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Emergent (anomalous) higher symmetries from topological orders and from dynamical electromagnetic field in condensed matter systems

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Global symmetry (0-symmetry) acts on the whole space while higher k -symmetry acts on all the codimension- k closed subspaces. The usual condensed matter lattice theories do not include dynamical electromagnetic (EM) field and do not have higher symmetries (unless we engineer fine-tuned toy models). However, for gapped systems, (anomalous) higher symmetries can emerge from the usual condensed matter theories at low energies (usually in a spontaneously broken form). We pointed out that the emergent spontaneously broken higher symmetries are nothing but a kind of topological orders. Thus the study of emergent spontaneously broken higher symmetries is a study of topological order. The emergent (anomalous) higher symmetries can be used to constrain possible phase transitions and possible phases induced by certain types of excitations in topological orders. (Anomalous) higher symmetry can also emerge in gapless systems if the gapless excitations contain gapless gauge fields. In particular, EM condensed matter systems that include the dynamical EM field have an emergent *anomalous* $U(1)$ -1-symmetry below the energy gap of the magnetic monopoles. So EM condensed matter systems can realize some physical phenomena of anomalous higher symmetry. In particular, any gapped liquid phase of an EM condensed matter system (induced by arbitrary fluctuations and condensations of electric charges and photons) must have a non-trivial bosonic topological order.

I. INTRODUCTION

A global symmetry acts on the whole space, and a local symmetry acts on all the points (*i.e.* all the sub manifolds of 0-dimension). A higher symmetry, such as a k -symmetry, acts on all the closed sub manifolds of codimension k . Thus a global symmetry is a 0-symmetry. A lattice Hamiltonian system has a k -symmetry, if the Hamiltonian is invariant under certain unitary transformations defined on all the closed codimension- k subspaces.

Higher symmetry had been studied before in lattice systems, where exactly soluble lattice Hamiltonians commuting with closed string and/or closed membrane operators were constructed¹⁻⁵ to realize topological orders⁶⁻⁸. A direct relation between higher symmetries and topological orders was pointed out in Ref. 9 and 10, under the name *low-dimensional gauge-like symmetries*.

The term *higher form symmetry* was first introduced in Ref. 11, where it was stressed that higher symmetry can be viewed as a generalization of the global symmetry (*i.e.* 0-symmetry) and many results and intuitions for global symmetry can be extended to higher symmetry. For a Lagrangian field theory in $(d+1)$ -dimensional spacetime, a 0-symmetry is generated by constant fields (closed 0-forms) in spacetime, such as $\varphi \rightarrow \varphi + c$, while a k -symmetry is generated by closed k -forms in spacetime. From the field theory definition of the k -symmetry generated by a closed k -form α_k , it appears that we require the field theory to have a k -form field a_k so that the k -symmetry transformation can be written as $a_k \rightarrow a_k + \alpha_k$. However, from the lattice point of view, a k -symmetry does not require a k -form field.

The emergence of higher symmetry from lattice model without higher symmetry was also studied before in 2005¹², under the name of emergent gauge symmetry.

A topological robustness of emergent higher symmetry against any local perturbation was discovered. Such a topological robustness was used to show the topological robustness of a Goldstone-like theorem for spontaneous broken continuous higher symmetry: the gapless $U(1)$ gauge bosons from spontaneous breaking of $U(1)$ higher symmetry remain to be gapless even after we explicitly break the higher symmetry by arbitrary perturbations. The result of Ref. 12 also suggests that every topological order containing Abelian gauge theory has emergent higher symmetry.

Recently, higher symmetry and higher anomaly, as well as the related higher symmetry protected trivial (SPT) orders and higher gauge theories, became an active topic in field theory.^{11,13-35} Ref. 18, 36, and 37 discussed higher symmetry and higher anomaly in lattice systems. Ref. 18 studied higher SPT phases and higher anomaly in-flow. Ref. 36 discussed how to construct a lattice Hamiltonian of a higher gauge theory. Ref. 37 obtained a Lieb-Schultz-Mattis type theorem from higher anomalies.

In this paper, we will concentrate on applications of higher symmetry to condensed matter systems, by studying the lattice aspect of higher symmetry, higher anomaly, and how higher symmetry in field theory can emerge from lattice models without higher symmetry. The following is a summary of the results:

1. The usual condensed matter theories (which do not include the dynamical electromagnetic field) do not have higher symmetries. However, higher symmetries can emerge in usual gapped condensed matter theories in a spontaneously broken form, at energies below the energy gap. In fact, the spontaneously broken finite higher symmetry is nothing but a special kind of topological orders (see Section IV).
2. If certain topological excitations in topological or-

- ders have very large energy gap Δ_{top} , while other topological excitations have small gaps of order $\Delta \ll \Delta_{\text{top}}$, we may have emergent higher symmetry or emergent anomalous higher symmetry at energies below Δ_{top} . Note that such an emergent (anomalous) higher symmetry can appear at energies above Δ , and is smaller than the emergent higher symmetry below Δ . Thus emergent (anomalous) higher symmetry in a topological order is characterized by a set \mathcal{C}_a of low energy allowed topological excitations. The set is closed under the fusion and braiding of the topological excitations (see Section IV).
3. For bosonic 2+1D Abelian topological orders, the emergent higher symmetry characterized by \mathcal{C}_a is anomaly free iff the topological excitations in \mathcal{C}_t are bosons with trivial mutual statistics. Here \mathcal{C}_t is the set of topological excitations that have trivial mutual statistics with all the topological excitations in \mathcal{C}_a (see Section IV).
 4. The emergent (anomalous) higher symmetry will constrain the possible phase transitions and phases induced by the low energy topological excitations (see Section VI). In particular, for a topological order with emergent *anomalous* higher symmetry characterized by \mathcal{C}_a , the topological order cannot change into a trivial phase with no topological order no matter how we condense the topological excitations in \mathcal{C}_a .
 5. Anomalous higher symmetries can be realized at boundary of higher symmetry protected topological states in one higher dimension protected by the corresponding anomaly-free higher symmetry.
 6. We find a spacetime lattice regularization of 3+1D $U^\kappa(1)$ gauge theory with 2π -quantized topological term
- $$Z = \int D[a_I] e^{i \int_{M^4} \frac{\kappa_{IJ}}{4\pi} f_I f_J - \int_{M^4} \frac{|f_I|^2}{g}}, \quad (1)$$
- where $f_I = da_I$ is the field-strength 2-form of the $U(1)$ gauge field a_I . We show that the lattice model is a local bosonic model with a $Z_{k_1} \times Z_{k_2} \times \cdots$ -1-symmetry which is determined by the even integer matrix K , where k_I are the diagonal elements of the Smith normal form of K . The lattice model (83) is exactly soluble on closed spacetime in $g \rightarrow \infty$ limit, and realizes a higher SPT phase (see Section VID).
7. The Abelian higher symmetry and the non-Abelian 0-symmetry can have a non-trivial mix described by a higher group.^{16,25} In this paper, we discuss a general way to construct lattice models with a combined 0-symmetry, 1-symmetry, etc (see Section VIII).

8. We systematically construct lattice bosonic models that realize higher SPT phases with higher symmetry described by higher group $\mathcal{B}(G, \Pi_2, \cdots)$ in any dimension.¹⁸ Our construction suggests a (many-to-one) classification of $(d+1)$ D bosonic higher SPT phases in terms of the cohomology of an extended higher group: $H^{d+1}[\mathcal{B}(G \rtimes SO_\infty, \Pi_2, \cdots), \mathbb{R}/\mathbb{Z}]$ (see Section IX C). Using fermion worldline decoration, we also systematically construct lattice fermionic models that realize higher SPT phases with higher symmetry $\mathcal{B}(G_f, \Pi_2, \cdots)$ in any dimension (see Section IX D).
9. The condensed matter theories that include the dynamical electromagnetic (EM) field (which will be called the EM condensed matter theories) can be viewed as local bosonic theories with an *anomalous* $U(1)$ -1-symmetry, since the magnetic monopoles can be ignored at low energies. Such an anomaly comes from the property that all fermions carry odd electric charges and all bosons carry even electric charges. Using the anomalous $U(1)$ -1-symmetry, we show that any gapped liquid phase^{38,39} of any EM condensed matter system must have a non-trivial bosonic topological order (see Section XI).

Exactly soluble models have been constructed to realize various topological orders¹⁻⁵. The toy model Hamiltonian commute with closed string and/or membrane operators, and thus has higher symmetries. However, the real condensed matter systems (after ignoring the dynamical electromagnetic field) do not have higher symmetries. In this paper we pointed out the topologically ordered phases and the states near the topologically ordered phases has emergent higher symmetries bellow the energy gap of *some* topological excitations. Thus if those topological excitations remain to be gapped, we can use the emergent higher symmetries to study the low energy dynamics and phase transition of the topologically ordered phases.

II. NOTATIONS AND CONVENTIONS

In part of this paper, we will use extensively the notion of cochain, cocycle, and coboundary, as well as their higher cup product \smile_k and Steenrod square Sq^k . A brief introduction can be found in Appendix A. We will abbreviate the cup product $a \smile_k b$ as ab by dropping \smile . We will use $\stackrel{n}{=}$ to mean equal up to a multiple of n , and use $\stackrel{d}{=}$ to mean equal up to $d\text{f}$ (*i.e.* up to a coboundary). We will use $\langle l, m \rangle$ to denote the greatest common divisor of l and m ($\langle 0, m \rangle \equiv m$). We will also use $[x]$ to denote the integer that is closest to x . (If two integers have the same distance to x , we will choose the smaller one, *eg* $[\frac{1}{2}] = 0$.)

In this paper, we will deal with many \mathbb{Z}_n -value quantities. We will denote them as, for example, $a^{\mathbb{Z}_n}$. However,

we will always lift the \mathbb{Z}_n -value to \mathbb{Z} -value, so the value of $a^{\mathbb{Z}_n}$ has a range from $-\lfloor \frac{N}{2} \rfloor$ to $\lfloor \frac{N}{2} \rfloor$. In this case, even the expression like $a^{\mathbb{Z}_n} + a^{\mathbb{Z}_m}$ makes sense.

We introduced a symbol \curlywedge to construct fiber bundle X from the fiber F and the base space B :

$$pt \rightarrow F \rightarrow X = F \curlywedge B \rightarrow B \rightarrow pt. \quad (2)$$

We will also use \curlywedge to construct group extension of H by N ⁴⁰:

$$1 \rightarrow N \rightarrow N \curlywedge_{e_2, \alpha} H \rightarrow H \rightarrow 1. \quad (3)$$

Here $e_2 \in H^2[H; Z(N)]$, $Z(N)$ is the center of N , and $\alpha : H \rightarrow \text{Aut}(N)$ is a non-trivial action of H on $Z(N)$. Thus e_2 and α characterize different group extensions.

We will use $K(\Pi_1, \Pi_2, \dots, \Pi_n)$ to denote a connected topological space with homotopy group $\pi_i(K(\Pi_1, \Pi_2, \dots, \Pi_n)) = \Pi_i$ for $1 \leq i \leq n$, and $\pi_i(K(\Pi_1, \Pi_2, \dots, \Pi_n)) = 0$ for $i > n$. If only one of the homotopy groups, say Π_d , is non-trivial, then $K(\Pi_1, \Pi_2, \dots, \Pi_n)$ is the Eilenberg-MacLane space, which is denoted as $K(\Pi_d, d)$. If only two of the homotopy groups, say $\Pi_d, \Pi_{d'}$, is non-trivial, then we denote the space as $K(\Pi_d, d; \Pi_{d'}, d')$, etc. We will use $\mathcal{B}(\Pi_1; \Pi_2; \dots; \Pi_n)$, $\mathcal{B}(\Pi_d, d)$, and $\mathcal{B}(\Pi_d, d; \Pi_{d'}, d')$ to denote the simplicial sets with only one vertex satisfying Kan conditions that describe a special triangulation of $K(\Pi_1, \Pi_2, \dots, \Pi_n)$, $K(\Pi_d, d)$, and $K(\Pi_d, d; \Pi_{d'}, d')$ respectively. Since simplicial sets satisfying Kan conditions are viewed as higher groupoids in higher category theory, the simplicial sets $\mathcal{B}(\Pi_1; \Pi_2; \dots; \Pi_n)$, $\mathcal{B}(\Pi_d, d)$, and $\mathcal{B}(\Pi_d, d; \Pi_{d'}, d')$, with only one vertex (unit), can be viewed as higher groups. In this paper, higher groups are treated therefore as this sort of special simplicial sets, as in Ref. 30.

III. SYMMETRY AND HIGHER SYMMETRY ON LATTICE

The notion of phase and phase transition play a major role in condensed matter physics in our attempts to understand properties of various materials. However, to mathematically define the concepts of quantum phase and quantum phase transition at zero temperature, we need to first introduce the notion Hamiltonian class. The notions of phase and phase transition can only be defined relative to a Hamiltonian class. (For example, the notion of phase is not a property of a single Hamiltonian.)

For example, we can define a Hamiltonian class as a set of local Hamiltonians for bosons on lattice. Relative to such a Hamiltonian class, we can define the notion bosonic topological orders.^{6,7} Such a precise definition of bosonic topological orders allows us to classify them in 1 spatial dimension^{41,42} where there is no non-trivial bosonic topological orders, as well as in 2 spatial dimensions^{8,43–46} and in 3 spatial dimensions^{47,48}, where very rich bosonic topological orders exist.

We may define another Hamiltonian class as a set of local Hamiltonians for fermions on lattice. Relative to such a Hamiltonian class, we can define the notion fermionic topological orders.⁴⁹ There is only one non-trivial fermionic topological order in 1 spatial dimension – the 1+1D topological p -wave superconductor.^{50,51} The fermionic topological orders in 2 spatial dimensions are classified in Ref. 46, and a proposal to classify fermionic topological orders in 3 spatial dimensions is given in Ref. 48.

We can also introduce a Hamiltonian class as a set of local Hamiltonians H on lattice with an on-site symmetry^{52,53} described by a group G :

$$HW_g = W_g H, \quad W_g = \prod_i W_g(i), \quad g \in G, \quad (4)$$

where $W_g(i)$ is a representation of the symmetry group G acting on the local Hilbert space on site- i . Relative to such a Hamiltonian class, we can define the notion of spontaneous symmetry breaking orders, symmetry protected topological (SPT) orders^{54,55} (also known as symmetry protected trivial orders^{56,57}), and symmetry enriched topological (SET) orders.^{58–61} The classification of spontaneous symmetry breaking orders is given by a pair of groups $G_{\text{grnd}} \subset G$, where G_{grnd} is the symmetry group of the ground state (the unbroken symmetry group). The classification of SPT and SET orders are given in Ref. 47, 48, 52, 60, 62–67, via group cohomology theory, cobordism theory, and (higher) category theory.

The on-site symmetry G is also called global symmetry. However, a global symmetry in field theory or in lattice theory may not be an on-site symmetry G . In this case, we say^{53,68} the global symmetry to have a 't Hooft anomaly.⁶⁹ We will also call the on-site symmetry G as an on-site 0-symmetry, where the symmetry transformation acts on codimension-0 space or spacetime.^{11,24}

Now, we are ready to define higher symmetry on lattice, by introducing a Hamiltonian class as a set of local Hamiltonians H on a triangulation of d -dimensional space (which is a more precise definition of lattice) that satisfy

$$HW_a(C^{d-k}) = W_a(C^{d-k})H, \quad (5)$$

for any $d-k$ -dimensional (*i.e.* codimension- k) closed subcomplex C^{d-k} of the triangulated d -dimensional space. Here $W_a(C^{d-k})$ is an operator that acts on the degrees of freedom on the closed subcomplex C^{d-k} with the following “on-site” property

$$W_a = \prod_{i \in C^{d-k}} W_a(i). \quad (6)$$

The operator $W_a(C^{d-k})$ labeled by a satisfy a so called pointed fusion rule

$$W_a(C^{d-k})W_b(C^{d-k}) = W_c(C^{d-k}), \quad (7)$$

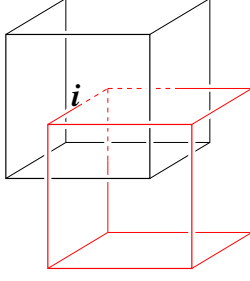


FIG. 1. (Color online) A cubic lattice (black) and its dual cubic lattice (red). The index i labels the faces of the cubic lattice and the links of the dual cubic lattice.

and they commute with each other

$$[W_a(C^{d-k}), W_b(\tilde{C}^{d-k})] = 0. \quad (8)$$

Such a pointed fusion rule makes the index set $\{a\}$ into an Abelian group Π . The Hamiltonian class defined above is referred as having an on-site k -symmetry since the symmetry acts on codimension- k subcomplex of the space.^{11,24}

IV. AN EXAMPLE OF ANOMALY-FREE (ON-SITE) Z_2 -1-SYMMETRY

A. The 3+1D bosonic lattice model

The simplest on-site 1-symmetry is a Z_2 -1-symmetry. Let us consider a qubit model on a 3-dimensional cubic lattice (see Fig. 1), where the qubits live on the square faces of the cubic lattice. We choose the closed subcomplex C^2 to the closed 2-dimensional surfaces formed by the square faces of the cubic lattice. The generator of the Z_2 -1-symmetry is given by

$$W(C^2) = \prod_{i \in C^2} \sigma_i^z \quad (9)$$

where i labels the square faces of the cubic lattice. We can check that $W(C^2)$'s all commute with each other and their pointed fusion is described by a Z_2 group. So we say $W(C^2)$ generates an on-site Z_2 1-symmetry.

A Hamiltonian with the above Z_2 1-symmetry is given by¹⁻³

$$H = -U_1 \sum_{\langle ijkl \rangle} \sigma_i^x \sigma_j^x \sigma_k^x \sigma_l^x - U_2 \sum_{\langle ijklmn \rangle} \sigma_i^z \sigma_j^z \sigma_k^z \sigma_l^z \sigma_m^z \sigma_n^z - B \sum_{\langle i \rangle} \sigma_i^z, \quad (10)$$

where $\sum_{\langle i \rangle}$ sums over all the square faces, $\sum_{\langle ijklmn \rangle}$ sums over all cubes, and $\sum_{\langle ijkl \rangle}$ sums over all squares formed by the links in the dual cubic lattice. (We note

that \mathbf{I} 's also label the links of the dual cubic lattice. See Fig. 1)

When $|B| \ll |U_1|, |U_2|$, the Hamiltonian (10) has a topologically ordered ground state. The topological order is described by a Z_2 gauge theory. A pair of Z_2 -charge e is created by an open string operator

$$S_{\text{str}}(\tilde{C}^1) = \prod_{i \in \text{string } \tilde{C}^1} \sigma_i^x, \quad (11)$$

where the string is formed by the links of the dual cubic lattice. Note that the open string creation operators break the Z_2 -1-symmetry. Thus we cannot even include the short open string operators in the Hamiltonian. This implies that in the presence of the Z_2 -1-symmetry, the Z_2 -charge is not mobile. A Z_2 -flux loop s is created by an open membrane operator bounded by the loop:

$$M_{\text{memb}}(C^2) = \prod_{i \in \text{membrane } C^2} \sigma_i^z, \quad (12)$$

where the membrane is formed by the square faces of the original cubic lattice. The Z_2 -1-symmetry allow the Hamiltonian to have such a open membrane operator. Thus the Z_2 -flux loop s is mobile even in the presence of the Z_2 -1-symmetry.

We also note that the closed membrane operator happen to be the generator (9) of the Z_2 -1-symmetry. Thus we say that the Z_2 -1-symmetry is generated by the topological excitations of the Z_2 -flux loops.

B. Z_2 -1-symmetry and unbreakable strings

In this subsection, we discuss a meaning of 1-symmetry in our lattice model eqn. (10). Let $|\uparrow\rangle, |\downarrow\rangle$ be the eigenstates of σ^z . We view $\otimes_i |\uparrow\rangle_i$ as a reference state. We create a closed-string state by changing $|\uparrow\rangle_i$ to $|\downarrow\rangle_i$ for i on closed strings. Here strings are formed by links of the dual cubic lattice.

Because of the Z_2 -1-symmetry, the ground state of eqn. (10) is a superposition of closed strings (assuming $U_2, B > 0$). When $B = 0$, the ground state is an equal weight superposition of all closed strings, and spontaneously break the Z_2 -1-symmetry (on space with non-trivial first homotopy group π_1). When $B \rightarrow +\infty$, the ground state has no strings and does not break the Z_2 -1-symmetry.

From this example, we see that the physical meaning of the Z_2 -1-symmetry is the appearance of unbreakable strings. Even when we force a string breaking, the Z_2 -1-symmetry requires that the end of the string (*i.e.* the Z_2 -charge) cannot move and have no dynamics.

To summarize, the Z_2 -1-symmetry is generated by the Z_2 -flux-line excitations. Such a Z_2 -1-symmetry forbid the excitations that have non-trivial mutual statistics with the Z_2 -flux lines, such as the Z_2 -charge excitations. This can be achieved by including the Z_2 -1-symmetry generator (9) in the Hamiltonian with a large coefficient

(the U_2 term in eqn. (10)), which gives the Z_2 -charge a large energy gap.

C. Emergence of generic higher symmetry in topological orders

The above physical understanding of Z_2 -1-symmetry can also be generalized:

If a topological order contains a topological excitation η of unit quantum dimension, then the topological order can have an emergent higher symmetry generated by η below an energy gap Λ , if all the topological excitations with non-trivial mutual statistics respect to any combination of η have a large gap beyond Λ .

We note that the emergent higher symmetry allows the topological excitations with trivial mutual statistics respect to η to have a small gap, or even become gapless and drive a phase transition via their condensation. The emergent higher symmetry will be present through the phase transition.

To understand the above result in more details, let us consider a dimension- n topological excitation in a topological order. We assume that the topological excitation can be created by a dimension- $n+1$ operator $W_1(C^{n+1})$ at its boundary ∂C^{n+1} . Here $W_1(C^{n+1})$ is an operator acts on the $(n+1)$ -dimensional subcomplex C^{n+1} in the space (not spacetime). If the quantum dimension of the topological excitation is 1, then $W_1(C^{n+1})$ on a closed subcomplex C^{n+1} generates a Z_N fusion

$$W_a(C^{n+1})W_b(C^{n+1}) = W_{a+b}(C^{n+1}), \quad W_N(C^{n+1}) = 1, \quad (13)$$

for a certain integer N . Now, we require the lattice Hamiltonians to commute with $W_1(C^{n+1})$ for all closed C^{n+1} . This way we obtain a Hamiltonian system that has a $Z_n D - n - 2$ -symmetry where D is the spacetime dimension. If a Hamiltonian with such a higher symmetry realizes the above topological order, then all the topological excitations with non-trivial mutual statistics with the n -dimensional topological excitation are not mobile. We can even make those topological excitations to have a large gap by adding the $W_a(C^{n+1})$ terms to the Hamiltonian with a large coefficient. Only topological excitations having trivial mutual statistics with the n -dimensional topological excitation are allowed to appear at low energies and to have non-trivial dynamics.

We stress that, in our construction, the higher symmetry is a property of the pair: the topological order plus the allowed low energy topological excitations: the higher symmetry is generated from the topological excitations of unit quantum dimension that have trivial mutual statistics with the allowed low energy topological excitations. Even in the same topological order, allowing different types of topological excitations to appear at low energy will lead to different (anomalous) higher

symmetries. Later, we will present several examples of this phenomenon.

For the 3+1D Z_2 topological order discussed above, if we only allow the Z_2 -flux line excitations at low energies (*i.e.* the Z_2 -charges associated with the string ends all have very high energies), then we will have the Z_2 1-symmetry at low energies, generated by the Z_2 -flux lines. Later we will show that if we only have the Z_2 -charge excitations at low energies, we will have a Z_2 -2-symmetry at low energies (see eqn. (62)), generated by the Z_2 -charge excitations. If we only allow the trivial excitations at low energies, then we will have the Z_2 -1-symmetry generated by the Z_2 -flux lines, and the Z_2 -2-symmetry, generated by the Z_2 -charge excitations. If we allow both Z_2 -flux and Z_2 -charge excitations at low energies, we will explicitly break the Z_2 -1-symmetry and the Z_2 -2-symmetry at low energies. In fact, there are no topological excitations that have trivial mutual statistics with both Z_2 -flux and Z_2 -charge excitations, and there is no higher symmetry.

We like to remark that the above Z_2 -1-symmetry and Z_2 -2-symmetry have mixed anomaly between them (see Section ??). A system with both of those higher symmetries cannot realize a trivial topological order.

D. Generalized higher symmetry

In the above, we have constructed a higher symmetry from one topological excitation of unit quantum dimension. Certainly, we can construct more general higher symmetry from several topological excitations of unit quantum dimension. We can even construct something from a topological excitations with higher quantum dimensions. We call the “something” generalized higher symmetry (see Ref. 70):

A topological order can have an emergent generalized higher symmetry generated by a topological excitation η , if all the topological excitations with non-trivial mutual statistics respect to any combination of η have a large gap.

Since a topological order, by definition, always has a finite energy gap, at energies much below the energy gap, the topological order always has an emergent generalized higher symmetry, and such a generalized higher symmetry is spontaneously broken. If the topological order contains excitations with unit quantum dimension, then, part of the *generalized higher symmetry* can be viewed as the *higher symmetry*. If certain topological excitations have a large gap and other topological excitations have a small gap, then below the large gap, the topological order may have an emergent generalized higher symmetry, which may be smaller than the emergent generalized higher symmetry below the small gap.

