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Thermal conductivity of the degenerate one-dimensional Fermi gas

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We study heat transport in a gas of one-dimensional fermions in the presence of a small temperature gradient. At temperatures well below the Fermi energy there are two types of relaxation processes in this system, with dramatically different relaxation rates. As a result, in addition to the usual thermal conductivity, one can introduce the thermal conductivity of the gas of elementary excitations, which quantifies the dissipation in the system in the broad range of frequencies between the two relaxation rates. We develop a microscopic theory of these transport coefficients in the limit of weak interactions between the fermions.

I. INTRODUCTION

Relaxation of one-dimensional systems toward equilibrium has a number of special features. The two-particle scattering processes, which control relaxation in higher dimensions, are strongly restricted in one dimension by the conservation laws, and do not lead to effective relaxation of the system. As a result, the relaxation is dominated by three-particle processes. In a quantum system at low temperature T these scattering processes are strongly suppressed, resulting in a slow relaxation toward equilibrium [1]. This leads to a different temperature dependence of the transport coefficients at low temperatures. For example, while the bulk viscosity of the three-dimensional Fermi liquid vanishes at $T \rightarrow 0$ as $\zeta \propto T^2$ [2], in one dimension it grows as $\zeta \propto T^{-3}$ [3].

Another important feature of one-dimensional systems is that each particle moves in one of only two directions. As a result, at low temperatures the dominant scattering processes with small momentum transfer are very inefficient at changing the direction of motion. This effect is best illustrated in the case of a one-dimensional Fermi gas. The most efficient three-particle process that changes the relative number of the right- and left-moving particles is shown in Fig. 1(a). In order for an electron to change the direction of motion, this process must involve a hole near the bottom of the band. Thus the rate of such processes is exponentially small, $\tau^{-1} \propto \exp(-\mu/T)$ [4–7], where μ is the chemical potential. On the other hand, the scattering processes shown in Fig. 1(b) and (c) do not change the numbers of the right- and left-moving particles, but rearrange excitations near the two Fermi points. The corresponding rate of scattering of thermal excitations τ_{ex}^{-1} scales as a power of temperature. In spinless one-dimensional systems $\tau_{\text{ex}}^{-1} \propto T^7$ [1, 8, 9], while for weakly interacting spin- $\frac{1}{2}$ fermions $\tau_{\text{ex}}^{-1} \propto T$ [10].

The presence of exponentially slow relaxation processes in the system results in a very large thermal conductivity κ in one dimension. Phenomenological treatment [11] of the transport in a spinless one-dimensional quantum liquid based on the Luttinger liquid theory [12] yields

$$\kappa = \frac{\pi T v \tau}{3 \hbar}. \quad (1)$$

Here v is the velocity of the bosonic excitations in the Luttinger model and \hbar is the Planck's constant. It is important to note that the above result applies to thermal conductivity measured at low frequencies $\omega \ll \tau^{-1}$. At higher frequencies the exponentially slow relaxation processes of Fig. 1(a) can be neglected. In this case one can assume that the numbers of the right- and left-moving fermions are conserved, and the relaxation in the system is due to the processes of Fig. 1(b) and (c). A small temperature gradient $\partial_x T$ still results in a dissipative contribution to the energy current $-\kappa_{\text{ex}} \partial_x T$ proportional to it, but with a different thermal conductivity κ_{ex} . The transport coefficient κ_{ex} was recently introduced in the two-fluid hydrodynamic theory of one-dimensional quantum liquids [13]. It describes the thermal conductivity of the gas of elementary excitations of the quantum liquid and appears in the expressions for damping of the sound modes in this system.

In this paper we develop a microscopic theory of thermal conductivity of a one-dimensional Fermi gas with weak interactions between the particles. Our main focus is on the case of spinless fermions, for which the relaxation processes have been studied in considerable detail [6, 14–16]. At $T \ll \mu$ our result for the thermal conductivity κ is consistent with the phenomenological expression (1), while also providing an expression for the relaxation time τ in terms of the microscopic interaction potential. More importantly, our approach enables us to obtain the thermal conductivity of the gas of excitations κ_{ex} , for which no phenomenological theory is available. Because the relaxation processes are sensitive to the form of interaction between fermions [14, 16], we find very different temperature dependence of κ_{ex} for the short-range and Coulomb interactions.

The paper is organized as follows. In Sec. II we use Boltzmann equation approach to obtain a microscopic expression for the thermal conductivity κ of the degenerate one-dimensional Fermi gas. The same technique is applied to the calculation of the thermal conductivity of the gas of excitations in Sec. III, where a general expression and the order of magnitude of κ_{ex} are obtained. A careful evaluation of κ_{ex} involves a detailed treatment of the relaxation processes shown in Fig. 1(b) and (c), which is presented in Sec. IV. We discuss our results in

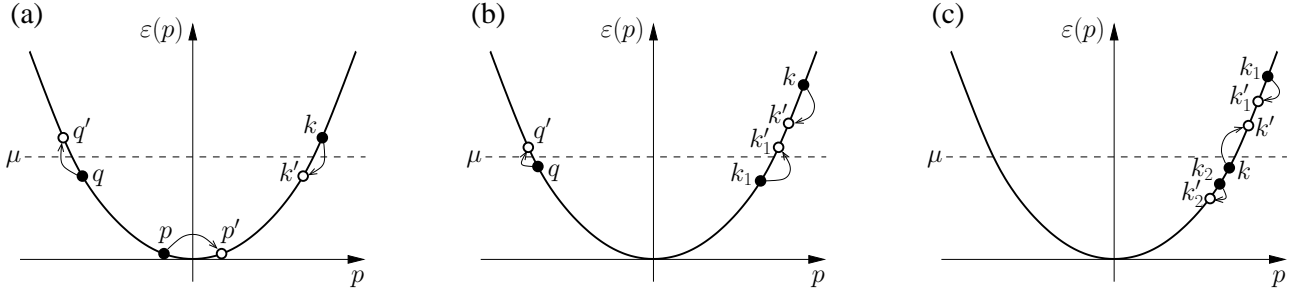


FIG. 1. The dominant three-particle scattering processes. Solid lines show energy of the particle as a function of momentum, while dashed lines show the position of the chemical potential. Processes of type (a) enable the backscattering of fermions; the resulting relaxation rate τ^{-1} is exponentially small. Processes of types (b) and (c) do not change the numbers of the right- and left-moving particles. Their rates scale as a power of temperature.

Sec. V.

II. THERMAL CONDUCTIVITY OF THE FERMION GAS

A. Boltzmann equation approach

We start by evaluating the thermal conductivity of the one-dimensional gas of spinless fermions with the energy spectrum

$$\varepsilon_p = \frac{p^2}{2m}, \quad (2)$$

where p is the momentum of the fermion and m is its mass. We will subject the system to an infinitesimal temperature gradient $\partial_x T$ and obtain the occupation numbers of the fermionic states n_p from the Boltzmann equation [17]

$$\partial_t n_p + \frac{p}{m} \partial_x n_p = I[n_p]. \quad (3)$$

Weak interactions between fermions give rise to the scattering processes accounted for by the collision integral $I[n_p]$. In the left-hand side of the Boltzmann equation interactions will be neglected. In this approximation the energy of the fermion (2) does not depend on its position x , which enabled us to omit an additional term $-(\partial_x \varepsilon_p) \partial_p n_p$ in the left-hand side of Eq. (3).

We are considering a translation-invariant system, in which collisions between the fermions conserve not only the total number of particles and energy of the system, but also its momentum. In this case even in thermodynamic equilibrium the system can move with respect to the lab frame with some velocity u , and the equilibrium occupation numbers of the fermionic states are given by

$$n_p^{(0)} = \frac{1}{e^{\beta \varepsilon_p - \gamma p - \alpha} + 1}, \quad (4)$$

where $\beta = 1/T$, $\alpha = \beta\mu$, and $\gamma = \beta u$. In the presence of the temperature gradient $\partial_x T$ the occupation numbers

deviate from the equilibrium form (4),

$$n_p = n_p^{(0)} + \delta n_p, \quad (5)$$

where the small non-equilibrium correction $\delta n_p \propto \partial_x T$. The distribution function n_p depends on the spatial coordinate x . We assume that the parameters $\alpha(x)$, $\beta(x)$, and $\gamma(x)$ of the equilibrium part $n_p^{(0)}$ of the distribution function (5) are chosen in such a way that the particle, energy, and momentum densities of the system can be evaluated by substituting $n_p^{(0)}$ for n_p . In other words, we impose the conditions

$$\int \delta n_p dp = 0, \quad \int \varepsilon_p \delta n_p dp = 0, \quad \int p \delta n_p dp = 0 \quad (6)$$

upon δn_p .

Our immediate goal is to obtain the thermal conductivity κ , defined by the relation

$$j_Q = -\kappa \partial_x T. \quad (7)$$

Here

$$j_Q = \int \frac{dp}{2\pi\hbar} \varepsilon_p \frac{p}{m} \delta n_p \quad (8)$$

is the dissipative part of the energy current [17].

We will obtain the correction δn_p to the equilibrium distribution function by solving the Boltzmann equation (3) written in the form

$$\dot{n}_p = I[n_p^{(0)} + \delta n_p], \quad (9)$$

where \dot{n}_p is given by the left-hand side of Eq. (3). Since the collision integral evaluated for the equilibrium distribution $I[n_p^{(0)}] = 0$, it is important to keep the infinitesimal correction δn_p in the right-hand side of Eq. (9). On the other hand, \dot{n}_p should be evaluated for $\delta n_p = 0$, i.e.,

$$\dot{n}_p = \partial_t n_p^{(0)} + \frac{p}{m} \partial_x n_p^{(0)}. \quad (10)$$

The thermal conductivity is defined in the steady-state regime, in which the parameters α , β , and γ of the equilibrium distribution function (4) do not depend on time.

In addition, the system is assumed to be stationary, i.e., $\gamma = 0$. The latter condition is satisfied due to the gradient of the chemical potential emerging in the system, which compensates for the force acting on the system as a result of the applied temperature gradient. Thus we substitute into Eq. (9) the equilibrium distribution (4) with the parameters

$$\alpha = \alpha(x), \quad \beta = \beta(x), \quad \gamma = 0. \quad (11)$$

This results in

$$\dot{n}_p = -g_p^2 [(\partial_x \beta) \varepsilon_p - \partial_x \alpha] \frac{p}{m}, \quad (12)$$

where we have introduced

$$g_p = \sqrt{n_p^{(0)} (1 - n_p^{(0)})} = \frac{1}{2 \cosh \frac{\beta \varepsilon_p - \alpha}{2}}. \quad (13)$$

The value of $\partial_x \alpha$ in Eq. (12) can be found by noticing that the scattering processes accounted for by the collision integral $I[n_p]$ conserve the total momentum of the system, i.e.,

$$\int \frac{dp}{2\pi\hbar} p \dot{n}_p = 0. \quad (14)$$

Imposing this condition on Eq. (12), we obtain

$$\dot{n}_p = -g_p^2 \frac{\partial_x \beta}{m} [\varepsilon_p - \tilde{\mu}] p, \quad \tilde{\mu} = \frac{1}{2m} \frac{I_4}{I_2}, \quad (15)$$

where

$$I_\lambda = \int_0^{+\infty} dp g_p^2 p^\lambda. \quad (16)$$

In the zero temperature limit $\tilde{\mu} \rightarrow \mu$.

