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# Correlation-induced localization

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A new paradigm of Anderson localization caused by correlations in the long-range hopping along with uncorrelated on-site disorder is considered which requires a more precise formulation of the basic localization-delocalization principles. A new class of random Hamiltonians with translation-invariant hopping integrals is suggested and the localization properties of such models are established both in the coordinate and in the momentum spaces alongside with the corresponding level statistics. Duality of translation-invariant models in the momentum and coordinate space is uncovered and exploited to find a full localization-delocalization phase diagram for such models. The crucial role of the spectral properties of hopping matrix is established and a new matrix inversion trick is suggested to generate a one-parameter family of equivalent localization/delocalization problems. Optimization over the free parameter in such a transformation together with the localization/delocalization principles allows to establish exact bounds for the localized and ergodic states in long-range hopping models. When applied to the random matrix models with deterministic power-law hopping this transformation allows to confirm localization of states at all values of the exponent in power-law hopping and to prove analytically the symmetry of the exponent in the power-law localized wave functions.

## I. INTRODUCTION.

The standard picture of Anderson localization in a three-dimensional single-particle system with short-range hopping [1] is represented by the phase transition between extended ergodic and localized phases at a certain critical disorder strength or energy with a sharp mobility edge separating ergodic and localized states. Exactly at the Anderson localization transition (AT) non-ergodic (multifractal) extended states have been proven to appear [2, 3]. It is well-known that in low dimensions  $d = 1, 2$  for any tight-binding (or short-range) Hamiltonian with uncorrelated disorder all states are localized.

However, delocalized states may appear even in one-dimensional systems if the hopping is long-ranged [4–7]. An archetypical example of such nominally one-dimensional systems is suggested in Ref. [6]. In this power-law random banded matrix (PLRBM) model the long-ranged hopping terms are completely uncorrelated and Gaussian distributed with a power-law decay of the variance  $\langle |H_{nm}|^2 \rangle \propto (b/|n-m|)^{2a}$  with the distance  $|m-n|$  that saturates  $\langle |H_{nm}|^2 \rangle \sim 1$  at  $|m-n| < b$ . The parameter that drives the localization transition in this system is the exponent  $a$ . For  $a > 1$  the states are power-law localized, while at  $a < 1$  they are extended. At the critical point  $a = 1$  multifractal states with variable (depending on the parameter  $b$ ) strength of multifractality are formed [6–8].

At the first glance this *delocalization* at long range hopping is natural and independent of the uncorrelated nature of the hopping integrals, as at one hop the particle can reach any point of the system. Yet, as we show in this paper, localization effects get *stronger* if the long-range hopping integrals are fully or partially correlated (see Fig. 1).

It is commonly believed that correlated disorder in the on-

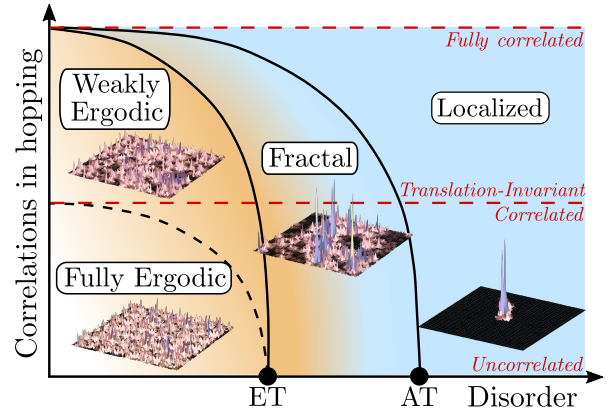


Figure 1. **Effect of correlations in long-range hopping on the Anderson localization.** The correlations in the long-range hopping results in the sequence (1) of phase transitions with degrading ergodicity. The parameter labeled as 'Disorder' is an *effective* disorder which determines the ratio of on-site disorder (fixed in our models) to the hopping integrals at large distances (controlled by the exponents  $a$  or  $\gamma$ ). The phase diagram is shown for the Rosenzweig-Porter (RP) family of ensembles, being an example of the system where all phases (fully and weakly ergodic, fractal, and localized) are present. The fractal phase separated by the localization (AT) and the ergodic (ET) transitions from the localized and the ergodic phases is present in these models even in the absence of correlations. Increasing the correlations (upwards along the vertical axis) sends the AT and ET to smaller values of disorder and stretches the critical point of ET into a whole weakly ergodic phase. Three-dimensional plots show cartoons of spatial distributions of wavefunction intensities in the corresponding localized, (multi)fractal, and ergodic phases. For a family of the power-law random banded matrices (PLRBM) considered in this work (not shown) the fractal phase is replaced by the weakly ergodic one.

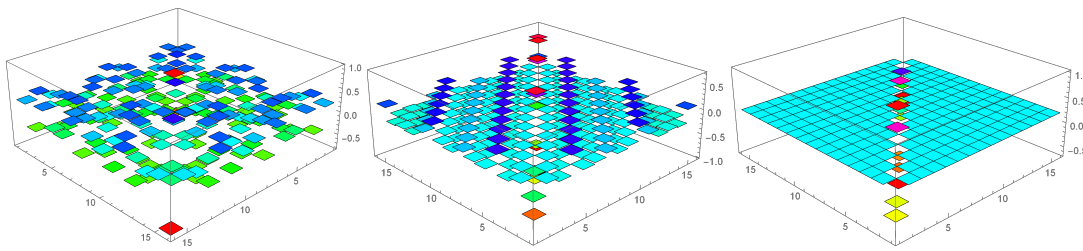


Figure 2. (Color online) **Matrix Hamiltonians (in the order of increasing correlations)** with fully random, random translation-invariant and deterministic hopping. Squares of different color and at different heights represent the value of matrix elements for different  $m, n = 1, \dots, 15$ .

site energies (“diagonal disorder”) tends to *delocalize* systems. An important example is the Aubry-Andre lattice model [9] with an incommensurate periodic potential that possesses delocalized states and exhibits AT. This type of *quasi-disorder* was widely used in recent experiments on localization of matter waves of cold atoms [10]. Our findings show that correlations in the long-range hopping produce an *opposite* effect. This effect is not restricted to the one-dimensional systems: the tendency towards localization at correlated long-range hopping is present in higher dimensional systems thus showing the universality of this phenomenon which we call ‘*correlation-induced localization*’.

The physics of long-range interacting systems is now an emerging field. Initially it was motivated by experiments on trapped cold atoms with dipole moments (see e.g. [11–13]). However, now the interest is being shifted towards many-body localization in systems with long-range (e.g. Coulomb) interaction [14]. Several models with such interactions have been suggested in the past [15] and recently [16–27] in the problem of entanglement dynamics at many-body localization. The models with fully-correlated hopping and interaction terms [15, 16, 18, 21–24, 26] show significantly different behavior as compared to the ones with uncorrelated hopping and interactions [16, 17, 19, 25, 27]. Some of the former works (see, e.g., [23]) even demonstrate explicitly that many-body properties are formed as superpositions of short-range and power-law decaying contributions in the complete agreement with the single-particle picture developed in this work.

Moreover in physics of classical dynamical systems long-range interactions also play a significant role leading to the formation of inhomogeneous spatial temperature distribution anti-correlated with the density profile after a global spatially homogeneous quench [28–30]. The relaxation times from these emergent inhomogeneous states to the equilibrium are very long and diverge with the system size [31–33]. This physics is relevant, in particular, to the explanation of the heating of the solar corona. Thus, we believe that the results of this paper are relevant for all above mentioned types of many-body problems as well.

The correlations in long-range hopping are barely studied. The earlier study of a single-particle system with *deterministic* (or fully-correlated) power-law decay of long-range hopping [34], has been nearly unnoticed until recently. Several recent works [35–39] reported about localization in such systems with fully-correlated long-range hopping, confirming (mostly numerically) the conclusion of the renormaliza-

tion group (RG) analysis done in Ref. [34]. Neither of the models [35–39] demonstrates a truly delocalized behavior of wave functions in the bulk of the spectrum for *all* strengths of disorder and *all* values of the exponent  $a$ .

Furthermore, very recently a striking duality  $\mu(a) = \mu(2 - a)$  in the spatial decay rate  $\mu \geq 2$  of the power-law localized wave functions  $|\psi|^2 \propto |r - r_0|^{-\mu}$  was discovered [39] in the models with algebraically decaying correlated hopping [34, 38, 39]. This implies enhancement of localization upon making hopping more long-ranged. In this work we prove this duality and analytically show the absence of delocalized phase in these models.

Despite all these facts spread in the literature, the systematic study of correlations at long-range hopping has not been done so far and importance and generality of the phenomenon of enhancement of localization by correlations have not been appreciated. This paper is aimed to fill in this gap in the theory of Anderson localization.

In all the above mentioned models the long-range hopping integrals are either uncorrelated or fully correlated (deterministic). The systematic study of the role of correlations requires a gradual increase of correlations. In this work we suggest a new class of models that bridge between the models with uncorrelated hopping and those with fully correlated hopping (see Fig. 2). These are the models with random long-ranged hopping integrals which are *translation-invariant* (TI). In a given realization the hopping integrals  $H_{nm} = H_{|n-m|}$  in TI models are fully correlated along a diagonal (see Fig. 2) but they are uncorrelated and sign-alternating for different diagonals [40]. Such models emerge naturally, e.g. in the case when hopping is caused by the RKKY interaction which oscillates with the period incommensurate with the lattice constant.