E. Spontaneous higher symmetry breaking and topological order

When $|B| \gg |U_1|, |U_2|$, the ground state of the Hamiltonian (10) is a product state without topological order. When $|B| \ll |U_1|, |U_2|$, the ground state has a non-trivial topological order. The small $|B|$ topologically ordered phase and the large $|B|$ trivial phase can also be distinguished by *spontaneous 1-symmetry breaking*. The Z_2 -1-symmetry is generated by $W(C^2)$ in eqn. (9).

On space $S^1 \times S^1 \times S^1$, the large $|B|$ trivial phase has an unique ground state, which is invariant under all the Z_2 -1-symmetry transformations. The Z_2 -1-symmetry is not broken. While the topologically ordered phase for small $|B|$ has 8 ground states on space $S^1 \times S^1 \times S^1$. Some the Z_2 -1-symmetry transformations act non-trivially in the 8-dimensional ground state subspace, *i.e.* are not proportional to an identity operator. Thus, the Z_2 -1-symmetry is spontaneously broken.

The spontaneous 1-symmetry broken state is nothing but a topologically ordered state. Since the 1-symmetry is not spontaneously broken in the large $|B|$ trivial product state, the transition from the trivial product state to the topologically ordered state can be viewed as a spontaneous breaking of the 1-symmetry. This result is general:

A spontaneous higher symmetry broken state always corresponds to a topologically ordered state.

Here we have assumed that the higher symmetry is finite. The spontaneous breaking of continuous higher symmetry is discussed in Ref. 11 and 24, and give rise to gapless states. However, even though the spontaneous breaking of continuous higher symmetry produce gapless excitations, the gaplessness of the excitations do not need higher symmetry. Even after we explicitly break the higher symmetry, the gapless excitations remain gapless (see Section VII).¹² This is very different from the gapless excitations from the spontaneous breaking of continuous 0-symmetry.

We also like to point out that a topologically ordered state can be more general and may not correspond to a spontaneous higher symmetry broken state. For example, we can break the higher symmetry explicitly. Even without higher symmetry, we can still have topologically order. Even though, some topological orders, such as Z_2 -gauge theory, can be viewed as spontaneous higher symmetry broken states. Some other topological orders, such as S_3 -gauge theory, cannot be viewed as spontaneous higher symmetry broken states, since spontaneous 1-symmetry broken states only give rise to Abelian gauge theory.

F. The usefulness of higher symmetry in condensed matter

We see that some topological orders can be understood as spontaneous higher symmetry breaking in the systems

with higher symmetry. But the usual condensed matter theories on lattice never have higher symmetry. (Here by “usual condensed matter theories” we mean the theories that do not include the dynamical electromagnetic fields.) So it appears that this way to understand topological order may not be very useful. Also, this way to understand topological order misses a key feature of topological order: topological order is robust against any local perturbation that can break all the symmetries and higher symmetries.

However, the notion of higher symmetry and their spontaneous breaking can still be useful in condensed matter in the following sense: A gapped liquid state^{38,39} may have many emergent symmetries and higher symmetries at low energies. If some of the emergent higher symmetries are spontaneously broken, then the corresponding gapped liquid state has topological orders. This allows us to use spontaneously broken emergent higher symmetry to characterize a subclass of topological orders.

The higher symmetries in low energy effective field theory may come from lattice model with exact higher symmetry, or may emerge from a lattice model that has no higher symmetry. Therefore, even though the usual condensed matter theories on lattice do not have higher symmetry, low energy effective theories with higher symmetries can still be used to describe condensed matter systems, since higher symmetry can be emergent.

Later, we will see that if we include the dynamical electromagnetic (EM) fields in condensed matter theories, the resulting EM condensed matter theories will have an approximate $U(1)$ -1-symmetry if we ignore the magnetic monopoles. (The appearance of magnetic monopoles in condensed matter energy scale will break the $U(1)$ higher symmetry.) In this sense, $U(1)$ -1-symmetry is useful for real condensed matter systems.

The notion of higher symmetry can also be useful for condensed matter in another way. We can construct toy models with higher symmetries, and make the ground states spontaneous break the higher symmetry. This way, we construct toy models that realize some topological orders. There are already many different ways to construct exactly soluble models to systematically realize topological orders, SPT orders, and SET orders.^{1,30,49,52,67,71–74} But it does not hurt to have one more construction. In this paper, we will construct some simple toy models with higher symmetry that realize some topological orders, SET orders, and SPT orders.

Last, higher symmetry can also be used to constrain possible phase transitions and possible phases.³⁷ In certain topological orders, if we only allow a certain type of topological excitations at low energies, the topological orders plus the topological excitations may have an emergent (anomalous) higher symmetry. Then any phase transition *induced by this particular type of topological excitations* must preserve the anomaly of the higher symmetry. This puts constraint on possible phase transitions and possible resulting phases.

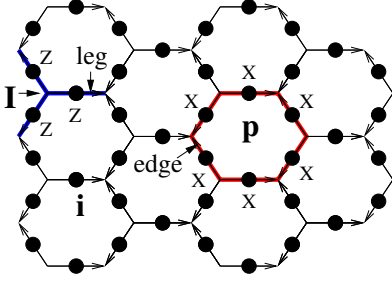


FIG. 2. (Color online) A qubit model, where qubits live on the links of a honeycomb lattice. I labels the vertices and i labels the links (the qubits) of the honeycomb lattice.

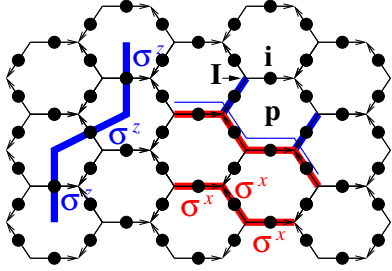


FIG. 3. (Color online) The string operators.

V. AN EXAMPLE OF ANOMALOUS Z_2 -1-SYMMETRY

A. The 2+1D bosonic lattice model

In this section, we are going to discuss another simple lattice that has several Z_2 -1-symmetries and one of them is an anomalous Z_2 -1-symmetry. In this model the qubits live on the links of a honeycomb lattice (see Fig. V A), with a Hamiltonian:¹

$$\begin{aligned} H &= -U \sum_I Q_I - g \sum_p F_p, \\ Q_I &= \prod_{\text{legs of } I} \sigma_i^z, \\ F_p &= \prod_{\text{edges of } p} \sigma_i^x, \end{aligned} \quad (14)$$

where \sum_I sum over the vertices and where \sum_p sum over the hexagons of the honeycomb lattice (see Fig. V A). Notice that H is a sum of commuting operators $[F_p, F_{p'}] = 0$, $[Q_I, Q_{I'}] = 0$, $[F_p, Q_I] = 0$, and $F_p^2 = Q_I^2 = 1$. Thus the ground state $|\Psi_{\text{grnd}}\rangle$ is given by $F_p |\Psi_{\text{grnd}}\rangle = Q_I |\Psi_{\text{grnd}}\rangle = |\Psi_{\text{grnd}}\rangle$.

There are two types of topological excitations above the ground state with $Q_I = F_p = 1$: e -type with $Q_I = -1$ and m -type with $F_p = -1$. Those excitations cannot be created individually. They can only be created in pairs by string operators.

We have type- e string operator $W_e(C_1) = \prod_{i \in C_1} \sigma_i^x$ where the string C_1 is formed by the links of the honeycomb lattice (see Fig. 3). An open e -string operator creates two e -type topological excitations at its ends. We also have type- m string operator $W_m(\tilde{C}_1) = \prod_{i \in \tilde{C}_1} \sigma_i^z$ where the string \tilde{C}_1 is formed by the links of dual of the honeycomb lattice (see Fig. 3). An open m -string operator creates two m -type topological excitations at its ends.

We can also fuse the e -string and m -string operators together to form a type- f string operator $W_f(C_1 \prod \tilde{C}_1) = \prod_{i \in C_1} \sigma_i^x \prod_{i \in \tilde{C}_1} \sigma_i^z$ where the string C_1 in the lattice and the string \tilde{C}_1 in the dual lattice closely follow each other (see Fig. 3). An open f -string operator creates two f -type topological excitations at its ends. It turns out that e and m are bosons, and f is a fermion. They all have a π mutual statistics respect to each other.

We find that H in eqn. (14) commutes with the above three types of string operators if the strings are closed:

$$[H, W_e^{\text{closed}}] = [H, W_m^{\text{closed}}] = [H, W_f^{\text{closed}}] = 0. \quad (15)$$

Therefore, our lattice model has two Z_2 -1-symmetries since

$$W_e^2 = W_m^2 = W_f^2 = 1, \quad W_e W_m = W_f. \quad (16)$$

On a torus, the model (14) has four degenerate ground states, and the closed string operators W_e^{closed} , W_m^{closed} , and W_f^{closed} act non-trivially in the ground state subspace when the closed strings are not contractible. Thus the ground state on eqn. (14) spontaneously breaks the two Z_2 -1-symmetries, and has a Z_2 topological order.

Now we consider the following model with $U \gg g, J$

$$\begin{aligned} H &= -U \sum_I Q_I - g \sum_p F_p - J \sum_i \sigma_i^z, \\ Q_I &= \prod_{i \in I} \sigma_i^z, \quad F_p = \prod_{i \in p} \sigma_i^x \end{aligned} \quad (17)$$

As we go from $g \gg J$ to $g \ll J$ the ground state undergoes a phase transition that changes the Z_2 topological order to trivial product state, driven by m particle condensation. This is because that the J -term is the hopping for m and can drive the m excitations to have a negative energy.

The above Hamiltonian and the transition has the Z_2^m -1-symmetry generated by the closed m -strings $W_m(\tilde{C}_1) = \prod_{i \in \tilde{C}_1} \sigma_i^z$, but does not have the Z_2^e -1-symmetry generated by the closed e -strings $W_e(C_1) = \prod_{i \in C_1} \sigma_i^x$ and the Z_2^f -1-symmetry generated by the closed f -strings $W_f(C_1 \prod \tilde{C}_1)$.

The end of W_m (the 1-symmetry generator) is the m particle. The low energy allowed excitations of the above Hamiltonian are the particles with trivial mutual statistics with the m particle. Thus the low energy allowed excitations include the m particles, but not include the e particles and the f particles (the fermions).

To summarize, the Z_2 topological order in 2+1D has three type of topological excitations:

1. the Z_2 -charge e – boson
2. the Z_2 -flux m – boson
3. the charge-flux bound state $f = m \otimes e$ – fermion

The three particles have mutual π statistics respect to each other. Below the minimal gap of the three particles $\Delta_e, \Delta_m, \Delta_f$, we have three Z_2 -1-symmetries generated by closed string operators $W_e(C_1)$, $W_m(\tilde{C}_1)$, and $W_f(C_1 \otimes \tilde{C}_1)$.

If $\Delta_m \ll \Delta_e, \Delta_f$, then below Δ_e, Δ_f (but may be above Δ_m), we have a Z_2 -1-symmetry generated by closed string operators $W_m(\tilde{C}_1)$, but not the ones from $W_e(C_1)$ and $W_f(C_1 \otimes \tilde{C}_1)$. The low energy allowed particles are $\mathcal{C}_a = \{m\}$. The 1-symmetry is generated by string operators for the particles $\mathcal{C}_t = \{m\}$. If we reduce Δ_m to make it negative, we will induce a Bose-condensation of the Z_2 -flux and a Z_2^m -1-symmetric confinement transition: the Z_2 topological order is changed the trivial product state.

If $\Delta_f \ll \Delta_e, \Delta_m$, then below Δ_e, Δ_m (but may be above Δ_f), we have a Z_2 -1-symmetry generated by closed string operators $W_f(C_1 \otimes \tilde{C}_1)$, but not the ones from $W_e(C_1)$ and $W_m(\tilde{C}_1)$. The low energy allowed particles is $\mathcal{C}_a = \{f\}$, and the 1-symmetry is generated by string operators for the particles $\mathcal{C}_t = \{f\}$.

If we reduce Δ_f to make it negative, can we still induce confinement transition to change the Z_2 topological order to a trivial product state with no topological order? Since f is a fermion, it cannot Bose-condense. But it can condense into some other topologically ordered state. Can the new topological order cancel the parent Z_2 topological order to produce a trivial phase without topological order?

The condensation of f is a Z_2^f -1-symmetric phase transition. We will show later that the Z_2^f -1-symmetry is anomalous, and the Z_2^f -1-symmetric phase transition cannot induce a trivial product state.

B. On-site/non-on-site higher symmetry

To understand the anomalous higher symmetry, let us first review the connection between non-on-site symmetry and anomalous symmetry.⁵³ An on-site symmetry (on-site 0-symmetry) of group G is generated by a transformation of the following form:

$$U(g) = \prod_i U_i(g), \quad U(g)U(h) = \prod_i U_i(gh) = U(gh), \quad (18)$$

where $g, h \in G$, i label the lattice site and $U_i(g)$ only acts on the degrees of freedom on site- i . The on-site symmetry

can be gauged to get a local $G^{\otimes N_{\text{site}}}$ symmetry

$$U(\{g_i\}) = \prod_i U_i(g_i), \\ U(\{g_i\})U(\{h_i\}) = \prod_i U_i(g_i h_i) = U(\{g_i h_i\}). \quad (19)$$

An on-site symmetry is also called anomaly-free symmetry.

Roughly speaking, an non-on-site symmetry of group G does have the product form

$$U(g) \neq [\prod_i U_i(g)], \quad U(g)U(h) = U(gh). \quad (20)$$

It cannot be gauged to get a $G^{\otimes N_{\text{site}}}$ symmetry:

$$U(\{g_i\}) = \prod_i U_{i,i+1}(g_i), \\ U(\{g_i\})U(\{h_i\}) \neq U(\{g_i h_i\}). \quad (21)$$

An non-on-site symmetry is also called anomalous symmetry. For a more accurate discussion of non-on-site and anomalous symmetry, see Ref. 53.

Similarly, an on-site k -symmetry of an Abelian group Π in d -dimensional space is given by

$$U(C_{d-k}, g) = \prod_{i \in C_{d-k}} U_i(g), \\ U(g)U(h) = \prod_{i \in C_{d-k}} U_i(gh) = U(gh), \quad g \in \Pi. \quad (22)$$

Here we stress that we have assumed that the space is a complex (a lattice) and there are independent degrees of freedom living on the $(d-k)$ -cells of the complex. The operator $U_i(g)$ only acts on the degrees of freedom on the $(d-k)$ -cell labeled by i . C_{d-k} is a collection of $(d-k)$ -cells and $\prod_{i \in C_{d-k}}$ is a product over all the $(d-k)$ -cells in C_{d-k} .

The on-site $(d-1)$ -symmetry can be gauged

$$U(C_{d-k}, \{g_i\}) = \prod_{i \in C_{d-k}} U_i(g_i), \quad (23) \\ U(C_{d-k}, \{g_i\})U(C_{d-k}, \{h_i\}) = U(C_{d-k}, \{g_i h_i\}).$$

An on-site higher symmetry is also called anomaly-free higher symmetry.

Non-on-site $(d-1)$ -symmetry for a group Π

$$U(C_{d-k}, g) \neq \prod_{i \in C_{d-k}} U_i(g), \\ U(C_{d-k}, g)U(C_{d-k}, h) = U(C_{d-k}, gh), \quad g \in \Pi. \quad (24)$$

The non-on-site higher symmetry cannot be gauged

$$U(C_{d-k}, \{g_i\})U(C_{d-k}, \{h_i\}) \neq U(C_{d-k}, \{g_i h_i\}). \quad (25)$$

A non-on-site higher symmetry is called anomalous higher symmetry, if we cannot make it on-site via some local unitary operations.⁵⁸ More precisely

A higher symmetry (which may be generated by operators in several different dimensions) is anomaly-free if it allows a symmetric ground state with trivial topological order. A higher symmetry is anomalous if it does not allow a symmetric ground state with trivial topological order.

For example, the Z_2 -1-symmetry generated by $W_e(C_1) = \prod_{i \in C_1} \sigma_i^x$ are on-site and anomaly-free. We note that the open string operator $W_e(C_1)$ creates two bosons at its ends.

Also, the Z_2 -1-symmetry generated by $W_m(\tilde{C}_1) = \prod_{j \in \tilde{C}_1} \sigma_j^z$ are on-site and anomaly-free. Again the open string operator $W_m(\tilde{C}_1)$ creates two bosons at its ends.

The Z_2 -1-symmetry generated by $W_f(C_1 \otimes \tilde{C}_1) = \prod_{i \in C_1} \sigma_i^x \prod_{j \in \tilde{C}_1} \sigma_j^z$ is not on-site and maybe anomalous. But how to determine if a higher symmetry is anomaly-free or anomalous? Later, we will show that an 1-symmetry generated by a string operator $W(C_1)$ is anomaly-free if and only if the end of the string is a boson. So the Z_2 -1-symmetry generated by $W_f(C_1 \otimes \tilde{C}_1) = \prod_{i \in C_1} \sigma_i^x \prod_{j \in \tilde{C}_1} \sigma_j^z$ is anomalous, since the end of string $W_f(C_1 \otimes \tilde{C}_1)$ is a fermion. In fact, one can show that the open string operators $W_f(C_1 \otimes \tilde{C}_1)$ satisfy the so-called fermion-hopping algebra, which make the string end to be a fermion. The string operators satisfying fermion-hopping algebra cannot be made into on-site operators.

The $Z_2 \times Z_2$ -1-symmetry generated by $W_e(C_1) = \prod_{i \in C_1} \sigma_i^x$ and by $W_m(\tilde{C}_1) = \prod_{j \in \tilde{C}_1} \sigma_j^z$ is also anomalous. This is because it contains the Z_2 -1-symmetry generated by $W_f(C_1 \otimes \tilde{C}_1) = \prod_{i \in C_1} \sigma_i^x \prod_{j \in \tilde{C}_1} \sigma_j^z$ which is anomalous. We also note that the end of string $W_e(C_1)$ and the end of string $W_m(\tilde{C}_1)$ have a non-trivial mutual statistics between them, which implies a mixed anomaly between the two Z_2 -1-symmetries.

If we have a higher symmetry of generated by operators in several dimensions defined on the same spacial complex, $U(C_{d-k_1}, g_1), U(C_{d-k_2}, g_2), \dots$ and if all those operators are on-site, then the higher symmetry is anomaly-free. We note that since all the operators are defined on the same spacial complex the higher symmetry generators with different dimensions act on different degrees of freedoms living on cells of different dimensions. So the higher symmetry generators with different dimensions always commute with each other. If some higher symmetry generators are defined on a complex while other higher symmetry generators are defined on the dual complex, then the higher symmetry generators may not commute and may be anomalous.

More generally, the boundary of higher symmetry generators can produce pointlike, stringlike, \dots , topological excitations. We can use a higher category \mathcal{C}_t with one object to describe their fusion and braiding (see Table I). In fact the higher category \mathcal{C}_t characterizes the higher symmetry completely. We like to conjecture that

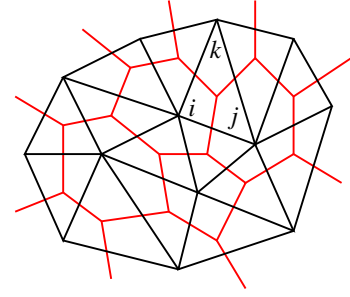


FIG. 4. (Color online) The black lines describe a 2-dimensional spacetime complex \mathcal{M}^2 . The red lines describe the dual complex $\tilde{\mathcal{M}}^2$.

The higher symmetry is anomaly-free if and only if all the morphisms in \mathcal{C}_t have a unit quantum dimension, and have no phase factor under exchange, braiding and fusion.

Here the statement “have no phase factor under exchange, braiding and fusion” need a more precise definition. For pointlike excitations in 2-dimensional space and higher, “no phase factor under exchange and braiding” means that the pointlike excitations are all bosons with trivial mutual statistics. If the fusion of some excitations is described by a pointed fusion category, the “no phase factor under fusion” means the F -symbol of the fusion category is equal to 1. For more details, see Ref. 70.

VI. SIMPLE LATTICE EXAMPLES THAT REALIZE HIGHER SYMMETRY PROTECTED TOPOLOGICAL PHASES

One way to show a higher symmetry in a system is anomalous is to show that the symmetric system can be regarded as a boundary of higher symmetry protected topological (hSPT) state in one-higher dimension, using the relation between anomaly and SPT state in one-higher dimension.⁵³ In this section, we will describe some examples of hSPT states. Using those examples, we will show that a higher symmetry generated by several types of closed string operator is anomaly-free only if the ends of string are bosons with trivial mutual statistics with each others.

To construct lattice models with 0-symmetries and higher symmetries, it is more convenient to do so in the spacetime Lagrangian formalism. We construct a spacetime lattice by first triangulating a D -dimensional spacetime manifold M^D . (In this paper, we will use D to denote spacetime dimensions and d to denote space dimensions.) So a spacetime lattice is a D -complex \mathcal{M}^D with vertices labeled by i , links labeled by ij , triangles labeled by ijk , etc(see Fig. 4). The D -complex \mathcal{M}^D also has a dual complex denoted as $\tilde{\mathcal{M}}^D$. The vertices of \mathcal{M}^D correspond to the D -cells in $\tilde{\mathcal{M}}^D$, The links of \mathcal{M}^D

TABLE I. The higher category theory is actually a theory of pointlike, stringlike, \dots , excitations in physics.⁷⁵ This table lists the corresponding concepts in mathematics and in physics.

Concepts in higher category	Concepts in physics
Unitary D -category \mathcal{M}	Topological excitations with their braiding fusion properties in a topologically ordered state in D -spacetime dimension
Objects (0-morphisms)	The ground states
Simple 1-morphisms	The codimension-1 topological excitations
Simple $D-2$ -morphisms	The stringlike topological excitations
Simple $D-1$ -morphisms	The pointlike topological excitations
Composite morphisms	The topological excitations with accidental degeneracy
The collection of simple $D-1$ -morphisms, simple $D-2$ -morphisms, etc	Topological excitations
Trivial morphisms	The excitations that can be created by local operators (non-topological excitations)

correspond to the $(D-1)$ -cells in $\tilde{\mathcal{M}}^D$, etc

Our spacetime lattice model may have a field living on the vertices, g_i . Such a field is called a 0-cochain. The model may also have a field living on the links, a_{ij} . Such a field is called a 1-cochain, etc. To construct spacetime lattice models, in particular, the topological spacetime lattice models,^{16,18,76,77} we will use extensively the mathematical formalism of cochains, coboundaries, and cocycles (see Appendix A). The relation between the spacetime path integral approach and the Hamiltonian approach is discussed in Appendix B.

A. A 3+1D model to realize a pure Z_n -1-SPT phase

1. The bulk theory and the boundary theory

In this section, we will consider a 3+1D bosonic model on a spacetime complex \mathcal{M}^4 , with Z_n -valued dynamic field $a_{ij}^{Z_n}$ on the links ij of the complex \mathcal{M}^4 . We also have a Z_n -valued non-dynamical background field $\hat{B}_{ijk}^{Z_n}$ on the triangles ijk of the complex \mathcal{M}^4 . The path integral of our bosonic model is given by

$$Z = \sum_{\{a^{Z_n}\}} e^{2\pi i \int_{\mathcal{M}^4} \frac{k}{n} \text{Sq}^2(\hat{B}^{Z_n} + da^{Z_n})}, \quad (26)$$

$(k, n) = (\text{integer}, \text{integer}),$

where $\sum_{\{a^{Z_n}\}}$ sums over Z_n -valued 1-cochains a^{Z_n} , and \hat{B}^{Z_n} is a Z_n -valued 2-cocycle

$$d\hat{B}^{Z_n} \stackrel{n}{=} 0. \quad (27)$$

Also Sq^2 is the generalized Steenrod square defined by eqn. (A21). We will show that the above model realizes a Z_n -1-SPT phase.

Since \hat{B}^{Z_n} and a^{Z_n} are Z_n -valued, we require the action amplitude $e^{2\pi i \int_{\mathcal{M}^4} \frac{k}{n} (\hat{B}^{Z_n} + da^{Z_n})^2}$ to be invariant under the transformation

$$\hat{B}^{Z_n} \rightarrow \hat{B}^{Z_n} + nb^Z, \quad a^{Z_n} \rightarrow a^{Z_n} + nu^Z, \quad (28)$$

where b^Z and u^Z are any Z valued 2-cochain and 1-cochain. (To do the addition $a^{Z_n} + nu^Z$, we have lifted the Z_n -value of a^{Z_n} to Z .) From eqn. (A24), we see that

$$\begin{aligned} \text{Sq}^2(nb^Z + \hat{B}^{Z_n}) &= n^2 \text{Sq}^2(b^Z) + \text{Sq}^2(\hat{B}^{Z_n}) + 2nb^Z \hat{B}^{Z_n} \\ &+ nd\hat{B}^{Z_n} \smile_2 db^Z - nd(\hat{B}^{Z_n} \smile_1 b^Z) - nd(d\hat{B}^{Z_n} \smile_2 b^Z) \\ &\stackrel{n}{=} \text{Sq}^2(\hat{B}^{Z_n}). \end{aligned} \quad (29)$$

We see that the action amplitude $e^{2\pi i \int_{\mathcal{M}^4} \frac{k}{n} (\hat{B}^{Z_n} + da^{Z_n})^2}$ is indeed invariant under eqn. (28) even when \mathcal{M}^4 has a boundary. The above result implies that the model has a Z_n -1-symmetry generated by

$$a^{Z_n} \rightarrow a^{Z_n} + \alpha^{Z_n}, \quad d\alpha^{Z_n} \stackrel{n}{=} 0, \quad (30)$$

even when \mathcal{M}^4 has a boundary.