B. Linearized collision integral

Our next step is to obtain δn_p by solving Eq. (9). Since we are interested in the linear response to an infinitesimal temperature gradient, we can linearize the collision integral in the right-hand side of Eq. (9). In addition, it is convenient to write the resulting integral equation in terms of function ϕ_p defined by

$$\delta n_p = g_p \phi_p. \quad (17)$$

As a result, Eq. (9) becomes a linear integral equation

$$\int \frac{dp'}{2\pi\hbar} K(p, p') \phi_{p'} = \frac{\dot{n}_p}{g_p} \quad (18)$$

with a real symmetric kernel $K(p, p')$. The latter property means that one can, in principle, obtain an orthonormal set of eigenfunctions of this integral operator,

$$\int \frac{dp'}{2\pi\hbar} K(p, p') \phi_{p'}^{(l)} = -\frac{1}{\tau_l} \phi_p^{(l)}, \quad \langle \phi_p^{(l)} | \phi_{p'}^{(l')} \rangle = \delta_{l, l'}, \quad (19)$$

with real eigenvalues $-1/\tau_l$. Here we defined the inner product by

$$\langle \phi_p | \psi_p \rangle = \int \frac{dp}{2\pi\hbar} \phi_p \psi_p. \quad (20)$$

Because the collisions result in the evolution of the distribution function toward equilibrium, the relaxation rates τ_l^{-1} are non-negative.

The full set of eigenfunctions $\phi_p^{(l)}$ includes three modes with zero eigenvalues. The existence of such zero modes is due to the conservation of the total number of particles N , energy E , and momentum P of the system. They are obtained by small variations of parameters α , β , and γ in the expression for the equilibrium distribution (4),

$$\phi_p^{(N)} = g_p, \quad \phi_p^{(E)} = \left(\varepsilon_p - \frac{I_2}{2mI_0} \right) g_p, \quad \phi_p^{(P)} = p g_p, \quad (21)$$

where we orthogonalized the modes, but omitted the normalization constants. Indeed, any such variation transforms $n_p^{(0)}$ to another equilibrium distribution. Collisions do not modify equilibrium distributions, resulting in vanishing eigenvalues in Eq. (19).

We can now expand ϕ_p in the basis of the eigenfunctions $\phi_p^{(l)}$ and obtain a formal solution of the integral equation (18),

$$\phi_p = -\frac{\partial_x T}{mT^2} \sum_{l \neq N, E, P} \tau_l \langle \phi_p^{(l)} | g_p (\varepsilon_p - \tilde{\mu}) p | \phi_p^{(l)} \rangle, \quad (22)$$

where we used Eq. (15) for \dot{n}_p . In the sum in Eq. (22) we excluded the zero modes (21) for which the corresponding τ_l is infinite. This is due to the fact that the overlaps of $g_p (\varepsilon_p - \tilde{\mu}) p$ with $\phi_p^{(N)}$ and $\phi_p^{(E)}$ vanish due to opposite symmetries with respect to $p \rightarrow -p$, while the overlap with $\phi_p^{(P)}$ vanishes because of the momentum conservation condition (14). As a result, the conditions (6) for δn_p are satisfied.

Next, we notice that due to the last of the conditions (6) one can replace $\varepsilon_p \rightarrow \varepsilon_p - \tilde{\mu}$ in the expression for the dissipative contribution (8) to the energy current, after which the latter becomes

$$j_Q = \frac{1}{m} \langle g_p (\varepsilon_p - \tilde{\mu}) p | \phi_p \rangle. \quad (23)$$

Substitution of Eq. (22) and comparison with Eq. (7) yield

$$\kappa = \frac{1}{m^2 T^2} \sum_{l \neq N, E, P} \tau_l \langle \phi_p^{(l)} | g_p (\varepsilon_p - \tilde{\mu}) p \rangle^2. \quad (24)$$

This expression gives the thermal conductivity of the one-dimensional spinless Fermi gas at any temperature. Significant further progress in understanding thermal conductivity can be made in the regime of low temperature, $T \ll \mu$.

C. Thermal conductivity at low temperatures

At low temperatures the relaxation of the one-dimensional Fermi gas is dominated by the processes shown in Fig. 1. The process of Fig. 1(a) is exponentially suppressed, whereas relaxation rate τ_{ex}^{-1} associated with the processes of Fig. 1(b) and (c) has a power-law temperature dependence. It is important to note, however, that understanding the full relaxation of the system to equilibrium requires accounting for the effect of the processes of Fig. 1(a), because the remaining processes do not change the numbers of the left- and right-moving particles in the system.

Let us consider a Fermi gas with zero total momentum, which relaxes to the equilibrium distribution (4) with $\gamma = 0$. Keeping in mind the presence of two very different relaxation times $\tau \gg \tau_{\text{ex}}$, one concludes that the relaxation of the degenerate one-dimensional Fermi gas proceeds in two steps. First, at the time scales of the order of τ_{ex} the particle-hole excitations come to equilibrium with each other, but the chemical potentials of the left- and right-moving particles remain different. This means that the distribution function takes the form

$$n_p = \frac{\theta(p)}{e^{\beta\varepsilon_p - \gamma p - \alpha - \delta\alpha} + 1} + \frac{\theta(-p)}{e^{\beta\varepsilon_p - \gamma p - \alpha + \delta\alpha} + 1}. \quad (25)$$

The presence of the new parameter $\delta\alpha$ in the distribution function accounts for the fact that in addition to the total number of particles, energy, and momentum of the system, the difference J of the numbers of the right- and left-moving particles is also conserved at time scales $t \ll \tau$.

To linear order in $\delta\alpha$ and γ , the momentum density of the Fermi gas with the distribution (25) is $(\delta\alpha I_1 + \gamma I_2)/\pi\hbar$. Thus the parameter γ in Eq. (25) is related to $\delta\alpha$ by $\gamma = -(I_1/I_2)\delta\alpha$, and the deviation of the distribution function (25) from the equilibrium form $(e^{\beta\varepsilon_p - \alpha} + 1)^{-1}$ is

$$\delta n_p = \delta\alpha g_p^2 \left(\text{sgn } p - \frac{I_1}{I_2} p \right). \quad (26)$$

At the second stage of the relaxation process the system slowly approaches equilibrium, $\delta\alpha \propto e^{-t/\tau}$, while the momentum dependence of δn_p retains the form (26). We therefore conclude that the eigenvalue problem (19) for the collision integral has the solution

$$\phi_p^{(J)} = C_J g_p \left(\text{sgn } p - \frac{I_1}{I_2} p \right) \quad (27)$$

with exponentially small eigenvalue $-1/\tau$. The constant

$$C_J = \sqrt{\frac{\pi\hbar I_2}{I_2 I_0 - I_1^2}} \quad (28)$$

is obtained from the normalization condition in Eq. (19).

The remaining relaxation modes $\phi_p^{(l)}$ describe how the non-equilibrium distribution function approaches the

form (25) at the first stage of the relaxation process, dominated by the processes of Fig. 1(b) and (c). The corresponding relaxation times $\tau_l \sim \tau_{\text{ex}}$ are much smaller than $\tau \propto e^{\mu/T}$. Thus, at low temperatures $T \ll \mu$, the thermal conductivity (24) is dominated by the mode (27). Evaluation of the corresponding matrix element is straightforward,

$$\langle \phi_p^{(J)} | g_p (\varepsilon_p - \tilde{\mu}) p \rangle = \frac{C_J}{2\pi\hbar m} \frac{I_3 I_2 - I_4 I_1}{I_2}. \quad (29)$$

At $T \ll \mu$ the integrals (16) can be approximated by the Sommerfeld expansion

$$I_\lambda \simeq mT(2m\mu)^{\frac{\lambda-1}{2}} \left(1 + \frac{(\lambda-1)(\lambda-3)}{24} \frac{\pi^2 T^2}{\mu^2} + \frac{7(\lambda-1)(\lambda-3)(\lambda-5)(\lambda-7)}{5760} \frac{\pi^4 T^4}{\mu^4} \right). \quad (30)$$

For the dominant contribution to the thermal conductivity (24) we then find

$$\kappa \simeq \frac{\tau}{m^2 T^2} \langle \phi_p^{(J)} | g_p (\varepsilon_p - \tilde{\mu}) p \rangle^2 \simeq \frac{\pi T \tau}{3\hbar} \sqrt{\frac{2\mu}{m}}. \quad (31)$$

The speed v of the low-energy elementary excitations in a Fermi gas at $T \ll \mu$ is the Fermi velocity $v_F = \sqrt{2\mu/m}$. Therefore the result (31) is consistent with the phenomenological expression (1) for the thermal conductivity of a spinless one-dimensional quantum liquid. We note that both Eqs. (1) and (31) express κ in terms of the relaxation time τ . The latter has been studied in some detail phenomenologically in Refs. [6, 7], where it was expressed in terms of the quasiparticle spectrum of the quantum liquid. In Sec. V we discuss how one can obtain a microscopic expression for τ in terms of interactions between fermions.

III. THERMAL CONDUCTIVITY OF THE GAS OF ELEMENTARY EXCITATIONS

In Sec. II we found that at $T \ll \mu$ the thermal conductivity of the Fermi gas (31) is proportional to the relaxation time τ and is, therefore, exponentially large. This result holds as long as thermal conductivity is measured at frequencies $\omega \ll \tau^{-1}$. On the other hand, interesting new behavior of one-dimensional systems is expected in the broad range of frequencies

$$\tau^{-1} \ll \omega \ll \tau_{\text{ex}}^{-1}. \quad (32)$$

It was shown recently [18, 19] that in this regime one-dimensional systems behave like superfluids and support two sound modes, in contrast to a single sound mode at $\omega \ll \tau^{-1}$.

In the presence of an ordinary sound wave the temperature of the system depends on position. This results in dissipation, which is proportional to the thermal conductivity κ and contributes to the attenuation of sound [20].

The same physics applies in the two-sound regime (32), but the resulting contribution to the sound attenuation is controlled by a different thermal transport coefficient κ_{ex} [13], which has the meaning of the dissipative part of the thermal conductivity at frequencies in the range (32).