In addition to the models with the typical long-range hopping integrals decreasing algebraically with the distance which physical realization is more or less obvious, the models with the typical hopping integrals being distance-independent but dependent on the system size (as  $N^{-\gamma/2}$ ) have recently come under the spotlight. The interest to such models emerged because of the discovery [41] of the new non-ergodic extended (multi-fractal) phase and the corresponding *ergodic transition* in the generalized Rosenzweig-Porter (RP) model. This model appeared to be relevant for several many-body problems such as the Quantum Random Energy Model [42] with implications for quantum computing [43], as well as for non-ergodic extended states in the Sachdev-Ye-Kitaev [44, 45] ( $SYK_4 + SYK_2$ ) many-body model [46–48]. The

presence of non-ergodic extended phase and of above mentioned ergodic transition puts on a solid ground the search for ergodic transition and non-ergodic extended phase on Random Regular Graphs (RRG) (initiated in Ref. [49, 50] and discussed in detail in [50]) and in real many-body systems [51, 52]. Slow dynamics on RRG [53–55] and in disordered spin chains [56–58] may be a signature of such a phase. In this work we suggest the translation-invariant extension of the RP model and study the localization properties of the RP family of models along with the PLRBM family as the correlations in the long-range hopping increase.

A remarkable feature of random TI models is the presence of the Poisson spectral statistics within the delocalized phase (see Fig. 3). This goes against the common wisdom that the Poisson statistics signals of localization. The reason for such a behavior is that the Poisson spectral statistics emerges in the parameter region where the states in the coordinate space are, indeed, extended and weakly ergodic [59] but those in the momentum space are localized. The common wisdom assumes by default that the states in momentum space are always chaotically extended. The TI models introduced in this paper constitute a class of models where this assumption fails. We formulated principles to identify the type of basis-invariant spectral statistics if the statistics of eigenstates in the coordinate and in the momentum spaces are known (see Fig. 3). One of them reads that the Poisson spectral statistics emerges each time when the eigenstates are localized either in the coordinate or in the momentum space [60]. These statement is checked numerically in the paper.

The results of this paper allow us to formulate a new phase diagram which is presented in Fig. 1. This figure shows a certain hierarchy of phases with respect to the extent of ergodicity of eigenstates. The *fully ergodic* (FE) phase corresponds to the Porter-Thomas eigenfunction statistics if it is basis-independent. The corresponding level statistics is Wigner-Dyson. We denote the states as *weakly ergodic* (WE) if the eigenfunction support set [50, 61] in a given basis scales like the matrix size  $N$  but the significant fraction of sites are not populated. The eigenfunction statistics in the WE phase is basis-dependent and deviates from the Porter-Thomas one. The non-ergodic extended, (*multi*)-*fractal* (F) states are characterized by the support sets which scale as  $N^D$ , where  $0 < D < 1$ . Finally the *localized* (L) states correspond to  $D = 0$ . Obviously, the ergodicity of the states decreases in the following sequence:

$$FE \rightarrow WE \rightarrow F \rightarrow L. \quad (1)$$

The main result of this paper illustrated by Fig. 1 is that with increasing correlations in the long-range hopping the sequence of phases at a certain fixed disorder strength is that of Eq. (1) where some phases of this sequence may be missing, i.e. with increasing the correlations in the long-range hopping the ergodicity of eigenstates progressively degrades. Simultaneously, the lines of localization or ergodic transitions are shifted to lower disorder. At fully correlated long-range hopping the delocalized states in the bulk of the spectrum disappear whatsoever.

It is important that the critical lines of all transitions bend

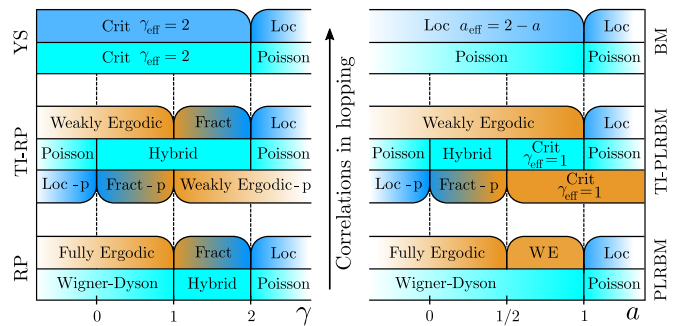


Figure 3. **Localization-delocalization phase diagrams for (left) RP- and (right) PLBM-families of ensembles.** Additionally to coordinate-space diagrams (above horizontal lines) and level-statistic diagrams (in the middle) for TI-models the momentum-space diagrams are shown below the lines. The phases in TI-RP model are symmetric with respect to duality point  $\gamma = 1$ . The type of spectral statistics (Wigner-Dyson, Poisson and hybrid) is indicated for each phase. Notice Poisson level statistics in delocalized phases of TI-models in accordance with general principles formulated in Sec. IV. The increase of correlations in the hopping (from bottom to top) first destroys the fully-ergodic phase in all models, making TI-systems weakly ergodic (WE), and then (in YS and BM models) localizes wave functions in the coordinate space.

to the left, i.e. the states which are localized in the absence of correlations remain localized when the correlations are present. However the former ergodic extended states may become weakly ergodic, non-ergodic or even localized in the presence of correlations in the long-range hopping. This is the essence of correlation-induced localization.

## II. LOCALIZATION CRITERIA FOR MODELS WITH LONG-RANGE HOPPING.

The most generic free-particle Hamiltonian is defined as follows:

$$H_{nm} = \varepsilon_m \delta_{nm} + j_{nm}, \quad (2)$$

where  $1 \leq m, n \leq N$  are lattice sites,  $\varepsilon_m$  are random on-site energies with zero mean  $\langle \varepsilon_m \rangle = 0$  and the variance  $\langle \varepsilon_m^2 \rangle = \Delta^2$  [62] represents uncorrelated *diagonal* disorder. The (possibly correlated) hopping integrals  $j_{mn} = j_{nm}^*$  can be deterministic or random and they are characterized by the averaged value  $\langle j_{nm} \rangle$  and the variance  $\langle |j_{nm}|^2 \rangle$ . Throughout the text we refer to the correlations in the hopping terms  $j_{nm}$  simply as correlations. For simplicity we restrict our consideration to  $d = 1$ , unless stated otherwise.

The basic localization principle originally suggested by Mott [63] states that the wave functions are localized (extended) when the disorder strength  $\Delta$  is larger (smaller) than the bandwidth  $\Delta_p$  in the absence of diagonal disorder. The results of this paper and other recent works [34–39, 41, 64, 65], however, show that this principle should be reformulated.

Let us first consider the case when the spectrum of the off-diagonal part of the Hamiltonian Eq. (2)  $\hat{j} = \hat{H}(\varepsilon_n = 0)$  is

bounded both from above and from below in the limit  $N \rightarrow \infty$ , and we are concerned with the statistics of eigenstates in the bulk of the spectrum. We claim that the Mott's criterion:

$$\Delta \lesssim \Delta_p, \Rightarrow \text{weak ergodicity}, \quad (3)$$

is the *sufficient condition* for (at least weakly) *ergodic delocalization* when  $\Delta$  is smaller than the bandwidth  $\Delta_p$  of  $\hat{j}$ . We will be using this criterion in a weak sense as the condition:

$$\lim_{N \rightarrow \infty} \frac{\Delta}{\Delta_p} = 0, \Rightarrow \text{weak ergodicity}, \quad (4)$$

In the absence of correlations in  $\hat{j}$  the bandwidth is given by:

$$\Delta_p^2 = \frac{1}{N} \sum_{n,m,m \neq n}^N \langle |H_{nm}|^2 \rangle. \quad (5)$$

For this particular case the criterion equivalent to Eq. (4) was mentioned in Ref. [64].

For correlated long-range hopping, specifically for the translation-invariant hopping described in Section III, the spectrum of  $\hat{j}$  is often non-compact, with an infinite support set in the energy space. In this case the bandwidth  $\Delta_p$  should be defined as the width of the energy domain where mean level spacing  $\delta(N)$  takes a typical value. The observation energy  $E$  should be also chosen from inside of this domain.

As was explained in Introduction, weak ergodicity [49, 50] defined in a given (e.g. coordinate) basis does not imply invariance of wave function statistics under basis rotation. In some models (see, e.g., [64]) weak ergodicity may survive beyond the condition (3), showing that (3) is only the sufficient but not the necessary condition of weak ergodicity and, thus,  $\Delta = \Delta_p$  is the lower bound for the ergodic transition between the weakly ergodic extended phase and the non-ergodic phases (localized or extended).