In eqn. (26), \hat{B}^{Z_n} is the Z_n background 2-connection to describe the twist of the Z_n -1-symmetry. The model has a Z_n gauge symmetry:

$$a^{Z_n} \rightarrow a^{Z_n} + \hat{a}^{Z_n}, \quad \hat{B}^{Z_n} \rightarrow \hat{B}^{Z_n} - d\hat{a}^{Z_n}. \quad (31)$$

Using eqn. (A26) we find that

$$\begin{aligned} \text{Sq}^2(\hat{B}^{Z_n} + da^{Z_n}) &= \text{Sq}^2\hat{B}^{Z_n} + 2\hat{B}^{Z_n} d\hat{a}^{Z_n} + d[\text{Sq}^2\hat{a}^{Z_n} - d\hat{a}^{Z_n} \smile_1 \hat{B}^{Z_n}] \\ &\stackrel{n}{=} \text{Sq}^2\hat{B}^{Z_n} + d[\hat{a}^{Z_n} d\hat{a}^{Z_n} - d\hat{a}^{Z_n} \smile_1 \hat{B}^{Z_n} + \hat{B}^{Z_n} \hat{a}^{Z_n}] \end{aligned} \quad (32)$$

Therefore

$$e^{2\pi i \int_{\mathcal{M}^4} \frac{k}{n} \text{Sq}^2(\hat{B}^{Z_n} + da^{Z_n})} = e^{2\pi i \int_{\mathcal{M}^4} \frac{k}{n} \text{Sq}^2(\hat{B}^{Z_n})} \quad (33)$$

for closed spacetime \mathcal{M}^4 . The model is exactly soluble and gapped for closed spacetime \mathcal{M}^4 .

Eqn. (26) has no topological order since on closed spacetime and for $\hat{B}^{Z_n} = 0$

$$Z = \sum_{\{a^{Z_n}\}} e^{2\pi i \int_{\mathcal{M}^4} \frac{k}{n} (da^{Z_n})^2} = \sum_{\{a^{Z_n}\}} 1 = n^{N_l}, \quad (34)$$

where N_l is the number of links in the spacetime complex \mathcal{M}^4 . n^{N_l} is the so called the volume term that is linear in

the spacetime volume. The topological partition function Z^{top} is given via

$$Z = e^{-\epsilon V} Z^{\text{top}} \quad (35)$$

where V is the spacetime volume. (For a detailed discussion of the non-universal volume term and the universal topological terms, see Ref. 75 and 78.) After removing the volume term, the topological partition function of the above model is $Z^{\text{top}}(\mathcal{M}^4) = 1$ for all closed 4-complex \mathcal{M}^4 . Thus the above model has no topological order. After we turn on the flat Z_n 2-connection \hat{B}^{Z_n} , the topological partition function of the model (26) is

$$Z^{\text{top}}(\mathcal{M}^4, \hat{B}^{Z_n}) = e^{2\pi i \int_{\mathcal{M}^4} \frac{k}{n} (\hat{B}^{Z_n})^2}, \quad d\hat{B}^{Z_n} \stackrel{n}{=} 0. \quad (36)$$

The above 1-SPT invariant looks different for different $k \bmod n$. But are they really different? If we gauge the Z_n -1-symmetry, we turn the above Z_n -1-SPT phase into a topological ordered phase described by a pure Z_n 2-gauge theory:³⁰

$$Z = \sum_{\{db^{Z_n} \stackrel{n}{=} 0\}} e^{2\pi i \int_{\mathcal{M}^4} \frac{k}{n} (\hat{b}^{Z_n})^2}. \quad (37)$$

It turns out that the same topological order is also described by a $Z_{\langle 2k, n \rangle}$ gauge theory. The $Z_{\langle 2k, n \rangle}$ gauge theory has emergent fermions iff $2kn/\langle 2k, n \rangle^2 = \text{odd}$. So the 1-SPT invariant is really different at least when the pairs $[\langle 2k, n \rangle, \text{mod}(2kn/\langle 2k, n \rangle^2, 2)]$ are different.

In Ref. 30, it was shown that $H^4(\mathcal{B}(Z_n, 2); \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n$ for $n = \text{odd}$, and $H^4(\mathcal{B}(Z_n, 2); \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_{2n}$ for $n = \text{even}$. Thus the above 1-SPT invariant is non-trivial. There are (at least) n distinct Z_n -1-SPT phases labeled by $k = 0, \dots, n-1$.

To see the physical properties of the Z_n -1-SPT phase, we consider its 2+1D boundary state described by

$$Z(\mathcal{B}^3) = \sum_{\{a^{Z_n}\}} e^{2\pi i \int_{\mathcal{B}^3} \frac{k}{n} a^{Z_n} da^{Z_n}} \quad (38)$$

where we have set the background 2-connection $\hat{B}^{Z_n} = 0$. The boundary theory also has the Z_n -1-symmetry which is generated by eqn. (30). We like to point out that the action amplitude $e^{2\pi i \int_{\mathcal{B}^3} \frac{k}{n} a^{Z_n} da^{Z_n}}$ for spacetime with boundary $\partial\mathcal{B}^3 \neq 0$ is actually not invariant under the Z_n -1-symmetry transformation. Only the action amplitude for closed spacetime $\partial\mathcal{B}^3 = 0$ has the Z_n -1-symmetry. This indicates that the Z_n -1-symmetry in the 2+1D model eqn. (38) is anomalous.

The 2+1D model is not exactly soluble. To have a soluble model, we restrict a^{Z_n} to be cocycles and obtain

$$Z(\mathcal{B}^3) = \sum_{\{da^{Z_n} \stackrel{n}{=} 0\}} e^{2\pi i \int_{\mathcal{B}^3} \frac{k}{n} a^{Z_n} da^{Z_n}} = \sum_{\{da^{Z_n} \stackrel{n}{=} 0\}} 1. \quad (39)$$

The above model actually describes a 2+1D untwisted Z_n gauge theory.

In the presence of the Z_n -flux described by 2-coboundary $d\hat{a}^{Z_n}$ and the Z_n charge described by worldline C^1 , the above path integral is modified. The new one is obtained by adding the term $e^{\frac{2\pi i}{n} \int_{C^1} a^{Z_n}}$ and then replacing a^{Z_n} by $a^{Z_n} + \hat{a}^{Z_n}$. We find

$$Z(\mathcal{B}^3) = \sum_{\{da^{Z_n} \stackrel{n}{=} 0\}} e^{2\pi i \int_{\mathcal{B}^3} a^{Z_n} \beta_n a^{Z_n} + \frac{2k\pi}{n} \int_{\mathcal{B}^3} (2a^{Z_n} d\hat{a}^{Z_n} + \hat{a}^{Z_n} d\hat{a}^{Z_n})} e^{\frac{2\pi i}{n} \int_{C^1} (a^{Z_n} + \hat{a}^{Z_n})}. \quad (40)$$

Let \tilde{c}_2 be the Poincaré dual of the cycle C^1 which is a 2-cycle on the dual complex $\tilde{\mathcal{B}}^3$. The above can be rewritten as

$$Z(\mathcal{B}^3) = \sum_{\{da^{Z_n} \stackrel{n}{=} 0\}} e^{+2\pi i \int_{\mathcal{B}^3} \frac{2k}{n} a^{Z_n} d\hat{a}^{Z_n} + \frac{k}{n} \hat{a}^{Z_n} d\hat{a}^{Z_n} + \frac{1}{n} \tilde{c}_2 (a^{Z_n} + \hat{a}^{Z_n})}. \quad (41)$$

Now let us consider a bound state of m_f Z_n -flux quanta and m_c unit of Z_n charges. Let 2-cocycle $d\hat{a}$ be the Poincaré dual of the worldline of such a bound state. The path integral in presence of such a bound state is obtained by setting $d\hat{a}^{Z_n} = m_f d\hat{a}$ and $\tilde{c}_2 = m_c d\hat{a}$. We get

$$Z(\mathcal{B}^3) = \sum_{\{da^{Z_n} \stackrel{n}{=} 0\}} e^{2\pi i \int_{\mathcal{B}^3} \frac{2k}{n} a^{Z_n} d\hat{a} + \frac{m_c}{n} a^{Z_n} d\hat{a} + \frac{km_f^2}{n} \hat{a} d\hat{a} + \frac{m_c m_f}{n} \hat{a} d\hat{a}}. \quad (42)$$

This suggests that the statistics of the bound state is given by $\theta = 2\pi(\frac{km_f^2}{n} + \frac{m_c m_f}{n}) = \pi \mathbf{m}^\top K^{-1} \mathbf{m}$, where

$$K = \begin{pmatrix} -2kn & n \\ n & 0 \end{pmatrix}, K^{-1} = \begin{pmatrix} 0 & \frac{1}{n} \\ \frac{1}{n} & \frac{2k}{n} \end{pmatrix}, \mathbf{m} = \begin{pmatrix} m_c \\ m_f \end{pmatrix}. \quad (43)$$

The above statistics can be reproduced by a $U(1) \times U(1)$ Chern-Simons (CS) theory. Thus the 2+1D bosonic model (40) can be described by $U(1) \times U(1)$ CS theory

$$\mathcal{L} = \pi K_{IJ} a_I da_J + \dots, \quad (44)$$

where a_I 's are 1-forms and $a_I da_J = a_I \wedge da_J$ is a wedge product of differential forms. The \dots term makes a_I to have a small curvature

$$da_I \approx 0 \quad (45)$$

We can choose a new basis to rewrite eqn. (44) as

$$\mathcal{L} = \pi K'_{IJ} a'_I da'_J + \dots, \quad (46)$$

where

$$K' = \begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix} = W^\top K W, \quad W = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \quad (47)$$

Thus, the CS theory eqn. (44) always describes the same Z_n gauge theory eqn. (46) regardless the value of k . In the new bases, the excitations are labeled by

$$\mathbf{m}' = (W^\top)^{-1} \mathbf{m} \quad \text{or} \quad \begin{pmatrix} m'_1 \\ m'_2 \end{pmatrix} = \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} m_c \\ m_f \end{pmatrix}. \quad (48)$$

The 2+1D bosonic model (40) without the C^1 term has a Z_n -1-symmetry (30). The model (40) can be described by a $U(1) \times U(1)$ CS theory (44) or (46). The low energy allowed excitations are described by $\mathbf{m}_a^\top = (0, 1)$, or $\mathbf{m}_a'^\top = (-k, 1)$ in the new basis. Thus the Z_n -1-symmetry is generated by the excitation $\mathbf{m}_t'^\top = (k, 1)$ which has a trivial mutual statistics with \mathbf{m}_a' . However, such a Z_n -1-symmetry is anomalous (or non-on-site) when $k \neq 0 \bmod n$, since the 2+1D theory is a boundary of the 3+1D hSPT phase. We cannot gauge it to obtain a Z_n 2-gauge theory. This is an example of emergent anomalous Z_n -1-symmetry in a topologically ordered state.

We note that the the excitation $\mathbf{m}_t'^\top = (k, 1)$ has a statistics $\theta = 2\pi \frac{k}{n}$, which is not bosonic when $k \neq 0 \bmod n$. Thus, the Z_n -1-symmetry (30) is anomalous when the associated excitation is not a boson.

2. A conjecture to detect anomalous 1-symmetry

In fact, the above discussions can be generalized to obtain emergent (anomalous) 1-symmetry in a $U^\kappa(1)$ CS theory (44) described by a general K -matrix.^{79–81} (For a more general discussion, see Ref. 28.) For a 2+1D bosonic topological order described by even K -matrix, an emergent higher symmetry is described by a set of low energy allowed topological excitations $\mathcal{C}_a = \{\mathbf{m}_a\}$ which form a lattice. All other non-trivial topological excitations not in the \mathcal{C}_a have very high energies above Λ . Then the K topological order plus the low energy allowed topological excitations \mathcal{C}_a has an emergent higher symmetry below the energy Λ . The emergent higher symmetry is generated by the topological excitations in $\mathcal{C}_t = \{\mathbf{m}_t\}$, which is formed by \mathbf{m}_t 's that satisfy

$$\mathbf{m}_a K^{-1} \mathbf{m}_t = \text{integer}, \quad \forall \mathbf{m}_a \in \mathcal{C}_a, \quad (49)$$

(i.e. \mathbf{m}_t has a trivial mutual statistics with all low energy allowed excitations \mathbf{m}_a in \mathcal{C}_a). Note that \mathcal{C}_t is also a lattice. We see that

an emergent higher symmetry in a topological order can be fully characterized by a subset \mathcal{C}_a of low energy allowed topological excitations, which is closed under fusion and braiding.

If we include those allowed low energy topological excitations, the action amplitude will become

$$e^{i\pi \int_{\mathcal{B}^3} K_{IJ} a_I da_J} e^{i2\pi \int_{C^1} \mathbf{m}_a^\top \mathbf{a}} \quad (50)$$

where C^1 is the worldline of the \mathbf{m}_a excitation. The above action amplitude actually describes a boundary of a 3+1D hSPT phase described by eqn. (83) with a 1-symmetry (see eqn. (73)). In Section VID, we show that such an 1-symmetry happen to be the one described by the lattice \mathcal{C}_t introduced above. Also in Section VID, we will show that the 3+1D hSPT order is trivial iff $\pi \mathbf{m}_t^\top K^{-1} \mathbf{m}_t = 2\pi \times \text{integer}$, $\forall \mathbf{m}_t, \mathbf{m}_t' \in \mathcal{C}_t$. Therefore,

for the higher symmetry characterized by low energy allowed topological excitations \mathcal{C}_a in a 2+1D Abelian topological order, the higher symmetry is anomaly free iff \mathcal{C}_t contains only bosons with trivial mutual statistics among them. (Here \mathcal{C}_t is formed by the topological excitations, that have trivial mutual statistics with all the topological excitations in \mathcal{C}_a).

For a more general and detailed discussion, see Ref. 28 and Ref. 70.

3. Higher anomaly and phase transition

In above, we see that the emergent (anomalous) higher symmetry is not a property of a topologically ordered state. It is a property of a pair: a topologically ordered state plus its allowed low energy topological excitations.

For example, a 2+1D untwisted Z_n -gauge theory has a Z_n -1-symmetry if we only allow Z_n -flux and their fluctuations, and do not allow, for example, any Z_n -charge and its fluctuations. Such a Z_n -1-symmetry

$$a^{Z_n} \rightarrow a^{Z_n} + \alpha^{Z_n}, \quad d\alpha^{Z_n} \stackrel{n}{=} 0. \quad (51)$$

is anomaly free (i.e. on-site and gaugable).

Now let us start with the deconfined phase of the 2+1D untwisted Z_n -gauge theory described by

$$Z(\mathcal{B}^3) = \sum_{\{da^{Z_n} \stackrel{n}{=} 0\}} 1. \quad (52)$$

The deconfined phase has an anomaly-free Z_n -1-symmetry (51). We then increase the fluctuations of the Z_n -flux to drive a phase transition to the confined phase. The confined phase is described by

$$Z(\mathcal{B}^3) = \sum_{\{a^{Z_n}\}} 1, \quad (53)$$

which is a product state. The product phase also has an anomaly-free Z_n 1-symmetry (51), which is the same as the deconfined phase. Thus the phase transition from the Z_n -gauge deconfined phase to the confined phase (the product state) is an allowed phase transition. Such a phase transition is induced by the boson condensation of the Z_n -flux quanta.

As a second example, let us consider the same 2+1D untwisted Z_n -gauge theory described by the mutual CS theory (46), but with different allowed topological excitations: the bound states of unit Z_n -flux and $-k$ Z_n -charges $\mathbf{m}_a'^\top = (-k, 1)$. In this case, the system has a different Z_n -1-symmetry. When $k \neq 0$, the Z_n -1-symmetry (51) is *anomalous* (i.e. non-on-site and not gaugable). This means that if we want increase the fluctuations of the $\mathbf{m}_a'^\top = (-k, 1)$ excitations to induce a phase transition, we get a phase described by eqn. (38). We cannot reach a product state described by eqn. (53) which has

a different *anomaly-free* Z_n -1-symmetry. This result is expected. When $k \neq 0$, the $\mathbf{m}'_a = (-k, 1)$ excitation has a statistics $\theta = -\frac{2k}{n}\pi$. The anyons cannot condense directly. However, anyon-pairs or other proper clusters of anyons may condense to drive a phase transition. Previously, we believe that those condensations lead to topologically ordered phases, but we are not totally sure.

The result from this paper provides a proof for the above general belief, by understanding it from a point of view of the anomaly matching of the Z_n 1-symmetry. But why do we need to match the anomaly of the Z_n -1-symmetry? This is because the theories with different anomalies of higher symmetry are boundaries of different hSPT states in one higher dimension. No matter how we change the boundary interaction, we cannot change the hSPT order in one higher dimension. Hence we cannot change the higher anomaly, unless we explicitly break the higher symmetry on the boundary. Thus for $k \neq 0 \bmod n$

no matter how we condense the $\mathbf{m}'_a = (-k, 1)$ excitation in a $2+1D$ Z_n -gauge theory (i.e. the CS theory (46)), we can never get the trivial confined phase with no topological order.

On the other hand, if we allow the fluctuations of the bound states of several different combinations of Z_n -flux and Z_n -charges, then we may be able to induce the trivial confined phase with no topological order. In this case, the Z_n -1-symmetry is explicitly broken and the anomaly matching of the Z_n -1-symmetry is invalidated.

In general

no matter how we condense the low energy allowed topological excitation that form C_a , we can never get the trivial confined phase with no topological order, if the higher symmetry characterize by C_a is anomalous (i.e. if C_t obtained from C_a are not formed by boson with trivial mutual statistics).

B. A D -dimensional model to realize a Z_2 k -SPT phase

1. The bulk theory and the boundary theory

In the above, we have constructed models to realize Z_n -1-SPT phases in $3+1D$. Here we will construct models to realize pure Z_2 k -SPT phases in any dimension:

$$Z(\mathcal{M}^D) = \sum_{\{a_k^{Z_2}\}} e^{m\pi i \int_{\mathcal{M}^D} \mathbb{S}q^{D-k-1}(\hat{B}_{k+1}^{Z_2} + da_k^{Z_2})}, \quad (54)$$

where $d\hat{B}_{k+1}^{Z_2} \stackrel{2}{=} 0$, $m = 0, 1$. The theory is well defined even for \mathcal{M}^D with boundary, since

$$\mathbb{S}q^{D-k-1}(\hat{B}_{k+1}^{Z_2} + 2c_{k+1}) \stackrel{2}{=} \mathbb{S}q^{D-k-1}\hat{B}_{k+1}^{Z_2} \quad (55)$$

where we have used eqn. (A24). So the model has a Z_2 k -symmetry

$$a_k^{Z_2} \rightarrow a_k^{Z_2} + \alpha_k^{Z_2}, \quad d\alpha_k^{Z_2} \stackrel{2}{=} 0. \quad (56)$$

even when \mathcal{M}^D has a boundary. We can also show that

$$\mathbb{S}q^{D-k-1}(\hat{B}_{k+1}^{Z_2} + da_k^{Z_2}) \stackrel{2, d}{=} \mathbb{S}q^{D-k-1}\hat{B}_{k+1}^{Z_2}, \quad (57)$$

using eqn. (A26). Thus, the hSPT phase is characterized by hSPT invariant

$$Z^{\text{top}}(\mathcal{M}^D, \hat{B}_{k+1}^{Z_2}) = e^{m\pi i \int_{\mathcal{M}^D} \mathbb{S}q^{D-k-1}\hat{B}_{k+1}^{Z_2}}. \quad (58)$$

for closed \mathcal{M}^D .

One boundary of the above hSPT state is described by (after setting $\hat{B}_{k+1}^{Z_2} = 0$)

$$Z(\mathcal{B}^{D_b}) = \sum_{\{a_k^{Z_2}\}} e^{m\pi i \int_{\mathcal{B}^{D_b}} \mathbb{S}q^{D_b-k}a_k^{Z_2}}, \quad (59)$$

where $D_b = D - 1$ is the spacetime dimension of the boundary. But such a boundary theory is not exactly soluble. An exactly soluble boundary is described by

$$Z(\mathcal{B}^{D_b}) = \sum_{\{da_k^{Z_2} \stackrel{2}{=} 0\}} e^{m\pi i \int_{\mathcal{B}^{D_b}} \mathbb{S}q^{D_b-k}a_k^{Z_2}} \quad (60)$$

which describes a Z_2 k -gauge theory twisted by the topological term $e^{m\pi i \int_{\mathcal{B}^{D_b}} \mathbb{S}q^{D_b-k}a_k^{Z_2}}$. In the presence of the higher Z_2 -flux, the path integral becomes (after replacing $a_k^{Z_2}$ by $a_k^{Z_2} + \hat{a}_k^{Z_2}$)

$$\begin{aligned} Z(\mathcal{B}^{D_b}) &= \sum_{\{da_k^{Z_2} \stackrel{2}{=} 0\}} e^{m\pi i \int_{\mathcal{B}^{D_b}} \mathbb{S}q^{D_b-k}(\hat{a}_k^{Z_2} + a_k^{Z_2})} \\ &= \sum_{\{da_k^{Z_2} \stackrel{2}{=} 0\}} e^{m\pi i \int_{\mathcal{B}^{D_b}} \mathbb{S}q^{D_b-k}\hat{a}_k^{Z_2} + \mathbb{S}q^{D_b-k}a_k^{Z_2}}, \end{aligned} \quad (61)$$

where $d\hat{a}_k^{Z_2}$ describes the higher Z_2 -flux on the boundary. The above model has a Z_2 k -symmetry (56). The Z_2 k -symmetry is anomalous for $m = 1$ and anomaly-free for $m = 0$.

2. Higher anomaly and phase transition

Now consider topologically ordered state in D_b space-time dimension described by the deconfined phase of the Z_2 k -gauge theory (60). We allow only the fluctuations of the higher Z_2 -flux, and try to use them to drive a phase transition. Such a system has the Z_2 k -symmetry (56). Using the anomaly matching condition, we find that the phase transition can nerve produce the confined phase with topological order, when $m = 1$. On the other hand, when $m = 0$, the tirivial confined phase can be reach by the phase transition.

We like to stress that here we only ask can we obtain the product state from the deconfined phase of the Z_2 k -gauge theory (60) by the fluctuations of the higher Z_2 -flux only. We find that we cannot obtain the product state from the deconfined phase when $m = 1$. However,

if we include both fluctuations of the higher Z_2 -charge and the higher Z_2 -flux, then we can always obtain the product state from the deconfined phase regardless the value of m .

As an application of the above result, let us consider the case with $D_b = 4$ and $k = 2$. The deconfined phase of the 3+1D Z_2 2-gauge theory is described by (with the Z_2 2-flux)

$$Z(\mathcal{B}^4) = \sum_{\{db^{\mathbb{Z}_2} \stackrel{=}{=} 0\}} e^{m\pi i \int_{\mathcal{B}^4} \mathbb{S}q^2(\hat{b}^{\mathbb{Z}_2} + b^{\mathbb{Z}_2})}, \quad (62)$$

where $d\hat{b}^{\mathbb{Z}_2}$ is a fixed 3-coboundary describing the Z_2 2-flux and $b^{\mathbb{Z}_2}$ is a dynamical 2-cochain.

It is well known that a Z_2 2-gauge theory in 3+1D is dual to a Z_2 gauge theory (see for example Ref. 76). The so called Z_2 2-flux in the Z_2 2-gauge theory correspond to the Z_2 -charge in the Z_2 gauge theory. In fact, the 3-coboundary $d\hat{b}^{\mathbb{Z}_2}$ is the Poincaré dual of the world-line of the Z_2 -charge in 3+1D spacetime. When $m = 0$, eqn. (62) corresponds to a untwisted Z_2 gauge theory where the Z_2 charge is a boson and the Z_2 2-symmetry is anomaly-free. When $m = 1$, eqn. (62) corresponds to a twisted Z_2 gauge theory where the Z_2 charge is a fermion³ and the Z_2 2-symmetry is anomalous. The result in this section implies that

any Z_2 charge fluctuation and condensations in the 3+1D bosonic topological order described by a twisted Z_2 gauge theory (62) cannot induce the trivial gapped phase with no topological order.

In contrast, the Z_2 -charge fluctuations and condensations in the *untwisted* Z_2 gauge theory can induce the trivial product state. Also, the Z_2 -charge and Z_2 -flux fluctuations and condensations in the twisted Z_2 gauge theory can induce the trivial product state. The Z_2 -flux fluctuations breaks the Z_2 -2-symmetry and invalidate the anomaly matching of the Z_2 -2-symmetry.

