$\Delta m_{\text{tr}} = \delta x \left[\frac{1}{2}Q_-^2(\rho_{j-1}^n - \rho_j^n) + Q_-\rho_j^n \right]. \quad (39)$$

The mass density at grid point j (averaged over cell j) and time step $n+1$ is therefore:

$$\begin{aligned}\rho_j^{n+1} &= (\Delta m_{\text{re}} + \Delta m_{\text{tr}})/\delta x \\ &= \frac{1}{2}Q_-^2(\rho_{j-1}^0 - \rho_j^0) + \frac{1}{2}Q_+^2(\rho_{j+1}^0 - \rho_j^0) \\ &\quad + (Q_+ + Q_-)\rho_j^0.\end{aligned}\quad (40)$$

For a uniform velocity, the above equation takes a simple form:

$$\rho_j^{n+1} = \rho_j^n - \frac{\epsilon}{2}(\rho_{j+1}^n - \rho_{j-1}^n) + \left(\frac{1}{8} + \frac{\epsilon^2}{2}\right)(\rho_{j+1}^n - 2\rho_j^n + \rho_{j-1}^n)$$

where $\epsilon = v\delta t/\delta x$. For zero velocity, it becomes

$$\rho_j^{n+1} = \rho_j^n + \frac{1}{8}(\rho_{j+1}^n - 2\rho_j^n + \rho_{j-1}^n) \quad (41)$$

In the above transport stage of SHASTA, the solution is monotonic and positive but has a large zero order diffusion.

B. Anti-diffusion Stage

To correct the diffusion, an explicit form of anti-diffusion can be used (for the zero velocity case):

$$\bar{\rho}_j^{n+1} = \rho_j^{n+1} - \frac{1}{8}(\rho_{j+1}^{n+1} - 2\rho_j^{n+1} + \rho_{j-1}^{n+1}). \quad (42)$$

One can illustrate the above diffusion and anti-diffusion stage with a square wave initial

field profile $(\dots, 1, 1, 0, 0, \dots)$. The field becomes $(\dots, 1, \frac{7}{8}, \frac{1}{8}, 0, \dots)$ after the transport stage, and becomes $(\dots, \frac{65}{64}, \frac{61}{64}, \frac{3}{64}, -\frac{1}{64}, \dots)$ after the explicit anti-diffusion stage. One can see that new maximum and minimum are created, and the positivity is destroyed. To solve this problem, the anti-diffusion terms are written in mass flux form:

$$\bar{\rho}_j^{n+1} = \rho_j^{n+1} - f_{j+1/2}^{n+1} + f_{j-1/2}^{n+1}, \quad (43)$$

where the anti-diffusion mass flux is defined as:

$$f_{j\pm 1/2}^{n+1} = \pm \frac{1}{8}(\rho_{j\pm 1}^{n+1} - \rho_j^{n+1}). \quad (44)$$

The above mass flux is further corrected as

$$f_{j+1/2}^{(c)n+1} = \sigma \max \left\{ 0, \min \left\{ \sigma \Delta_{j-1/2}, \frac{1}{8} \Delta_{j+1/2}, \sigma \Delta_{j+3/2} \right\} \right\} \quad (45)$$

which is limited term by term so that no anti-diffusive-flux transfer of mass can push the density at any grid point beyond the density value at neighboring points. Here $\Delta_{j+1/2} = \rho_{j+1}^{n+1} - \rho_j^{n+1}$ and $\sigma = \text{sgn} \Delta_{j+1/2}$. This is the origin of the name ‘‘Flux Corrected Transport’’. The final mass density at grid j and time step $n+1$ after corrected anti-diffusion stage is then

$$\bar{\rho}_j^{n+1} = \rho_j^{n+1} - f_{j+1/2}^{(c)n+1} + f_{j-1/2}^{(c)n+1}, \quad (46)$$

where the diffusion is corrected by the anti-diffusion and the dispersion is corrected by the limitation on the mass flux. The formula can be checked with several examples.

In our (3+1)D hydrodynamics, we use a limiter described by Zalesak [68] which is proved better than the original one given by Boris and Book [67].

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