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Projector-based renormalization approach to electron-hole-photon systems in nonequlibrium steady-state

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We present an extended version of the projector-based renormalization method that can be used to address not only equilibrium but also non-equilibrium situations in coupled fermion-boson systems. The theory is applied to interacting electrons, holes and photons in a semiconductor microcavity, where the loss of cavity photons into vacuum is of particular importance. The method incorporates correlation and fluctuation processes beyond mean-field theory in a wide parameter range of detuning, Coulomb interaction, light-matter coupling and damping, even in the case when the number of quasiparticle excitations is large. This enables the description of exciton and polariton formation, and their possible condensation through spontaneous phase symmetry breaking by analyzing the ground-state, steady-state and spectral properties of a rather generic electron-hole-photon Hamiltonian, which also includes the coupling to two fermionic baths and a free-space photon reservoir. Thereby, the steady-state behavior of the system is obtained by evaluating expectation values in the long-time limit by means of the Mori-Zwanzig projection technique. Tracking and tracing different order parameters, the fully renormalized single-particle spectra and the steady-state luminescence, we demonstrate the Bose-Einstein condensation of excitons and polaritons and its smooth transition when the excitation density is increased.

I. INTRODUCTION

Semiconductor microcavity systems with quantum well potentials have created fascinating possibilities with regard to the formation of diverse condensed phases [1, 2]. These condensates constitute a macroscopic, long-range quantum phase-coherent state that exhibits unconventional transport and luminescence properties in particular. Coupled electron-hole-photon (e-h-p) systems have led to very early speculations about a Bose-Einstein condensation of excitons, i.e., of electron-hole pairs formed by the attractive Coulomb interaction, at low but sufficient large particle densities [3].

While the short life time of optically generated excitons seems to be a serious problem establishing a Bose-Einstein condensate (BEC) in bulk semiconductors, such as Cu_2O, even in potential traps [4], quantum wells realized in layered semiconductors significantly reduce the rate at which electrons and holes recombine into photons (albeit there is not yet compelling evidence for an exciton BEC in these systems). Increasing the excitation density, phase-space (Pauli-blocking) and Fermi-surface effects become important and, as a result, the exciton BEC may cross over into an e-h BCS phase [5, 6]. In response to a specific electronic band structure, such as those near a semiconductor-semimetal transition, the exciton condensate can also exist in equilibrium whereby it typifies an excitonic insulator phase [7–9].

Of course the e-h-p system is also influenced by its interaction with the surroundings. In the case of a semiconductor microcavity the loss of cavity photons into the vacuum space is of particular importance. This means that the microcavity system is essentially in a non-equilibrium state. To maintain the system in a stationary quasi-equilibrium state one has to supply continuously electrons and holes to the e-h-p system which compensates the decay of photons into the environment. Unfortunately, however, only for low excitation densities, when photon effects are still irrelevant, the properties of the e-h-p system reduce to the equilibrium physics. At large excitation density, the photonic effects play a predominant role, and the condensate turns from excitonic to polaritonic. Polaritons in semiconductor microcavities have also been observed to exhibit BEC [1, 10]. At even higher excitation densities, the excitonic component saturates, whereas the photonic order parameter continues to increase. Here, the relationship between a polariton BEC and photon lasing has to be clarified [11, 12].

The main objective of this paper is to describe both the equilibrium and the non-equilibrium properties of the e-h-p system on an equal footing. To this end, we employ a minimal model for the e-h-p gas that includes attractive interactions between electrons and holes as

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well as between cavity photons and electron-hole excitations [13–17]. Moreover the decay of cavity photons to an external vacuum and the pumping from two fermionic baths to the electrons and holes of the e-h-p system are taken into account. The major difficulty results from the lack of reliable techniques to tackle such a model in the whole parameter regime. So far most theoretical approaches [18–21] have addressed the equilibrium properties separately from those of the steady state [22–24]. Only recently a steady-state framework [25, 26] based on a non-equilibrium Green's function approach[13, 14] was formulated which allows to treat equally the equilibrium BEC and BCS phases at low excitation densities just as the non-equilibrium state at high excitation densities.

In this work, we utilize an alternative theoretical tool, the projector-based renormalization method (PRM) [27–29]. The PRM was applied before exclusively to equilibrium phenomena, and also to describe the equilibrium properties of e-h-p systems [30] at small–to–moderate excitation densities, where the leakage of photons to the vacuum is not important. We show that the PRM can be extended to non-equlibrium situations, and applied to the model under consideration even in the case when the number of excitations is large. Here, the steady-state properties can be found from time-dependent expectation values for long times which will be evaluated by means of the Mori-Zwanzig projection technique. Thereby, in contrast to the work [25, 26], the PRM incorporates fluctuation processes beyond mean field theory for all excitation densities. This allows us to address the great variety of e-h-p condensation phenomena mentioned above.