The criterion of *localization* suggested for systems with long-range hopping by Levitov [4, 5] following Anderson's ideas of locator expansion, reads:

$$\delta_R > |j_R|, \Rightarrow \text{localization}. \quad (6)$$

The key point of [4, 5] is that one should compare the mean level spacing  $\delta_R \sim \Delta/R^d$  of a  $d$ -dimensional system at a certain length scale  $1 \lesssim |m-n| \sim R \lesssim N$  with the width of a resonance governed by the average absolute value of hopping integrals  $j_R$  within the same scale. Then most eigenstates (except measure zero) are localized if (6) holds for almost all  $R$ . Indeed, in order to find the eigenstates one can use the perturbation theory in the small parameter  $j_R/\delta_R$ . The inequality (6) means convergence of the perturbation series and thus localization. A more strict condition

$$\frac{|j_R|}{\delta_R} < R^{-\epsilon}, \quad \epsilon > 0 \quad (7)$$

as  $R \rightarrow \infty$  implies convergence of the series  $\sum_R |j_R|$ . For random matrices the corresponding criterion of convergence

reads as follows:

$$\lim_{N \rightarrow \infty} S/\Delta < \infty, \Rightarrow \text{localization}, \quad (8)$$

$$S = \frac{1}{N} \sum_{n,m,n \neq m} \langle |j_{nm}| \rangle.$$

If the criterion (6) is violated, both a multifractal [41] and a weakly ergodic [64] extended phases may emerge. More surprisingly, violation of (6) does not exclude localization either, provided that the hopping integrals are correlated. Indeed, the presence of correlations cannot destroy localization if the condition (6) is fulfilled and the perturbation series is convergent. Under this condition the main contribution to the eigenfunction amplitude away from the localization center comes from the first-order perturbation theory which knows nothing about correlations in the hopping matrix elements. The situation changes completely when the perturbation theory diverges. In this case all orders in perturbation theory contribute to the eigenfunction amplitude on an equal footing and correlations come into play. As recent examples [35–39] show, the effect of correlations when (6) is violated may be localization of states which were extended in the absence of correlations. These examples, in which hopping is deterministic, prove that (6) is a *sufficient* but not a necessary condition of localization.

The Anderson and Mott criteria may be made *sufficient and necessary* criterion of localization by means of the *matrix inversion trick* described in Section V. This trick converts the initial Shrödinger problem into a *family* of equivalent problems with modified Hamiltonians  $\hat{H}_{\text{eq}}(E_0)$  parameterized by a continuous parameter  $E_0 \propto N^\beta$ . The effective disorder strength saturating Eq. (6) to an equality is a function of this parameter  $\beta$ . The true border of the localized phase then corresponds to the optimal  $\beta$  that minimizes this effective disorder.

The domains of validity of (3) and (6) are in general non-complimentary. This is the reason why the non-ergodic extended phase may exist [41] in the parameter region where neither (3) nor (6) holds true.

### III. TRANSLATION-INVARIANT (TOEPLITZ) MODELS.

An important sub-class of the models (2) is a family of translation-invariant (TI) models with the hopping term  $j_{nm} = j_{n-m}$  [66] depending only on the directed distance  $m-n$  between coupled sites [40] (Toeplitz random matrix models). For such models a special role is played by the momentum basis. An equivalent dual form of the Hamiltonian  $H_{nm} = \varepsilon_m \delta_{nm} + j_{n-m}$  in the momentum basis is  $H_{pq} = \tilde{E}_p \delta_{p,q} + \tilde{J}_{p-q}$ , with new on-site energies

$$\tilde{E}_p = \tilde{E}_p^* = \sum_{m=0}^{N-1} j_m e^{-2\pi i \frac{pm}{N}}, \quad (9)$$



and new hopping integrals

$$\tilde{J}_p = \tilde{J}_{-p}^* = \frac{1}{N} \sum_{m=0}^{N-1} \varepsilon_m e^{-2\pi i \frac{pm}{N}}, \quad (10)$$

exchanging their roles after Fourier transforming (FT). It allows one to generalize the Levitov's localization principle (6) to the momentum space, with  $|p - q| \sim P$

$$\tilde{\delta}_P > \tilde{J}_P, \quad \tilde{\delta}_P = \frac{\langle |\tilde{E}_p - \tilde{E}_q| \rangle_P}{P}, \quad \tilde{J}_P = \langle |\tilde{J}_{p-q}| \rangle_P. \quad (11)$$

The dual counterpart of the weak ergodicity criterion (4) for TI-models differs from (4) only by the opposite sign of the inequality:

$$\lim_{N \rightarrow \infty} \frac{\Delta_p}{\Delta} = 0, \quad \Delta_p = \left\langle \max_{p,p'} |\tilde{E}_p - \tilde{E}_{p'}| \right\rangle. \quad (12)$$

Equations (4, 6, 11, 12) form the basis of our qualitative analysis throughout the manuscript [60]. Below we consider two families of random matrix models as examples and show the effect of correlations on their localization properties applying two dual pairs of these localization-delocalization principles.

#### IV. PRINCIPLES OF LEVEL STATISTICS.

It is frequently taken for granted that the level and eigenfunction statistics are in one-to-one correspondence: delocalized states correspond to the Wigner-Dyson level statistics and the localized ones correspond to the Poisson level statistics. However, there is a big problem with this statement: the level statistics is basis-invariant while the eigenfunction statistics is generically basis-dependent. We will show in this paper that in the random TI ensembles eigenfunctions may be extended in the coordinate space and localized in the momentum one.

The key phenomenological principles to identify the level statistics in such a situation are the following:

- (i) if there is a basis in which the states are localized and uncorrelated with the corresponding eigenvalues then the level statistics is Poisson,
- (ii) the Wigner-Dyson (WD) level statistics hold if and only if the eigenfunction statistics is fully ergodic and basis-invariant, and
- (iii) when neither (i) or (ii) holds, the level statistics is of the hybrid nature that shares the features of both WD and the Poisson statistics.

It follows from these principles that a coexistence of Poisson levels statistics and the delocalized (but not fully ergodic!) character of eigenstates is possible in a certain (e.g., coordinate) basis. Indeed, according to Eqs. (9 - 11) in TI models at small enough disorder  $\Delta$  there exist states localized in the momentum space (p-localized states). At the same time in the coordinate space these states must be extended due to the dual

criterion (4). Then using the principles (i) and (ii) we come to the conclusion that the level statistics in TI models at small disorder must be Poisson, despite the states are delocalized in the coordinate space.

Below we consider the level statistics in TI models in more detail.

#### V. MATRIX INVERSION TRICK.

In this section we describe a useful trick that allows to reduce the problem with the spectrum of the hopping matrix  $\hat{j}$  which is unbounded from above or from below to the equivalent problem with the bounded spectrum.

The initial problem is given by the Schrödinger equation:

$$(E - \varepsilon_n) \psi_E(n) = \sum_{\substack{m=1 \\ m \neq n}}^N j_{n-m} \psi_E(m). \quad (13)$$

Let us introduce the matrix  $M_{n-m}$ :

$$\hat{M} = (\hat{1} + \hat{j}/E_0)^{-1} = \sum_p \frac{|p\rangle \langle p|}{1 + \frac{\tilde{E}_p}{E_0}}, \quad (14)$$

where  $|p\rangle$  is the momentum-space basis vector and  $E_0 \propto N^\beta$  is a certain energy, such that  $(-E_0)$  lies outside the spectrum of  $\tilde{E}(p)$  of  $\hat{j}$  or inside the gaps in this spectrum.

Singling out the diagonal term  $M_{nn} = M_0$  and symmetrizing the hopping matrix, one arrives at the equivalent eigenvalue/eigenfunction problem:

$$(\tilde{E} - \varepsilon_n) (E + E_0 - \varepsilon_n) \psi_E(n) = \sum_{\substack{m=1 \\ m \neq n}}^N J_{nm} \psi_E(m), \quad (15)$$

where  $\tilde{E} = E + E_0 (1 - M_0^{-1})$ , and

$$J_{nm} = -(E + E_0 - \varepsilon_n) \frac{M_{n-m}}{M_0} (E + E_0 - \varepsilon_m). \quad (16)$$

The spectrum of the matrix  $M_{n-m}$  ( $n - m \neq 0$ ) is given by Eq. (14), and it is bounded in the limit  $N \rightarrow \infty$  even if  $\tilde{E}_p$  is unbounded. The same is true for the constant  $M_0$ . For an unbounded  $\tilde{E}_p \gg E_0$  one obtains at large  $|n - m|$ :

$$M_{n-m} \approx \frac{E_0}{N} \sum_p \frac{e^{2\pi i(n-m)p/N}}{\tilde{E}_p}, \quad (17)$$

in contrast to

$$j_{n-m} = \frac{1}{N} \sum_p \tilde{E}_p e^{2\pi i(n-m)p/N}. \quad (18)$$

Eq. (17, 18) will be useful to prove the duality discovered in Ref. [39]. However, the main idea of introducing the matrix inversion trick is applying the localization/delocalization criteria Eqs. (4, 8) to a new hopping matrix  $J_{nm}(E_0)$ , Eq. (16),

and to find the true borders of the localized and ergodic phases by optimization over  $E_0$ .

In the next Sections we show how does it work for RP-family, Sec. VI, and PLRBM-family, Sec. VII, of matrix ensembles.