To evaluate κ_{ex} we will assume that the exponentially long relaxation time $\tau \rightarrow \infty$ and then take the limit $\omega \rightarrow 0$. In other words, we will find the thermal conductivity assuming that the relaxation processes conserve not only the number of particles, energy, and momentum of the system, but also the difference J of the numbers of right- and left-moving particles. We now adapt the evaluation of the thermal conductivity in Sec. II A and II B to account for this additional conservation law.

First of all, the equilibrium distribution $n_p^{(0)}$ now takes the form (25) and depends on four parameters, α , β , γ , and $\delta\alpha$. A small temperature gradient $\partial_x T$ results in a small deviation of the distribution function from the equilibrium form, see Eq. (5). Because of the fourth conservation law, in addition to Eq. (6) we impose the condition

$$\int \text{sgn } p \delta n_p dp = 0 \quad (33)$$

to ensure that the value of J is determined by the equilibrium part of n_p .

We next obtain δn_p in the expression for the dissipative energy current (8) by solving the Boltzmann equation in the form (9) with \dot{n}_p given by Eq. (10). The conditions (11) must be modified to account for the extra conservation law. First, the condition of zero total momentum of the system now takes the form $\gamma = -(I_1/I_2)\delta\alpha$, see Sec. II C. Second, in the absence of the relaxation processes of Fig. 1(a) one can no longer exclude the possibility of a time-dependent difference of the chemical potentials of the right- and left-moving particles. This results in the following assumptions regarding the parameters of the equilibrium distribution (25)

$$\alpha = \alpha(x), \quad \beta = \beta(x), \quad \gamma = -\frac{I_1}{I_2} \delta\alpha, \quad \delta\alpha = \delta\alpha(t). \quad (34)$$

Substituting the expression (25) for $n_p^{(0)}$ in Eq. (10) we get

$$\dot{n}_p = -g_p^2 \left\{ [(\partial_x \beta) \varepsilon_p - \partial_x \alpha] \frac{p}{m} - (\partial_t \delta\alpha) \left(\text{sgn } p - \frac{I_1}{I_2} p \right) \right\} \quad (35)$$

The values of $\partial_x \alpha$ and $\partial_t \delta\alpha$ are determined by imposing the condition (14) of conservation of momentum in collisions along with the new condition of conservation of J ,

$$\int \frac{dp}{2\pi\hbar} \text{sgn } p \dot{n}_p = 0. \quad (36)$$

This yields

$$\dot{n}_p = -g_p^2 \frac{\partial_x \beta}{2m^2} \nu_p, \quad (37)$$

where

$$\nu_p = p^3 - \frac{I_4 I_0 - I_3 I_1}{I_2 I_0 - I_1^2} p + \frac{I_4 I_1 - I_3 I_2}{I_2 I_0 - I_1^2} \text{sgn } p. \quad (38)$$

We note that by imposing the conservation laws (14) and (36) we ensured that the function $\phi_p = g_p \nu_p$ is orthogonal to both the usual zero modes (21) and the additional zero mode (27) corresponding to the conservation of J .

The next step is to find the non-equilibrium correction δn_p to the distribution function by solving the integral equation (18). Writing δn_p as prescribed in Eq. (17), and expanding ϕ_p in the basis of the eigenfunctions $\phi_p^{(l)}$ of the linearized collision integral with non-zero eigenvalues, we obtain

$$\phi_p = -\frac{\partial_x T}{2m^2 T^2} \sum_{l \neq N, E, P, J} \tau_l \langle \phi_p^{(l)} | g_p \nu_p \rangle \phi_p^{(l)}. \quad (39)$$

Because ϕ_p is orthogonal to the four zero modes (21) and (27), the conditions (6) and (33) imposed on δn_p are satisfied.

Using Eq. (33) and the last of the conditions (6), it is convenient to replace $\varepsilon_p p \rightarrow \nu_p/2m$ in the definition (8) of j_Q , which results in

$$j_Q = \frac{1}{2m^2} \langle g_p \nu_p | \phi_p \rangle. \quad (40)$$

Substitution of the expression (39) for ϕ_p yields $j_Q = -\kappa_{\text{ex}} \partial_x T$ with

$$\kappa_{\text{ex}} = \frac{1}{4m^4 T^2} \sum_{l \neq N, E, P, J} \tau_l \langle \phi_p^{(l)} | g_p \nu_p \rangle^2. \quad (41)$$

Unlike the similar expression (24) for κ , the result (41) assumes the low-temperature regime, $T \ll \mu$, because the transport coefficient κ_{ex} is defined only at $\tau \gg \tau_{\text{ex}}$.

At $T \ll \mu$ the particle and hole excitations are confined to the vicinities of the two Fermi points $p = \pm p_F$, where $p_F = \sqrt{2m\mu}$. For such values of momentum we can use Eq. (30) to approximate ν_p defined by Eq. (38) as

$$\nu_p = 3p_F \left[(|p| - p_F)^2 - \frac{\pi^2 T^2}{3v_F^2} \right] \text{sgn } p. \quad (42)$$

For typical values of momentum, $|p| - p_F \sim T/v_F$, the leading order correction to Eq. (42) scales as T^3 and can be easily shown to give a subleading contribution to Eq. (41). Thus, the dominant contribution to κ_{ex} can be obtained by combining Eqs. (41) and (42).

We now obtain an order of magnitude estimate of the transport coefficient κ_{ex} using Eqs. (41) and (42). The typical relaxation time $\tau_l \sim \tau_{\text{ex}}$. Evaluation of the inner product (20) adds a factor of order $T/\hbar v_F$. Thus the normalization of the eigenfunctions prescribed by Eq. (19) gives $\phi_p^{(l)} \sim \sqrt{\hbar v_F/T}$. Combining these estimates we find

$$\kappa_{\text{ex}} \sim \frac{T^3 v_F \tau_{\text{ex}}}{\hbar \mu^2}. \quad (43)$$

To evaluate τ_{ex} and obtain the numerical prefactor in Eq. (43), one has to carefully consider the collision integral of the Boltzmann equation (3). We present this treatment in the next section.

IV. RELAXATION OF THE DEGENERATE FERMION GAS TO EQUILIBRIUM

We now evaluate the transport coefficient κ_{ex} in terms of the two-particle interaction potential $U(x)$ between the fermions. Our prescription (41) requires one to find the full spectrum of the relaxation rates τ_l^{-1} in the system as well as the respective relaxation modes $\phi_p^{(l)}$ by solving the eigenvalue problem (19) for the linearized collision integral. We will show that the problem simplifies considerably for the interaction potentials that decay slowly with the distance between particles, such as the Coulomb interaction. In this case one of the relaxation modes coincides with ν_p given by Eq. (42) up to a normalization factor. As a result the sum in Eq. (41) includes just one term, and the evaluation of κ_{ex} simplifies considerably. This is not the case for interaction potentials that decay rapidly with the distance between fermions, which will be considered separately.

A. Coulomb and dipole-dipole interactions

In the case of charged particles, their interactions are usually dominated by Coulomb repulsion. For particles with charge e confined to a narrow channel the interaction potential takes the form $U(x) = e^2/|x|$ at distances x that are large compared to the width of the channel w . The behavior of $U(x)$ at $x \lesssim w$ is determined by the nature of the confining potential. The study of the relaxation spectrum requires evaluation of the Fourier transform of the interaction potential

$$V(p) = \int_{-\infty}^{+\infty} U(x) e^{ipx/\hbar} dx. \quad (44)$$

Substitution of $U(x) = e^2/|x|$ into Eq. (44) results in a logarithmic singularity. We therefore account properly for the short-distance behavior of $U(x)$ for particles confined to a channel of width w , see Appendix A. At low momenta, $|p| \ll \hbar/w$, we find

$$V(p) = 2e^2 \ln \left(\frac{\hbar}{|p|w} \right) \left(1 + \frac{w^2 p^2}{\hbar^2} \right). \quad (45)$$

A numerical factor in the argument of the logarithm in Eq. (45) depends on the details of the confinement and is not included in the above expression.

Another important special case is dipole-dipole interaction $U(x) = \Upsilon/|x|^3$. It can be realized, for example, in a quantum wire in the vicinity of a metal gate parallel to it. In this case $\Upsilon = 2e^2 d^2$, where d is the distance between the wire and the gate. Fourier transform

of dipole-dipole interaction is

$$V(p) = -\Upsilon \frac{p^2}{\hbar^2} \ln \left(\frac{\hbar}{|p|w} \right). \quad (46)$$

Similarly to Eq. (45) for Coulomb interaction, Eq. (46) is written within the logarithmic accuracy and is restricted to small momenta, $|p| \ll \hbar/w$. In Eq. (46) we omitted a large constant term that corresponds to the contact interaction. For spinless fermions, Pauli principle forbids two particles to occupy the same position in space, and thus the contact interaction does not affect this system.

A special feature of slowly decaying potentials, such as Coulomb and dipole-dipole ones, is that the relaxation processes which involve co-propagating particles, shown in Fig. 1(c), occur at a higher rate than the ones that involve counter-propagating particles, see Fig 1(b). The estimate of the corresponding rates can be obtained from the results of Ref. [16], which studied the relaxation of quasiparticles with energies much greater than T . The typical decay rate of thermal quasiparticles is $\tau_c^{-1} \propto T^2$ for Coulomb interaction (45), and $\tau_c^{-1} \propto T^6$ for dipole-dipole interaction (46). These decay rates are larger than the ones involving processes depicted in Fig. 1(b), which occur at rates $\tau_b^{-1} \propto T^3$ and $\tau_b^{-1} \propto T^7$, respectively. As a result, at time scales longer than τ_c but shorter than τ_b , each branch of excitations independently achieves equilibrium characterized by its own parameters α , β , and γ . Thus, the distribution function takes a partially-equilibrated form [5]

$$n_p = \frac{\theta(p)}{e^{\beta_R \varepsilon_p - \gamma_R p - \alpha_R} + 1} + \frac{\theta(-p)}{e^{\beta_L \varepsilon_p - \gamma_L p - \alpha_L} + 1}. \quad (47)$$

Further relaxation of the distribution (47) toward the form (25) is controlled by the processes shown in Fig. 1(b), with the relaxation time $\tau_b \gg \tau_c$.

In the following we will calculate the eigenmodes and the corresponding decay rates for relaxation of the distribution (47), which will be sufficient to obtain the thermal conductivity (41). We note that this is a much simpler problem than the full solution of the eigenvalue problem (19). Instead of diagonalizing the full collision integral, which has an infinite number of eigenmodes, the problem is reduced to the study of the evolution of only six parameters in Eq. (47).