The paper is organized as follows. In Sec. II we introduce our theoretical model for a pumped-decaying exciton-polariton system and briefly discuss its adaption to a steady-state situation. Since the present theoretical study is based on the PRM, we outline this technique and its improvements in Sec. III. More details of the PRM approach can be found in the Appendices A–C. The steady-state expectation values are evaluated in Sec. IV, the single-particle spectral function in Sec. V and the steady-state luminescence in Sec. VI. Finally, in Sec. VII some characteristic numerical results will be presented and discussed. Section VIII contains a brief summary and our main conclusions.

II. MODELING OF PUMPED-DECAYING EXCITON-POLARITON SYSTEMS

As a typical example of an e-h-p system we will consider electrons and holes, confined in a semiconductor quantum well structure, are exposed to photons, entrapped in a microcavity. In such a setup Bose-Einstein condensates of bound electron-hole pairs (excitons) and polaritons may appear, which possibly can cross over into a BCS-like coherent state under quasi-equilibrium conditions at high particle densities, in case the quasiparticle lifetime is larger than the thermalization time [26]. In general, however, these system are driven out of equilibrium by coupling to multiple baths, and such nonequilibrium electron-hole condensates in the solid state are subject to dissipation, dephasing and decay. Therefore pump and loss channels have to be taken into account. In the following we introduce appropriate microscopic models for the system and for the reservoirs to which it is coupled in order to include these effects.

A. System Hamiltonian

Our starting point is the e-h-p Hamiltonian [11, 30] of an isolated semiconductor quantum-well/microcavity system,

$$\check{\mathcal{H}}_{\rm S} = \check{\mathcal{H}}_{0,\rm S} + \check{\mathcal{H}}_{\rm el-ph} + \check{\mathcal{H}}_{\rm el-el} \tag{1}$$

with

$$\check{\mathcal{H}}_{0,\rm S} = \sum_{\mathbf{k}} \check{\varepsilon}^e_{\mathbf{k}} \check{e}^\dagger_{\mathbf{k}} \check{e}_{\mathbf{k}} + \sum_{\mathbf{k}} \check{\varepsilon}^h_{\mathbf{k}} \check{h}^\dagger_{\mathbf{k}} \check{h}_{\mathbf{k}} + \sum_{\mathbf{q}} \check{\omega}_{\mathbf{q}} \check{\psi}^\dagger_{\mathbf{q}} \check{\psi}_{\mathbf{q}} \,, \tag{2}$$

$$\check{\mathcal{H}}_{\rm el-ph} = -\frac{g}{\sqrt{N}} \sum_{\mathbf{qk}} [\check{e}^\dagger_{\mathbf{k}+\mathbf{q}} \check{h}^\dagger_{-\mathbf{k}} \check{\psi}_{\mathbf{q}} + \text{H.c.}] \,, \tag{3}$$

$$\check{\mathcal{H}}_{\rm el-el} = -\frac{U}{N} \sum_{\mathbf{k}} \check{\rho}^e_{\mathbf{k}} \check{\rho}^h_{-\mathbf{k}} \,, \tag{4}$$

describing free particles (electrons created by $\check{e}^\dagger_{\mathbf{k}}$, holes by $\check{h}^\dagger_{\mathbf{k}}$, and photons by $\check{\psi}^\dagger_{\mathbf{q}}$), the coupling ($\propto g$) of electron-hole pairs to the radiation field, and the local Coulomb interaction ($\propto U$) between electrons (density operators $\check{\rho}^e_{\mathbf{k}} = \sum_{\mathbf{k}_1} \check{e}^\dagger_{\mathbf{k}+\mathbf{k}_1} \check{e}_{\mathbf{k}_1}$) and holes ($\check{\rho}^h_{\mathbf{k}} = \sum_{\mathbf{k}_1} \check{h}^\dagger_{\mathbf{k}+\mathbf{k}_1} \check{h}_{\mathbf{k}_1}$), respectively. In $\check{\mathcal{H}}_{0,\rm S}$, $\check{\varepsilon}^e_{\mathbf{k}}$ ($\check{\varepsilon}^h_{\mathbf{k}}$) denotes the dispersion of electrons (holes),