## VI. ROSENZWEIG-PORTER FAMILY

### A. Yuzbashyan-Shastry model.

The simplest model with long-range *fully-correlated* hopping is the model with the constant  $j_{nm} = N^{-\gamma/2}$  [35–37] which we will refer to as the Yuzbashyan-Shastry (YS) model. It is a particular case of a wider class of exactly solvable random matrix ensembles with the rank-1 hopping matrix  $j_{nm} = g_n g_m^*$  suggested in Ref. [35, 36]. We use this exactly solvable model to illustrate the general method of identifying the eigenfunction and spectral statistics developed in this paper.

In the absence of (diagonal) disorder in this model the single zero-momentum ( $p = 0$ ) level  $\tilde{E}_0$  is decoupled from the degenerate set of the rest of states in the momentum space:

$$\tilde{E}_0 = N^{1-\gamma/2}, \quad \tilde{E}_{p \neq 0} = 0. \quad (19)$$

Thus for  $\gamma < 2$  the spectrum of  $\hat{j}$  is unbounded from above. Then applying the matrix inversion trick Eq. (14) and taking into account that  $|0\rangle\langle 0| = N^{-1}$  and  $\sum_p |p\rangle\langle p| = \hat{1}$  one obtains for the matrix  $\hat{M}$ :

$$\hat{M} = \hat{1} - \frac{N^{-\gamma/2}}{E_0 + N^{1-\gamma/2}}, \quad (20)$$

where  $\hat{1}$  is the unit matrix and  $E_0 \propto N^\beta$ . Then the matrix elements  $M_{n \neq m}$  are independent of  $n, m$  and scale with  $N$  like:

$$M_{n \neq m} \propto \begin{cases} N^{-\gamma/2-\beta}, & \text{if } \gamma > 2(1-\beta) \\ N^{-1}, & \text{if } \gamma < 2(1-\beta). \end{cases} \quad (21)$$

The ratio of the hopping matrix  $J_{nm}(E_0)$  to effective diagonal disorder

$$\Delta(E_0) \sim N^{\max(0, \beta)} \quad (22)$$

of the equivalent problem, Eq. (15), scales as:

$$\frac{\langle |J_{nm}(E_0)| \rangle}{\Delta(E_0)} \propto \begin{cases} N^{-\gamma/2-\min(0, \beta)}, & \text{if } \gamma > 2(1-\beta) \\ N^{-(1-\max(0, \beta))}, & \text{if } \gamma < 2(1-\beta). \end{cases} \quad (23)$$

As the result, the border line for Eq. (7) with  $j_{nm} \Rightarrow J_{nm}(E_0)$  is (see Fig. 7(a)):

$$\gamma(\beta) = \begin{cases} 2(1-\beta), & \text{if } \beta \leq 0 \\ 2, & \text{if } \beta > 0. \end{cases} \quad (24)$$

The minimal value  $\gamma_{\min} = 2$  of  $\gamma(\beta)$  is reached at the optimal value of  $\beta = \beta_{\text{opt}} = 0$ . Thus we conclude that the true border of localization for YS model is  $\gamma = 2$ .

At  $\beta = \beta_{\text{opt}} = 0$ , Eq. (23) gives:

$$\left( \frac{\langle |J_{nm}| \rangle}{\Delta} \right)_{\text{opt}} \propto \begin{cases} N^{-\gamma/2}, & \text{if } \gamma > 2 \\ N^{-1}, & \text{if } \gamma \leq 2. \end{cases} \quad (25)$$

Eq. (25) implies that  $(S/\Delta)_{\text{opt}} \sim N^0$  for *all*  $\gamma \leq 2$ . It corresponds to the critical state similar to the one in the point of Anderson transition on the Bethe lattice [49]. In many respects this state may be considered as the limiting localized state which we refer to as the 'critically localized' state.

The absence of truly extended states in YS model can be further confirmed by the Mott's criterion Eq. (4). Indeed, the spectrum of  $\hat{j}$  given by Eq. (19) consists of the  $(N-1)$ -fold degenerate band and a single level. Thus the *typical* level spacing of  $\hat{j}$  is exactly zero, the same as the corresponding bandwidth  $\Delta_p$ . This means that the Mott's criterion of delocalization Eq. (4) is never fulfilled.

We come to the conclusion that for YS model the delocalized phase in the coordinate space is absent, in agreement with the results in the literature [35–37], despite infinitely long-ranged hopping integrals. This is the most spectacular effect of destructive interference of long-range hopping trajectories on Anderson localization.

### B. Rosenzweig-Porter ensemble.

The destructive interference in long-range hopping is drastically sensitive to correlations in the hopping integrals. The best studied relative of YS model is the Rosenzweig-Porter (RP) ensemble [41, 65, 67–81].

The Hamiltonian of the RP-ensemble takes the form (2) with *totally uncorrelated* hopping matrix elements  $j_{nm}$  with zero mean and the variance  $\langle |j_{nm}|^2 \rangle = \Delta^2 N^{-\gamma}$  scaling with the matrix size  $N$  in the same way as  $|j_{nm}|^2$  in YS model. The diagonal elements are characterized by  $\langle \varepsilon_m^2 \rangle = \Delta^2$ .

In contrast to YS model, there are three phases in RP model [41]: fully ergodic (FE) for  $\gamma < 1$ , fractal(F) for  $1 < \gamma < 2$  and localized (L) for  $\gamma > 2$ , of which two (FE and F) are extended. They are separated by two phase transitions: the Anderson localization transition (AT) at  $\gamma = 2$  and the ergodic transition (ET) at  $\gamma = 1$ . At  $\gamma = 2$  the eigenfunctions are critically localized like in the corresponding point of YS model, while at  $\gamma = 1$  a different type of critical statistics emerges.

The level statistics of RP-ensemble [41, 65, 68–72, 75] is of Wigner-Dyson form for  $\gamma < 1$  and Poisson for  $\gamma > 2$ . For  $1 < \gamma < 2$  it shows the Wigner-Dyson-like level repulsion at small level spacings  $s_k = E_{n+k} - E_n < k\delta$  and the Poisson statistics at  $s_k \gg k\delta$  [65]. Further on we refer to this level statistics as the *hybrid* one.

Low-energy level repulsion is well-represented by a so-called ratio- or  $r$ -statistics, see Fig. 4:

$$r = \left\langle \min \left( r_n, \frac{1}{r_n} \right) \right\rangle, \quad r_n = \frac{E_n - E_{n-1}}{E_{n+1} - E_n}, \quad (26)$$

showing the value  $r \approx 0.5307$  for Gaussian orthogonal ensemble (GOE),  $r \approx 0.5996$  for Gaussian unitary ensemble

(GUE), and  $r = 2 \ln 2 - 1 \simeq 0.3863$  for Poisson level statistics [82].

We would like to stress once again that despite the  $r$ -statistics is widely used to locate the localization transition, it is not capable to distinguish between the WD level statistics of fully ergodic phases and the hybrid statistics. In order to distinguish between them one should study the spectral statistics at energy scale much larger than the mean level spacing  $\delta$ . An example of such statistics is the level number variance  $\langle n^2 \rangle - \langle n \rangle^2$  at a large average number  $\langle n \rangle \gtrsim \Gamma/\delta \sim N^{2-\gamma}$  of levels in the studied energy interval (here  $\Gamma \sim N^{1-\gamma}$  and  $\delta \sim N^{-1}$ ) [41] which for the hybrid statistics should show the quasi-Poisson behavior  $\langle n^2 \rangle - \langle n \rangle^2 = \chi \langle n \rangle$  ( $0 < \chi < 1$ ). Another possibility is to study the probability distributions of several consecutive level spacings  $s_k = E_{n+k} - E_n$  [65, 83].

A relevant measure of eigenfunction statistics is the distribution of amplitudes  $P(|\psi_E(n)|^2)$  encoded in the *spectrum of fractal dimensions* [7]

$$f(\alpha) = 1 - \alpha + \lim_{N \rightarrow \infty} \ln[P(|\psi_E(n)|^2 = N^{-\alpha})] / \ln N. \quad (27)$$

As shown in [41] for RP  $f(\alpha)$  takes a simple linear form

$$f(\alpha) = \begin{cases} 1 + (\alpha - \gamma)/2, & \max(0, 2 - \gamma) < \alpha < \gamma \\ -\infty, & \text{otherwise} \end{cases}, \quad (28)$$

for  $\gamma \geq 1$  with an additional point  $f(0) = 0$  for  $\gamma > 2$ . The  $f(\alpha)$  in the ergodic phase,  $\gamma < 1$ , coincides with the one at  $\gamma = 1$  and is represented by a single point  $f(1) = 1$ , see Fig. 5.

Simple arguments based on the Mott's and Anderson's criteria, Eq. (4), (6) allow to locate the localized and ergodic phase without going into cumbersome mathematics. Indeed, the Anderson's criterion  $\delta \sim N^{-1} \gtrsim \langle |j_{nm}| \rangle \sim N^{-\gamma/2}$  predicts localization for  $\gamma > 2$ . At the same time, the Mott's criterion  $\Delta \lesssim \Delta_p$  predicts ergodic delocalized states for  $\gamma < 1$ , as  $\Delta \sim 1$ , and  $\Delta_p \sim N^{(1-\gamma)/2}$  according to Eq. (5).