At small deviations from equilibrium, we expand Eq. (47) as $n_p = (e^{\beta \varepsilon_p - \alpha} + 1)^{-1} + \delta n_p$, where

$$\delta n_p = \theta(p) g_p^2 [-(\beta_R - \beta) \varepsilon_p + \gamma_R p + \alpha_R - \alpha] + \theta(-p) g_p^2 [-(\beta_L - \beta) \varepsilon_p + \gamma_L p + \alpha_L - \alpha]. \quad (48)$$

The decay of δn_p of Eq. (48) is constrained by the four conditions given by Eqs. (6) and (33). This leads to four equations for the six parameters of Eq. (48). We use them to express four parameters as a function of the two remaining ones, which we select to be $\beta_R - \beta_L$ and $\gamma_R - \gamma_L$. The correction to the distribution function (48) then

takes the form

$$\delta n_p = -\frac{g_p^2}{4m}(\beta_R - \beta_L)\eta_p - \frac{g_p^2}{4p_F}(\gamma_R - \gamma_L)\sigma_p, \quad (49)$$

where

$$\eta_p = p^2 \operatorname{sgn} p - \frac{I_3 I_0 - I_2 I_1}{I_2 I_0 - I_1^2} p + \frac{I_3 I_1 - I_2^2}{I_2 I_0 - I_1^2} \operatorname{sgn} p, \quad (50)$$

$$\sigma_p = 2p_F \left(\frac{I_3 I_0 - I_2 I_1}{I_4 I_0 - I_2^2} p^2 - |p| + \frac{I_4 I_1 - I_3 I_2}{I_4 I_0 - I_2^2} \right). \quad (51)$$

We note that η_p is an odd function of p , whereas σ_p is an

even one. At low temperature we find

$$\eta_p = \left[(|p| - p_F)^2 - \frac{\pi^2 T^2}{3v_F^2} \right] \operatorname{sgn} p, \quad (52)$$

$$\sigma_p = (|p| - p_F)^2 - \frac{\pi^2 T^2}{3v_F^2}. \quad (53)$$

In this regime η_p and σ_p are equal in absolute value.

To find the evolution of the distribution function (47), we consider the rate of change of the occupation number of the state p due to three-particle collisions

$$\begin{aligned} \dot{n}_p = & -\frac{1}{12} \sum_{\substack{p_1, p_2 \\ p', p'_1, p'_2}} \frac{2\pi}{\hbar} \left| \mathcal{A}_{p, p_1, p_2}^{p', p'_1, p'_2} \right|^2 \delta(\varepsilon_p + \varepsilon_{p_1} + \varepsilon_{p_2} - \varepsilon_{p'} - \varepsilon_{p'_1} - \varepsilon_{p'_2}) \\ & \times [n_p n_{p_1} n_{p_2} (1 - n_{p'}) (1 - n_{p'_1}) (1 - n_{p'_2}) - (1 - n_p) (1 - n_{p_1}) (1 - n_{p_2}) n_{p'} n_{p'_1} n_{p'_2}]. \end{aligned} \quad (54)$$

Here the scattering matrix element $\mathcal{A}_{p, p_1, p_2}^{p', p'_1, p'_2}$ depends on the details of the two-body interaction potential [1, 16] and will be discussed below. The factor 1/12 accounts for 12 identical configurations that exist due to unrestricted summations over the two initial (p_1 and p_2) and three final (p' , p'_1 , and p'_2) states.

For systems close to the thermal equilibrium, we can linearize the occupation factors by using Eqs. (5) and (17). This yields

$$\dot{\phi}_p = -\frac{1}{12g_p} \sum_{\substack{p_1, p_2 \\ p', p'_1, p'_2}} W_{p, p_1, p_2}^{p', p'_1, p'_2} \left(\frac{\phi_p}{g_p} + \frac{\phi_{p_1}}{g_{p_1}} + \frac{\phi_{p_2}}{g_{p_2}} - \frac{\phi_{p'}}{g_{p'}} - \frac{\phi_{p'_1}}{g_{p'_1}} - \frac{\phi_{p'_2}}{g_{p'_2}} \right), \quad (55)$$

where we introduced

$$W_{p, p_1, p_2}^{p', p'_1, p'_2} = \frac{2\pi}{\hbar} \left| \mathcal{A}_{p, p_1, p_2}^{p', p'_1, p'_2} \right|^2 \delta(\varepsilon_p + \varepsilon_{p_1} + \varepsilon_{p_2} - \varepsilon_{p'} - \varepsilon_{p'_1} - \varepsilon_{p'_2}) g_p g_{p_1} g_{p_2} g_{p'} g_{p'_1} g_{p'_2}. \quad (56)$$

Using the separation of variables we transform Eq. (55) into the eigenvalue problem [cf. Eq. (19)]

$$\frac{1}{\tau_l} \phi_p^{(l)} = \frac{1}{12g_p} \sum_{\substack{p_1, p_2 \\ p', p'_1, p'_2}} W_{p, p_1, p_2}^{p', p'_1, p'_2} \left(\frac{\phi_p^{(l)}}{g_p} + \frac{\phi_{p_1}^{(l)}}{g_{p_1}} + \frac{\phi_{p_2}^{(l)}}{g_{p_2}} - \frac{\phi_{p'}^{(l)}}{g_{p'}} - \frac{\phi_{p'_1}^{(l)}}{g_{p'_1}} - \frac{\phi_{p'_2}^{(l)}}{g_{p'_2}} \right). \quad (57)$$

Here the eigenvalue $1/\tau_l$ represents the decay rate of the eigenfunction $\phi_p^{(l)}$ of the collision operator. Multiplying both sides in Eq. (57) by $\phi_p^{(l)}$ and performing the summation over p , we find

$$\frac{1}{\tau_l} \sum_p (\phi_p^{(l)})^2 = \frac{1}{72} \sum_{\substack{p, p_1, p_2 \\ p', p'_1, p'_2}} W_{p, p_1, p_2}^{p', p'_1, p'_2} \left(\frac{\phi_p^{(l)}}{g_p} + \frac{\phi_{p_1}^{(l)}}{g_{p_1}} + \frac{\phi_{p_2}^{(l)}}{g_{p_2}} - \frac{\phi_{p'}^{(l)}}{g_{p'}} - \frac{\phi_{p'_1}^{(l)}}{g_{p'_1}} - \frac{\phi_{p'_2}^{(l)}}{g_{p'_2}} \right)^2. \quad (58)$$

Here we used the symmetries of W to extend the summation over six equivalent terms, thereby conveniently making the right-hand side symmetric. As a result we have obtained an extra 1/6 prefactor and the square of the expression in parentheses in the right-hand side. We notice that Eqs. (54)–(58) do not have restriction on the

summation range and therefore apply for all three kinds of processes shown in Fig. 1.

The momentum inversion symmetry of W enables one to classify the eigenfunctions of Eq. (57) with respect to parity. This guarantees that the eigenfunctions in Eq. (57) will be either odd or even in p . Thus, the odd

and even components of ϕ_p obtained from Eq. (49) as $\delta n_p/g_p$, which are proportional to $g_p\eta_p$ and even $g_p\sigma_p$, respectively, relax as two independent eigenmodes. We now normalize them and introduce

$$\phi_p^{(\eta)} = C_\eta g_p \eta_p, \quad (59)$$

$$\phi_p^{(\sigma)} = C_\sigma g_p \sigma_p, \quad (60)$$

where to leading order in T/μ the normalization constants are equal,

$$C_\eta = C_\sigma = \sqrt{\frac{45\hbar v_F^5}{16\pi^3 T^5}}. \quad (61)$$

Next, we calculate the decay rates of $\phi_p^{(\eta)}$ and $\phi_p^{(\sigma)}$ using Eq. (58). We split the six-fold summation in that expression into $2^6 = 64$ six-fold summations over momenta that are either positive or negative. Out of 64 different terms, 44 describe particle backscattering, i.e., the numbers of particles with positive (negative) momenta are different in the initial and the final states [see, for example, Fig. 1(a)]. At low temperatures, such processes occur at exponentially long time scales as discussed in Sec. II. They determine the thermal conductivity of the system κ but must be neglected in evaluating the thermal conductivity of the gas of excitations κ_{ex} .

Out of the remaining 20 terms, two have all the six momenta of the same sign. The expression in parentheses in Eq. (58) evaluated for such restricted range of momenta, i.e., all positive or all negative, nullifies for the two eigenfunctions (59) and (60) due to the conservation laws of energy and momentum. Therefore, for the processes represented in Fig. 1(c), in addition to four zero modes (21) and (27), Eqs. (59) and (60) define two additional zero modes. This conclusion is consistent with six-parameter partially equilibrated distribution (47).

The remaining 18 terms of the sum in Eq. (58) are equivalent and contain configurations of six momenta with two near one Fermi point, and the remaining four near the opposite one, in both the initial and final states, see Fig. 1(b). Using the notation of the figure where q and q' are negative while k, k_1, k', k'_1 positive, the expression in parentheses of Eq. (58) becomes

$$\begin{aligned} & \frac{\phi_k^{(\eta)}}{g_k} + \frac{\phi_{k_1}^{(\eta)}}{g_{k_1}} + \frac{\phi_q^{(\eta)}}{g_q} - \frac{\phi_{k'}^{(\eta)}}{g_{k'}} - \frac{\phi_{k'_1}^{(\eta)}}{g_{k'_1}} - \frac{\phi_{q'}^{(\eta)}}{g_{q'}} \\ &= 2C_\eta(q'^2 - q^2), \end{aligned} \quad (62)$$

$$\begin{aligned} & \frac{\phi_k^{(\sigma)}}{g_k} + \frac{\phi_{k_1}^{(\sigma)}}{g_{k_1}} + \frac{\phi_q^{(\sigma)}}{g_q} - \frac{\phi_{k'}^{(\sigma)}}{g_{k'}} - \frac{\phi_{k'_1}^{(\sigma)}}{g_{k'_1}} - \frac{\phi_{q'}^{(\sigma)}}{g_{q'}} \\ &= 4C_\sigma p_F(q - q'), \end{aligned} \quad (63)$$

where we accounted for the conservation laws of momentum and energy. Since $\sum_p (\phi_p^{(\eta)})^2 = \sum_p (\phi_p^{(\sigma)})^2 = L$,

where L is the system size, we obtain

$$\frac{1}{\tau_\eta} = \frac{C_\eta^2}{L} \sum_{\substack{k, k_1, k', k'_1 > 0 \\ q, q' < 0}} W_{k, k_1, q}^{k', k'_1, q'} (q'^2 - q^2)^2, \quad (64)$$

$$\frac{1}{\tau_\sigma} = \frac{4p_F^2 C_\sigma^2}{L} \sum_{\substack{k, k_1, k', k'_1 > 0 \\ q, q' < 0}} W_{k, k_1, q}^{k', k'_1, q'} (q' - q)^2. \quad (65)$$

At low temperature, the momenta in the summation in Eqs. (64) and (65) are confined near the corresponding Fermi points. We can thus linearize $(q'^2 - q^2)^2 \simeq 4p_F^2(q' - q)^2$. Since $C_\eta = C_\sigma$, the two decay rates (64) and (65) are equal at the leading order in small T/μ .