C. A D -dimensional model to realize a Z_n k -SPT phase

We can also construct models to realize more general pure Z_n hSPT phases. For $D - k = \text{even}$, the following model realizes a Z_n k -SPT phase.

$$Z = \sum_{\{a_k^{\mathbb{Z}_n}\}} e^{2\pi i \int_{\mathcal{M}^D} \frac{m}{n} \mathbb{S}q^{D-k-1}(\hat{B}_{k+1}^{\mathbb{Z}_n} + da_k^{\mathbb{Z}_n})}, \quad (63)$$

The theory is well defined when $d\hat{B}_{k+1}^{\mathbb{Z}_n} \stackrel{=}{=} 0$, since

$$\mathbb{S}q^{D-k-1}(\hat{B}_{k+1}^{\mathbb{Z}_n} + nc_{k+1}) \stackrel{=}{=} \mathbb{S}q^{D-k-1}\hat{B}_{k+1}^{\mathbb{Z}_n} \quad (64)$$

where we have used eqn. (A24). We can also show that

$$\mathbb{S}q^{D-k-1}(\hat{B}_{k+1}^{\mathbb{Z}_n} + da_k^{\mathbb{Z}_n}) \stackrel{=}{=} \mathbb{S}q^{D-k-1}\hat{B}_{k+1}^{\mathbb{Z}_n}, \quad (65)$$

using eqn. (A26) and $D - k - 1 = \text{odd}$. The model has a Z_n k -symmetry

$$a_k^{\mathbb{Z}_n} \rightarrow a_k^{\mathbb{Z}_n} + \alpha_k^{\mathbb{Z}_n}, \quad d\alpha_k^{\mathbb{Z}_n} \stackrel{=}{=} 0. \quad (66)$$

and Z_n $(k+1)$ -gauge symmetry

$$\hat{B}_{k+1}^{\mathbb{Z}_n} \rightarrow \hat{B}_{k+1}^{\mathbb{Z}_n} + da_{k+1}^{\mathbb{Z}_n}. \quad (67)$$

The Z_n k -symmetry is anomaly free since it can be gauged.

Such a hSPT phase is characterized by hSPT invariant

$$Z^{\text{top}}(\mathcal{M}^D, \hat{B}_{k+1}^{\mathbb{Z}_n}) = e^{2\pi i \int_{\mathcal{M}^D} \frac{m}{n} \mathbb{S}q^{D-k-1}\hat{B}_{k+1}^{\mathbb{Z}_n}}, \quad (68)$$

The hSPT state can have a boundary described by

$$Z = \sum_{\{a_k^{\mathbb{Z}_n}\}} e^{2\pi i \int_{\mathcal{B}^{D-1}} \frac{m}{n} \mathbb{S}q^{D-k-1}(a_k^{\mathbb{Z}_n})}, \quad (69)$$

after setting $\hat{B}_{k+1}^{\mathbb{Z}_n} = 0$. The boundary theory (69) also has the Z_n k -symmetry (66) when \mathcal{B}^{D-1} has no boundary. This can be shown by using eqn. (A26).

We may choose $D = 6$ and $k = 2$

$$Z = \sum_{\{a_2^{\mathbb{Z}_n}\}} e^{2\pi i \int_{\mathcal{M}^4} \frac{m}{n} \mathbb{S}q^3(\hat{B}_3^{\mathbb{Z}_n} + da_2^{\mathbb{Z}_n})}. \quad (70)$$

The model has a Z_n -2-symmetry

$$a_2^{\mathbb{Z}_n} \rightarrow a_2^{\mathbb{Z}_n} + \alpha_2^{\mathbb{Z}_n}, \quad d\alpha_2^{\mathbb{Z}_n} \stackrel{=}{=} 0, \quad (71)$$

and realizes a Z_n -2-SPT phase. A Z_n -2-symmetric boundary of such a 2-SPT phase is described by (after setting $\hat{B}_3^{\mathbb{Z}_n} = 0$):

$$Z(\mathcal{B}^5) = \sum_{\{da_2^{\mathbb{Z}_n} \stackrel{=}{=} 0\}} e^{2\pi i \int_{\mathcal{B}^5} \frac{m}{n} \mathbb{S}q^3(a_2^{\mathbb{Z}_n})} = \sum_{\{da_2^{\mathbb{Z}_n} \stackrel{=}{=} 0\}} e^{2\pi i \int_{\mathcal{B}^5} \frac{m}{n} a_2^{\mathbb{Z}_n} da_2^{\mathbb{Z}_n}}. \quad (72)$$

The Z_n -2-symmetry on \mathcal{B}^5 is anomalous when $m \neq 0 \pmod n$. The model can not reach to trivial gapped phase with no topological order even if we allow fluctuations with $da_2^{\mathbb{Z}_n} \neq 0$, but do not allow the fluctuations of the charges of Z_n 2-gauge theory (which are closed strings).

D. A 3+1D $U^\kappa(1)$ bosonic model to realize a $Z_{k_1} \times Z_{k_2} \times \dots$ -1-SPT phase

In this section, we will use a 3+1D $U^\kappa(1)$ “gauge theory” in the confined phase to realize some hSPT phase. Our model is a bosonic model defined on a triangulated spacetime (with vertices labeled by i, j, \dots). On each link ij , we have bosonic degrees of freedom described by $(a_I^{\mathbb{R}/\mathbb{Z}})_{ij} \in (-\frac{1}{2}, \frac{1}{2}]$, $I = 1, \dots, \kappa$. To write down the

path integral of the bosonic model, we start with 2+1D $U^\kappa(1)$ Chern-Simons theory on spacetime lattice \mathcal{B}^3 :⁸²

$$Z = \int D[a_I^{\mathbb{R}/\mathbb{Z}}] e^{i2\pi \int_{\mathcal{B}^3} \sum_{I < J} k_{IJ} d(a_I^{\mathbb{R}/\mathbb{Z}} - \lfloor a_J^{\mathbb{R}/\mathbb{Z}} \rfloor)} e^{i2\pi \int_{\mathcal{B}^3} \sum_{I \leq J} k_{IJ} a_I^{\mathbb{R}/\mathbb{Z}} (da_J^{\mathbb{R}/\mathbb{Z}} - \lfloor da_J^{\mathbb{R}/\mathbb{Z}} \rfloor) - \lfloor da_I^{\mathbb{R}/\mathbb{Z}} \rfloor a_J^{\mathbb{R}/\mathbb{Z}}} e^{-\int_{\mathcal{B}^3} \sum_I \frac{|da_I^{\mathbb{R}/\mathbb{Z}} - \lfloor da_I^{\mathbb{R}/\mathbb{Z}} \rfloor|^2}{g_3}} \quad (73)$$

where $a_I^{\mathbb{R}/\mathbb{Z}}$ is a \mathbb{R}/\mathbb{Z} -valued 1-cochain, $\int D[a_I^{\mathbb{R}/\mathbb{Z}}] = \prod_{ij,I} \int_{\frac{1}{2}}^{\frac{1}{2}} d(a_I^{\mathbb{R}/\mathbb{Z}})_{ij}$, and k_{IJ} integers. Since $a_I^{\mathbb{R}/\mathbb{Z}}$ is \mathbb{R}/\mathbb{Z} -valued, we require eqn. (73) to have the following gauge symmetry

$$a_I^{\mathbb{R}/\mathbb{Z}} \rightarrow a_I^{\mathbb{R}/\mathbb{Z}} + u_I^{\mathbb{Z}} \quad (74)$$

for any \mathbb{Z} -valued 1-cochain $u_I^{\mathbb{Z}}$. Eq. (73) satisfies this condition even for \mathcal{B}^3 with boundary, as shown in Ref. 82.

The path integral of the 3+1D bosonic model (for spacetime \mathcal{M}^4 with or without boundary) is obtained from eqn. (73) by taking a derivative and setting $g_3 = \infty$:

$$e^{i2\pi \int_{\mathcal{M}^4} k_{IJ} d[a_I^{\mathbb{R}/\mathbb{Z}} (da_J^{\mathbb{R}/\mathbb{Z}} - \lfloor da_J^{\mathbb{R}/\mathbb{Z}} \rfloor) - \lfloor da_I^{\mathbb{R}/\mathbb{Z}} \rfloor a_J^{\mathbb{R}/\mathbb{Z}}]} = e^{i2\pi \int_{\mathcal{M}^4} k_{IJ} (da_I^{\mathbb{R}/\mathbb{Z}} - \lfloor da_I^{\mathbb{R}/\mathbb{Z}} \rfloor) (da_J^{\mathbb{R}/\mathbb{Z}} - \lfloor da_J^{\mathbb{R}/\mathbb{Z}} \rfloor)} \quad (75)$$

We obtain a 3+1D bosonic model on spacetime lattice

$$Z = \int D[a_I^{\mathbb{R}/\mathbb{Z}}] e^{-\int_{\mathcal{M}^4} \sum_I \frac{|da_I^{\mathbb{R}/\mathbb{Z}} - \lfloor da_I^{\mathbb{R}/\mathbb{Z}} \rfloor|^2}{g}} e^{i2\pi \int_{\mathcal{M}^4} \sum_{I \leq J} k_{IJ} (da_I^{\mathbb{R}/\mathbb{Z}} - \lfloor da_I^{\mathbb{R}/\mathbb{Z}} \rfloor) (da_J^{\mathbb{R}/\mathbb{Z}} - \lfloor da_J^{\mathbb{R}/\mathbb{Z}} \rfloor)}. \quad (76)$$

In the above, we have included an extra term $e^{-\sum_I \int_{\mathcal{M}^4} \frac{|da_I^{\mathbb{R}/\mathbb{Z}} - \lfloor da_I^{\mathbb{R}/\mathbb{Z}} \rfloor|^2}{g}}$. Without such a term, eqn. (76) reduces to eqn. (73) when \mathcal{M}^4 has a boundary $\mathcal{B}^3 = \partial\mathcal{M}^4$.

When \mathcal{M}^4 has no boundary, by its construction from eqn. (73), eqn. (76) can be simplified to

$$Z(\mathcal{M}^4) = \int D[a_I^{\mathbb{R}/\mathbb{Z}}] e^{-\int_{\mathcal{M}^4} \frac{|da_I^{\mathbb{R}/\mathbb{Z}} - \lfloor da_I^{\mathbb{R}/\mathbb{Z}} \rfloor|^2}{g}}, \quad (77)$$

We find that when $g \sim 0$, $da_I^{\mathbb{R}/\mathbb{Z}}$ fluctuate weakly and the above model describes the deconfined phase of the $U^\kappa(1)$ gauge theory. In this case, the model is gapless. In this limit, $da_I^{\mathbb{R}/\mathbb{Z}} \sim 0$ or $\lfloor da_I^{\mathbb{R}/\mathbb{Z}} \rfloor = 0$, and we can reduce eqn. (76) to a familiar $U^\kappa(1)$ gauge theory with 2π quantized topological terms $2\pi \int_{\mathcal{M}^4} \sum_{I \leq J} k_{IJ} da_I^{\mathbb{R}/\mathbb{Z}} da_J^{\mathbb{R}/\mathbb{Z}}$ and the Maxwell terms $\int_{\mathcal{M}^4} \frac{|da_I^{\mathbb{R}/\mathbb{Z}}|^2}{g}$:

$$Z = \int D[a_I^{\mathbb{R}/\mathbb{Z}}] e^{i2\pi \int_{\mathcal{M}^4} \sum_{I \leq J} k_{IJ} da_I^{\mathbb{R}/\mathbb{Z}} da_J^{\mathbb{R}/\mathbb{Z}} - \int_{\mathcal{M}^4} \frac{|da_I^{\mathbb{R}/\mathbb{Z}}|^2}{g}}. \quad (78)$$

In particular, when $\kappa = 1$, the above becomes

$$Z = \int D[a] e^{i2\pi \int_{\mathcal{M}^4} da da - \int_{\mathcal{M}^4} \frac{|da|^2}{g}}, \quad (79)$$

where $k = k_{11}$ is an integer.

When $g \sim \infty$, $da_I^{\mathbb{R}/\mathbb{Z}}$ fluctuate strongly and the above model describes the confined phase of the $U^\kappa(1)$ gauge theory. The model is fully gapped. For any closed \mathcal{M}^4 and when $g = \infty$, the partition function $Z(\mathcal{M}^4) = \int D[a_I^{\mathbb{R}/\mathbb{Z}}] = 1$ since $\int_{-\frac{1}{2}}^{\frac{1}{2}} d(a_I^{\mathbb{R}/\mathbb{Z}})_{ij} = 1$. Thus the topological partition function $Z^{\text{top}}(\mathcal{M}^4) = 1$ is trivial for any closed \mathcal{M}^4 . This implies that the $g = \infty$ confined phase is a gapped phase with trivial topological order.

Regardless the value of g , let us include low energy allowed excitations described by charges of the $U^\kappa(1)$ gauge field. The values of the charges are encoded in integer vectors \mathbf{m}_a . In the $U(1)$ confined phase, the so-called low energy allowed excitations becomes the particle-hole fluctuations for the charges in \mathcal{C}_a . Since the set of allowed excitations is closed under the fusion, the allowed integer vectors \mathbf{m}_a form a lattice \mathcal{C}_a . We like to point out that \mathcal{C}_a includes the column vector of the K matrix, which is given by

$$K_{II} = 2k_{II}, \quad K_{IJ} = K_{JI} = k_{IJ}, \quad I < J. \quad (80)$$

To see this point, we note that for closed \mathcal{M}^4

$$e^{i2\pi \int_{\mathcal{M}^4} \sum_{I \leq J} k_{IJ} (da_I^{\mathbb{R}/\mathbb{Z}} - \lfloor da_I^{\mathbb{R}/\mathbb{Z}} \rfloor) (da_J^{\mathbb{R}/\mathbb{Z}} - \lfloor da_J^{\mathbb{R}/\mathbb{Z}} \rfloor)} = e^{-i2\pi \int_{\mathcal{M}^4} \sum_{I,J} K_{IJ} a_I^{\mathbb{R}/\mathbb{Z}} d\lfloor da_J^{\mathbb{R}/\mathbb{Z}} \rfloor}, \quad (81)$$

where $d\lfloor da_J^{\mathbb{R}/\mathbb{Z}} \rfloor$ can be viewed as the Poincaré dual of the worldline of the $U(1)$ monopoles. This implies that the effect of the topological term $e^{i2\pi \int_{\mathcal{M}^4} \sum_{I \leq J} k_{IJ} (da_I^{\mathbb{R}/\mathbb{Z}} - \lfloor da_I^{\mathbb{R}/\mathbb{Z}} \rfloor) (da_J^{\mathbb{R}/\mathbb{Z}} - \lfloor da_J^{\mathbb{R}/\mathbb{Z}} \rfloor)}$ is bind $U(1)$ monopoles with the $U(1)$ charges. In particular, the monopole of the I^θ $U(1)$ field carries the J^θ $U(1)$ charge K_{IJ} . For large g , the monopoles described by $d\lfloor da_I^{\mathbb{R}/\mathbb{Z}} \rfloor$ are low energy allowed excitations. Those monopole excitations carry charges given by the column vector of the K matrix. So the column vectors of the K matrix are the charges for the allowed low energy excitations.

Let \mathbf{m}_a^μ , $\mu = 1, 2, \dots, \kappa$ be a basis of the \mathcal{C}_a lattice, and let M_a is a square matrix whose columns are \mathbf{m}_a^μ vectors. If we do not have any extra low energy allowed charge excitation, M_a will be given by K . In this case, our $U^\kappa(1)$ model has maximal 1-symmetry. Using Smith normal form, we can always choose a basis such are that the square matrix M_a is diagonal, *i.e.*

$$(\mathbf{m}_a^\mu)_I = k_I \delta_{\mu I}. \quad (82)$$

The allowed charge excitations can be included in the path integral via the Wilson-loop C^1

$$Z = \int D[a_I^{\mathbb{R}/\mathbb{Z}}] e^{-\int_{\mathcal{M}^4} \sum_I \frac{|da_I^{\mathbb{R}/\mathbb{Z}} - \lfloor da_I^{\mathbb{R}/\mathbb{Z}} \rfloor|^2}{g}} e^{i2\pi \int_{C^1} \mathbf{m}_a^\top \mathbf{a}} e^{i2\pi \int_{\mathcal{M}^4} \sum_{I \leq J} k_{IJ} (da_I^{\mathbb{R}/\mathbb{Z}} - \lfloor da_I^{\mathbb{R}/\mathbb{Z}} \rfloor) (da_J^{\mathbb{R}/\mathbb{Z}} - \lfloor da_J^{\mathbb{R}/\mathbb{Z}} \rfloor)} \quad (83)$$

Such a model has 1-symmetries generated by

$$a_I^{\mathbb{R}/\mathbb{Z}} \rightarrow a_I^{\mathbb{R}/\mathbb{Z}} + s_I \alpha^{\mathbb{Z}} \quad (84)$$

for \mathcal{M}^4 with or without boundary. Here, $\alpha^{\mathbb{Z}}$ are arbitrary \mathbb{Z} -valued 1-cocycles, and $\mathbf{s} = (s_1, s_2, \dots)^\top$ is an arbitrary rational vector that satisfies

$$\mathbf{s}^\top \mathbf{m}_a \stackrel{!}{=} 0, \quad \forall \mathbf{m}_a \in \mathcal{C}_a. \quad (85)$$

or

$$M_a^\top \mathbf{s} \stackrel{!}{=} 0. \quad (86)$$

The above choices of \mathbf{s} ensure the invariance of $e^{i2\pi \int_{C^1} \mathbf{m}_a^\top \mathbf{a}}$.

The above implies that $K\mathbf{s} \stackrel{!}{=} 0$ since \mathcal{C}_a contains the columns of K . It is more convenient to introduce integer vectors

$$\mathbf{m}_t = K\mathbf{s} \quad (87)$$

to describe the 1-symmetries. \mathbf{m}_t 's satisfy

$$M_a^\top K^{-1} \mathbf{m}_t \stackrel{!}{=} 0. \quad (88)$$

In fact, the integer vectors \mathbf{m}_t satisfying the above conditions form a lattice \mathcal{C}_t . Let \mathbf{m}_t^μ be a basis of the lattice \mathcal{C}_t . The 1-symmetry is characterized by this \mathcal{C}_t lattice. For the special basis eqn. (82), \mathbf{m}_t^μ and $\mathbf{s}^\mu = K^{-1} \mathbf{m}_t^\mu$ are given by

$$(\mathbf{m}_t^\mu)_I = \frac{K_{I\mu}}{k_\mu} \in \mathbb{Z}, \quad (\mathbf{s}^\mu)_I = \frac{\delta_{I\mu}}{k_\mu}, \quad (89)$$

where μ is not summed.

The above 1-symmetry is a $Z_{k_1} \times Z_{k_2} \times \dots$ -1-symmetry, with k_I given by eqn. (82). Such a 1-symmetry is defined on the spacetime lattice with or without boundary, and are expected to be anomaly-free. Thus for large g , the gapped state described by eqn. (83) is a state with 1-symmetry but no topological order. In the following, we will try to determine the hSPT order in such a gapped state.

To do so, let us gauge the 1-symmetries by replacing $da_I^{\mathbb{R}/\mathbb{Z}}$ with

$$b_I \equiv da_I^{\mathbb{R}/\mathbb{Z}} - s_I^\mu \hat{B}_\mu^{\mathbb{Z}}, \quad (90)$$

where $\mathbf{s}^\mu = K^{-1} \mathbf{m}_t^\mu$ and $\hat{B}_\mu^{\mathbb{Z}}$ are \mathbb{Z} -valued 2-cocycles:

$$Z = \int D[a_I^{\mathbb{R}/\mathbb{Z}}] e^{i2\pi \int_{\mathcal{M}^4} \sum_{I \leq J} k_{IJ} (b_I - \lfloor b_I \rfloor)(b_J - \lfloor b_J \rfloor)} e^{i2\pi \int_{C^1} \mathbf{m}_a^\top \mathbf{a}} e^{-\int_{\mathcal{M}^4} \frac{|b_I - \lfloor b_I \rfloor|^2}{g}}. \quad (91)$$

In the above, we have replaced $e^{i2\pi \int_{C^1} \mathbf{m}_a^\top \mathbf{a}}$ by $e^{i2\pi \int_{D^2} (\mathbf{m}_a^\top d\mathbf{a} - \mathbf{m}_a^\top \mathbf{s}^\mu \hat{B}_\mu^{\mathbb{Z}})} = e^{i2\pi (\int_{C^1} \mathbf{m}_a^\top \mathbf{a} - \int_{D^2} \mathbf{m}_a^\top \mathbf{s}^\mu \hat{B}_\mu^{\mathbb{Z}})}$ where $\partial D^2 = C^1$. Since $\mathbf{m}_a^\top \mathbf{s}^\mu$ are integers, $e^{i2\pi (\int_{C^1} \mathbf{m}_a^\top \mathbf{a} - \int_{D^2} \mathbf{m}_a^\top \mathbf{s}^\mu \hat{B}_\mu^{\mathbb{Z}})} = e^{i2\pi \int_{C^1} \mathbf{m}_a^\top \mathbf{a}}$, and $e^{i2\pi \int_{C^1} \mathbf{m}_a^\top \mathbf{a}}$ is unchanged under the gauging of the 1-symmetry. Note that the 1-symmetries are discrete symmetries, and can be probed by flat 2-gauge connections.

The 2-gauged $U^\kappa(1)$ theory (91) still have the 1-symmetries (84) for \mathcal{M}^4 with or without boundary. In fact, it has the following 2-gauge symmetries that include the 1-symmetries:

$$a_I^{\mathbb{R}/\mathbb{Z}} \rightarrow a_I^{\mathbb{R}/\mathbb{Z}} + s_I^\mu u_\mu^{\mathbb{Z}}, \quad \hat{B}_\mu^{\mathbb{Z}} \rightarrow \hat{B}_\mu^{\mathbb{Z}} + du_\mu^{\mathbb{Z}} \quad (92)$$

for \mathcal{M}^4 with or without boundary. Here $u_\mu^{\mathbb{Z}}$ are arbitrary \mathbb{Z} -valued 1-cochains. This is because $b_I = da_I^{\mathbb{R}/\mathbb{Z}} - s_I^\mu \hat{B}_\mu^{\mathbb{Z}}$ is invariant under the 2-gauge transformation (92). We also note that the 2-gauged theory (91) has the gauge symmetry eqn. (74) for \mathcal{M}^4 with or without boundary.

When \mathcal{M}^4 has no boundary and $g = \infty$, eqn. (91) can be rewritten as

$$Z = \int D[a_I^{\mathbb{R}/\mathbb{Z}}] e^{i\pi \int_{\mathcal{M}^4} K_{IJ} (da_I^{\mathbb{R}/\mathbb{Z}} - s_I^\mu \hat{B}_\mu^{\mathbb{Z}})(da_J^{\mathbb{R}/\mathbb{Z}} - s_J^\mu \hat{B}_\mu^{\mathbb{Z}})} e^{i2\pi \int_{C^1} \mathbf{m}_a^\top \mathbf{a}}. \quad (93)$$

Without the charged excitations described by \mathbf{m}_a , the partition function becomes

$$Z(\mathcal{M}^4, \hat{B}_\mu^{\mathbb{Z}}) = Z^{\text{top}}(\mathcal{M}^4, \hat{B}_\mu^{\mathbb{Z}}) = e^{i\pi \int_{\mathcal{M}^4} (\mathbf{s}^\nu)^\top K \mathbf{s}^\mu \hat{B}_\mu^{\mathbb{Z}} \hat{B}_\nu^{\mathbb{Z}}} = e^{i\pi \int_{\mathcal{M}^4} (\mathbf{m}_t^\nu)^\top K^{-1} \mathbf{m}_t^\mu \hat{B}_\mu^{\mathbb{Z}} \hat{B}_\nu^{\mathbb{Z}}} \quad (94)$$

where $\mathbf{m}_t^\mu \in \mathcal{C}_t$. We see that if \mathcal{C}_t satisfies

$$\mathbf{m}_t^\top K^{-1} \mathbf{m}_t' \stackrel{!}{=} 0, \quad \forall \mathbf{m}_t, \mathbf{m}_t' \in \mathcal{C}_t \quad (95)$$

then the hSPT invariant $e^{i\pi \int_{\mathcal{M}^4} (\mathbf{m}_t^\nu)^\top K^{-1} \mathbf{m}_t^\mu \hat{B}_\mu^{\mathbb{Z}} \hat{B}_\nu^{\mathbb{Z}}} = 1$ and is trivial. The confined phase of our $U^\kappa(1)$ model is a trivial hSPT phase protected by the 1-symmetry characterized by \mathcal{C}_t . Otherwise, the confined phase of our $U^\kappa(1)$ model is a non-trivial hSPT phase.

To summarize, for our $U^\kappa(1)$ model with low energy allowed charges in \mathcal{C}_a (91), the 1-symmetry is characterized by a lattice

$$\mathcal{C}_t = \{\mathbf{m}_t | \mathbf{m}_t^\top K^{-1} \mathbf{m}_a \stackrel{!}{=} 0 \ \forall \mathbf{m}_a \in \mathcal{C}_a\}, \quad (96)$$

which is a $Z_{k_1} \times Z_{k_2} \times \dots$ -1-symmetry. The confined phase of our $U^\kappa(1)$ model can be a non-trivial hSPT phase protected by the 1-symmetry \mathcal{C}_t . The confined phase is a trivial hSPT phase iff \mathcal{C}_t satisfies eqn. (95). This supports our conjecture in Section VIA 2.

We like to remark that for \mathcal{M}^4 without boundary, our model reduces to eqn. (77). Such a model have a $U^\kappa(1)$ -1-symmetry generated by shifting $a_I^{\mathbb{R}/\mathbb{Z}}$ by \mathbb{R} -valued cocycles. However, the $U^\kappa(1)$ 1-symmetry is broken for the model with boundary and with the charge excitations, *i.e.* eqn. (83) does not have the $U^\kappa(1)$ -1-symmetry. However, the model (83) has an anomaly-free discrete 1-symmetry generated by a subset of the $U^\kappa(1)$ 1-transformations, *i.e.* eqn. (84). The model (83) realizes a hSPT phase for such an anomaly-free discrete and finite 1-symmetry. The finite 1-symmetry is a $Z_{k_1} \times Z_{k_2} \times \dots$ -1-symmetry where k_I is given in eqn. (82).