Note that using the spectral properties of the hopping term of the RP-model in its eigenbasis and the optimization procedure for Eqs. (4) and (6) one may show that the latter are not only sufficient but also necessary conditions for weak ergodicity and localization, respectively. The corresponding analysis in the translation-invariant model is given in the next subsection.

### C. TI-RP ensemble and the coordinate-momentum space duality

We extend the *Rosenzweig-Porter family* of random matrix ensembles by introducing a translation-invariant RP ensemble (TI-RP). It is described by the Hamiltonian

$$H_{nm} = \varepsilon_m \delta_{nm} + j_{n-m}, \quad \langle |j_{n-m}|^2 \rangle = \Delta^2 N^{-\gamma}, \quad (29)$$

with independent identically distributed (i.i.d.) Gaussian random (GR) hopping integrals  $j_{n-m}$  with zero mean and the variance independent of  $m$  and  $n$ .

Because of translation invariance  $j_{nm} = j_{n-m}$ , the TI-RP model possesses the duality of properties in the coordinate and the momentum spaces [84]. Indeed, FT of i.i.d. real  $\{\varepsilon_n\}$  or complex  $\{j_n = j_{-n}^*\}$  GR numbers are i.i.d. complex  $\{\tilde{J}_p = \tilde{J}_{-p}^*\}$  or real  $\{\tilde{E}_p\}$  GR numbers with the *dual* variances [85]. Then from Eqs. (9), (10) one obtains:

$$\langle \tilde{E}_p^2 \rangle \simeq N \langle |j_n|^2 \rangle \propto N^{1-\gamma}, \quad (30)$$

$$\langle |\tilde{J}_p|^2 \rangle \simeq N^{-1} \langle \varepsilon_n^2 \rangle. \quad (31)$$

To avoid complications related to the correlations (degeneracy)  $\{\tilde{E}_p = \tilde{E}_{-p}\}$  of FT of real symmetric GR  $\{j_n = j_{-n}^* = j_n^*\}$  here and further we consider the class of Gaussian unitary ensembles. For discussion of orthogonal class of ensembles see [85].

Thus the ratio  $\langle |\tilde{J}_p|^2 \rangle / \langle \tilde{E}_p^2 \rangle \propto N^{-\gamma_p}$  determines a parameter  $\gamma_p$  dual to  $\gamma$  in the momentum space

$$\gamma_p = 2 - \gamma. \quad (32)$$

Eq. (32) implies that in TI-RP model the phases in the coordinate and momentum spaces are symmetric with respect to the point  $\gamma = 1$  (see Fig. 3).

The Mott's criterion Eqs. (3, 4) ensures existence of *weakly ergodic* phase for  $\gamma < 1$ , since according to Eq. (30)  $\Delta_p \sim N^{(1-\gamma)/2}$  and  $\Delta \sim N^0$  [86]. This result is corroborated by numerics (see Fig. 5(a)).

In order to use efficiently the Anderson localization criterion we first apply the matrix-inversion trick. Consider first the case when  $E_0 \propto N^\beta \gg \Delta_p \sim N^{(1-\gamma)/2}$ . Then expanding Eq. (14) in  $\hat{j}/E_0$  one obtains:

$$\hat{M} = \hat{1} - \hat{j}/E_0. \quad (33)$$

The new hopping matrix  $J_{nm}(E_0)$ , Eq. (14), is estimated as:

$$\frac{\langle |J_{nm}(E_0)| \rangle}{\Delta(E_0)} \sim \begin{cases} |j_{nm}| = N^{-\gamma/2}, & \text{for } \beta > 0 \\ |j_{nm}|/E_0 \sim N^{-\gamma/2-\beta}, & \text{for } \beta \leq 0. \end{cases} \quad (34)$$

Then the border line  $\gamma(\beta)$  for the Anderson localization criterion Eq. (8) takes the form (see Fig. 7) identical to Eq. (24) [87].

Thus we find the same optimal  $\beta_{\text{opt}} = 0$  as for the YS model. However, being substituted into Eq. (34), this optimal  $\beta$  results in a *different* optimal  $\langle |J_{nm}^{\text{(opt)}}| \rangle = \langle |J_{nm}(E_0 \sim N^{\beta_{\text{opt}}})| \rangle$  (cf. Eq. (25)):

$$\langle |J_{nm}^{\text{(opt)}}| \rangle / \Delta(E_0) \sim N^{-\gamma/2}. \quad (35)$$

With the optimal  $\hat{j}^{\text{(opt)}}$  Eq. (8) becomes the *necessary and sufficient* criterion of localization. Thus we conclude that in the TI-RP model the localized phase in the coordinate space corresponds only to  $\gamma > 2$ . Numerics fully confirms this conclusion (see Fig. 5(c)). Due to the duality Eq. (32) the localized phase in the momentum space (which corresponds to the ballistic propagation) is realized for  $\gamma < 0$ .



To establish the character of wave function statistics in the remaining interval  $1 < \gamma < 2$  in the coordinate space and in the dual interval  $0 < \gamma < 1$  in the momentum space we apply the Mott's criterion to the equivalent problem Eq. (15). The bandwidth  $\Delta_p(\beta)$  for this problem determined by Eqs. (5), (34) is given by:

$$\Delta_p(\beta) \sim \begin{cases} N^{(1-\gamma)/2-\beta}, & \text{if } (1-\gamma)/2 < \beta \leq 0 \\ N^{(1-\gamma)/2}, & \text{if } \beta > 0 \end{cases} \quad (36)$$

The bandwidth is small  $\Delta_p(\beta) \ll 1$  in the entire region  $\gamma > 1$  and  $\beta > 0$ , and thus the borderline for the Mott's criterion is  $\gamma(\beta) = 1$  for all  $\beta > 0$ . For  $\beta \leq 0$  the domain of validity of the Mott's criterion seems to be wider, as the bandwidth increases with decreasing  $\beta$ . This is, however, not true. The reason is that the l.h.s. of Eq. (15) is sign definite  $(E - \varepsilon_n)^2 \geq 0$  at  $E_0 = N^\beta \ll 1$ . For the usual Schrödinger equation this corresponds to the energy  $E$  outside the band of on-site energies or on the border of it. If, in addition, the hopping matrix bandwidth is small, the true eigenstates will be either absent (as in band insulator) or localized as in the Lifshitz tail. We conclude that the special structure of l.h.s. of Eq. (15) prohibits extended states in the region  $\beta \leq 0$ . Thus optimization over  $\beta$  helps to establish a true border line  $\gamma = 1$  of the WE states in the TI-RP model. In the region  $1 < \gamma < 2$  in the coordinate space and the dual region  $0 < \gamma < 1$  in the momentum space wave functions are neither ergodic nor localized, i.e. they are *non-ergodic extended*. Numerics confirms (see Fig.5b) that they are *fractal*, like in the RP model.

A non-trivial property that follows from the above analysis which is also confirmed by numerics, see Figs. 3 and 5) is that the sequence of phases in the *coordinate space* of RP and TI-RP ensembles and the positions of phase transitions are *the same* along with the spectra of fractal dimensions (see Fig. 5). The only difference is that *fully* ergodic phase is not realized for the TI-RP ensembles, as due to duality Eq. (32) the phases in the coordinate and in the momentum space never coincide at the same value of the disorder parameter  $\gamma$ .

In agreement with the principles formulated in Section IV, the level statistics of TI-RP is *symmetric* with respect to the dual point  $\gamma = \gamma_p = 1$ , see Fig. 4. It shows the hybrid behavior (the same as for RP in the interval  $1 < \gamma < 2$ ) in the entire interval  $0 < \gamma < 2$  and the Poisson behavior outside of it.

Note that for  $\gamma < 0$  the Poisson level statistics coexists (because of localization in the momentum space) with the weakly ergodic delocalized wave function statistics in the coordinate space. This is fully confirmed by numerics presented in Fig. 4. In contrast to RP model, the Wigner-Dyson level statistics in TI-RP model do not occur, as the eigenfunction statistics in the coordinate and in the momentum spaces never coincide.

## VII. POWER-LAW BANDED MATRIX FAMILY.

The next family we consider is the one of the power-law random banded matrices (PLRBM) [6, 7]. The Hamiltonian of the conventional (fully random) PLRBM is of the form of (2), with  $\langle j_{nm} \rangle = 0$  and  $\langle |j_{nm}|^2 \rangle = [1 + (|n - m|/b)^{2a}]^{-1}$ .

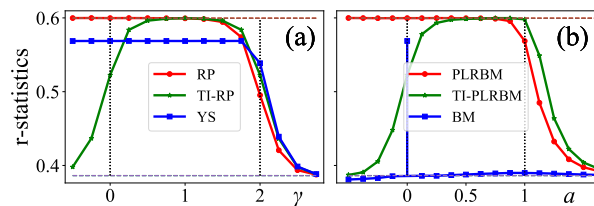


Figure 4. *r*-statistics (average level-spacing ratio) for (a) RP- and (b) PLBM-families of models numerically calculated for Random Matrix Ensembles of *unitary* symmetry for the system size  $N = 2^{14}$  and  $N_r = 10^3$  disorder realizations. In both cases deterministic models (YS and BM) show only the localized or the critical behavior, while TI-models (TI-RP and TI-PLRBM) demonstrate delocalized behavior in a finite range of parameters turning to Poisson statistics both at small and large hopping integrals, corresponding to localization in real and momentum space. Upper (lower) horizontal line shows the *r*-values for Wigner-Dyson (Poisson) statistics. Right (left) vertical line shows the Anderson localization transition in real (momentum) space for TI-models.