1. Evaluation of τ_η^{-1}

To find τ_η^{-1} we need an expression for the three-particle scattering matrix element \mathcal{A} that enters Eq. (64) via W [see Eq. (56)]. The matrix element \mathcal{A} was calculated in Ref. [16] for an arbitrary configuration of momenta. In the special case of momenta that corresponds to the process shown in Fig. 1(b), the result of Ref. [16] for Coulomb interaction takes the form

$$\begin{aligned} \mathcal{A}_{k, k_1, q}^{k', k'_1, q'} &= \frac{12e^4}{L^2} \frac{mw^2}{\hbar^2} \ln\left(\frac{\hbar}{p_F w}\right) \ln\left|\frac{k_1 - k + k' - k'_1}{k - k_1 + k' - k'_1}\right| \\ &\times \delta_{k+k_1+q, k'+k'_1+q'}. \end{aligned} \quad (66)$$

Similarly, for dipole-dipole interaction we have

$$\begin{aligned} \mathcal{A}_{k, k_1, q}^{k', k'_1, q'} &= -\frac{5\Upsilon^2 m}{L^2 \hbar^4} \ln\left(\frac{\hbar}{p_F w}\right) (k_1 - k)(k'_1 - k') \\ &\times \ln\left(\frac{p_F}{\sqrt{(k_1 - k)^2 + (k'_1 - k')^2}}\right) \delta_{k+k_1+q, k'+k'_1+q'}. \end{aligned} \quad (67)$$

We now proceed to the evaluation of the expression (64). Using the conservation laws of momentum and energy we find

$$q' - q = \frac{(k' - k)(k' - k_1)}{k'_1 - q} \simeq \frac{(k' - k)(k' - k_1)}{2p_F}, \quad (68)$$

because for the process shown in Fig. 1(b), $k'_1 - q \simeq 2p_F$. Therefore, we substitute

$$(q'^2 - q^2)^2 \simeq (k' - k)^2 (k' - k_1)^2 \quad (69)$$

in Eq. (64). For thermally excited quasiparticles, $|k' - k| \sim |k' - k_1| \sim T/v_F$. On the other hand, from Eq. (68) we find $|q' - q| \sim T^2/\mu v_F$. We therefore find $q' = q + O(T^2/\mu v_F)$. After the substitution (69), the remainder of the expression (64) does not depend on the difference $q' - q$ apart from the delta functions contained in W . We

therefore approximate

$$\begin{aligned} & \delta_{k+k_1+q, k'+k'_1+q'} \delta(\varepsilon_k + \varepsilon_{k_1} + \varepsilon_q - \varepsilon_{k'} - \varepsilon_{k'_1} - \varepsilon_{q'}) \\ & \simeq \frac{1}{2v_F} \delta_{k+k_1, k'+k'_1} \delta(q - q'). \end{aligned} \quad (70)$$

The integration over q and q' is now straightforward, resulting in $\int dq dq' \delta(q - q') g_q g_{q'} = T/v_F$. Here we linearized the spectrum at low temperature, such that

$$g_q = \frac{1}{2 \cosh \frac{v_F(q+p_F)}{2T}}. \quad (71)$$

The remaining four integrations involve one delta function. For Coulomb interaction (45) we were able to perform analytically one more integration and found the decay rate

$$\frac{1}{\tau_\eta} = c \frac{w^4}{a_B^4} \ln^2 \left(\frac{\hbar}{p_F w} \right) \frac{T^3}{\hbar \mu^2}. \quad (72)$$

Here $a_B = \hbar^2/m_e^2$ is the Bohr radius and

$$c = -\frac{405}{4\pi^7} \int_{-\infty}^{+\infty} \frac{u^2 w^2 (u+w) \ln^2 \left| \frac{w}{u} \right| du dw}{\sinh u \sinh w \sinh(u+w)} \approx 0.2306. \quad (73)$$

For dipole-dipole interaction (46) we have been able to evaluate the numerical prefactor in the decay rate analytically,

$$\frac{1}{\tau_\eta} = \frac{225\pi^3}{616} \left[\frac{p_F^2 m^2 \Upsilon^2}{\hbar^6} \ln \left(\frac{\hbar}{p_F w} \right) \ln \left(\frac{\mu}{T} \right) \right]^2 \frac{T^7}{\hbar \mu^6}. \quad (74)$$

Here the momentum-dependent logarithm originating from the scattering matrix element (67) is replaced by $\ln(\mu/T)$, in accordance with the logarithmic accuracy adopted earlier.

2. Evaluation of κ_{ex}

As a result of separation of time scales $\tau_c \ll \tau_b$, at scales longer than τ_c there exist only two relaxation modes of the distribution function given by Eqs. (59) and (60). Up to a normalization constant, the former one, $\phi_p^{(\eta)}$ coincides with $g_p \nu_p$ of Eq. (42). Therefore the eigenmode (59) actually exhausts the sum in Eq. (41) yielding

$$\kappa_{\text{ex}} = \frac{1}{4m^4 T^2} \tau_\eta \langle \phi_p^{(\eta)} | g_p \nu_p \rangle^2. \quad (75)$$

The overlap entering the latter expression can now be easily obtained by a comparison between Eqs. (42) and (52):

$$\langle \phi_p^{(\eta)} | g_p \nu_p \rangle = \frac{3p_F}{C_\eta}, \quad (76)$$

which gives

$$\kappa_{\text{ex}} = \frac{\pi^3}{5} \frac{T^3 v_F \tau_\eta}{\hbar \mu^2}. \quad (77)$$

This is our final expression for the thermal conductivity of the gas of elementary excitations. It applies to systems with long-range two-body interactions. For the Coulomb and dipole-dipole interactions, the relaxation time τ_η is given by Eqs. (72) and (74).

We stress that the simplification (77) corresponding to just one eigenmode of the linearized collision integral contributing to the general expression (41) holds only for the long-range interactions. A different scenario occurs in systems where the interaction potential decays rapidly with the distance. In this case the evaluation of κ_{ex} requires more involved study of the relaxation modes of the collision integral, which we turn to next.

B. Short-range interactions

1. Scattering matrix element

In the case of short-range interactions, the rates associated with the processes of Figs. 1(b) and (c) scale with the temperature as $\tau_b^{-1} \propto T^7$ [1, 8, 9] and $\tau_c^{-1} \propto T^{14}$ [9], respectively. Thus, unlike the cases of Coulomb and dipole-dipole interactions, at low temperatures $\tau_b \ll \tau_c$. As a result, only the processes of Fig. 1(b) need to be taken into consideration [21].

At low temperature $T \ll \mu$, when all the states are close to the respective Fermi points, the scattering matrix element takes the form

$$\mathcal{A}_{k, k_1, q}^{k', k'_1, q'} = \frac{\Lambda}{L^2} (k - k_1)(k' - k'_1) \delta_{k+k_1+q, k'+k'_1+q'}, \quad (78)$$

where L is the system size. This expression was obtained in Ref. [15] for a spinless quantum liquid with arbitrarily strong interactions. It is consistent with the matrix element used in Ref. [14] provided that

$$\Lambda = \frac{3V(0)V''(0)}{8m v_F^2}, \quad (79)$$

where $V(p)$ is the Fourier transform (44) of the interaction potential. We note that the calculation of Ref. [14] assumes that $V(p)$ falls off rapidly away from the peak at $p = 0$, such that $V(2p_F)$ is negligible compared with $V(0)$.

In Ref. [15] the parameter Λ was expressed in terms of the properties of the quasiparticle spectrum of the spinless quantum liquid. In Appendix B we apply that prescription to the weakly-interacting spinless Fermi gas and

find

$$\Lambda = m \left(\frac{3(V(0) - V(2p_F))V''(0)}{8p_F^2} + \frac{1}{2}V''(0)V''(2p_F) \right. \\ \left. + \frac{3V''(0)V'(2p_F)}{4p_F} - \frac{V'(2p_F)V''(2p_F)}{4p_F} \right. \\ \left. - \frac{(V'(2p_F))^2}{4p_F^2} - \frac{(V(0) - V(2p_F))V''(2p_F)}{8p_F^2} \right. \\ \left. - \frac{(V(0) - V(2p_F))V'''(2p_F)}{12p_F} \right). \quad (80)$$

This expression recovers Eq. (79) at $V(2p_F) = 0$.

2. Linearized collision integral in the low-temperature limit

Collision processes shown in Fig. 1(b) change the occupation number of the state k on the right-moving branch with the rate

$$\dot{n}_k = -\frac{2\pi}{\hbar} \frac{1}{2} \sum_{\substack{k_1, k', k'_1 \\ q, q'}} \left| \mathcal{A}_{k, k_1, q}^{k', k'_1, q'} \right|^2 \delta(\varepsilon_k + \varepsilon_{k_1} + \varepsilon_q - \varepsilon_{k'} - \varepsilon_{k'_1} - \varepsilon_{q'}) \\ \times [n_k n_{k_1} n_q (1 - n_{k'}) (1 - n_{k'_1}) (1 - n_{q'}) - (1 - n_k) (1 - n_{k_1}) (1 - n_q) n_{k'} n_{k'_1} n_{q'}]. \quad (81)$$

Here we assume that the sums over k' , k_1 , and k'_1 are limited to the right-moving branch, while those over q and q' are limited to the left-moving one, Fig. 1(b). The factor $1/2$ compensates for the double counting in the sum due to the permutation of k' and k'_1 . Note, that there is an additional contribution to \dot{n}_k due to the processes involving one right-moving and two left-moving particles. We will see below that at low temperatures this contribution is negligible.

We now focus on systems close the thermal equilibrium by substituting Eqs. (5) and (17) and linearizing the collision integral in small ϕ_p . This yields

$$\dot{\phi}_k = -\frac{\pi}{\hbar} \sum_{\substack{k_1, k', k'_1 \\ q, q'}} \left| \mathcal{A}_{k, k_1, q}^{k', k'_1, q'} \right|^2 \delta(\varepsilon_k + \varepsilon_{k_1} + \varepsilon_q - \varepsilon_{k'} - \varepsilon_{k'_1} - \varepsilon_{q'}) g_{k_1} g_q g_{k'} g_{k'_1} g_{q'} \left(\frac{\phi_k}{g_k} + \frac{\phi_{k_1}}{g_{k_1}} + \frac{\phi_q}{g_q} - \frac{\phi_{k'}}{g_{k'}} - \frac{\phi_{k'_1}}{g_{k'_1}} - \frac{\phi_{q'}}{g_{q'}} \right). \quad (82)$$

At low temperature the typical values of momentum of thermally excited quasiparticles measured from the nearest Fermi point are of the order of T/v_F . On the other hand, as we saw in Sec. IV A, for the processes of Fig. 1(b) the difference of the momenta on the left branch is $|q - q'| \sim T^2/\mu v_F \ll T/v_F$. Thus, when solving Eq. (82) to leading order in $T/\mu \ll 1$, after substituting the expression (78) for the matrix element one can apply the approximation (70). This yields

$$\dot{\phi}_k = -\frac{\Lambda^2 T}{32\pi^3 \hbar^5 v_F^2} \int dk_1 dk' dk'_1 (k - k_1)^2 (k' - k'_1)^2 \\ \times \delta(k + k_1 - k' - k'_1) g_{k_1} g_{k'} g_{k'_1} \\ \times \left(\frac{\phi_k}{g_k} + \frac{\phi_{k_1}}{g_{k_1}} - \frac{\phi_{k'}}{g_{k'}} - \frac{\phi_{k'_1}}{g_{k'_1}} \right). \quad (83)$$

We note that because Eq. (83) holds only in the low-temperature limit, the same approximation must be used in Eq. (13), resulting in

$$g_k = \frac{1}{2 \cosh \frac{v_F(k - p_F)}{2T}}. \quad (84)$$

Equation (83) shows that to leading order in T/μ the processes involving two right-moving and one left-moving

particles [Fig. 1(b)] affect only the distribution function on the right-moving branch. This justifies our earlier approximation that neglected the contribution to \dot{n}_k from the processes involving one right-moving and two left-moving particles.