E. A model to realize a hSPT phase with a $U(1)$ k -symmetry

In this section, we consider a model to realize a hSPT phase with a continuous $U(1)$ k -symmetry:

$$Z(\mathcal{M}^D) = \sum_{\{a_k^{\mathbb{R}/\mathbb{Z}}\}} e^{\pi i \int_{\mathcal{M}^D} \text{Sq}^{D-k-2} d[da_k^{\mathbb{R}/\mathbb{Z}}]}, \quad (97)$$

where $a_k^{\mathbb{R}/\mathbb{Z}}$ is a \mathbb{R}/\mathbb{Z} -valued k -cochain. Since $a_k^{\mathbb{R}/\mathbb{Z}}$ is \mathbb{R}/\mathbb{Z} -valued, the theory must also have the following gauge symmetry, even for \mathcal{M}^D that has a boundary

$$a_k^{\mathbb{R}/\mathbb{Z}} \rightarrow a_k^{\mathbb{R}/\mathbb{Z}} + u_k^{\mathbb{Z}}, \quad (98)$$

where $u_k^{\mathbb{Z}}$ is an arbitrary \mathbb{Z} -valued k -cochain. We find that eqn. (97) indeed has such a gauge symmetry.

The above theory has the following $U(1)$ k -symmetry, even when \mathcal{M}^D has a boundary

$$a_k^{\mathbb{R}/\mathbb{Z}} \rightarrow a_k^{\mathbb{R}/\mathbb{Z}} + \alpha_k^{\mathbb{R}/\mathbb{Z}}, \quad d\alpha_k^{\mathbb{R}/\mathbb{Z}} \stackrel{!}{=} 0 \quad (99)$$

where $\alpha_k^{\mathbb{R}/\mathbb{Z}}$ is an arbitrary \mathbb{R}/\mathbb{Z} -valued k -cocycle. This implies that the model (97) has an anomaly-free $U(1)$ k -symmetry.

Using eqn. (A23), we can show that when \mathcal{M}^D is closed

$$e^{\pi i \int_{\mathcal{M}^D} \text{Sq}^{D-k-2} d[da_k^{\mathbb{R}/\mathbb{Z}}]} = 1, \quad \partial\mathcal{M}^D = 0. \quad (100)$$

Therefore, the corresponding topological partition function $Z^{\text{top}}(\mathcal{M}^D) = 1$ for any closed \mathcal{M}^D . The model (97) describes a phase with trivial topological order.

Here we would like to mention that when $D - k - 2 = \text{odd}$ or when $D - k - 2 \geq k + 2$, we have (see (A22))

$$\text{Sq}^{D-k-2} d[da_k^{\mathbb{R}/\mathbb{Z}}] = d\text{Sq}^{D-k-2} [da_k^{\mathbb{R}/\mathbb{Z}}], \quad (101)$$

and

$$e^{\theta i \int_{\mathcal{M}^D} \text{Sq}^{D-k-2} d[da_k^{\mathbb{R}/\mathbb{Z}}]} = 1, \quad \partial\mathcal{M}^D = 0, \quad (102)$$

for any θ . Thus in this case, we can tune π in eqn. (97) continuously to 0 without encounter phase transitions. We see that when $D - k - 2 = \text{odd}$ or $D - k - 2 \geq k + 2$, eqn. (97) describes a trivial hSPT phases.

When $D - k - 2 = 0$

$$\text{Sq}^0 d[da_k^{\mathbb{R}/\mathbb{Z}}] \stackrel{!}{=} d[da_k^{\mathbb{R}/\mathbb{Z}}]. \quad (103)$$

In this case

$$e^{\pi i \int_{\mathcal{M}^D} \text{Sq}^0 d[da_k^{\mathbb{R}/\mathbb{Z}}]} = e^{\pi i \int_{\mathcal{M}^D} d[da_k^{\mathbb{R}/\mathbb{Z}}]} \quad (104)$$

even when $\partial\mathcal{M}^D \neq 0$. When $\partial\mathcal{M}^D = 0$, $e^{\theta i \int_{\mathcal{M}^D} d[da_k^{\mathbb{R}/\mathbb{Z}}]} = 1$ for any θ . So we can tune π to 0 without phase transitions. We see that when $D - k - 2 = 0$, eqn. (97) also describes a trivial hSPT phase.

When $D - k - 2 = \text{even}$ and $0 < D - k - 2 < k + 2$,

$$\begin{aligned} & \text{Sq}^{D-k-2} d[da_k^{\mathbb{R}/\mathbb{Z}}] \\ &= d\text{Sq}^{D-k-2} [da_k^{\mathbb{R}/\mathbb{Z}}] - 2(-)^{k+1} \text{Sq}^{D-k-1} [da_k^{\mathbb{R}/\mathbb{Z}}]. \end{aligned} \quad (105)$$

Therefore

$$\begin{aligned} e^{\theta i \int_{\mathcal{M}^D} \text{Sq}^{D-k-2} d[da_k^{\mathbb{R}/\mathbb{Z}}]} &= e^{-2\theta i \int_{\mathcal{M}^D} (-)^{k+1} \text{Sq}^{D-k-1} [da_k^{\mathbb{R}/\mathbb{Z}}]}, \\ \partial\mathcal{M}^D &= 0. \end{aligned} \quad (106)$$

Since $\text{Sq}^{D-k-2} [da_k^{\mathbb{R}/\mathbb{Z}}]$ is not a coboundary in general, the action amplitude is $e^{\theta i \int_{\mathcal{M}^D} \text{Sq}^{D-k-2} d[da_k^{\mathbb{R}/\mathbb{Z}}]} = 1$ only when $\theta = 0, \pi$. For other θ the action amplitude has a non-trivial phase, and the model may be gapless. In this case, $\theta = 0$ and $\theta = \pi$ may correspond to two different hSPT phases.

To see if the model (97) for $D - k - 2 = \text{even}$ and $0 < D - k - 2 < k + 2$ describes a phase with a non-trivial hSPT order or not, we gauge the $U(1)$ k -symmetry to obtain

$$Z = \sum_{\{a_k^{\mathbb{R}/\mathbb{Z}}\}} e^{\pi i \int_{\mathcal{M}^D} \text{Sq}^{D-k-2} (d[da_k^{\mathbb{R}/\mathbb{Z}}] + \hat{B}_{k+1}^{\mathbb{R}/\mathbb{Z}})}. \quad (107)$$

where the \mathbb{R}/\mathbb{Z} valued 2-cochain $\hat{B}_{k+1}^{\mathbb{R}/\mathbb{Z}}$ is the background 2-connection for the twisted $U(1)$ -1-symmetry. Since $\hat{B}_{k+1}^{\mathbb{R}/\mathbb{Z}}$ is \mathbb{R}/\mathbb{Z} -valued, the action amplitude should have the following gauge symmetry, even for \mathcal{M}^D that has a boundary,

$$\hat{B}_{k+1}^{\mathbb{R}/\mathbb{Z}} \rightarrow \hat{B}_{k+1}^{\mathbb{R}/\mathbb{Z}} + u_{k+1}^{\mathbb{Z}} \quad (108)$$

where $u_{k+1}^{\mathbb{Z}}$ is an arbitrary \mathbb{Z} -valued $(k+1)$ -cochain. But the above the action amplitude does not have this gauge symmetry. This problem can be fixed by including an additional term which vanishes when $\hat{B}_{k+1}^{\mathbb{R}/\mathbb{Z}} = 0$:

$$Z = \sum_{\{a_k^{\mathbb{R}/\mathbb{Z}}\}} e^{\pi i \int_{\mathcal{M}^D} \text{Sq}^{D-k-2} (d[da_k^{\mathbb{R}/\mathbb{Z}}] + \hat{B}_{k+1}^{\mathbb{R}/\mathbb{Z}} - [d(da_k^{\mathbb{R}/\mathbb{Z}} + \hat{B}_{k+1}^{\mathbb{R}/\mathbb{Z}})])}. \quad (109)$$

Such a theory has the following 2-gauge symmetry, even when \mathcal{M}^D has a boundary

$$\begin{aligned} a_k^{\mathbb{R}/\mathbb{Z}} &\rightarrow a_k^{\mathbb{R}/\mathbb{Z}} + u_k^{\mathbb{R}/\mathbb{Z}} \\ \hat{B}_{k+1}^{\mathbb{R}/\mathbb{Z}} &\rightarrow \hat{B}_{k+1}^{\mathbb{R}/\mathbb{Z}} - du_k^{\mathbb{R}/\mathbb{Z}} \end{aligned} \quad (110)$$

where $u_k^{\mathbb{R}/\mathbb{Z}}$ is an arbitrary \mathbb{R}/\mathbb{Z} -valued k -cochain.

Using eqn. (A26), we can show that, for closed \mathcal{M}^D ,

$$\begin{aligned} & e^{\pi i \int_{\mathcal{M}^D} \text{Sq}^{D-k-2} (d[da_k^{\mathbb{R}/\mathbb{Z}}] + \hat{B}_{k+1}^{\mathbb{R}/\mathbb{Z}} - [d(da_k^{\mathbb{R}/\mathbb{Z}} + \hat{B}_{k+1}^{\mathbb{R}/\mathbb{Z}})])} \\ &= e^{\pi i \int_{\mathcal{M}^D} \text{Sq}^{D-k-2} [d\hat{B}_{k+1}^{\mathbb{R}/\mathbb{Z}}]}. \end{aligned} \quad (111)$$

Therefore, the corresponding topological partition function of the gauged model is given by

$$Z^{\text{top}}(\mathcal{M}^D, \hat{B}_{k+1}^{\mathbb{R}/\mathbb{Z}}) = e^{\pi i \int_{\mathcal{M}^D} \text{Sq}^{D-k-2} [d\hat{B}_{k+1}^{\mathbb{R}/\mathbb{Z}}]} \quad (112)$$

for any closed \mathcal{M}^D . This non-trivial hSPT invariant implies that the model (97) or (109) describes a phase with a non-trivial hSPT order, when $D - k - 2 = \text{even}$ and $0 < D - k - 2 < k + 2$ or when $D - k = \text{even}$ and $k + 3 \leq D \leq 2k + 3$.

When $k = 1$, we have a model to realize a non-trivial 4+1D $U(1)$ -1-SPT phase

$$Z(\mathcal{M}^5) = \sum_{\{a^{\mathbb{R}/\mathbb{Z}}\}} e^{\pi i \int_{\mathcal{M}^5} \mathbb{S}q^2 d[da^{\mathbb{R}/\mathbb{Z}}]}. \quad (113)$$

When $k = 2$, we have a model to realize a non-trivial 5+1D $U(1)$ -2-SPT phase

$$Z(\mathcal{M}^6) = \sum_{\{b^{\mathbb{R}/\mathbb{Z}}\}} e^{\pi i \int_{\mathcal{M}^6} \mathbb{S}q^2 d[db^{\mathbb{R}/\mathbb{Z}}]}. \quad (114)$$

It turns out that the 4+1D hSPT phase described by eqn. (113) is very important for condensed matter. This is because all the EM condensed matter systems with dynamical EM fields must be a boundary of such a 4+1D hSPT phase (see Section XI).

VII. THE TOPOLOGICAL ROBUSTNESS OF EMERGENT HIGHER SYMMETRY

A. Translation invariant systems

The lattice model (10) has an exact Z_2 -1-symmetry generate by the membrane operator (12), since *the Z_2 charges are not mobile*. We can make the Z_2 charges mobile and break the Z_2 -1-symmetry by adding the term

$$\delta H = -J \sum_{\langle i \rangle} \sigma_i^x. \quad (115)$$

However when U_2 is very large, the Z_2 charges have a large energy gap of order $|U_2|$. The Z_2 charges do not even appear at low energies. In this case, we expect an emergence of Z_2 -1-symmetry at low energies even when $\delta H \neq 0$.

Indeed, it was shown in Ref. 12 that even though membrane operator (12) does not commute with the perturbed Hamiltonian $H + \delta H$, we can define fattened membrane operators

$$M_{\text{fat-memb}} = U_{\text{LU}} \left(\prod_{i \in \text{closed membrane}} \sigma_i^z \right) U_{\text{LU}}^\dagger, \quad (116)$$

where U_{LU} is the local unitary operator defined in Ref. 58. We can choose U_{LU} such that the low energy eigenstates are also the eigenstates of the fattened membrane operators. This indicates an emergence of Z_2 -1-symmetry at low energies.

Ref. 12 shows that such fattened membrane operators can be found for any local perturbation δH that can break any symmetries and higher symmetries. Thus the emergence of Z_2 -1-symmetry at low energies is robust

against any local perturbation. This represents a *topological robustness* of emergent of higher symmetry. In general, we believe the emergence of higher symmetry to be always topological, reflecting the topological robustness of topological orders.

In fact U_{LU} can be constructed using adiabatic evolution:¹²

$$U_{\text{LU}} = T[e^{-i \int_0^1 dt H(t)}], \quad H(t) \equiv H + t\delta H. \quad (117)$$

The degenerate ground states $|\psi'_\alpha\rangle$ of $H + \delta H$ can be obtained from the degenerate ground states $|\psi_\alpha\rangle$ of H :

$$|\psi'_\alpha\rangle = U_{\text{LU}} |\psi_\alpha\rangle. \quad (118)$$

We see that fattened membrane operators $M_{\text{fat-memb}}$ acts within the ground state subspace of $H + \delta H$, and generates the low energy emergent Z_2 -1-symmetry.

We like to remark that the generators of higher symmetry discussed in this paper (regardless on-site or not) are always finite-depth local quantum circuits. The fattened generators of higher symmetry are also finite-depth local quantum circuits. It is known that string operator that create a pair of non-Abelian anyons are not finite-depth local quantum circuits.^{83,84} The topological excitations associated with the string operators that generate higher symmetry are always Abelian anyons. However, it is not proven that string operators that generate Abelian anyons are always finite-depth local quantum circuits. We like to remark that string operators (linear-depth local quantum circuits) that generate non-Abelian anyons correspond to generalized higher symmetry, which is always anomalous.⁷⁰

B. Emergent higher symmetry and many-body localization

The lattice model (10) has an exact Z_2 -1-symmetry for systems of *any size and at any energy*. In the presence of a small perturbation δH , the model has an emergent Z_2 -1-symmetry for *large systems at low energies*. Since the essence of Z_2 -1-symmetry is that the pointlike topological excitations are not mobile, we can use many-body localization to realize a stronger emergent Z_2 -1-symmetry for *large systems at any energy*.⁸⁵⁻⁸⁷

We first consider the model

$$H = - \sum_{\langle ijkl \rangle} U_1(\langle ijkl \rangle) \sigma_i^x \sigma_j^x \sigma_k^x \sigma_l^x - \sum_{\langle ijklmn \rangle} U_2(\langle ijklmn \rangle) \sigma_i^z \sigma_j^z \sigma_k^z \sigma_l^z \sigma_m^z \sigma_n^z \quad (119)$$

where $U_1(\langle ijkl \rangle)$ and $U_2(\langle ijklmn \rangle)$ strongly random positive numbers. The random $U_1(\langle ijkl \rangle)$ make the Z_2 -flux-loop s to have a random tension. The random $U_2(\langle ijklmn \rangle)$ make the Z_2 -charge e to have a random energy. In such a model, there is no Z_2 -flux-loop hopping term nor Z_2 -charge hopping term. The Z_2 -flux-loop s

cannot change its shape and the Z_2 -charge e cannot move around. As a result, eqn. (119) has a Z_2 -1-symmetry generated by (see eqn. (9))

$$W(C^2) = \prod_{i \in C^2} \sigma_i^z \quad (120)$$

and a Z_2 -2-symmetry generated by

$$W(\tilde{C}^1) = \prod_{i \in \tilde{C}^1} \sigma_i^x \quad (121)$$

where \tilde{C}^1 is a closed string formed by the links of the dual cubic lattice.

After we add a small perturbation δH , due to the strong randomness of the energies of the Z_2 -charge and the Z_2 -flux, many-body localization may happen, and the Z_2 -charge and the Z_2 -flux are still not mobile. In this case, there are emergent Z_2 -1-symmetry and Z_2 -2-symmetry for *large systems at any energy*.

C. Continuous higher symmetry and gapless cases

Next, we briefly discuss continuous higher symmetry and gapless cases. The emergence of 3+1D gapless $U(1)$ gauge theory is also accompanied with an emergence of $U(1)$ -1-symmetries, if the $U(1)$ -charges and the $U(1)$ -monopoles have a large energy gap. It was shown that the emergence of such higher symmetries to be topological.¹² The topological robustness of the emergent $U(1)$ -1-symmetries (which was called the local $U(1)$ gauge symmetries in Ref. 12) is used to show the topological robustness of the gapless $U(1)$ gauge theory:

*There are no local perturbations that can open an energy gap for the gapless $U(1)$ gauge bosons.*¹²

VIII. GENERIC HIGHER SYMMETRY IN SPACETIME LATTICE MODELS

In this section, we will construct lattice model with a combined 0-symmetry and 1-symmetry. The mixture of the 0-symmetry and 1-symmetry can be quite non-trivial. We also like to include background gauge field and higher gauge field that describe the spacetime twist of the 0-symmetry and 1-symmetry. But before describing the mixture of the 0-symmetry and 1-symmetry, we will first review a particular construction of spacetime lattice models with global on-site symmetry G (*i.e.* 0-symmetry). This particular construction can be generalized to obtain lattice model with a combined 0-symmetry and 1-symmetry.

A. Models with 0-symmetry and 0-symmetry twist

To describe a 0-symmetry described by a finite group G , we consider a spacetime lattice model with a field g_i

living on vertices. The 0-symmetry lives on the closed D -subcomplex of the dual spacetime complex $\tilde{\mathcal{M}}^D$ (*i.e.* the dual of the vertices of \mathcal{M}^D), which generate the following transformation

$$g_i \rightarrow gg_i, \quad g \in G. \quad (122)$$

The 0-symmetry invariant lattice model

$$Z = \sum_{\{g_i\}} e^{-\int_{\mathcal{M}^D} L(g_i)} \quad (123)$$

satisfies

$$L(g_i) = L(hg_i), \quad h \in G. \quad (124)$$

The Lagrangian $L(g_i)$ (a D -cochain) can be “gauged” to obtain $L(g_i, \hat{A}_{ij})$ with a non-dynamical flat gauge connection $\hat{A}_{ij} \in G$:

$$\hat{A}_{ij} \hat{A}_{jk} = \hat{A}_{ik}. \quad (125)$$

\hat{A}_{ij} is also called the symmetry twist. The “gauged” Lagrangian has a 1-gauge symmetry

$$L(g_i, \hat{A}_{ij}) = L(h_i g_i, h_i \hat{A}_{ij} h_i^{-1}), \quad h_i \in G. \quad (126)$$

In the following, we will choose the value of the g_i field to be the symmetry group G . Using the above symmetry, we can rewrite

$$L(g_i, \hat{A}_{ij}) = L(1, g_i^{-1} \hat{A}_{ij} g_j) = L(A_{ij}), \quad (127)$$

where A_{ij} is the effective field

$$A_{ij} = g_i^{-1} \hat{A}_{ij} g_j. \quad (128)$$

The partition function now can be written as

$$Z = \sum_{\{g_i\}} e^{-\int_{\mathcal{M}^D} L(A_{ij})}, \quad A_{ij} = g_i^{-1} \hat{A}_{ij} g_j. \quad (129)$$

We remark that eqn. (129) describes a G symmetric system in a background of twisted 0-symmetry. The twisted 0-symmetry is described by a connection \hat{A}_{ij} . We may also view the connection \hat{A}_{ij} as a probe of the G 0-symmetry.

We also like to remark that the effective field A_{ij} in eqn. (129) describes a flat connection

$$A_{ij} A_{jk} = A_{ik}. \quad (130)$$

The summation $\sum_{\{g_i\}}$ sums over all gauge equivalent configuration that correspond to the same flat G -bundle. In fact, we can view g_i as the gauge transformation, and thus $\sum_{\{g_i\}}$ sums over all gauge transformations.

Last, we note that $L(A_{ij})$ can be viewed as a Lagrangian of a lattice gauge theory (*i.e.* 1-gauge theory). Here we construct a lattice theory with a 0-symmetry twist by starting with a Lagrangian for lattice 1-gauge theory, and doing the path integral by only summing over the 1-gauge configurations within one gauge equivalent class. We will use the similar approach to construct lattice model with higher symmetry, with a higher symmetry twist.

B. Models with a combined 0-symmetry and 1-symmetry and their twist

To construct a model with a combined 0-symmetry and 1-symmetry, we include an extra bosonic field a_{ij} living on the links ij . The value of a_{ij} is taken from an Abelian group Π_2 . We start with the Lagrangian in terms of the effective fields A_{ij} and B_{ijk} . Here B_{ijk} is a Π_2 -valued 2-cochain field living on the triangles ijk . The 1-cochain field A_{ij} is flat as before (see eqn. (130)). The 2-cochain field B_{ijk} may not be flat

$$dB = n_3(A), \quad (131)$$

To understand $n_3(A)$, we note that, as explained in Ref. 30, the field A on the links satisfying (130) define a map $\mathcal{M}^D \xrightarrow{\phi} \mathcal{BG}$ (or more precisely a homomorphism of simplicial complexes). Then $n_3(A)$ is given by $n_3(A) = \phi^* \bar{n}_3$, where $\bar{n}_3 \in H^3(\mathcal{BG}, \Pi_2)$. Note that \bar{n}_3 is a cocycle on the classifying space \mathcal{BG} , while $n_3(A)$ lives on \mathcal{M}^D . Thus $n_3(A)$ is the pullback of \bar{n}_3 on \mathcal{BG} by the homomorphism ϕ . We see that the map ϕ must satisfy a property that the pullback of \bar{n}_3 is a coboundary on \mathcal{M}^D . (For details, see Ref. 30 and 77.)

The higher gauge transformations are generated by g_i, a_{ij} :

$$\begin{aligned} A_{ij} &\rightarrow g_i^{-1} A_{ij} g_j, \\ B_{ijk} &\rightarrow B_{ijk} + a_{ij} + a_{jk} - a_{ik} + \xi_{ijk}(A_{ij}, g_i), \end{aligned} \quad (132)$$

where $\xi_{ijk}(A_{ij}, g_i)$ is given by

$$d\xi(A_{ij}, g_i) = n_3(g_i^{-1} A_{ij} g_j) - n_3(A_{ij}). \quad (133)$$

Here eqn. (132) is called a 2-gauge transformation.

Let $L(A_{ij}, B_{ijk})$ be a D -cochain that depends on A and B . Then summing over all the 2-gauge transformations (132)

$$Z = \sum_{\{g_i, a_{ij}\}} e^{-\int_{\mathcal{M}^D} L(A_{ij}, B_{ijk})} \quad (134)$$

will give us a bosonic model with a non-trivially combined 0-symmetry and 1-symmetry. Here

$$\begin{aligned} A_{ij} &= g_i^{-1} \hat{A}_{ij} g_j, \\ B_{ijk} &= \hat{B}_{ijk} + a_{ij} + a_{jk} - a_{ik} + \xi_{ijk}(\hat{A}_{ij}, g_i), \end{aligned} \quad (135)$$

g_i, a_{ij} are dynamical fields, and $\hat{A}_{ij}, \hat{B}_{ijk}$ are non-dynamical background 2-gauge connections satisfying

$$\hat{A}_{ij} \hat{A}_{jk} = \hat{A}_{ik}, \quad d\hat{B} = n_3(\hat{A}). \quad (136)$$

Note that here $L(A_{ij}, B_{ijk})$ can be any function of A_{ij}, B_{ijk} . In particular, it does not have to be invariant under the higher gauge transformation (135). The model eqn. (134) has a combined global 0-symmetry and

1-symmetry when $\hat{A}_{ij} = 1$ and $\hat{B}_{ijk} = 0$. The combined 0-symmetry and 1-symmetry is generated by

$$g_i \rightarrow gg_i, \quad a_{ij} \rightarrow a_{ij} + \alpha_{ij}; \quad g \in G, \quad \alpha_{ij} \in \Pi_2, \quad d\alpha = 0. \quad (137)$$

(Note that $\xi_{ijk}(\hat{A}_{ij} = 1, g_i) = \xi_{ijk}(\hat{A}_{ij} = 1, gg_i)$.) In particular, the global 1-symmetry transformation changes the 1-cochain field a by a cocycle. (We can view the 1-cochain field a as a field on $(D-1)$ -cells of the dual complex $\tilde{\mathcal{M}}^D$. The global 1-symmetry transformation changes the 1-cochain field a by a constant on the $(D-1)$ -cells of a closed the $(D-1)$ -dimensional (or codimension-1) complex in the dual complex $\tilde{\mathcal{M}}^D$.)

We point out that eqn. (134) describes a system with 0-symmetry and 1-symmetry on a background of twisted 0-symmetry and 1-symmetry. The twisted 0-symmetry is described by the 1-connection $\hat{A}_{ij} \in G$. The twisted 1-symmetry is described by the 2-connection $\hat{B}_{ijk} \in \Pi_2$, which is a Π_2 -valued 2-cochain satisfying $d\hat{B} = n_3(\hat{A})$.