Its fully correlated counterpart, to which we refer further as the Burin-Maksimov (BM) model [34], is characterized by the *deterministic* sign-fixed power-law decaying hopping integrals [34, 38, 39, 88–91]  $j_{nm} = j_0(1 - \delta_{nm})/|n - m|^a$ .

PLRBM shows ALT at  $a = 1$ , with ergodic states for  $a < 1$  and localized states for  $a > 1$ . The parameter  $b$  matters only at the transition point  $a = 1$  and determines the strength of multifractality [6, 7].

By contrast, the BM-model demonstrates the *power-law localization* for most of the states (except measure zero) not only at  $a > 1$ , but also at  $a < 1$  [39] with an intriguing symmetry of the exponents in the power-law decay of wave functions.

The level statistics of PLRBM is of the Wigner-Dyson form at least for  $a < 1/2$  and Poisson for  $a > 1$  [6, 7]. Recently it has been shown [64] that in PLRBM the wavefunction statistics is not Porter-Thomas for  $1/2 < a < 1$  (see also Fig. 8(a-d)) implying the presence of weakly ergodic phase in this interval. In contrast, for the BM-model the level statistics is always Poisson, except for an integrable point  $a = 0$  coinciding with the YS-model with  $\gamma = 0$ , where the statistics is critical, see right panel in Fig. 4.

Both power-law models have a built-in spatial structure. Therefore the eigenstate statistics allows more detailed characterization than in RP-family. Indeed, considering the typical decay of the wave function intensity  $|\psi_E(n)|^2$  with the distance  $|n - n_0|$  [40] from its maximal value  $|\psi_E(n_0)|^2$ , one finds at large distances:

$$|\psi_E(n)|_{typ}^2 \equiv \exp[\langle \ln |\psi_E(n)|^2 \rangle] \sim |n - n_0|^{-\mu}, \quad (37)$$

with  $\mu = 2a$  for  $a > 1$  both in PLRBM and in the BM-model by the perturbation theory. At  $a < 1$  the fully random model shows  $\mu = 0$ , while the deterministic one gives  $\mu = 2a_{\text{eff}} = 2(2 - a)$ , as shown numerically in [39], see also Fig. 6.

A random TI relative of the PLRBM model, namely TI-PLRBM, is described by  $H_{nm} = \varepsilon_m \delta_{nm} + j_{n-m}$ , with i.i.d.

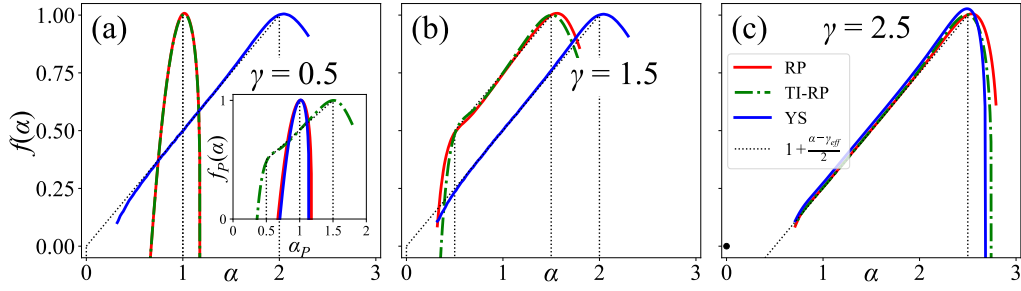


Figure 5. **Spectrum of fractal dimensions  $f(\alpha)$  for the Rosenzweig-Porter family of models (RP, TI-RP, YS) for (a)  $\gamma = 0.5$ , (b) 1.5, (c) 2.5 numerically extrapolated from system sizes  $N = 2^9 \dots 2^{14}$  with  $N_r = 10^3$  disorder realizations in each. Dashed lines show analytical predictions (28) for  $f(\alpha)$ . (inset) Spectrum of fractal dimensions in the momentum space  $f_p(\alpha_p)$  with analytical predictions (28) (dashed lines) and  $\gamma_p = 2 - \gamma$  for TI-RP, demonstrating the difference between RP and TI-RP ensembles in their delocalized phases.**

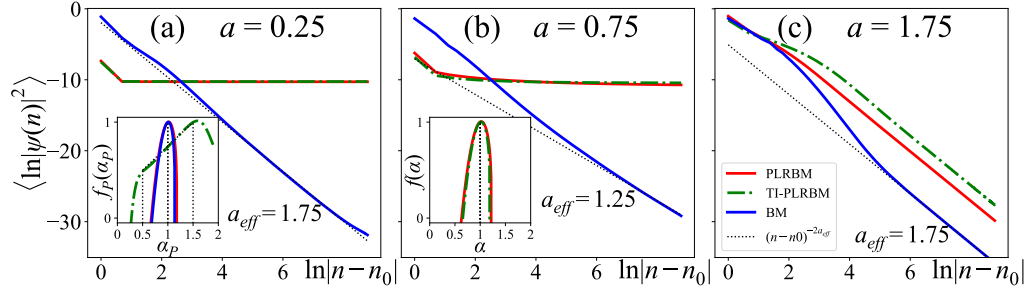


Figure 6. **Average  $\langle \ln(|\psi_E(n)|^2) = \ln |\psi_E(n)|_{\text{typ}}^2$  for power-law banded matrix family (PLRBM, TI-PLRBM, BM) for different exponents in the power-law decay of hopping (a)  $a = 0.25$ , (b) 0.75, (c) 1.75 numerically calculated for the system size  $N = 2^{14}$  and  $N_r = 10^3$  disorder realizations. All models are power-law localized for  $a > 1$ , while for  $a < 1$  only BM shows localization with effective exponent  $a_{\text{eff}} = 2 - a$ . Dashed lines show analytical prediction (37) of this power-law decay. (inset) (a) spectrum of fractal dimensions in the momentum space  $f_p(\alpha_p)$  with analytical predictions (28) (dashed lines) demonstrating the difference between PLRBM and TI-PLRBM ensembles in their delocalized phases  $a < 1$ ; (b) spectrum of fractal dimensions in the coordinate space for TI-PLRBM coinciding with that of PLRBM for  $1/2 < a < 1$ .**

GR hopping integrals with zero mean and the variance:

$$\langle |j_{n-m}|^2 \rangle = (1 - \delta_{nm})/|n - m|^{2a}. \quad (38)$$

In the momentum space both BM and TI-PLRBM ensemble are characterized by i.i.d. GR hopping integrals which can be found from Eq. (10):  $\tilde{J}_{p-q}$  with

$$\langle |\tilde{J}_p|^2 \rangle \simeq \Delta^2/N. \quad (39)$$

The momentum-space on-site energies  $\tilde{E}_p$  (which coincide with the spectrum of the corresponding  $\hat{j}$ ) are given by Eq. (9) and depend crucially on correlations in the hopping matrix elements.

For TI-PLRBM the spectrum of  $\tilde{E}_p$  is random with zero mean and the variance:

$$\langle |\tilde{E}_p|^2 \rangle = \sum_m \langle |j_{n-m}|^2 \rangle \sim \begin{cases} N^{1-2a}, & \text{if } a < 1/2 \\ \ln N, & \text{if } a = 1/2 \\ 1, & \text{if } a > 1/2. \end{cases} \quad (40)$$

In contrast, for BM model with fully correlated hopping  $j_0 |n - m|^{-a}$  ( $a \neq 0$ ) one obtains:

$$\tilde{E}_p/(2j_0) \simeq \zeta_a + A_a \left( \frac{N}{|p|} \right)^{1-a}, \quad \text{for } |p| \ll N, \quad (41)$$

$$\tilde{E}_p/(2j_0) \simeq \tilde{E}_{\min} + B_a \left( \frac{2q}{N} \right)^2, \quad \text{for } |N/2 - p| \ll N. \quad (42)$$

$\zeta_a$  is the Riemann zeta-function  $\zeta_a$ , and dimensionless constants  $A_a$ ,  $B_a$ , and  $\tilde{E}_{\min}$  given in [85].

One can see that the spectrum  $\tilde{E}_p$  for TI-PLRBM is either bounded ( $a > 1/2$ ) or unbounded from both sides ( $a \leq 1/2$ ). In contrast, for BM model the spectrum, while also bounded for  $a > 1$ , is unbounded *only from one side* for all  $a < 1$ .

This difference appears to have crucial consequences for the eigenfunctions statistics.