3. Dimensionless form of the collision integral

Let us now bring Eq. (83) to a dimensionless form by introducing dimensionless momentum ξ and time θ ,

$$\xi = \frac{v_F}{2\pi T}(k - p_F), \quad \theta = \frac{2\pi^3}{\tau_{\text{ex}}} t, \quad (85)$$

where the relaxation time τ_{ex} is defined by

$$\frac{1}{\tau_{\text{ex}}} = \frac{\Lambda^2 T^7}{\hbar^5 v_F^8} \quad (86)$$

and has the expected power-law scaling $\tau_{\text{ex}}^{-1} \propto T^7$ [1, 8, 9]. In these units, Eq. (83) takes the form

$$\frac{\partial \Phi(\xi)}{\partial \theta} = -\widehat{M} \Phi(\xi), \quad (87)$$

where

$$\widehat{M}\Phi(\xi) = \int d\xi_1 d\xi' d\xi'_1 (\xi - \xi_1)^2 (\xi' - \xi'_1)^2 \delta(\xi + \xi_1 - \xi' - \xi'_1) G(\xi_1) G(\xi') G(\xi'_1) \left(\frac{\Phi(\xi)}{G(\xi)} + \frac{\Phi(\xi_1)}{G(\xi_1)} - \frac{\Phi(\xi')}{G(\xi')} - \frac{\Phi(\xi'_1)}{G(\xi'_1)} \right). \quad (88)$$

Here $\Phi(\xi) = \phi_k$ and $G(\xi) = g_k$, i.e.,

$$G(\xi) = \frac{1}{2 \cosh(\pi \xi)}. \quad (89)$$

After some algebra the integral operator (88) can be rewritten in the form

$$\widehat{M}\Phi(\xi) = A(\xi)\Phi(\xi) + \int d\xi' [B_1(\xi, \xi') + B_2(\xi, \xi')]\Phi(\xi'), \quad (90)$$

where

$$A(\xi) = \frac{(1 + 4\xi^2)(9 + 4\xi^2)(5 + 44\xi^2)}{5760}, \quad (91)$$

$$B_1(\xi, \xi') = \frac{1}{6}(\xi - \xi')^2 \frac{(\xi + \xi')[1 + (\xi + \xi')^2]}{\sinh(\pi(\xi + \xi'))}, \quad (92)$$

$$B_2(\xi, \xi') = -\frac{\xi - \xi'}{240 \sinh(\pi(\xi - \xi'))} \times (7 + 120\xi\xi' + 128\xi^4 - 752\xi^3\xi' + 1488\xi^2\xi'^2 - 752\xi\xi'^3 + 128\xi'^4). \quad (93)$$

As expected, the kernel of the integral operator (90) is symmetric with respect to permutation $\xi \leftrightarrow \xi'$. This property of the operator \widehat{M} ensures that the eigenvalue problem

$$\widehat{M}\Phi_l(\xi) = \lambda_l \Phi_l(\xi) \quad (94)$$

has an orthonormal set of solutions $\Phi_l(\xi)$ with real eigenvalues λ_l .

The eigenvalue problem (94) was derived from Eq. (83), which describes time evolution of the distribution function of fermions near the right Fermi point. Particles near the left Fermi point can be treated in the same way. Therefore, each eigenfunction $\Phi_l(\xi)$ gives a solution $\phi_p^{(l)}$ of the full eigenvalue problem (19) that is confined to either right- or left-moving part of the quasi-particle spectrum. The corresponding eigenvalue is

$$\frac{1}{\tau_l} = \frac{2\pi^3}{\tau_{\text{ex}}} \lambda_l. \quad (95)$$

Alternatively, one can symmetrize and antisymmetrize the eigenfunctions, resulting in two sets of solutions $\phi_p^{(l)}$ that are either even or odd in p with the same eigenvalues (95). Evaluation of the transport coefficient κ_{ex} given by Eqs. (41) and (42) requires odd solutions, which take the form

$$\phi_p^{(l)} = \left(\frac{\hbar v_F}{2T} \right)^{1/2} \Phi_l \left(\frac{v_F(|p| - p_F)}{2\pi T} \right) \text{sgn } p, \quad (96)$$

where the prefactor assumes that the eigenfunctions $\Phi_l(\xi)$ are normalized according to

$$\int_{-\infty}^{+\infty} \Phi_l^2(\xi) d\xi = 1. \quad (97)$$

Using Eq. (96), we can evaluate the matrix element in our expression (41) for κ_{ex} ,

$$\langle \phi_p^{(l)} | g_p \nu_p \rangle = \frac{6\sqrt{2}\pi^2 m T^{5/2}}{\hbar^{1/2} v_F^{3/2}} \int_{-\infty}^{\infty} \frac{\xi^2 - \frac{1}{12}}{\cosh(\pi \xi)} \Phi_l(\xi) d\xi. \quad (98)$$

Substituting this matrix element into Eq. (41), we obtain

$$\kappa_{\text{ex}} = \frac{9\pi c}{4} \frac{T^3 v_F \tau_{\text{ex}}}{\hbar \mu^2}, \quad (99)$$

where τ_{ex} is given by Eq. (86) and the numerical coefficient c is defined as

$$c = \sum_l' \frac{1}{\lambda_l} \left[\int_{-\infty}^{\infty} \frac{\xi^2 - \frac{1}{12}}{\cosh(\pi \xi)} \Phi_l(\xi) d\xi \right]^2. \quad (100)$$

Summation in Eq. (100) excludes the zero modes for which $\lambda_l = 0$.

Because the dimensionless eigenvalue problem (94) describes relaxation of the right-moving particles, which at $T/\mu \rightarrow 0$ is decoupled from the relaxation of the left movers, operator \widehat{M} has only two zero modes:

$$\Phi_0(\xi) = \sqrt{2\pi} G(\xi), \quad (101)$$

$$\Phi_1(\xi) = \sqrt{24\pi} \xi G(\xi). \quad (102)$$

The modes Φ_0 and Φ_1 correspond to the conservation of the particle number and momentum, respectively. Their forms are easily verified analytically using Eq. (88). The integral operator (90) can be diagonalized numerically. Excluding the zero modes (101) and (102) from the sum (100), we obtained $c \approx 0.41088$.

V. DISCUSSION OF THE RESULTS

In this paper we have developed a microscopic theory of the thermal transport coefficients of one-dimensional Fermi gas at low temperature $T \ll \mu$. A special feature of one-dimensional quantum systems is that in addition to the usual thermal conductivity of the system κ , one can introduce thermal conductivity of the gas of elementary excitations κ_{ex} . This is a consequence of separation of scales of the rates of various processes responsible for the relaxation of the system to equilibrium. Specifically, the rates of the processes of Fig. 1(a), which are responsible for the equilibration of the chemical potentials of the

left- and right-moving particles, are exponentially small, $\tau^{-1} \propto e^{-\mu/T}$. On the other hand, the remaining scattering processes illustrated in Fig. 1(b) and (c) occur at rates τ_{ex}^{-1} that scale as a power law of T/μ and are therefore much faster.

In general, transport coefficients are proportional to the relevant relaxation times. As a result, the thermal conductivity $\kappa \propto \tau$ is exponentially large. Our microscopic theory gives the result (31), which is consistent with the phenomenological expression (1) obtained within the Luttinger liquid theory. These results relate κ to the relaxation time τ , which requires special evaluation. A phenomenological theory [6, 7] expresses τ in terms of the properties of the excitation spectrum of the quantum liquid. For the system of weakly interacting fermions further progress can be made. In Appendix C we obtained an expression for τ in terms of the interaction potential. For example, in the case of spinless fermions with dipole-dipole interaction (46) we found

$$\frac{1}{\tau} = \frac{36(\ln 4 - 1)^2}{5\pi^{3/2}} \frac{\Upsilon^4 p_F^9}{\hbar^{13} v_F^3} \ln^2 \left(\frac{\hbar}{p_F w} \right) \left(\frac{T}{\mu} \right)^{3/2} e^{-\mu/T}. \quad (103)$$

A more general expression given by Eqs. (C3) and (C4) applies to any interaction $U(x)$ that falls off at large distances faster than $1/x$.

In one-dimensional systems the transport coefficient κ describes the thermal conductivity only at low frequencies $\omega \ll \tau^{-1}$. In particular, it gives the dominant contribution to the attenuation of sound at low frequencies [13]. On the other hand, at $\omega \gg \tau^{-1}$ the exponentially slow relaxation processes can be neglected, which leads to a very different behavior of the system. Instead of the conventional sound, the system now supports two sound modes [18, 19], whose attenuation is no longer exponentially strong. Instead, the thermal transport coefficient that enters the expression for sound attenuation is κ_{ex} [13], which can be thought of as the thermal conductivity at frequencies $\omega \gg \tau^{-1}$.

Our microscopic theory relates κ_{ex} to the relaxation modes of the system, see Eqs. (41) and (42). This enables one to find the order of magnitude estimate (43), which expresses κ_{ex} in terms of the relaxation time τ_{ex} . To obtain a full microscopic expression for κ_{ex} , a more detailed treatment of the relaxation processes is required. We performed such a treatment for two kinds of interactions between fermions. For Coulomb and dipole-dipole interactions, the long range of the interaction potential results in processes of the type shown in Fig. 1(c) having a higher rate than those of Fig. 1(b). The relaxation rate τ_{η}^{-1} that controls the thermal conductivity of the gas of excitations is due to the latter type of processes and is given by Eqs. (72) and (74). Our microscopic result for κ_{ex} is given by Eq. (77). It is consistent with the earlier estimate (43) provided that the rate τ_{ex}^{-1} of relaxation of the gas of excitations is identified with τ_{η}^{-1} . On the other hand, in the case of short-range interactions, the processes shown in Fig. 1(c) are negligible. The result-

ing κ_{ex} is expressed in terms of the interaction potential using Eqs. (80), (86) and (99).