We like to remark that in our above construction of lattice models, we started with a lattice 2-gauge theory. However, in our construction, the 2-gauge invariant field strength is a non-dynamical background field. The pure 2-gauge transformations are our dynamical fields. Such a lattice model has a combined global 0-symmetry and 1-symmetry. We point out that the above construction can also be used to construct lattice models with a combined global 0-symmetry, 1-symmetry, and 2-symmetry, by starting with 3-gauge theories. In general, lattice models with higher symmetry can be constructed by starting from lattice higher gauge theories,³⁰ where the higher field strength corresponds to fixed higher symmetry twist, and the dynamical fields come from the higher gauge transformations.

IX. LATTICE MODELS THAT REALIZE HIGHER SPT PHASES – SYSTEMATIC CONSTRUCTIONS

A. Models realizing bosonic SPT phases

After constructing models with on-site 0-symmetry eqn. (129), we can choose $L(A)$ to be a $2\pi i \mathbb{R}/\mathbb{Z}$ -valued cocycle

$$Z(\hat{A}) = \sum_{\{g_i\}} e^{2\pi i \int_{\mathcal{M}^D} \omega_D(A_{ij})}, \quad A_{ij} = g_i^{-1} \hat{A}_{ij} g_j, \quad (138)$$

where $\omega_D(A_{ij}) = \phi^* \bar{\omega}_D$, $\bar{\omega}_D$ is a cocycle $\bar{\omega}_D \in H^D(\mathcal{BG}, \mathbb{R}/\mathbb{Z})$ and \mathcal{BG} is the classifying space of G . Note that $\bar{\omega}_D$ lives on \mathcal{BG} , while $\omega_D(A_{ij})$ lives on \mathcal{M}^D . Thus $\omega_D(A_{ij})$ is the pullback of $\bar{\omega}_D$ by the map $\mathcal{M}^D \xrightarrow{\phi} \mathcal{BG}$ determined by the 1-cochain field A_{ij} : $\omega_D(A_{ij}) = \phi^* \bar{\omega}_D$. The above exactly soluble model realizes a bosonic G -SPT state characterized by cocycle $\bar{\omega}_D \in H^D(\mathcal{BG}, \mathbb{R}/\mathbb{Z})$. For more details and a more precise description of the

above model and the notations, see, for examples, Ref. 76 and Ref. 30.

B. Models realizing bosonic higher SPT phases

We have seen that using $\bar{\omega}_D \in H^D(\mathcal{B}(G, \mathbb{R}/\mathbb{Z}))$, we construct exactly soluble bosonic models that realize SPT phases protected by symmetry G . Similarly, using $\bar{\omega}_D \in H^D[\mathcal{B}(G, \Pi_2); \mathbb{R}/\mathbb{Z}]$, we construct exactly soluble bosonic models that realize hSPT phases protected by a combined 0-symmetry and 1-symmetry described by $\mathcal{B}(G, \Pi_2)$.

Starting with the model (134) with a combined 0-symmetry and 1-symmetry, we can choose $L(A, B)$ to obtain an exactly soluble model

$$\begin{aligned} Z &= \sum_{\{g_i, a_{ij}\}} e^{2\pi i \int_{\mathcal{M}^D} \omega_D(A, B)}, \\ A_{ij} &= g_i^{-1} \hat{A}_{ij} g_j, \\ B_{ijk} &= \hat{B}_{ijk} + a_{ij} + a_{jk} - a_{ik} + \xi_{ijk}(\hat{A}, g), \end{aligned} \quad (139)$$

where the dynamical field on vertices is $g_i \in G$ and the dynamical field on links is $a_{ij} \in \Pi_2$. Here $\omega_D(A, B) = \phi^* \bar{\omega}_D$, $\bar{\omega}_D \in H^D(\mathcal{B}(G, \Pi_2), \mathbb{R}/\mathbb{Z})$ and $\mathcal{B}(G, \Pi_2)$ is the classifying space of a 2-group.³⁰ We call $\bar{\omega}_D$ a 2-group cocycle. Also ϕ is the map $\mathcal{M}^D \xrightarrow{\phi} \mathcal{B}(G, \Pi_2)$ as determined by the fields A_{ij}, B_{ijk} . In fact, the homomorphism ϕ and the fields A, B on \mathcal{M}^D are directly related in the following way

$$A = \phi^* \bar{A}, \quad B = \phi^* \bar{B} \quad (140)$$

where \bar{A} is the G -valued canonical 1-cochain on $\mathcal{B}(G, \Pi_2)$ and \bar{B} is the Π_2 -valued canonical 2-cochain on $\mathcal{B}(G, \Pi_2)$. For more details, see, for example, Ref. 30.

The model (134) realizes a hSPT phase with a higher symmetry described by 2-group $\mathcal{B}(G, \Pi_2)$.¹⁸ The hSPT phases are systematically constructed via \mathbb{R}/\mathbb{Z} -valued D -cocycles ω_D on the classifying space $\mathcal{B}(G, \Pi_2)$.

For more general higher group $\mathcal{B}(G, \Pi_2, \Pi_3, \dots)$, we note that higher group admits a special one-vertex triangulation. The resulting complex is a simplicial set (see for example Ref. 30). We will use the same symbol $\mathcal{B}(G, \Pi_2, \Pi_3, \dots)$ to denote such a simplicial set. Using the simplicial set, an exactly soluble local bosonic model that realize a higher gauge theory with gauge group $\mathcal{B}(G, \Pi_2, \Pi_3, \dots)$ can be constructed³⁰

$$Z(\mathcal{M}^D) = \sum_{\phi} e^{2\pi i \int_{\mathcal{M}^D} \phi^* \bar{\omega}_D}, \quad (141)$$

where \sum_{ϕ} sums over all the simplicial-complex homomorphisms $\mathcal{M}^D \xrightarrow{\phi} \mathcal{B}(G, \Pi_2, \Pi_3, \dots)$. Here $\bar{\omega}_D$ is a \mathbb{R}/\mathbb{Z} valued D -cocycle on $\mathcal{B}(G, \Pi_2, \Pi_3, \dots)$:

$$\bar{\omega}_D \in H^D[\mathcal{B}(G, \Pi_2, \Pi_3, \dots); \mathbb{R}/\mathbb{Z}], \quad (142)$$

and $\phi^* \bar{\omega}_D$ is the pullback of the cocycle on $\mathcal{B}(G, \Pi_2, \Pi_3, \dots)$ to \mathcal{M}^D . We note that the model (141) realizes a topologically ordered phase described by a higher gauge theory (which is not a hSPT phase).

To obtain a model that realizes a hSPT phase with trivial topological order, we note that the simplicial-complex homomorphisms $\mathcal{M}^D \xrightarrow{\phi} \mathcal{B}(G, \Pi_2, \Pi_3, \dots)$ can be divided into many different homotopy classes. Each class correspond to gauge equivalent configurations. So we can label the homotopy classes as $[\phi]$, which are formed by all the configurations that are homotopic to ϕ . We may also label the homotopy classes as $[A, B, \dots]$ where A, B, \dots are the higher gauge connections, and $[A, B, \dots]$ are formed by all the configurations that are gauge equivalent (*i.e.* homotopic) to A, B, \dots . $[\phi]$ and $[A, B, \dots]$ are just two notations for the same thing.

Now we generate the gauge equivalent configurations in the class $[A, B, \dots]$ by gauge transformations g, a_{ij}, \dots

$$A^g, B^{a, g}, \dots \in [A, B, \dots]. \quad (143)$$

We may also rewrite the above as

$$\phi^{g, a, \dots} \in [\phi], \quad (144)$$

where $\phi^{g, a}$ is the homomorphism obtained from ϕ by 2-gauge transformation g, a (see eqn. (132)). We note that the number of gauge transformations g, a, \dots and the number of configurations in $[\phi]$ may not be the same, since some different gauge transformations may give rise to the same homomorphism $\phi^{g, a, \dots} = \phi^{g', a', \dots}$.

With the above notation, we can write down the local bosonic model that realize a hSPT phase

$$Z(\mathcal{M}^D, \hat{\phi}) = \sum_{g, a, \dots} e^{2\pi i \int_{\mathcal{M}^D} \hat{\phi}^{g, a, \dots} \bar{\omega}_D}. \quad (145)$$

Comparing to the higher gauge theory eqn. (141) here we just change the dynamics of the field ϕ by restricting it to a homotopy class $[\hat{\phi}]$. We note that ϕ in each homotopy class $[\hat{\phi}]$ give rise to the same $e^{2\pi i \int_{\mathcal{M}^D} \phi^* \bar{\omega}_D}$ for closed \mathcal{M}^D . Thus

$$Z(\mathcal{M}^D, \hat{\phi}) = (|G|^{N_v} |\Pi_2|^{N_l} |\Pi_3|^{N_t} \dots) e^{2\pi i \int_{\mathcal{M}^D} \hat{\phi}^* \bar{\omega}_D}, \quad (146)$$

where N_v, N_l, N_t, \dots are the numbers of vertices, links, triangles, \dots , in the spacetime complex \mathcal{M}^D . We see that the topological partition function is given by

$$Z^{\text{top}}(\mathcal{M}^D, \hat{\phi}) = e^{2\pi i \int_{\mathcal{M}^D} \hat{\phi}^* \bar{\omega}_D}, \quad (147)$$

which is the hSPT invariant charactering the hSPT phase.

C. More general bosonic hSPT phases

The bosonic model eqn. (145) does not realize all possible bosonic hSPT phases. To obtain more general bosonic

hSPT phases protected by higher symmetry described by higher group $\mathcal{B}(G, \Pi_2, \dots)$, we can replace the symmetry group G by G^{SO} , as proposed in Ref. 56 and Ref. 77:

$$G_{SO} = G \rtimes SO_\infty. \quad (148)$$

We arrive at the following local bosonic model

$$Z(\mathcal{M}^D, \hat{\phi}_{G_{SO}}) = \sum_{g^{G_{SO}}, a^{\Pi_2}} e^{2\pi i \int_{\mathcal{M}^D} \hat{\phi}_{G_{SO}}^{*,g^{G_{SO}}, a^{\Pi_2}} \bar{\omega}_D^{G_{SO}}}, \quad (149)$$

where $\bar{\omega}_D^{G_{SO}} \in H^D[\mathcal{B}(G_{SO}, \Pi_2, \dots); \mathbb{R}/\mathbb{Z}]$, $\hat{\phi}_{G_{SO}}$ is a simplicial-complex homomorphism $\mathcal{M}^D \xrightarrow{\hat{\phi}_{G_{SO}}} \mathcal{B}(G_{SO}, \Pi_2, \dots)$, and $\sum_{g^{G_{SO}}, a^{\Pi_2}}$ sums over all the gauge transformations described by $g^{G_{SO}}, a^{\Pi_2}, \dots$. Also, $\hat{\phi}_{G_{SO}}^{g^{G_{SO}}, a^{\Pi_2}}$ is the homomorphism obtained from the $\hat{\phi}_{G_{SO}}$ by gauge transformation $g^{G_{SO}}, a^{\Pi_2}, \dots$. We stress that here $\hat{\phi}_{G_{SO}}$ is not an arbitrary homomorphism from \mathcal{M}^D to $\mathcal{B}(G_{SO}, \Pi_2, \dots)$. We note that a homomorphism $\hat{\phi}_{G_{SO}}$ give rise to a G_{SO} gauge configuration $\hat{A}^{G_{SO}} = \hat{\phi}_{G_{SO}}^* \bar{A}^{G_{SO}}$ on \mathcal{M}^D , where $\bar{A}^{G_{SO}}$ is the canonical 1-cochain on $\mathcal{B}(G_{SO}, \Pi_2, \dots)$. Since $\hat{A}_{ij}^{G_{SO}} \in G_{SO}$, we can use the natural projection $G_{SO} \xrightarrow{\pi} SO_\infty$ to obtain $\hat{A}_{ij}^{SO} = \pi(\hat{A}_{ij}^{G_{SO}}) \in SO_\infty$. We require \hat{A}_{ij}^{SO} to be the connection of the tangent bundle of \mathcal{M}^D . The resulting model (149) realizes more general bosonic hSPT phases.

In the presence of time-reversal symmetry, the symmetry group is given by $G = G_0 \rtimes Z_2^T$, where Z_2^T is the time-reversal symmetry group. In this case we replace G by G_O , as proposed in Ref. 56 and Ref. 77:

$$G_O = G \rtimes SO_\infty = G_0 \rtimes Z_2^T \rtimes SO_\infty = G_0 \rtimes O_\infty \quad (150)$$

since $O_\infty = Z_2^T \rtimes SO_\infty$. We obtain the following local bosonic model

$$Z(\mathcal{M}^D, \hat{\phi}_{G_O}) = \sum_{g^{G_O}, a^{\Pi_2}} e^{2\pi i \int_{\mathcal{M}^D} \hat{\phi}_{G_O}^{*,g^{G_O}, a^{\Pi_2}} \bar{\omega}_D^{G_O}}, \quad (151)$$

where $\bar{\omega}_D^{G_O} \in H^D[\mathcal{B}(G_O, \Pi_2, \dots); \mathbb{R}/\mathbb{Z}]$, $\hat{\phi}_{G_O}$ is a simplicial-complex homomorphism $\mathcal{M}^D \xrightarrow{\hat{\phi}_{G_O}} \mathcal{B}(G_O, \Pi_2, \dots)$, and $\sum_{g^{G_O}, a^{\Pi_2}}$ sums over all the higher gauge transformations of the higher group $\mathcal{B}(G_O, \Pi_2, \dots)$. Again, $\hat{\phi}_{G_O}$ is not an arbitrary homomorphism from \mathcal{M}^D to $\mathcal{B}(G_O, \Pi_2, \dots)$. We require $\hat{A}_{ij}^O = \pi(\hat{A}_{ij}^{G_O}) \in O_\infty$ to be the connection of the tangent bundle of \mathcal{M}^D , where $\hat{A}^{G_O} = \hat{\phi}_{G_O}^* \bar{A}^{G_O}$. The resulting model (151) realizes more general bosonic hSPT phases.

D. Fermionic hSPT phases

With the above general construction of bosonic models to realize bosonic hSPT phases, we can use higher

dimensional bosonization^{65,76} to obtain fermionic models to realize fermionic hSPT phases. Such a construction is closely related to the fermion worldline decoration⁷⁷, and does not produce all possible fermionic hSPT phases.

Without time reversal symmetry, we consider the following bosonized local fermion model

$$Z(\mathcal{M}^D, \hat{\phi}_{G_{fSO}}) = \sum_{g^{G_{fSO}}, a^{\Pi_2}} e^{2\pi i \int_{\mathcal{M}^D} \hat{\phi}_{G_{fSO}}^{*,g^{G_{fSO}}, a^{\Pi_2}} \bar{\nu}_D^{G_{fSO}}} \times e^{2\pi i \int_{\mathcal{N}^{D+1}} \hat{\phi}_{SO}^{*,g^{G_{fSO}}, a^{\Pi_2}} \bar{\omega}_{D+1}^{SO}}. \quad (152)$$

The model is build using the following data

1. A fermion higher symmetry described by higher group $\mathcal{B}(G_{fSO}, \Pi_2, \dots)$, where

$$G_{fSO} = G_f \rtimes SO_\infty \quad (153)$$

and $G_f = Z_2^f \rtimes G_b$ is the fermion 0-symmetry group. The higher group $\mathcal{B}(G_{fSO}, \Pi_2, \dots)$ has the canonical G_{fSO} -valued 1-cochain $\bar{A}^{G_{fSO}}$, the canonical Π_2 -valued 2-cochain $\bar{B}^{G_{fSO}}$ that satisfy $d\bar{B}^{G_{fSO}} = \bar{n}_3(\bar{A}^{G_{fSO}})$, etc (see Ref. 30 and Ref. 77).

2. A higher group $\mathcal{B}_f(G_{fSO}, 1; Z_2, D-1)$ with the canonical SO_∞ -valued 1-cochain \bar{A}^{SO} , the canonical \mathbb{Z}_2 -valued $(D-1)$ -cochain \bar{f}_{D-1} that satisfy $d\bar{f}_{D-1} \stackrel{=}{=} 0$ (see Ref. 30 and Ref. 77).
3. A \mathbb{R}/\mathbb{Z} -valued $(D+1)$ -cocycle

$$\bar{\omega}_{D+1}^{SO} \stackrel{=}{=} \frac{1}{2} \text{Sq}^2 \bar{f}_{D-1} + \frac{1}{2} \bar{f}_{D-1} \bar{w}_2(\bar{A}^{SO}) \quad (154)$$

on the higher group $\mathcal{B}_f(SO_\infty, 1; Z_2, D-1)$.

4. A trivialization homomorphisms $\varphi : \mathcal{B}(G_{fSO}, \Pi_2, \dots) \rightarrow \mathcal{B}_f(SO_\infty, 1; Z_2, D-1)$.
5. A choice of trivialization, i.e. a \mathbb{R}/\mathbb{Z} -valued D -cochain on $\mathcal{B}(G_{fSO}, \Pi_2, \dots)$ that satisfies

$$-d\bar{\nu}_D^{G_{fSO}} \stackrel{=}{=} \varphi^* \bar{\omega}_{D+1}^{SO}. \quad (155)$$

The above data, in fact, gives us a partial classification of fermionic hSPT phases without time reversal symmetry.

Now let us explain the compact notation eqn. (152) that describes the model

1. The model is defined on a spacetime complex \mathcal{M}^D .
2. The model has a higher symmetry described by a higher group $\mathcal{B}(G_{fSO}, \Pi_2, \dots)$. However, there is a twist of the higher symmetry described by the background higher connection on \mathcal{M}^D . Such a background higher connection is encoded in $\hat{\phi}^{G_{fSO}}$, which is a simplicial-complex homomorphism $\mathcal{M}^D \xrightarrow{\hat{\phi}^{G_{fSO}}} \mathcal{B}(G_{fSO}, \Pi_2, \dots)$. $\hat{\phi}_{G_{fSO}}$ is not

an arbitrary homomorphism. We require $\hat{A}_{ij}^{SO} = \pi(\hat{A}_{ij}^{GfSO}) \in SO_\infty$ to be the connection of the tangent bundle of \mathcal{M}^D , where $\hat{A}^{GfSO} = \hat{\phi}_{GfSO}^* \bar{A}^{GfSO}$ and π is the natural projection $G_{fSO} \xrightarrow{\pi} SO_\infty$.

3. $\sum_{g^{GfSO}, a^{\Pi_2}, \dots}$ is a summation of all the higher gauge transformations described by $g^{GfSO}, a^{\Pi_2}, \dots$ (see eqn. (132)). Here g^{GfSO} lives on the vertices of \mathcal{M}^D : $g_i^{GfSO} \in G_{fSO}$, a^{Π_2} lives on the links of \mathcal{M}^D : $a_{ij}^{GfSO} \in \Pi_2$, etc. $g^{GfSO}, a^{\Pi_2}, \dots$ are the dynamical fields in our model.
4. $\hat{\phi}_{GfSO}^{g^{GfSO}, a^{\Pi_2}}$ is the higher connection obtained from the background higher connection $\hat{\phi}_{GfSO}$ via the higher gauge transformation $g^{GfSO}, a^{\Pi_2}, \dots$.
5. \mathcal{N}^{D+1} is a $(D+1)$ -dimensional complex whose boundary is \mathcal{M}^D : $\partial\mathcal{N}^{D+1} = \mathcal{M}^D$.
6. $\hat{\phi}_{SO}^{g^{GfSO}, a^{\Pi_2}}$ is a simplicial-complex homomorphism $\mathcal{N}^{D+1} \xrightarrow{\hat{\phi}_{SO}^{g^{GfSO}, a^{\Pi_2}}} \mathcal{B}_f(SO_\infty, 1; \mathbb{Z}_2, D-1)$. When restricted to the boundary $\mathcal{M}^D = \partial\mathcal{N}^{D+1}$, it satisfies $\hat{\phi}_{SO}^{g^{GfSO}, a^{\Pi_2}} = \varphi \hat{\phi}_{GfSO}^{g^{GfSO}, a^{\Pi_2}}$:

$$\begin{array}{ccc} & \mathcal{B}(G_{fSO}, \Pi_2, \dots) & \\ \hat{\phi}_{GfSO}^{g^{GfSO}, a^{\Pi_2}} \nearrow & \downarrow \varphi & \\ \partial\mathcal{N}^{D+1} \xrightarrow{\hat{\phi}_{SO}^{g^{GfSO}, a^{\Pi_2}}} & \mathcal{B}_f(SO_\infty, 1; \mathbb{Z}_2, D-1) & \end{array}$$

The resulting model (152) realizes fermionic hSPT phases without time reversal symmetry.

When we have only the usual global symmetry, *i.e.* when $\mathcal{B}(G_{fSO}, \Pi_2, \dots) = \mathcal{B}G_{fSO}$, the above model (152) reduces to the one discussed in Ref. 77, which realizes fermionic SPT phases. When $\bar{\omega}_{D+1}^{SO} = 0$, the model (152) reduces to eqn. (149) which realizes bosonic hSPT phases.

To include time reversal symmetry, we simply replace SO_∞ by O_∞ in the above construction. However, now $\bar{\omega}_{D+1}^O$ has two choices: If the fermions are Kramers singlet, we have

$$\bar{\omega}_{D+1}^{SO} \stackrel{!}{=} \frac{1}{2} \text{Sq}^2 \bar{f}_{D-1} + \frac{1}{2} \bar{f}_{D-1} \bar{w}_2(\bar{A}^O) \quad (156)$$

If the fermions are Kramers doublets, we have

$$\bar{\omega}_{D+1}^{SO} \stackrel{!}{=} \frac{1}{2} \text{Sq}^2 \bar{f}_{D-1} + \frac{1}{2} \bar{f}_{D-1} [\bar{w}_2(\bar{A}^O) + \bar{w}_1^2(\bar{A}^O)] \quad (157)$$

E. The usefulness of hSPT phases

But hSPT phases do not exist in natural condensed matter systems. This is because, similar to the usual SPT phase, a hSPT phase requires a higher symmetry.

Without higher symmetry, we simply do not have distinct hSPT phases. But in the usual condensed matter systems, we do not have higher symmetry. In the EM condensed matter systems with dynamical electromagnetic field, we either have gapless photon modes or have non-trivial topological orders (where the higher symmetry is spontaneously broken due to the superconductivity in the EM condensed matter systems). This is why we do not have hSPT phases in natural condensed matter systems. (But we may construct fine-tuned toy models experimentally to realize higher symmetries and hSPT phases.)

However, understanding the hSPT phases is still important in condensed matter. This is because the emergent higher symmetries in topological orders may be anomalous, which has physical consequences. We need to understand hSPT phases in order to understand the anomalous higher symmetry via the boundary of hSPT states.^{18,53} In the following, we will study a few simple hSPT phases. In particular, we will give examples of topological orders with emergent (anomalous) higher symmetries.

X. LATTICE MODELS THAT REALIZE TOPOLOGICAL ORDERS WITH HIGHER SYMMETRY

A. The first type of constructions

We have constructed models to realize a hSPT phase with a combined 0-symmetry and 1-symmetry, where we have summed over all 1-cochains a_{ij} in (134). To realize a topologically ordered phase with a combined 0-symmetry and 1-symmetry, we can change the dynamics of the a_{ij} field, by instead summing over only all 1-cocycles a_{ij} that satisfy $da = 0$:

$$Z = \sum_{\{g_i, da=0\}} e^{2\pi i \int_{\mathcal{M}^D} \omega_D(A, B)}, \quad A_{ij} = g_i^{-1} \hat{A}_{ij} g_j, \quad B = \hat{B} + da + \xi(\hat{A}, g), \quad (158)$$

Since $a_{ij} \in \Pi_2$ (*i.e.* a is a Π_2 -valued 1-cocycle), the above model realizes a topologically ordered state described by Abelian gauge theory with Π_2 gauge group. The model also has a combined 0-symmetry and 1-symmetry. In particular, the 1-symmetry is generated by shifting a by Π_2 -valued 1-cocycles.

The flux of the Π_2 gauge theory is described by a $D-2$ -cycle \tilde{F}_{D-2} in the dual spacetime complex \mathcal{M}^D . The charge of the Π_2 gauge theory is described by a 1-cycle C^1 (the worldline of the charge) in the spacetime complex \mathcal{M}^D . In the presence of the flux and charge, the path integral becomes

$$Z = \sum_{\{g_i, da=* \tilde{F}_{D-2}\}} e^{2\pi i \int_{\mathcal{M}^D} \omega_D(A, B) + 2\pi i \int_{C^1} a}, \quad A_{ij} = g_i^{-1} \hat{A}_{ij} g_j, \quad B = \hat{B} + da + \xi(\hat{A}, g), \quad (159)$$

where $*\tilde{F}_{D-2}$ is a two cocycle – the Poincaré dual of $D-2$ cycle \tilde{F}_{D-2} .

We note that the gauge charges are not mobile in the exactly soluble model eqn. (158). To make the gauge charges mobile, we need to add terms like $e^{-\lambda \int_{I_1} a}$ where I_1 is a 1-chain in the spacetime complex \mathcal{M}^D . Shifting a by a cocycle will change $e^{-\lambda \int_{I_1} a}$. So the term $e^{-\lambda \int_{I_1} a}$ will break the 1-symmetry. The term $e^{2\pi i \int_{C^1} a}$ will also break the 1-symmetry if C^1 is not a 1-boundary. Thus if we only allow the gauge flux, the Π_2 -gauge theory (159) (without the C^1 term) will have a combined 0-symmetry and 1-symmetry.