#### A. Wave function statistics in BM model

In this section we consider the wave function statistics of BM model in the coordinate space. Before employing the Mott and Anderson localization/delocalization criteria to BM model at  $a < 1$  we have to define the effective bandwidth of a highly stretched spectrum  $\tilde{E}_p$  in this case. Eqs. (41), (42) show that the *typical* level spacing  $\delta(N) = d\tilde{E}_p/dp \sim N^{-1}$  corresponds to  $|p| \sim N/2$  and  $|N/2 - p| \sim N$ . The corresponding  $\tilde{E}_p \sim 1$  gives the right estimation of the effective

bandwidth:

$$\Delta_p^{(\text{eff})} \sim 1, \quad (43)$$

for *typical states* of BM problem with  $a < 1$ . The remaining part of the spectrum of  $\tilde{E}_p$  has an increasing mean level spacing up to the maximal level spacing of the order of  $\delta_{\text{max}} \sim N^{1-a}$  at  $\tilde{E}_p \sim N^{1-a}$ . This part of spectrum, as well as the properties of the separate state in the YS model, requires a special study [88–91]. In this paper we limit ourselves by the case when the energy  $E \sim 1$  lies inside of the band of typical states.

For  $a > 1$ , the spectrum is bounded with the bandwidth of order 1, so that Eq. (43) is valid for all  $a$ .

Eq. (43) implies that the Mott's delocalization criterion is never fulfilled in the sense of Eq. (4) and thus ergodic delocalization is nowhere guaranteed.

To apply the Anderson localization criterion Eq. (8) we first compute the “inverted” hopping matrix  $\hat{M}(E_0)$  given by Eq. (14) with  $E_0 \sim N^\beta$ . We start by the case  $a > 1 - \beta$ , where  $|\tilde{E}_p|/E_0 \ll 1$ , and the analysis may be carried out similar to the case of TI-RP model. One obtains:

$$\frac{\langle |J_{nm}(E_0)| \rangle}{\Delta(E_0)} \sim \begin{cases} j_R, & \text{if } \beta > 0 \\ N^{-\beta} j_R, & \text{if } 1 - a < \beta < 0 \end{cases}, \quad (44)$$

where  $j_R = R^{-a}$ .

For  $a > 1$ , the sum  $S(E_0) = N^{-1} \sum_{\substack{n,m \\ n \neq m}} \langle |j_{nm}| \rangle$  in Eq. (8) converges and one obtains:

$$\frac{S(E_0)}{\Delta(E_0)} = \begin{cases} N^0, & \text{if } \beta > 0 \\ N^{-\beta}, & \text{if } 1 - a < \beta < 0 \end{cases} \quad (45)$$

For  $1 - \beta < a < 1$  and  $\beta > 0$  one obtains  $S(E_0)/\Delta(E_0) \sim N^{1-a}$ .

Now consider the case  $a < 1 - \beta$ , where  $\tilde{E}_p/E_0 \gg 1$  and the matrix inversion trick, Eq. (17), applies. Then we obtain:

$$\frac{M_{n-m}}{E_0} = C_1 e^{-\varkappa|n-m|} + C_2 \frac{(1 - \delta_{nm})}{j_0 |n-m|^{2-a}}. \quad (46)$$

where dimensionless constants  $C_1$ ,  $C_2$ , and  $\varkappa$  can be found in [85].

Notice that due to inverted position of  $\tilde{E}_p$  in Eq. (17) compared to Eq. (18) a new exponent:

$$a_{\text{eff}} = 2 - a \quad (47)$$

emerges in the place of  $a$ .

With this modification, Eq. (44), takes the form:

$$\frac{\langle |J_{nm}(E_0)| \rangle}{\Delta(E_0)} \sim \begin{cases} N^{2\beta} R^{-a_{\text{eff}}}, & \text{if } 1 - a > \beta > 0 \\ N^\beta R^{-a_{\text{eff}}}, & \text{if } \beta < 0 \end{cases}. \quad (48)$$

The corresponding expression for  $S(E_0)/\Delta(E_0)$  reads as follows:

$$\frac{S(E_0)}{\Delta(E_0)} = \begin{cases} N^{2\beta}, & \text{if } a < 1; 1 - a > \beta > 0 \\ N^\beta, & \text{if } a < 1; \beta < 0 \\ N^{a-1+\beta}, & \text{if } a > 1; \beta < 1 - a \end{cases}. \quad (49)$$

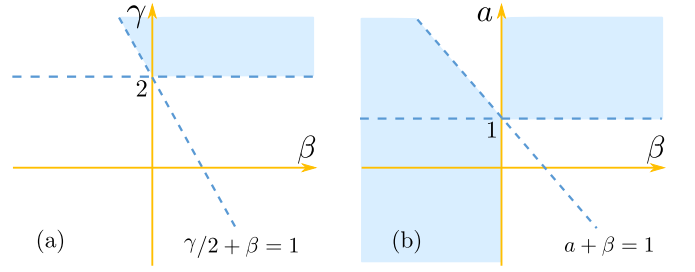


Figure 7. **Optimized Anderson localization criterion.** Domains of validity of Eq. (8): (a) for the YS and TI-RP models and (b) for BM model. The true domain of the localized states corresponds to the optimal  $\beta_{\text{opt}}$  at which the domain of validity of Eq. (8) is the widest possible. In both cases  $\beta_{\text{opt}} = 0$  but for the BM model the typical states are localized at all values of the exponent  $a$ , while for YS and TI-RP models the truly localized states exist only at  $\gamma > 2$ .

As the result of this analysis we obtain a diagram which shows the domains on the plane  $(\beta, a)$  where the *sufficient* condition for localization, Eq. (8), is fulfilled in BM ensemble (see Fig. 7(b)). The optimal  $\beta$  corresponds to the widest domain of validity of Eq. (8) which is the *true* domain of the localized phase. In Fig. 7 such domains are shown in blue for the BM model (Fig. 7(b)) and for the YS and TI-RP models (Fig. 7(a)). It is seen that for BM model at the optimal  $\beta_{\text{opt}} = 0$  the states inside the band Eq. (43) are localized at *all* values of the parameter  $a$ . The corresponding spectral statistics is therefore Poisson.

Note that the above analysis corresponding to the energy  $E \sim 1$  does not apply to the states outside the effective band Eq. (43) (i.e. in the stretched part of the spectrum) though the method itself is applicable everywhere.

## B. Duality of the exponent $\mu$ in BM model.

The fact that the typical states in BM ensemble are localized at all values of  $a$  can be traced back to the divergence of the spectrum  $\tilde{E}_p$  and as a consequence to a possibility to use the matrix inversion trick and Eq. (14) to derive Eq. (46) for  $a < 1$  and define  $a_{\text{eff}}$  as in Eq. (47). The same Eq. (46) helps to prove the duality of the exponents  $\mu(a)$ , Eq. (37), of the power-law localization:

$$\mu(a) = \mu(2 - a), \quad (50)$$

suggested recently in Ref. [39].

At  $a > 1$  the conventional representation of the eigenproblem

$$E\psi_E(n) = \varepsilon_n \psi_E(n) + j_0 \sum_{m \neq n} \psi_E(m) / |m - n|^a \quad (51)$$

gives the standard solution from the locator expansion method [1]. It converges to the power-law decaying large-distance asymptotics of the eigenstate

$$|\psi_E(n)|_{\text{typ}} \sim 1/|n - n_0|^a, \quad |n - n_0| \gg 1, (a > 1), \quad (52)$$

with the decay exponent coinciding with the matrix element exponent  $a$  due to the convergence of the sum in the r.h.s. of Eq. (51). Note that this method applies to all PLBRM-model at  $a > 1$  irrespectively to their hopping correlations.

At  $a < 1$  the usual locator expansion fails to converge. However, the locator expansion can be applied to the equivalent eigenproblem Eq. (15) with the “inverted” hopping matrix given by Eq. (46). The latter contains the power-law decaying part characterized by the exponent  $a_{\text{eff}} = 2 - a$ .

Thus by the same token as Eq. (52) we obtain a similar expression for  $|\psi_E(n)|_{\text{typ}}$  at  $a < 1$  but with  $a_{\text{eff}} = (2 - a)$  instead of  $a$ . Thus we conclude that:

$$\mu = \begin{cases} 2a, & \text{if } a > 1 \\ 2(2 - a), & \text{if } a < 1 \end{cases}, \quad (53)$$

which proves the duality Eq. (50).

Note that the duality concerns only the exponents in the power-law tail of the localized wave functions and not to the amplitude of this tail and the length scale at which the power-law asymptotics sets in (see Fig. 6).

### C. TI-PLBRM ensemble

Finally, we turn to statistics of eigen-data for the translation-invariant PLBRM. We start by the statistics of wave functions in the *momentum* space. Using Eqs. (39, 40) one finds:

$$\frac{\langle |J_p|^2 \rangle}{\langle |\tilde{E}_p|^2 \rangle} \propto N^{-\gamma_p^{(\text{eff})}}, \quad (54)$$

where

$$\gamma_p^{(\text{eff})} = \begin{cases} 2(1 - a), & \text{if } a < 1/2 \\ 1, & \text{if } a > 1/2 \end{cases} \quad (55)$$

Now the problem of wave function statistics of TI-PLBRM ensemble in the momentum space is reduced to the one for TI-RP ensemble in the coordinate space with the replacement  $\gamma \rightarrow \gamma_p^{(\text{eff})}$ . The result is presented on Fig. 3 where we denote by *Loc* -  $p$ , *Frac* -  $p$  and *Crit*  $\gamma_{\text{eff}} = 1$  the localized, fractal and critical phase at the point of ergodic transition, respectively.