In our treatment we considered the model of spinless fermions. Spins can be added to the Boltzmann equation treatment of thermal transport in a straightforward way, but they may strongly affect the relaxation processes. The processes of Fig. 1(a) remain exponentially suppressed in the presence of spins, and we again expect $\kappa \propto \tau \propto e^{\mu/T}$. The rate of processes shown in Fig. 1(b) was studied in Ref. [10], where $\tau_{\text{ex}}^{-1} \propto T$ was obtained, while the processes of Fig. 1(c) have not been explored. In the case of spinless fermions the relaxation rate relevant for the evaluation of κ_{ex} was that of the processes shown in Fig. 1(b), and this should hold for fermions with spin. Thus, using the estimate (43) we expect that for weakly interacting spin- $\frac{1}{2}$ fermions $\kappa_{\text{ex}} \propto T^2$. We leave more careful microscopic treatment of this problem for future study.

Finally, we note that our treatment of the thermal conductivity assumes that at every point in space the Fermi gas is near local thermal equilibrium. This assumption is violated if one takes into account the sound modes in the system, which are weakly damped at low frequencies [22]. This effect is particularly strong in one dimension, where it leads to power-law scaling of sound absorption [23]. Violation of the assumption of local equilibrium gives rise to power-law scaling of thermal conductance of one-dimensional classical systems with the system size, see, e.g., Ref. [24]. This effect should also manifest itself in the low-frequency behavior of thermal conductivity of one-dimensional Fermi gas at low temperatures. On the other hand, because the local contribution (1) to the thermal conductivity is exponentially amplified by the long relaxation time τ , one should expect the non-local effects to become significant only at frequencies that are exponentially smaller than τ^{-1} . Frequency dependence of thermal conductivity of one-dimensional electronic fluid was recently studied by R. Samanta *et al.* [25]. In addition to the crossover from the result (1) to a power-law scaling at $\omega \rightarrow 0$, they obtained several additional parametric frequency regions, including one in which $\kappa(\omega)$ is consistent with our result for κ_{ex} .

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Appendix A: Fourier transform of the Coulomb interaction potential

Let us consider two three-dimensional particles with charge e confined to a one-dimensional channel. The position of each particle will be described by its coordinate along the channel and the vector $\boldsymbol{\rho}$ in the transverse direction. Coulomb interaction between the two particles at the distance x along the channel is given by

$$U(x) = e^2 \iint d\boldsymbol{\rho}_1 d\boldsymbol{\rho}_2 \frac{|\Psi(\boldsymbol{\rho}_1)|^2 |\Psi(\boldsymbol{\rho}_2)|^2}{\sqrt{x^2 + |\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|^2}}, \quad (\text{A1})$$

where $\Psi(\boldsymbol{\rho})$ is the wavefunction of the transverse motion of a particle in the channel. Performing the Fourier transform (44), we find

$$V(p) = 2e^2 \iint d\boldsymbol{\rho}_1 d\boldsymbol{\rho}_2 |\Psi(\boldsymbol{\rho}_1)|^2 |\Psi(\boldsymbol{\rho}_2)|^2 K_0\left(\frac{p}{\hbar} |\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|\right), \quad (\text{A2})$$

where $K_0(z)$ is the Macdonald function. Its asymptotic behavior at small z is given by

$$K_0(z) = \ln \frac{2}{z} - \mathbf{C} + \frac{z^2}{4} \left(\ln \frac{2}{z} + 1 - \mathbf{C} \right) + o(z^3), \quad (\text{A3})$$

where $\mathbf{C} \approx 0.5772$ is the Euler's constant. Substituting Eq. (A3) into Eq. (A2) and omitting coefficients of order unity in the argument of the logarithms, we obtain the result (45), provided that the channel width w is defined by

$$w^2 = 8(\overline{\boldsymbol{\rho}^2} - \bar{\boldsymbol{\rho}}^2). \quad (\text{A4})$$

Here the averaging

$$\overline{f(\boldsymbol{\rho})} = \int d\boldsymbol{\rho} |\Psi(\boldsymbol{\rho})|^2 f(\boldsymbol{\rho}) \quad (\text{A5})$$

is performed over the distribution of the particle density in the transverse direction.

Appendix B: Evaluation of Λ in the limit of weak interactions

In this Appendix we evaluate the parameter Λ in the expression for the scattering matrix element (78) characterizing the process shown in Fig. 1(b). We will express Λ in terms of the Fourier components $V(p)$ of the interaction potential and obtain the result (80). Our calculation is based on the approach suggested in Ref. [15] where the matrix element (78) was used to study the decay rate of quasiparticle excitations of a spinless quantum liquid at zero temperature. The prescription of Ref. [15] is

$$\Lambda = - \lim_{p \rightarrow p_F} \frac{Y_p}{(p - p_F)^2}, \quad (\text{B1})$$

where Y_p is expressed in terms of the quasiparticle energies ϵ_p as follows

$$Y_p = \partial_{LR}^2 \epsilon_p - \frac{1}{m_p^*} \frac{\partial_L \epsilon_p}{v + v_p} \frac{\partial_R \epsilon_p}{v - v_p} + \partial_L v_p \frac{\partial_R \epsilon_p}{v - v_p} - \partial_R v_p \frac{\partial_L \epsilon_p}{v + v_p} + \frac{v \partial_n K}{\sqrt{K}} \left(\frac{\partial_R \epsilon_p}{v - v_p} + \frac{\partial_L \epsilon_p}{v + v_p} \right). \quad (\text{B2})$$

Here the quasiparticle velocity is defined as a derivative of quasiparticle energy, $v_p = \epsilon'_p$, the effective mass $m_p^* = 1/\epsilon''_p$, the speed of the low energy excitations $v = v_{p_F}$, and for Galilean invariant systems the Luttinger liquid parameter $K = v_F/v$. In an interacting system the quasiparticle energies depend on the density of particles n and momentum per particle χ . The partial derivatives in Eq. (B2) are defined by

$$\partial_R = \sqrt{K} \partial_n + \frac{\pi \hbar}{\sqrt{K}} \partial_\chi, \quad (\text{B3a})$$

$$\partial_L = \sqrt{K} \partial_n - \frac{\pi \hbar}{\sqrt{K}} \partial_\chi, \quad (\text{B3b})$$

$$\partial_{LR}^2 = K \partial_n^2 - \frac{(\pi \hbar)^2}{K} \partial_\chi^2. \quad (\text{B3c})$$

Finally, since the sign of Λ has no physical significance, for convenience, in Eq. (B1) we changed the sign of the expression used in Ref. [15].

A quasiparticle can be added to the ground state of the system in two ways. First, one can move a fermion from a Fermi point p_F to a state with momentum $p > p_F$. In this approach, the quasiparticle is essentially a particle-hole pair, with the hole remaining at the Fermi point. Alternatively, an additional fermion with momentum $p > p_F$ can be added to the system. The former approach was used in Ref. [15], but the prescription (B1) and (B2) applies in both cases. In this Appendix we will use the second approach. In particular, the energy of the excitation in the system of non-interacting fermions is

$$\epsilon_p^{(0)} = \frac{p^2}{2m}. \quad (\text{B4})$$

This energy, of course, depends on neither the density n nor the momentum per particle χ of the system, and therefore yields $\Lambda = 0$. In a weakly interacting system, the energies of both the ground state and the state with the additional particle with momentum p change. In this case the quasiparticle energy should be understood as the difference of the energies of those many-body states. The energy ϵ_p defined this way does depend on n and χ , resulting in a nonvanishing Λ .

Instead of treating quasiparticle energies as functions of n and χ , it will be convenient to think of the ground state of a moving system in terms of the positions of the left and right Fermi points, p_L and p_R . The two sets of variables are related by

$$n = \frac{p_R - p_L}{2\pi \hbar}, \quad \chi = \frac{p_R + p_L}{2}. \quad (\text{B5})$$

Then the derivatives (B3) can be written as

$$\partial_R = \pi\hbar \left(\frac{1}{\sqrt{K}} + \sqrt{K} \right) \frac{\partial}{\partial_{p_R}} + \pi\hbar \left(\frac{1}{\sqrt{K}} - \sqrt{K} \right) \frac{\partial}{\partial_{p_L}}, \quad (\text{B6a})$$

$$\partial_L = -\pi\hbar \left(\frac{1}{\sqrt{K}} - \sqrt{K} \right) \frac{\partial}{\partial_{p_R}} - \pi\hbar \left(\frac{1}{\sqrt{K}} + \sqrt{K} \right) \frac{\partial}{\partial_{p_L}}, \quad (\text{B6b})$$

$$\partial_{LR}^2 = -2\pi^2\hbar^2 \left(\frac{1}{K} + K \right) \frac{\partial^2}{\partial_{p_R} \partial_{p_L}} - \pi^2\hbar^2 \left(\frac{1}{K} - K \right) \left(\frac{\partial^2}{\partial_{p_R^2}} + \frac{\partial^2}{\partial_{p_L^2}} \right). \quad (\text{B6c})$$

In the following, we evaluate the quasiparticle energy $\epsilon_p(p_L, p_R)$ up to terms quadratic in interactions, substitute the resulting expressions into Eq. (B2), take the limits $p_L \rightarrow -p_F$ and $p_R \rightarrow p_F$, and obtain Λ from Eq. (B1).

We start by evaluating the first order correction $\delta\epsilon_k^{(1)}$ to the quasiparticle energy (B4). The interaction between fermions is given by

$$\hat{V} = \frac{1}{2L} \sum_{p_1, p_2, q} V(q) a_{p_1+q}^\dagger a_{p_2-q}^\dagger a_{p_2} a_{p_1}, \quad (\text{B7})$$

where $V(q)$ is defined by Eq. (44). The first-order correction to the energy of a many-body state

$$\delta E^{(1)} = \frac{1}{2L} \sum_{p_1, p_2} [V(0) - V(p_1 - p_2)] n_{p_1} n_{p_2}, \quad (\text{B8})$$

where $n_p = \langle a_p^\dagger a_p \rangle$ is the occupation number of the state with momentum p . In a moving ground state

$$n_p = \theta(p - p_L) - \theta(p - p_R), \quad (\text{B9})$$

where $\theta(x)$ is the unit step function, and we assumed $p_R > p_L$. The extra contribution to $\delta E^{(1)}$ from an additional fermion at state p is given by

$$\delta\epsilon_p^{(1)} = \frac{1}{2\pi\hbar} \int_{p_L}^{p_R} dp_1 [V(0) - V(p - p_1)]. \quad (\text{B10})$$

Differentiating this expression with respect to p , we find the first order correction to quasiparticle velocity $v_p^{(0)} = p/m$ in the form

$$\delta v_p^{(1)} = \frac{V(p - p_R) - V(p - p_L)}{2\pi\hbar}. \quad (\text{B11})$$

In a stationary system, $p_R = -p_L = p_F$, the velocity at the Fermi point is

$$v = v_F + \frac{V(0) - V(2p_F)}{2\pi\hbar}. \quad (\text{B12})$$

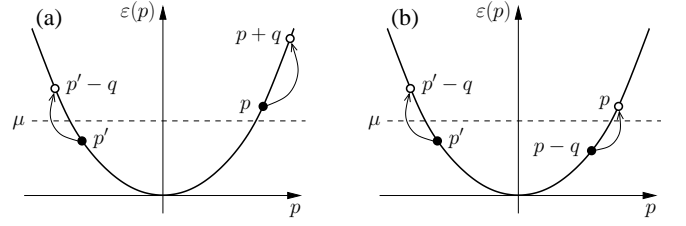


FIG. 2. Second-order contributions to the quasiparticle energy. Processes (a) and (b) correspond to the first and second terms in Eq. (B17), respectively.