We like to stress that the above 1-symmetry can be emergent in the weak coupling phase of the gauge theory. As long as the gauge flux to be the only low energy excitations, we will have the emergent 1-symmetry in the in the weak coupling gauge theory, and we will have a combined 0-symmetry and 1-symmetry at low energies.

B. The second type of constructions

In the second type of constructions, we start with a model that realize a hSPT state with no topological order (see eqn. (139))

$$\begin{aligned} Z &= \sum_{\{g_i, a_{ij}\}} e^{2\pi i \int_{\mathcal{M}^D} \omega_D(A, B)}, \\ A_{ij} &= g_i^{-1} \hat{A}_{ij} g_j, \\ B_{ijk} &= \hat{B}_{ijk} + a_{ij} + a_{jk} - a_{ik} + \xi_{ijk}(\hat{A}, g), \end{aligned} \quad (160)$$

where \hat{A}_{ij}, g_i belong to a group G and \hat{B}_{ijk}, a_{ij} belong to an Abelian group Π_2 . Also ω_D is a \mathbb{R}/\mathbb{Z} -valued 2-group cocycle. We note that the above model is construct by starting with a fixed background 2-connection described by \hat{A}_{ij} and \hat{B}_{ijk} , and then include the “pure” 2-gauge transformations described by g_i and a_{ij} as dynamical fields.

To obtain a model that realizes a topological order, we can partially gauge the 2-group. This way, we obtain a combined 1-gauge and a 2-gauge theory with a combined 0-symmetry and 1-symmetry. To do so, we assume $G = G^g \rtimes G^s$ and $\Pi_2 = \Pi_2^g \rtimes \Pi_2^s$. G^s will be our 0-symmetry group and G^g the gauge group of the 1-gauge theory. We label the group elements of G by a pair

$$g = (g^g, g^s), \quad g^g \in G^g, \quad g^s \in G^s, \quad g_i \in G. \quad (161)$$

So we can denote the effective field A_{ij} as a pair $A_{ij} = (A_{ij}^g, A_{ij}^s)$. We have

$$\begin{aligned} A_{ij} &= (g_i^g, g_i^s)^{-1} (a_{ij}^g, \hat{A}_{ij}^s) (g_j^g, g_j^s) = g_i^{-1} (a_{ij}^g, \hat{A}_{ij}^s) g_j, \\ g_i^g, a_{ij}^g &\in G^g, \quad g_i^s, \hat{A}_{ij}^s \in G^s, \quad g_i \in G. \end{aligned} \quad (162)$$

Here \hat{A}_{ij}^s is the flat connection describing the G^s symmetry twist. a_{ij}^g is the dynamical gauge field satisfying

$$a_{ij}^g a_{jk}^g = a_{ik}^g, \quad (163)$$

so that a_{ij}^g describes the deconfined phase of the G^g 1-gauge theory. $g_i \in G$ is a dynamical scalar field carrying both symmetry charge and gauge charge.

Similarly, we denote the elements in Π_2 with a pair

$$h = (h^g, h^s), \quad h^g \in \Pi_2^g, \quad h^s \in \Pi_2^s, \quad h \in \Pi_2. \quad (164)$$

So we can write the effective dynamical field B as

$$B_{ijk} = (b_{ijk}^g, \hat{B}_{ijk}^s) + a_{ij} + a_{jk} - a_{ik} + \xi_{ijk}[(a_{ij}^g, \hat{A}_{ij}^s), g_i], \quad (165)$$

Here $\hat{B}_{ijk}^s \in \Pi_2^s$ is the background 2-connection describing the 1-symmetry twist of Π_2^s . $b_{ijk}^g \in \Pi_2^g$ is the dynamical 2-gauge field of Π_2^g , that satisfy

$$d(b^g, \hat{B}^s) = n_3[(a^g, \hat{A}^s)]. \quad (166)$$

$a_{ij} \in \Pi_2$ is a dynamical 1-cochain field.

Now, we can write down our partially gauged hSPT model:

$$Z = \sum_{\{g_i, a_{ij}, a_{ij}^g, b_{ijk}^g\}} e^{2\pi i \int_{\mathcal{M}^D} \omega_D(A, B)}, \quad (167)$$

where the 1-cochain effective field A and the 2-cochain effective field B are given by eqn. (162) and eqn. (165). The dynamical fields $g_i \in G$ and $a_{ij} \in \Pi_2$ can fluctuate arbitrarily. The dynamical fields $a^g \in G^g$ and $b^g \in \Pi_2^g$ cannot fluctuate arbitrarily and should satisfy the conditions eqn. (163) and eqn. (166), so that they describe deconfined phase of a 2-gauge theory.

We see that partially gauging is simply making part of the background connections dynamical

$$\begin{aligned} \hat{A}_{ij} &= (\hat{A}_{ij}^g, \hat{A}_{ij}^s) \rightarrow (a_{ij}^g, \hat{A}_{ij}^s), \\ \hat{B}_{ijk} &= (\hat{B}_{ijk}^g, \hat{B}_{ijk}^s) \rightarrow (b_{ijk}^g, \hat{B}_{ijk}^s). \end{aligned} \quad (168)$$

plus “pure” 2-gauge fluctuations described by dynamical fields g_i and a_{ij} . We note that, in our construction, the combined 1-gauge and 2-gauge theory plus its combined 0-symmetry and 1-symmetry together is described a 2-group $\mathcal{B}(G, \Pi_2)$ and a 2-group cocycle ω_D .

More generally, we may consider a simplicial-complex homomorphism φ between two higher groups

$$\mathcal{B}(G, \Pi_2, \Pi_3, \dots) \xrightarrow{\varphi} \mathcal{B}(G^s, \Pi_2^s, \Pi_3^s, \dots). \quad (169)$$

We assume φ to be surjective. Then using a \mathbb{R}/\mathbb{Z} -valued cocycle $\bar{\omega}_D$ on $\mathcal{B}(G, \Pi_2, \Pi_3, \dots)$, we can construct the following local bosonic model (see Section IX B)

$$Z(\mathcal{M}^D, \hat{\phi}^s) = \sum_{\{g, a, \dots\}} \sum_{\{\phi | \varphi \phi = \hat{\phi}^s\}} e^{2\pi i \int_{\mathcal{M}^D} \phi^{*; g, a, \dots} \bar{\omega}_D}. \quad (170)$$

The above model realize a topological order with a higher symmetry described by the higher group $\mathcal{B}(G^s, \Pi_2^s, \Pi_3^s, \dots)$. Here ϕ is a complex homomorphism

from \mathcal{M}^D to $\mathcal{B}(G, \Pi_2, \Pi_3, \dots)$ and $\hat{\phi}^s$ is a fixed simplicial-complex homomorphism from \mathcal{M}^D to $\mathcal{B}(G^s, \Pi_2^s, \Pi_3^s, \dots)$:

$$\begin{array}{ccc} & \mathcal{B}(G, \Pi_2, \dots) & \\ \nearrow \phi & & \downarrow \varphi \\ \mathcal{M}^D & \xrightarrow{\hat{\phi}^s} & \mathcal{B}(G^s, \Pi_2^s, \dots) \end{array}$$

$\hat{\phi}^s$ is the background connection that describes the higher symmetry twist on \mathcal{M}^D . The summation $\sum_{\{\phi|\varphi\phi=\hat{\phi}^s\}}$ sums over all the homomorphisms ϕ such that $\varphi\phi = \hat{\phi}^s$. The summation $\sum_{\{g,a,\dots\}}$ sums over all the higher gauge transformations of the higher group $\mathcal{B}(G, \Pi_2, \Pi_3, \dots)$.

XI. EM CONDENSED MATTER SYSTEMS AND THEIR HIGHER SYMMETRY

In our theoretical descriptions of condensed matter systems, we usually ignore the dynamical electromagnetic (EM) field. Those usual condensed matter theories in general do not have higher symmetries. However, more accurate theoretical descriptions of condensed matter systems should contain the dynamical EM field. In this section, we point out that those EM condensed matter theories with dynamical EM field actually have an *anomalous* higher symmetry if we ignore the magnetic monopoles.

The reason is very simple. A dynamical $U(1)$ gauge theory in 3+1D does not have higher symmetry if the mobile $U(1)$ charges and $U(1)$ monopoles appear in the interested energy scales (which are about 1eV for condensed matter physics). However, for the dynamical $U(1)$ EM gauge theory in condensed matter, although the mobile $U(1)$ charges appear at energy scales of 1eV (the energy gap of an insulator), the $U(1)$ monopoles can appear only beyond 100GeV. So the dynamical $U(1)$ EM gauge theory in condensed matter can be viewed as a dynamical $U(1)$ gauge theory without mobile $U(1)$ monopoles. Such a dynamical $U(1)$ gauge theory has a higher symmetry.¹¹

Using the general picture of higher symmetry and its relation to topological excitations developed in Section VIA 1, we see that the emergent higher symmetry in the EM condensed matter systems is characterized by low energy allowed EM charge excitations. It is generated by the topological excitations with trivial mutual statistics with the EM charge excitations. In other words, the higher symmetry is generated by the electric charged charge excitations. This appearance of higher symmetry in condensed matter systems has been noticed in some recent papers.^{88,89}

To see the higher symmetry in EM condensed matter systems more explicitly, let us consider a dynamical $U(1)$ gauge lattice gauge theory described by a lattice rotor model where the rotor angles correspond to the gauge

connection of the *dual* $\tilde{U}(1)$ gauge field:^{90–92}

$$\begin{aligned} H = & U \sum_{\tilde{\mathbf{i}}} \left(\sum_{\tilde{\mathbf{j}} \text{ next to } \tilde{\mathbf{i}}} \tilde{L}_{\tilde{\mathbf{i}}\tilde{\mathbf{j}}} \right)^2 + g \sum_{\langle \tilde{\mathbf{i}}\tilde{\mathbf{j}} \rangle} \tilde{L}_{\tilde{\mathbf{i}}\tilde{\mathbf{j}}} \\ & - J \sum_{\langle \tilde{\mathbf{i}}\tilde{\mathbf{j}}\tilde{\mathbf{k}}\tilde{\mathbf{l}} \rangle} \left(\tilde{L}_{\tilde{\mathbf{i}}\tilde{\mathbf{j}}}^+ \tilde{L}_{\tilde{\mathbf{j}}\tilde{\mathbf{k}}}^+ \tilde{L}_{\tilde{\mathbf{k}}\tilde{\mathbf{l}}}^+ \tilde{L}_{\tilde{\mathbf{l}}\tilde{\mathbf{i}}}^+ + h.c. \right) \end{aligned} \quad (171)$$

where $\tilde{\mathbf{i}}, \tilde{\mathbf{j}}, \tilde{\mathbf{k}}, \tilde{\mathbf{l}}$ label the sites of a dual cubic lattice $\tilde{\mathcal{M}}^3$, and $\tilde{L}_{\tilde{\mathbf{i}}\tilde{\mathbf{j}}} = -i\partial_{\tilde{\theta}_{\tilde{\mathbf{i}}\tilde{\mathbf{j}}}}$ is the angular momentum of the rotor $\tilde{\theta}_{\tilde{\mathbf{i}}\tilde{\mathbf{j}}} = -\tilde{\theta}_{\tilde{\mathbf{j}}\tilde{\mathbf{i}}}$ living on the link $\tilde{\mathbf{i}}\tilde{\mathbf{j}}$. Also $\tilde{L}_{\tilde{\mathbf{i}}\tilde{\mathbf{j}}}^\pm = e^{\pm i\tilde{\theta}_{\tilde{\mathbf{i}}\tilde{\mathbf{j}}}}$. The summation $\sum_{\langle \tilde{\mathbf{i}}\tilde{\mathbf{j}}\tilde{\mathbf{k}}\tilde{\mathbf{l}} \rangle}$ sums over all the square faces $\langle \tilde{\mathbf{i}}\tilde{\mathbf{j}}\tilde{\mathbf{k}}\tilde{\mathbf{l}} \rangle$ of the cubic lattice. The charge of the dual $\tilde{U}(1)$ corresponds to the monopole of the EM $U(1)$. The monopole of the dual $\tilde{U}(1)$ corresponds to the charge of the EM $U(1)$.

The higher symmetry is given by

$$W_{C^2} = e^{i\varphi \sum_{\langle \tilde{\mathbf{i}}\tilde{\mathbf{j}} \rangle \in C^2} \tilde{L}_{\tilde{\mathbf{i}}\tilde{\mathbf{j}}}}. \quad (172)$$

Where C^2 is a closed surface formed by the square faces the cubic lattice \mathcal{M}^3 . Since \mathcal{M}^3 and $\tilde{\mathcal{M}}^3$ are dual to each other, the links $\langle \tilde{\mathbf{i}}\tilde{\mathbf{j}} \rangle$ in the dual lattice $\tilde{\mathcal{M}}^3$ correspond to the square faces of the lattice \mathcal{M}^3 . Since C^2 has a codimension-1 in the 3-dimensional space, the higher symmetry is a $\tilde{U}(1)$ -1-symmetry. The above $\tilde{U}(1)$ 1-higher symmetry is generated by

$$W_{\tilde{\mathbf{i}}} = e^{i\varphi \sum_{\tilde{\mathbf{j}} \text{ next to } \tilde{\mathbf{i}}} \tilde{L}_{\tilde{\mathbf{i}}\tilde{\mathbf{j}}}}. \quad (173)$$

which leaves the above Hamiltonian invariant. $W_{\tilde{\mathbf{i}}}$ is also called the local gauge symmetry. Thus the $U(1)$ -1-symmetry is simply the dual $\tilde{U}(1)$ gauge symmetry. Such an $U(1)$ -1-symmetry forbids the term $\sum_{\langle \tilde{\mathbf{i}}\tilde{\mathbf{j}} \rangle} \tilde{L}_{\tilde{\mathbf{i}}\tilde{\mathbf{j}}}^+$ in the Hamiltonian. So the charge of the dual $\tilde{U}(1)$ (*i.e.* the monopole of the EM $U(1)$) is not mobile.

We can also describe the above dual $\tilde{U}(1)$ gauge theory using path integral of cochain fields on spacetime complex \mathcal{M}^4 :

$$Z = \sum_{\{\tilde{a}^{\mathbb{R}/\mathbb{Z}}, \dots\}} e^{-\int_{\mathcal{M}^4} L(d\tilde{a}^{\mathbb{R}/\mathbb{Z}}, \dots)} \quad (174)$$

where $\sum_{\{\tilde{a}^{\mathbb{R}/\mathbb{Z}}, \dots\}}$ sums over \mathbb{R}/\mathbb{Z} -valued 1-cochains $\tilde{a}^{\mathbb{R}/\mathbb{Z}}$ on spacetime dual complex $\tilde{\mathcal{M}}^4$, and possibly some other EM neutral bosonic fields represented by \dots . Since $\tilde{a}^{\mathbb{R}/\mathbb{Z}}$ is \mathbb{R}/\mathbb{Z} -valued, $L(d\tilde{a}^{\mathbb{R}/\mathbb{Z}}, \dots)$ is invariant under

$$\tilde{a}^{\mathbb{R}/\mathbb{Z}} \rightarrow \tilde{a}^{\mathbb{R}/\mathbb{Z}} + \tilde{\alpha}^{\mathbb{Z}}, \quad (175)$$

where $\tilde{\alpha}^{\mathbb{Z}}$ is any \mathbb{Z} -valued 1-cochain. The model has a $\tilde{U}(1)$ 1-symmetry, which is generated by shifting $\tilde{a}^{\mathbb{R}/\mathbb{Z}}$ by \mathbb{R}/\mathbb{Z} -valued 1-cocycles $\tilde{\alpha}$:

$$\tilde{a}^{\mathbb{R}/\mathbb{Z}} \rightarrow \tilde{a}^{\mathbb{R}/\mathbb{Z}} + \tilde{\alpha}^{\mathbb{R}/\mathbb{Z}}, \quad d\tilde{\alpha}^{\mathbb{R}/\mathbb{Z}} = 0. \quad (176)$$

When $L(d\tilde{a}^{\mathbb{R}/\mathbb{Z}}, \dots)$ restricts the fluctuations to be $d\tilde{a}^{\mathbb{R}/\mathbb{Z}} \approx$ integer cochain, then $\tilde{a}^{\mathbb{R}/\mathbb{Z}}$ will describe a $\tilde{U}(1)$ gauge field (the EM field) in semiclassical limit.

We note that $d[d\tilde{a}^{\mathbb{R}/\mathbb{Z}}]$ is a \mathbb{Z} -valued 3-coboundary that corresponds to the Poincaré dual of the worldlines of monopoles of the $\tilde{U}(1)$ gauge field. The monopoles of the $\tilde{U}(1)$ gauge field is the electric charges. So the Poincaré dual of $d[d\tilde{a}^{\mathbb{R}/\mathbb{Z}}]$ is the world line of the electric charges. We also note that $d[d\tilde{a}^{\mathbb{R}/\mathbb{Z}}]$ is invariant under eqn. (175), and is thus physical.

Now we see that if $L(d\tilde{a}^{\mathbb{R}/\mathbb{Z}}, \dots)$ further restricts the fluctuations to be $d\tilde{a}^{\mathbb{R}/\mathbb{Z}} \approx 0$, in this case $\tilde{a}^{\mathbb{R}/\mathbb{Z}}$ will describes a deconfined phase of $\tilde{U}(1)$ gauge field $\tilde{a}^{\mathbb{R}/\mathbb{Z}}$, which corresponds to an EM insulator. On the other hand, if $L(d\tilde{a}^{\mathbb{R}/\mathbb{Z}}, \dots)$ restricts $d\tilde{a}^{\mathbb{R}/\mathbb{Z}} \approx$ integer cochain, and allow strong fluctuations of $d[d\tilde{a}^{\mathbb{R}/\mathbb{Z}}]$, the system will be in a metallic or a superconducting phase.

However, in the model (174) worldlines of electric charges, described by $*d[d\tilde{a}^{\mathbb{R}/\mathbb{Z}}]$, are bosons. So the model (174) does not describe the EM condensed matter systems, where the odd charges of EM $U(1)$ are always fermions. To make the electric charge worldline $*d[d\tilde{a}^{\mathbb{R}/\mathbb{Z}}]$ to describe the worldlines of fermions, we can use the high dimensional bosonization.^{65,76} Thus the correct models that describe EM condensed matter systems are given by

$$Z = \sum_{\{\tilde{a}^{\mathbb{R}/\mathbb{Z}}, \dots\}} e^{-\int_{\mathcal{M}^4} L(d\tilde{a}^{\mathbb{R}/\mathbb{Z}}, \dots) + \pi i \int_{\mathcal{M}^4} \hat{A}^{\mathbb{Z}_2} d[d\tilde{a}^{\mathbb{R}/\mathbb{Z}}]} e^{\pi i \int_{\mathcal{N}^5} \mathbb{S}q^2 d[d\tilde{a}^{\mathbb{R}/\mathbb{Z}}] + w_2 d[d\tilde{a}^{\mathbb{R}/\mathbb{Z}}]}, \quad (177)$$

where $\partial\mathcal{N}^5 = \mathcal{M}^4$, w_k is the k^{th} Stiefel-Whitney class of the tangent bundle of \mathcal{N}^5 , and $\hat{A}^{\mathbb{Z}_2}$ is a spin structure

$$d\hat{A}^{\mathbb{Z}_2} \stackrel{2}{=} w_2. \quad (178)$$

The term $e^{\pi i \int_{\mathcal{N}^5} \mathbb{S}q^2 d[d\tilde{a}^{\mathbb{R}/\mathbb{Z}}]}$ makes $*d[d\tilde{a}^{\mathbb{R}/\mathbb{Z}}]$ to describe the worldlines of fermions. For details, see Ref. 77.

We like to remark that even though

$$e^{\pi i \int_{\mathcal{N}^5} \mathbb{S}q^2 d[d\tilde{a}^{\mathbb{R}/\mathbb{Z}}]} = e^{\pi i \int_{\mathcal{M}^4} \mathbb{S}q^2 [d\tilde{a}^{\mathbb{R}/\mathbb{Z}}]}, \quad (179)$$

the term $e^{\pi i \int_{\mathcal{N}^5} \mathbb{S}q^2 d[d\tilde{a}^{\mathbb{R}/\mathbb{Z}}]}$ cannot come from a local Lagrangian on \mathcal{M}^4 such as $e^{\pi i \int_{\mathcal{M}^4} \mathbb{S}q^2 [d\tilde{a}^{\mathbb{R}/\mathbb{Z}}]}$. This is because $\mathbb{S}q^2 [d\tilde{a}^{\mathbb{R}/\mathbb{Z}}] \bmod 2$ is not invariant under the gauge transformation (175), and is not allowed in a 3+1D Lagrangian. Thus $e^{\pi i \int_{\mathcal{N}^5} \mathbb{S}q^2 d[d\tilde{a}^{\mathbb{R}/\mathbb{Z}}]}$ is an intrinsic \mathcal{N}^5 term like the Wess-Zumino-Witten term.^{93,94}

The model (177) still has the $\tilde{U}(1)$ -1-symmetry (176). To find out if such a $\tilde{U}(1)$ -1-symmetry is anomalous or not, we note that $e^{\pi i \int_{\mathcal{N}^5} \mathbb{S}q^2 d[d\tilde{a}^{\mathbb{R}/\mathbb{Z}}]}$ is the action amplitude (113), describing a 4+1D local bosonic model with the $\tilde{U}(1)$ 1-symmetry (176). A EM condensed matter system eqn. (177) is always a boundary of such a 4+1D $\tilde{U}(1)$ -1-symmetric model. It was shown that the action amplitude (113) describes a non-trivial hSPT phase of

$\tilde{U}(1)$ 1-symmetry. Thus the $\tilde{U}(1)$ -1-symmetry in the EM condensed matter systems is always anomalous.

Eqn. (177) describe a generic EM condensed matter system without time reversal symmetry. In the presence of time reversal symmetry, the EM $U(1)$ odd-charged fermions must also be a Kramers doublet. In this case EM condensed matter systems are described by⁷⁷

$$Z = \sum_{\{\tilde{a}^{\mathbb{R}/\mathbb{Z}}, \dots\}} e^{-\int_{\mathcal{M}^4} L(d\tilde{a}^{\mathbb{R}/\mathbb{Z}}, \dots) + \pi i \int_{\mathcal{M}^4} \hat{A}^{\mathbb{Z}_2} d[d\tilde{a}^{\mathbb{R}/\mathbb{Z}}]} e^{\pi i \int_{\mathcal{N}^5} \mathbb{S}q^2 d[d\tilde{a}^{\mathbb{R}/\mathbb{Z}}] + (w_2 + w_1^2) d[d\tilde{a}^{\mathbb{R}/\mathbb{Z}}]}, \quad (180)$$

where the spin structure $\hat{A}^{\mathbb{Z}_2}$ now satisfies

$$d\hat{A}^{\mathbb{Z}_2} \stackrel{2}{=} w_2 + w_1^2. \quad (181)$$

Similarly, the $\tilde{U}(1)$ -1-symmetry in eqn. (180) is also anomalous.

We note that the EM condensed matter systems described by eqn. (177) or eqn. (180) are actually a bosonic theory, since all the charge neutral excitations are bosons. In fact eqn. (177) and eqn. (180) themselves are bosonic theories, which are boundaries of a bosonic theory with trivial topological order. So eqn. (177) and eqn. (180) are really 3+1D local bosonic theories.

The anomalous $\tilde{U}(1)$ -1-symmetry in eqn. (177) or eqn. (180) implies that

all the gapped liquid phases^{38,39} of all the EM condensed matter systems must have a bosonic topological orders.

Here “bosonic topological orders” means topological orders in local bosonic systems. The 3+1D bosonic topological orders are classified in Ref. 47, Ref. 48, and Ref. 30.

We know that any condensation of even-number of electrons can only break the $U(1)$ gauge symmetry down to Z_n gauge symmetry ($n = \text{even} \neq 0$). So the induced gapped phases (the superconducting phases) have non-trivial topological orders described by Z_n gauge theory. However, in addition to the boson-condensation of the electron clusters, the electrons may also form a non-trivial topological state. One may wonder whether this extra topological order can cancel the topological order of the Z_n gauge theory, and give rise to a trivial product state. Using the higher anomaly, we find that no matter how electric charges fluctuate and condense, their induced gapped liquid phase of EM condensed matter systems must have a non-trivial bosonic topological order.

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Appendix A: Space-time complex, cochains, and cocycles

In this paper, we consider models defined on a space-time lattice. A spacetime lattice is a triangulation of the D -dimensional spacetime M^D , which is denoted by \mathcal{M}^D . We will also call the triangulation \mathcal{M}^D as a space-time complex, which is formed by simplices – the vertices, links, triangles, *etc.* We will use i, j, \dots to label vertices of the spacetime complex. The links of the complex (the 1-simplices) will be labeled by $(i, j), (j, k), \dots$. Similarly, the triangles of the complex (the 2-simplices) will be labeled by $(i, j, k), (j, k, l), \dots$.

In order to define a generic lattice theory on the space-time complex \mathcal{M}^D using local Lagrangian term on each simplex, it is important to give the vertices of each simplex a local order. A nice local scheme to order the vertices is given by a branching structure.^{52,95,96} A branching structure is a choice of orientation of each link in the d -dimensional complex so that there is no oriented loop on any triangle (see Fig. 5).

The branching structure induces a *local order* of the vertices on each simplex. The first vertex of a simplex is the vertex with no incoming links, and the second vertex is the vertex with only one incoming link, *etc.* So the simplex in Fig. 5a has the following vertex ordering: 0, 1, 2, 3.