As for the phases in the coordinate space, one can easily see from the Mott’s criterion Eq. (4) with  $\Delta_p \sim \sqrt{\langle |\tilde{E}_p|^2 \rangle}$  that for  $a < 1/2$  (when  $\Delta_p \propto N^{1/2-a}$  according to Eq. (40)) there is a weakly ergodic (WE) extended state. The Anderson localization criterion Eq. (8) ensures existence of the localized phase for  $a > 1$ .

The most difficult is the characterization of phase in the interval  $1/2 < a < 1$ . The matrix inversion also does not help to establish a true border for the localized phase. The reason is that for the case  $a > 1/2$  we are concerned, the bandwidth  $\Delta_p \sim \tilde{E}_p \sim 1$  and thus  $E_0$  could be chosen only to be  $E_0 \gtrsim 1$  which according to Eq. (34) leaves the scaling of effective hopping matrix unchanged.

In this situation we can rely only on numerics presented in Fig. 6(b) and Fig. 8. Indeed, Fig. 6(b) demonstrates a narrow  $f(\alpha)$  in the coordinate space of TI-PLBRM at  $a = 0.75$  which is typical for weakly ergodic states and identical to the one of non-TI PLBRM for the same value of  $a$ .

Additionally, Fig. 8(g) shows much smaller deviation from the Porter-Thomas distribution of the distribution function of  $|\psi_E|^2$  for TI-PLBRM at  $a = 0.75$  than that for the known multifractal case of  $a = 1$  of PLBRM on Fig. 8(d). This makes us to conclude that in the interval  $1/2 < a < 1$  of TI-PLBRM a weakly ergodic phase is realized, as well as for the non-TI PLBRM.

We note also that in contrast to the TI-RP case, the phases in the TI-PLBRM are *not* symmetric with respect to the point  $a = 1/2$ . The reason is that the typical off-diagonal matrix elements have a power-law decay in the coordinate space of TI-PLBRM ensemble, while in the momentum space they have no structure, similar to the coordinate space of TI-RP ensemble. In contrast, for TI-RP ensemble the typical off-diagonal elements are similar in a sense that they do not have structure both in the coordinate and in the momentum space. This allows to apply the duality relation Eq. (32) and establish the symmetry of phases with respect to  $\gamma = 1$ .

The level statistics in TI-PLBRM (see Fig. 3) can be easily identified using the three principles formulated Sec. IV and checked numerically, see Fig. 4. It is Poisson at  $a < 0$  and  $a > 1$ , a hybrid one at  $0 < a < 1/2$  and an ergodically-critical, like in the point  $\gamma = 1$  of RP ensemble, at  $1/2 < a < 1$ . As mentioned above, the latter interval in TI-PLBRM corresponds to  $\gamma_p^{(\text{eff})} = 1$ . Therefore the behavior of  $\langle n^2 \rangle - \langle n \rangle^2 = \chi \langle n \rangle$  (with level compressibility  $0 < \chi \leq 1$ ) should be quasi-Poisson, as in the point  $\gamma = 1$  of ergodic transition in RP ensemble [41]. In the interval  $0 < a < 1/2$  the hybrid character of level statistics follows from the lack of basis invariance of the eigenfunction statistics (see Fig. 3).

## VIII. CONCLUSION AND DISCUSSIONS.

The main result of this paper is the picture of *correlation-induced localization* which is presented in Fig. 1 and Fig. 3. We demonstrate that the correlations in long-range hopping may change drastically the localization-delocalization phase diagram of many models turning extended phase into the (multi)fractal or even localized one.

We show that the well-known localization principles (3) and (6) are not complimentary and are in fact the sufficient (but not the necessary) conditions for weakly ergodic delocalization and localization, respectively. Thus they are not able to determine exact bounds for localization/delocalization and may leave room for non-ergodic delocalized phases. However, by applying the matrix inversion trick and the optimization procedure suggested in this paper one can make Eqs. (4) and (6) also necessary conditions for ergodicity and localization and may in certain cases determine exact phase diagram. We believe that the same arguments apply to the models with sign-alternating non-random hopping integrals [92, 93] as the general applicability of the matrix inversion method is related



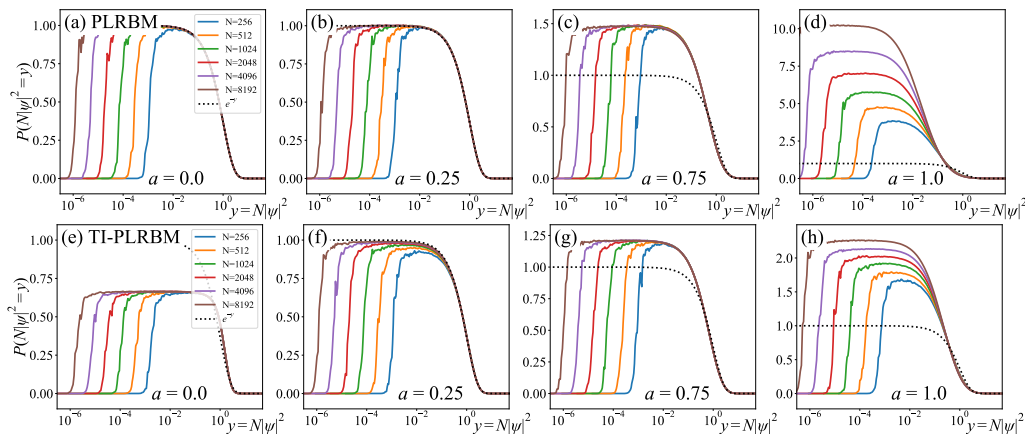


Figure 8. **Comparison of eigenstate probability distributions**  $P(N|\psi|^2 = y)$  in (a-d) PLRBM and (e-h) TI-PLRBM models (solid lines for different system sizes  $N$ ) with the GUE Porter-Thomas distribution  $P(y) = e^{-y}$  (black dashed line).

with the presence of the finite gaps or edges of the spectrum, but the consideration of these models is out of scope of the current paper.

We suggest a natural extension of the class of models with correlated long-range hopping integrals by introducing the translation-invariant (TI) random matrix models, where hopping integrals are fully correlated *along* the diagonals but the correlations *between* the diagonals are absent. We identify phases with different character of localization/delocalization in these models both in the coordinate and in the momentum spaces together with the spectral statistics. The results are summarized in Fig. 3. It is shown that at moderately weak disorder the delocalized phases in TI-models are never fully ergodic, as the eigenfunction statistics are different in the momentum and the coordinate spaces.

We formulated the principles to identify the level statistics in the considered models as belonging to the Wigner-Dyson, Poisson or the new hybrid class. In particular, the spectral statistics is Poisson if the eigenfunction statistics shows localization in either coordinate or in the momentum basis. This implies that in TI random matrix ensembles the spectral statistics may be Poisson despite the states are extended in the coordinate basis (but localized in the momentum one). This statement is confirmed by numerics.

The considered models with fully-correlated, TI-correlated, and uncorrelated hopping can be easily generalized to a whole class of matrix models with the continuous parameter correlations in the hopping integrals, see Fig. 1. Indeed, in TI-models hopping integrals are fully correlated *along* the diagonals, while in uncorrelated models they are statistically independent. In between one can consider, e.g., the models with hopping terms in each diagonal to be correlated in such a way that  $M_1$  elements in each diagonal are equal, where  $M_1$  changes from 1 for uncorrelated models to  $N$  for TI-models. In the similar way one can consider the continuous correlation parameter from TI- to fully-correlated models. Indeed, as in TI-models the correlations *between* the diagonals are absent one can partially add them by considering blocks of  $M_2$  diagonals to be equal, where  $M_2$  changes from 1 for TI-models to  $N$

for fully-correlated models. The overall number of independent hopping terms in the matrix that scales as  $N^2/(M_1 M_2)$  can be considered as a continuous hopping correlation parameter. Of course this is not a unique way to include hopping correlations in uncorrelated models, but this kind of correlations is natural as it emerges in physical models such as the RKKY where hopping integrals deterministically oscillate as a function of  $|n - m|$  with the period incommensurate with the lattice constant.

Within the same method, for the random matrix models with deterministic power-law decaying hopping integrals  $j_{n-m} \sim |n - m|^{-a}$  we confirm that both for  $a > 1$  (see [88–91]) and  $a < 1$  [39] the typical states are localized with the power-law tails  $\psi_{E_n}(m) \sim |n - m|^{-a_{\text{eff}}}$  at  $a \neq 0$  and analytically prove the duality  $a_{\text{eff}} = \max(a, 2 - a)$ .

It is also worth noticing that our arguments are not restricted only to the one-dimensional case,  $d = 1$ . Recent work [93] has shown the presence of localized states for isotropic deterministic power-law hopping with  $a < d = 3$  in three-dimensional cubic lattices. This problem might be understood within our formalism.

Another intriguing direction of research is the interplay between correlations in the hopping integrals and in the on-site energies. As recently shown the correlated on-site “disorder” (quasi-periodic potential [9]) may destroy localization and produce a whole bunch of (multi)fractal phases depending on the power  $a$  [94] in the BM-model with deterministic power-law hopping integrals.

Finally, the most challenging problem motivated by our paper is the effect of correlations on Many Body Localization in the long-range interacting models (see, e.g., [15–27]).

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