As a result, to first order in interaction the Luttinger liquid parameter $K = v_F/v$ given by

$$K = 1 - \frac{V(0) - V(2p_F)}{2\pi\hbar v_F}. \quad (\text{B13})$$

Because we have $K = 1$ in the absence of interactions, the derivative of K with respect to the particle density appears in the first order in interactions,

$$\partial_n K = \frac{1}{2p_F v_F} [V(0) - V(2p_F) + 2p_F V'(2p_F)]. \quad (\text{B14})$$

Similarly, the derivatives of ϵ_p and v_p in Eq. (B2) appear only in the first order. These leading contributions are found by using Eq. (B6), (B10), and (B11),

$$\partial_R \epsilon_p^{(1)} = V(0) - V(p - p_R), \quad (\text{B15a})$$

$$\partial_L \epsilon_p^{(1)} = V(0) - V(p - p_L), \quad (\text{B15b})$$

$$\partial_R v_p^{(1)} = -V'(p - p_R), \quad (\text{B15c})$$

$$\partial_L v_p^{(1)} = -V'(p - p_L). \quad (\text{B15d})$$

On the other hand, the second derivative $\partial_{LR}^2 \epsilon_p^{(1)}$ appears only in the second order in interactions

$$\begin{aligned} \partial_{LR}^2 \delta\epsilon_p^{(1)} &= -\pi^2\hbar^2 \left(\frac{1}{K} - K \right) \left(\frac{\partial^2}{\partial_{p_R^2}} + \frac{\partial^2}{\partial_{p_L^2}} \right) \delta\epsilon_p^{(1)} \\ &\simeq -\frac{V(0) - V(2p_F)}{2v_F} [V'(p - p_R) - V'(p - p_L)]. \end{aligned} \quad (\text{B16})$$

Substituting Eqs. (B14)–(B16) into Eq. (B2), we conclude that to first order in interactions Y_p and, therefore, Λ vanish. This is an expected outcome because the three-particle scattering matrix element (78) cannot be generated from the two-particle interaction (B7) in the first order.

To obtain Y_p in the second order in interaction strength, in addition to Eqs. (B14)–(B16) we need to find the second-order correction $\delta\epsilon_p^{(2)}$ to the quasiparticle energy, which will contribute to the first term in the right-hand side of Eq. (B2). Applying the standard second-order perturbation theory, we obtain

$$\delta\epsilon_p^{(2)} = \frac{1}{(2\pi\hbar)^2} \iint dp' dq \left(n_{p'}(1-n_{p'-q})(1-n_{p+q}) \frac{[V(q) - V(p+q-p')]^2}{\epsilon_p^{(0)} + \epsilon_{p'}^{(0)} - \epsilon_{p+q}^{(0)} - \epsilon_{p'-q}^{(0)}} \right. \\ \left. - n_{p'}(1-n_{p'-q})n_{p-q} \frac{[V(q) - V(p-p')]^2}{\epsilon_{p-q}^{(0)} + \epsilon_{p'}^{(0)} - \epsilon_p^{(0)} - \epsilon_{p'-q}^{(0)}} \right) \quad (\text{B17})$$

The first term in the integral accounts for the second-order contribution to the many-body state due to an additional fermion in state p , assuming $p > p_F$. The second term subtracts the contribution to the ground state energy due to processes involving state p , which are not allowed in the presence of the additional fermion. The two contributions are illustrated in Fig. 2.

The second-order correction (B17) contributes to the first term in Eq. (B2) via $\partial_{LR}^2 \epsilon_p$. To leading order in interactions, $\partial_{LR}^2 \delta\epsilon_p^{(2)} = -(2\pi\hbar)^2 \partial^2 \epsilon_p^{(2)} / \partial_{p_R} \partial_{p_L}$, see Eq. (B6c). The positions p_L and p_R of the two Fermi points enter Eq. (B17) via the occupation numbers (B9).

Differentiation of Eq. (B17) with respect to p_L and p_R yields

$$\partial_{LR}^2 \delta\epsilon_p^{(2)} = - \frac{[V(2p_F) - V(p+p_F)]^2}{\epsilon_p^{(0)} - \epsilon_{p+2p_F}^{(0)}} - \frac{[V(p-p_F) - V(p+p_F)]^2}{2(\epsilon_{p_F}^{(0)} - \epsilon_p^{(0)})} + \frac{[V(2p_F) - V(p-p_F)]^2}{\epsilon_{p-2p_F}^{(0)} - \epsilon_p^{(0)}}. \quad (\text{B18})$$

Here the first term originates from the first term in Eq. (B17), whereas the second and third ones originate from the second term in (B17).

We are now in a position to evaluate Y_p in the second order in interaction strength. To this end we substitute Eqs. (B14) and (B15) for the corresponding derivatives in the first order and add expressions (B16) and (B18) evaluated in the second order. (We replace $p_L \rightarrow -p_F$ and $p_R \rightarrow p_F$.) The remaining parameters need not account for interactions, i.e., we substitute $m_p^* = m$, $v = v_F$, $v_p = p/m$, and $K = 1$. The result has the form

$$Y_p = m \left[\frac{[V(2p_F) - V(p+p_F)]^2}{2p_F(p+p_F)} - \frac{[V(2p_F) - V(p-p_F)]^2}{2p_F(p-p_F)} + \frac{[V(p-p_F) - V(p+p_F)]^2}{(p+p_F)(p-p_F)} \right. \\ \left. - \frac{V(0) - V(2p_F)}{2p_F} [V'(p-p_F) - V'(p+p_F)] + \frac{[V(0) - V(p-p_F)][V(0) - V(p+p_F)]}{(p+p_F)(p-p_F)} \right. \\ \left. + \frac{V(0) - V(p-p_F)}{p-p_F} V'(p+p_F) + \frac{V(0) - V(p+p_F)}{p+p_F} V'(p-p_F) \right. \\ \left. + \left(\frac{V(0) - V(2p_F)}{2p_F} + V'(2p_F) \right) \left(\frac{V(0) - V(p+p_F)}{p+p_F} - \frac{V(0) - V(p-p_F)}{p-p_F} \right) \right]. \quad (\text{B19})$$

Substitution of the above result into Eq. (B1) yields Eq. (80).

Appendix C: Relaxation rate τ^{-1} in a system of weakly interacting spinless fermions

A phenomenological expression for the relaxation rate of in a spinless quantum liquid at low temperatures was obtained in Refs. [6, 7]. In this Appendix we apply that result to find the rate τ^{-1} for the special case of weakly interacting spinless fermions. To this end it is convenient to express the relaxation rate as [26]

$$\frac{1}{\tau} = \frac{3B}{\pi^{5/2} p_F^2} \left(\frac{v p_F}{T} \right)^3 \left(\frac{p_F^2}{2m^* T} \right)^{1/2} e^{-\Delta/T}. \quad (\text{C1})$$

Here Δ is the maximum energy of a hole-like excitation in the quantum liquid, m^* is the effective mass of the hole at the maximum of energy, and v is the velocity of

the low-energy excitations in the system. In a weakly interacting Fermi gas $\Delta = \mu$, $m^* = m$, and $v = v_F$.

The quantity B was expressed in Ref. [6] in terms of Δ , m^* , and v as functions of the particle density. An alternative expression

$$B = \frac{4\pi}{15} \frac{Y_0^2}{\hbar^5 v^6} T^5 \quad (\text{C2})$$

was obtained in Refs. [7, 26]. Here Y_0 is a function of the spectrum of holes in the quantum liquid, which is analogous to the Y_p for particle-like excitations given by Eq. (B2). Because of the difference in the type of excitations, the effective mass m^* in Eq. (B2) should be replaced with $-m^*$. The momentum p in the resulting Y_p should correspond to the maximum of the energy of the hole. If the hole is formed by moving a fermion from a state below the Fermi level to the right Fermi point, the maximum of energy corresponds to $p = p_F$. Alternatively, one may create a hole by removing a particle from the system, in which case the maximum of energy corresponds to $p = 0$. Here we adopt the latter approach.

Evaluation of Y_p for a hole-like excitation can be performed by retracing the steps leading from Eq. (B2) to Eq. (B19). We find that for the hole Y_p is given by Eq. (B19) with the opposite sign. Thus one can substitute for Y_0 into Eq. (C2) the result (B19) taken at $p = 0$. This yields

$$\frac{1}{\tau} = \frac{2}{5\pi^{3/2}} \frac{\mu Z^2}{\hbar^5 v_F^4} \left(\frac{T}{\mu}\right)^{3/2} e^{-\mu/T}, \quad (\text{C3})$$

where

$$\begin{aligned} Z = & [V(0) - V(2p_F)][V(p_F) - V(2p_F)] \\ & + 2p_F[V(0) - V(p_F)]V'(2p_F) \\ & - p_F[V(0) - 2V(p_F) + V(2p_F)]V'(p_F). \end{aligned} \quad (\text{C4})$$

The derivation of the expression (C1) in Refs. [6, 7] as-

sumed that the interactions between the particles fall off sufficiently fast at the long distances for the velocity v of the elementary excitations to be well defined. In practice this means that the interaction potential $U(x)$ falls off at $x \rightarrow \infty$ faster than $1/x$. In particular, the result (C3) does not apply in the case of Coulomb interactions, for which $V(0)$ in Eq. (C4) is ill-defined. On the other hand, the dipole-dipole interaction with the short-distance cut-off w does have a well-defined $V(0)$. Substituting Eq. (46) into Eq. (C4), we obtain

$$Z = -6(\ln 4 - 1)\Upsilon^2 \frac{p_F^4}{\hbar^4} \ln \frac{\hbar}{p_F w}. \quad (\text{C5})$$

This expression is obtained for $p_F w/\hbar \ll 1$ within logarithmic accuracy.

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