The branching structure also gives the simplex (and its sub-simplices) a canonical orientation. Fig. 5 illustrates two 3-simplices with opposite canonical orientations compared with the 3-dimension space in which they are embedded. The blue arrows indicate the canonical orientations of the 2-simplices. The black arrows indicate the canonical orientations of the 1-simplices.

Given an Abelian group $(\mathbb{M}, +)$, an n -cochain f_n is an assignment of values in \mathbb{M} to each n -simplex, for example a value $f_{n;i,j,\dots,k} \in \mathbb{M}$ is assigned to n -simplex (i, j, \dots, k) . So a cochain f_n can be viewed as a bosonic field on the spacetime lattice.

\mathbb{M} can also be viewed a \mathbb{Z} -module (*i.e.* a vector space with integer coefficient) that also allows scaling by an integer:

$$\begin{aligned} x + y &= z, & x * y &= z, & mx &= y, \\ x, y, z &\in \mathbb{M}, & m &\in \mathbb{Z}. \end{aligned} \quad (\text{A1})$$

The direct sum of two modules $\mathbb{M}_1 \oplus \mathbb{M}_2$ (as vector spaces) is equal to the direct product of the two modules (as sets):

$$\mathbb{M}_1 \oplus \mathbb{M}_2 \stackrel{\text{as set}}{=} \mathbb{M}_1 \times \mathbb{M}_2 \quad (\text{A2})$$

We like to remark that a simplex (i, j, \dots, k) can have two different orientations. We can use (i, j, \dots, k) and $(j, i, \dots, k) = -(i, j, \dots, k)$ to denote the same simplex with opposite orientations. The value $f_{n;i,j,\dots,k}$ assigned to the simplex with opposite orientations should differ by a sign: $f_{n;i,j,\dots,k} = -f_{n;j,i,\dots,k}$. So to be more precise f_n is a linear map $f_n : n\text{-simplex} \rightarrow \mathbb{M}$. We can denote the

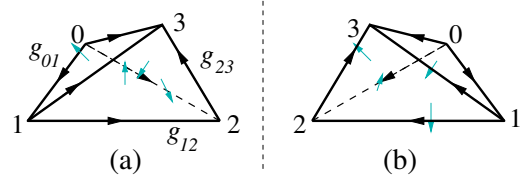


FIG. 5. (Color online) Two branched simplices with opposite orientations. (a) A branched simplex with positive orientation and (b) a branched simplex with negative orientation.

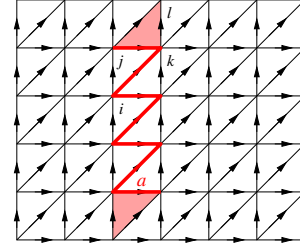


FIG. 6. (Color online) A 1-cochain a has a value 1 on the red links: $a_{ik} = a_{jk} = 1$ and a value 0 on other links: $a_{ij} = a_{kl} = 0$. da is non-zero on the shaded triangles: $(da)_{jkl} = a_{jk} + a_{kl} - a_{jl}$. For such 1-cochain, we also have $a \cup a = 0$. So when viewed as a \mathbb{Z}_2 -valued cochain, $\beta_2 a \neq a \cup a \pmod{2}$.

linear map as $\langle f_n, n\text{-simplex} \rangle$, or

$$\langle f_n, (i, j, \dots, k) \rangle = f_{n;i,j,\dots,k} \in \mathbb{M}. \quad (\text{A3})$$

More generally, a cochain f_n is a linear map of n -chains:

$$f_n : n\text{-chains} \rightarrow \mathbb{M}, \quad (\text{A4})$$

or (see Fig. 6)

$$\langle f_n, n\text{-chain} \rangle \in \mathbb{M}, \quad (\text{A5})$$

where a *chain* is a composition of simplices. For example, a 2-chain can be a 2-simplex: (i, j, k) , a sum of two 2-simplices: $(i, j, k) + (j, k, l)$, a more general composition of 2-simplices: $(i, j, k) - 2(j, k, l)$, *etc.* The map f_n is linear respect to such a composition. For example, if a chain is m copies of a simplex, then its assigned value will be m times that of the simplex. $m = -1$ correspond to an opposite orientation.

We will use $C^n(\mathcal{M}^D; \mathbb{M})$ to denote the set of all n -cochains on \mathcal{M}^D . $C^n(\mathcal{M}^D; \mathbb{M})$ can also be viewed as a set all \mathbb{M} -valued fields (or paths) on \mathcal{M}^D . Note that $C^n(\mathcal{M}^D; \mathbb{M})$ is an Abelian group under the $+$ -operation.

The total spacetime lattice \mathcal{M}^D correspond to a D -chain. We will use the same \mathcal{M}^D to denote it. Viewing f_D as a linear map of D -chains, we can define an “integral” over \mathcal{M}^D :

$$\int_{\mathcal{M}^D} f_D \equiv \langle f_D, \mathcal{M}^D \rangle \quad (\text{A6})$$

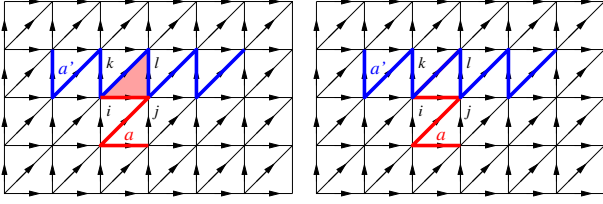


FIG. 7. (Color online) A 1-cochain a has a value 1 on the red links. Another 1-cochain a' has a value 1 on the blue links. On the left, $a \smile a'$ is non-zero on the shade triangles: $(a \smile a')_{ijl} = a_{ij}a'_{jl} = 1$. On the right, $a' \smile a$ is zero on every triangle. Thus $a \smile a' + a' \smile a$ is not a coboundary.

$$= \sum_{(i_0, i_1, \dots, i_D)} s_{i_0 i_1 \dots i_D} (f_D)_{i_0, i_1, \dots, i_D}.$$

Here $s_{i_0 i_1 \dots i_D} = \pm 1$, such that a D -simplex in the D -chain \mathcal{M}^D is given by $s_{i_0 i_1 \dots i_D} (i_0, i_1, \dots, i_D)$.

We can define a derivative operator d acting on an n -cochain f_n , which give us an $(n+1)$ -cochain (see Fig. 6):

$$\begin{aligned} & \langle df_n, (i_0 i_1 i_2 \dots i_{n+1}) \rangle \\ &= \sum_{m=0}^{n+1} (-)^m \langle f_n, (i_0 i_1 i_2 \dots \hat{i}_m \dots i_{n+1}) \rangle \end{aligned} \quad (A7)$$

where $i_0 i_1 i_2 \dots \hat{i}_m \dots i_{n+1}$ is the sequence $i_0 i_1 i_2 \dots i_{n+1}$ with i_m removed, and $i_0, i_1, i_2 \dots i_{n+1}$ are the ordered vertices of the $(n+1)$ -simplex $(i_0 i_1 i_2 \dots i_{n+1})$.

A cochain $f_n \in C^n(\mathcal{M}^D; \mathbb{M})$ is called a *cocycle* if $df_n = 0$. The set of cocycles is denoted by $Z^n(\mathcal{M}^D; \mathbb{M})$. A cochain f_n is called a *coboundary* if there exist a cochain f_{n-1} such that $df_{n-1} = f_n$. The set of coboundaries is denoted by $B^n(\mathcal{M}^D; \mathbb{M})$. Both $Z^n(\mathcal{M}^D; \mathbb{M})$ and $B^n(\mathcal{M}^D; \mathbb{M})$ are Abelian groups as well. Since $d^2 = 0$, a coboundary is always a cocycle: $B^n(\mathcal{M}^D; \mathbb{M}) \subset Z^n(\mathcal{M}^D; \mathbb{M})$. We may view two cocycles differ by a coboundary as equivalent. The equivalence classes of cocycles, $[f_n]$, form the so called cohomology group denoted by

$$H^n(\mathcal{M}^D; \mathbb{M}) = Z^n(\mathcal{M}^D; \mathbb{M}) / B^n(\mathcal{M}^D; \mathbb{M}), \quad (A8)$$

$H^n(\mathcal{M}^D; \mathbb{M})$, as a group quotient of $Z^n(\mathcal{M}^D; \mathbb{M})$ by $B^n(\mathcal{M}^D; \mathbb{M})$, is also an Abelian group.

For the \mathbb{Z}_N -valued cocycle x_n , $dx_n \stackrel{N}{=} 0$. Thus

$$\beta_N x_n \equiv \frac{1}{N} dx_n \quad (A9)$$

is a \mathbb{Z} -valued cocycle. Here β_N is Bockstrin homomorphism.

We notice the above definition for cochains still makes sense if we have a non-Abelian group (G, \cdot) instead of an Abelian group $(\mathbb{M}, +)$, however the differential d defined by eqn. (A7) will not satisfy $d \circ d = 1$, except for the first two d 's. That is, one may still make sense of 0-cocycle

and 1-cocycle, but no more further naively by formula eqn. (A7). For us, we only use non-Abelian 1-cocycle in this article. Thus it is ok. Non-Abelian cohomology is then thoroughly studied in mathematics motivating concepts such as gerbes to enter.

From two cochains f_m and h_n , we can construct a third cochain p_{m+n} via the cup product (see Fig. 7):

$$\begin{aligned} p_{m+n} &= f_m \smile h_n, \\ \langle p_{m+n}, (0 \rightarrow m+n) \rangle &= \langle f_m, (0 \rightarrow m) \rangle \times \\ & \quad \langle h_n, (m \rightarrow m+n) \rangle, \end{aligned} \quad (A10)$$

where $i \rightarrow j$ is the consecutive sequence from i to j :

$$i \rightarrow j \equiv i, i+1, \dots, j-1, j. \quad (A11)$$

Note that the above definition applies to cochains with global.

The cup product has the following property

$$d(h_n \smile f_m) = (dh_n) \smile f_m + (-)^n h_n \smile (df_m) \quad (A12)$$

for cochains with global or local values. We see that $h_n \smile f_m$ is a cocycle if both f_m and h_n are cocycles. If both f_m and h_n are cocycles, then $f_m \smile h_n$ is a coboundary if one of f_m and h_n is a coboundary. So the cup product is also an operation on cohomology groups $\smile: H^m(M^D; \mathbb{M}) \times H^n(M^D; \mathbb{M}) \rightarrow H^{m+n}(M^D; \mathbb{M})$. The cup product of two *cocycles* has the following property (see Fig. 7)

$$f_m \smile h_n = (-)^{mn} h_n \smile f_m + \text{coboundary} \quad (A13)$$

We can also define higher cup product $f_m \smile_k h_n$ which gives rise to a $(m+n-k)$ -cochain⁹⁷:

$$\begin{aligned} & \langle f_m \smile_k h_n, (0, 1, \dots, m+n-k) \rangle \\ &= \sum_{0 \leq i_0 < \dots < i_k \leq m+n-k} (-)^p \langle f_m, (0 \rightarrow i_0, i_1 \rightarrow i_2, \dots) \rangle \times \\ & \quad \langle h_n, (i_0 \rightarrow i_1, i_2 \rightarrow i_3, \dots) \rangle, \end{aligned} \quad (A14)$$

and $f_m \smile_k h_n = 0$ for $k < 0$ or for $k > m$ or n . Here $i \rightarrow j$ is the sequence $i, i+1, \dots, j-1, j$, and p is the number of permutations to bring the sequence

$$0 \rightarrow i_0, i_1 \rightarrow i_2, \dots; i_0+1 \rightarrow i_1-1, i_2+1 \rightarrow i_3-1, \dots \quad (A15)$$

to the sequence

$$0 \rightarrow m+n-k. \quad (A16)$$

For example

$$\begin{aligned} \langle f_m \smile_1 h_n, (0 \rightarrow m+n-1) \rangle &= \sum_{i=0}^{m-1} (-)^{(m-i)(n+1)} \times \\ & \quad \langle f_m, (0 \rightarrow i, i+n \rightarrow m+n-1) \rangle \langle h_n, (i \rightarrow i+n) \rangle. \end{aligned} \quad (A17)$$

We can see that $\smile_0 = \smile$. Unlike cup product at $k = 0$, the higher cup product of two cocycles may not be a cocycle. For cochains f_m, h_n , we have

$$\begin{aligned} d(f_m \smile_k h_n) &= df_m \smile_k h_n + (-)^m f_m \smile_k dh_n + \\ &(-)^{m+n-k} f_m \smile_{k-1} h_n + (-)^{mn+m+n} h_n \smile_{k-1} f_m \end{aligned} \quad (\text{A18})$$

Let f_m and h_n be cocycles and c_l be a chain, from eqn. (A18) we can obtain

$$\begin{aligned} d(f_m \smile_k h_n) &= (-)^{m+n-k} f_m \smile_{k-1} h_n \\ &+ (-)^{mn+m+n} h_n \smile_{k-1} f_m, \\ d(f_m \smile_k f_m) &= [(-)^k + (-)^m] f_m \smile_{k-1} f_m, \\ d(c_l \smile_{k-1} c_l + c_l \smile_k dc_l) &= dc_l \smile_k dc_l \\ &- [(-)^k - (-)^l] (c_l \smile_{k-2} c_l + c_l \smile_{k-1} dc_l). \end{aligned} \quad (\text{A19})$$

From eqn. (A19), we see that, for \mathbb{Z}_2 -valued cocycles z_n ,

$$\text{Sq}^{n-k}(z_n) \equiv z_n \smile_k z_n \quad (\text{A20})$$

is always a cocycle. Here Sq is called the Steenrod square. More generally $h_n \smile_k h_n$ is a cocycle if $n+k = \text{odd}$ and h_n is a cocycle. Usually, the Steenrod square is defined only for \mathbb{Z}_2 -valued cocycles or cohomology classes. Here, we like to define a generalized Steenrod square for \mathbb{M} -valued cochains c_n :

$$\text{Sq}^{n-k} c_n \equiv c_n \smile_k c_n + c_n \smile_{k+1} dc_n. \quad (\text{A21})$$

From eqn. (A19), we see that

$$\begin{aligned} d\text{Sq}^k c_n &= d(c_n \smile_{n-k} c_n + c_n \smile_{n-k+1} dc_n) \\ &= \text{Sq}^k dc_n + (-)^n \begin{cases} 0, & k = \text{odd} \\ 2\text{Sq}^{k+1} c_n & k = \text{even} \end{cases}. \end{aligned} \quad (\text{A22})$$

In particular, when c_n is a \mathbb{Z}_2 -valued cochain, we have

$$d\text{Sq}^k c_n \stackrel{2}{=} \text{Sq}^k dc_n. \quad (\text{A23})$$

Next, let us consider the action of Sq^k on the sum of two \mathbb{M} -valued cochains c_n and c'_n :

$$\begin{aligned} \text{Sq}^k(c_n + c'_n) &= \text{Sq}^k c_n + \text{Sq}^k c'_n + \\ &c_n \smile_{n-k} c'_n + c'_n \smile_{n-k} c_n + c_n \smile_{n-k+1} dc'_n + c'_n \smile_{n-k+1} dc_n \\ &= \text{Sq}^k c_n + \text{Sq}^k c'_n + [1 + (-)^k] c_n \smile_{n-k} c'_n \\ &- (-)^{n-k} [(-)^{n-k} c'_n \smile_{n-k} c_n + (-)^n c_n \smile_{n-k} c'_n] \\ &+ c_n \smile_{n-k+1} dc'_n + c'_n \smile_{n-k+1} dc_n \\ &= \text{Sq}^k c_n + \text{Sq}^k c'_n + [1 + (-)^k] c_n \smile_{n-k} c'_n \end{aligned}$$

$$\begin{aligned} &+ (-)^{n-k} [dc'_n \smile_{n-k+1} c_n + (-)^n c'_n \smile_{n-k+1} dc_n] \\ &- (-)^{n-k} d(c'_n \smile_{n-k+1} c_n) + c_n \smile_{n-k+1} dc'_n + c'_n \smile_{n-k+1} dc_n \\ &= \text{Sq}^k c_n + \text{Sq}^k c'_n + [1 + (-)^k] c_n \smile_{n-k} c'_n \\ &+ [1 + (-)^k] c'_n \smile_{n-k+1} dc_n - (-)^{n-k} d(c'_n \smile_{n-k+1} c_n) \\ &- [(-)^{n-k+1} dc'_n \smile_{n-k+1} c_n - c_n \smile_{n-k+1} dc'_n] \\ &= \text{Sq}^k c_n + \text{Sq}^k c'_n + [1 + (-)^k] c_n \smile_{n-k} c'_n \\ &+ [1 + (-)^k] c'_n \smile_{n-k+1} dc_n - (-)^{n-k} d(c'_n \smile_{n-k+1} c_n) \\ &- d(dc'_n \smile_{n-k+2} c_n) + dc'_n \smile_{n-k+2} dc_n \\ &= \text{Sq}^k c_n + \text{Sq}^k c'_n + dc'_n \smile_{n-k+2} dc_n \\ &+ [1 + (-)^k] [c_n \smile_{n-k} c'_n + c'_n \smile_{n-k+1} dc_n] \\ &- (-)^{n-k} d(c'_n \smile_{n-k+1} c_n) - d(dc'_n \smile_{n-k+2} c_n). \end{aligned} \quad (\text{A24})$$

We see that, if one of the c_n and c'_n is a cocycle,

$$\text{Sq}^k(c_n + c'_n) \stackrel{2,d}{=} \text{Sq}^k c_n + \text{Sq}^k c'_n. \quad (\text{A25})$$

We also see that

$$\begin{aligned} &\text{Sq}^k(c_n + df_{n-1}) \\ &= \text{Sq}^k c_n + \text{Sq}^k df_{n-1} + [1 + (-)^k] df_{n-1} \smile_{n-k} c_n \\ &- (-)^{n-k} d(c_n \smile_{n-k+1} df_{n-1}) - d(dc_n \smile_{n-k+2} df_{n-1}) \\ &= \text{Sq}^k c_n + [1 + (-)^k] [df_{n-1} \smile_{n-k} c_n + (-)^n \text{Sq}^{k+1} f_{n-1}] \\ &+ d[\text{Sq}^k f_{n-1} - (-)^{n-k} c_n \smile_{n-k+1} df_{n-1} - dc_n \smile_{n-k+2} df_{n-1}] \\ &= \text{Sq}^k c_n + [1 + (-)^k] [c_n \smile_{n-k} df_{n-1} + (-)^n \text{Sq}^{k+1} f_{n-1}] \\ &+ d[\text{Sq}^k f_{n-1} - (-)^{n-k} df_{n-1} \smile_{n-k+1} c_n]. \end{aligned} \quad (\text{A26})$$

Using eqn. (A27), we can also obtain the following result if $dc_n = \text{even}$

$$\begin{aligned} &\text{Sq}^k(c_n + 2c'_n) \\ &\stackrel{4}{=} \text{Sq}^k c_n + 2d(c_n \smile_{n-k+1} c'_n) + 2dc_n \smile_{n-k+1} c'_n \\ &\stackrel{4}{=} \text{Sq}^k c_n + 2d(c_n \smile_{n-k+1} c'_n) \end{aligned} \quad (\text{A27})$$

As another application, we note that, for a \mathbb{M} -valued cochain m_d and using eqn. (A18),

$$\begin{aligned} \text{Sq}^1(m_d) &= m_d \smile_{d-1} m_d + m_d \smile_d dm_d \\ &= \frac{1}{2}(-)^d [d(m_d \smile_d m_d) - dm_d \smile_d m_d] + \frac{1}{2} m_d \smile_d dm_d \\ &= (-)^d \beta_2(m_d \smile_d m_d) - (-)^d \beta_2 m_d \smile_d m_d + m_d \smile_d \beta_2 m_d \\ &= (-)^d \beta_2 \text{Sq}^0 m_d - 2(-)^d \beta_2 m_d \smile_{d+1} \beta_2 m_d \\ &= (-)^d \beta_2 \text{Sq}^0 m_d - 2(-)^d \text{Sq}^0 \beta_2 m_d \end{aligned} \quad (\text{A28})$$

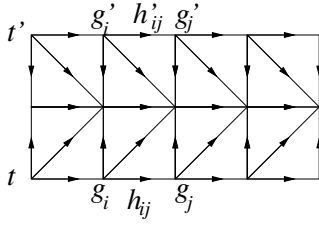


FIG. 8. Each time-step of evolution is given by the path integral on a particular form of branched graph. Here is an example in 1+1D.

This way, we obtain a relation between Steenrod square and Bockstein homomorphism, when m_d is a \mathbb{Z}_2 -valued cochain

$$\text{Sq}^1(m_d) \stackrel{2}{=} \beta_2 m_d, \quad (\text{A29})$$

where we have used $\text{Sq}^0 m_d = m_d$ for \mathbb{Z}_2 -valued cochain.

For a k -cochain a_k , k = odd, we find that

$$\begin{aligned} \text{Sq}^k a_k &= a_k a_k + a_k \smile_1 da_k \\ &= \frac{1}{2} [da_k \smile_1 a_k - a_k \smile_1 da_k - d(a_k \smile_1 a_k)] + a_k \smile_1 da_k \\ &= \frac{1}{2} [da_k \smile_2 da_k - d(da_k \smile_2 a_k)] - \frac{1}{2} d(a_k \smile_1 a_k) \\ &= \frac{1}{4} d(da_k \smile_3 da_k) - \frac{1}{2} d(a_k \smile_1 a_k + da_k \smile_2 a_k) \end{aligned} \quad (\text{A30})$$

Thus $\text{Sq}^k a_k$ is always a \mathbb{Q} -valued coboundary, when k is odd.

Appendix B: Path integral and Hamiltonian

Consider a spacetime complex of topology $M_{\text{space}} \times I$ where $I = [t, t']$ represents the time dimension and M_{space} is a closed space complex (see Fig. 8). The spacetime complex $M_{\text{space}} \times I$ has two boundaries: one at time t and another at time t' . A path integral on the spacetime complex $M_{\text{space}} \times I$ give us an amplitude $Z[\{g'_i, h'_{ij}, \dots\}, \{g_i, h_{ij}, \dots\}]$ from a configuration $\{g_i, h_{ij}, \dots\}$ at t to another configuration $\{g'_i, h'_{ij}, \dots\}$ at t' . Here, $\{g_i, h_{ij}, \dots\}$ and $\{g'_i, h'_{ij}, \dots\}$ are the degrees of freedom on the boundaries (see Fig. 8). We like to interpret $Z[\{g'_i, h'_{ij}, \dots\}, \{g_i, h_{ij}, \dots\}]$ as the amplitude of an

evolution in imaginary time by a Hamiltonian:

$$\begin{aligned} &Z[\{g'_i, h'_{ij}, \dots\}, \{g_i, h_{ij}, \dots\}] \\ &= \langle g'_i, h'_{ij}, \dots | e^{-(t'-t)H} | g_i, h_{ij}, \dots \rangle. \end{aligned} \quad (\text{B1})$$

However, such an interpretation may not be valid since $Z[\{g'_i, h'_{ij}, \dots\}, \{g_i, h_{ij}, \dots\}]$ may not give rise to a hermitian matrix.

To have a path integral that give rise to a hermitian matrix H , we require the path integral defined on the branched graphs to have a “reflection” property. The imaginary-time path integral (or partition function) has a form

$$Z = \sum_{\{g_i\}, \{h_{ij}\}, \dots} e^{-S(\{g_i\}, \{h_{ij}\}, \dots)} \quad (\text{B2})$$

where the total action amplitude e^{-S} for a configuration $\{g_i\}, \{h_{ij}\}, \dots$ (or a path) is given by

$$e^{-S} = \prod_{(ij \dots k)} [V_{ij \dots k}(\{g_i\}, \{h_{ij}\}, \dots)]^{s_{ij \dots k}}. \quad (\text{B3})$$

Here $\prod_{(ij \dots k)}$ is the product over all the n -cells $(ij \dots k)$. Note that the contribution from an n -cell $(ij \dots k)$ is $V_{ij \dots k}(\{g_i\}, \{h_{ij}\}, \dots)$ or $V_{ij \dots k}^*(\{g_i\}, \{h_{ij}\}, \dots)$ depending on the orientation $s_{ij \dots k}$ of the cell (see Fig. 5).

Such a path integral give rise to the hermitian Hamiltonian evolution. The key is to require that each time-step of evolution is given by branched graphs of the form in Fig. 8. One can show that $Z[\{g'_i, h'_{ij}, \dots\}, \{g_i, h_{ij}, \dots\}]$ obtained by summing over all in the internal indices in the branched graphs Fig. 8 has a form

$$\begin{aligned} &Z[\{g'_i, h'_{ij}, \dots\}, \{g_i, h_{ij}, \dots\}] \\ &= \sum_{\{g''_i, h''_{ij}, \dots\}} U^*[\{g''_i, h''_{ij}, \dots\}, \{g'_i, h'_{ij}, \dots\}] \\ &\quad U[\{g''_i, h''_{ij}, \dots\}, \{g_i, h_{ij}, \dots\}] \end{aligned} \quad (\text{B4})$$

and represents a positive-definite hermitian matrix. Thus the path integral of the form (B3) always correspond to a Hamiltonian evolution in imaginary time. In fact, the above $Z[\{g'_i, h'_{ij}, \dots\}, \{g_i, h_{ij}, \dots\}]$ can be viewed as an imaginary-time evolution $T = e^{-\Delta\tau H}$ for a single time step.

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