$$\check{\varepsilon}^e_{\mathbf{k}} = -2t \sum_i^D \cos k_i + \frac{E_g + 4tD}{2} = \check{\varepsilon}^h_{\mathbf{k}} \,, \tag{5}$$

where D is the dimension of the hypercubic lattice, t is the particle transfer amplitude between neighboring sites, E_g gives the minimum distance (gap) between the bare electron and hole bands, and $\check{\varepsilon}^e_{\mathbf{k}} = \check{\varepsilon}^h_{\mathbf{k}}$ is set for simplicity. The photon field is characterized by

$$\check{\omega}_{\mathbf{q}} = \sqrt{(c\mathbf{q})^2 + \omega_c^2} \tag{6}$$

with the zero-point cavity frequency ω_c.

B. Coupling to reservoirs

Next we model the coupling of the e-h-p system, being an open quantum system in reality, to its environment. In the first place, two pumping baths for electrons and holes made possible the injection of free fermions into the system. In addition, the cavity photons are connected to a free-space photon reservoir, allowing for a leakage of photons into the surroundings. To maintain a steady state,

the loss of cavity photons to the external reservoir must be compensated by bringing in fermionic carriers. Then for the total system the following Hamiltonian seems to be adequate

$$\check{\mathcal{H}} = \check{\mathcal{H}}_{\rm S} + \check{\mathcal{H}}_{\rm R} + \check{\mathcal{H}}_{\rm SR}\,, \tag{7}$$

where $\check{\mathcal{H}}_{\rm S}$ is given by Eq. (1), and $\check{\mathcal{H}}_{\rm R}$ and $\check{\mathcal{H}}_{\rm SR}$ are defined as:

$$\begin{aligned}\check{\mathcal{H}}_{\rm R} = &\sum_{\mathbf{p}} \check{\omega}^{e}_{\mathbf{p}}\, \check{b}^{\dagger}_{e,\mathbf{p}} \check{b}_{e,\mathbf{p}} + \sum_{\mathbf{p}} \check{\omega}^{h}_{\mathbf{p}}\, \check{b}^{\dagger}_{h,\mathbf{p}} \check{b}_{h,\mathbf{p}} \\ &+ \sum_{\mathbf{p}} \check{\omega}^{\varphi}_{\mathbf{p}}\, \check{\varphi}^{\dagger}_{\mathbf{p}} \check{\varphi}_{\mathbf{p}}\,, \end{aligned} \tag{8}$$

$$\begin{aligned}\check{\mathcal{H}}_{\rm SR} = &\frac{1}{N}\sum_{\mathbf{kp}} (\Gamma^{e}_{\mathbf{kp}} \check{e}^{\dagger}_{\mathbf{k}} \check{b}_{e,\mathbf{p}} + \text{H.c.}) \\ &+ \frac{1}{N}\sum_{\mathbf{kp}} (\Gamma^{h}_{\mathbf{kp}} \check{h}^{\dagger}_{-\mathbf{k}} \check{b}_{h,-\mathbf{p}} + \text{H.c.}) \\ &+ \frac{1}{N}\sum_{\mathbf{qp}} (\Gamma^{\psi}_{\mathbf{qp}} \check{\psi}^{\dagger}_{\mathbf{q}} \check{\varphi}_{\mathbf{p}} + \text{H.c.})\,. \end{aligned} \tag{9}$$

$\check{\mathcal{H}}_{\rm R}$ is the Hamiltonian for the two fermionic baths and the free-space photon reservoir which are interacting with the e-h-p system via $\check{\mathcal{H}}_{\rm SR}$. The quantities $\check{b}^{(\dagger)}_{e,\mathbf{p}}$ and $\check{b}^{(\dagger)}_{h,\mathbf{p}}$ are the fermion creation/annihilation operators of the two pumping baths, and $\check{\varphi}^{(\dagger)}_{\mathbf{p}}$ are the boson creation and annihilation operators of the free-space photons. Finally, $\Gamma^{e,h}_{\mathbf{kp}}$ and $\Gamma^{\psi}_{\mathbf{qp}}$ in Eq. (9) are the coupling constants between the system and the respective reservoirs.

Let us also define the particle number of the total system by

$$\begin{aligned}\mathcal{N} = &\frac{1}{2}\sum_{\mathbf{k}} (\check{e}^{\dagger}_{\mathbf{k}} \check{e}_{\mathbf{k}} + \check{h}^{\dagger}_{\mathbf{k}} \check{h}_{\mathbf{k}}) + \sum_{\mathbf{q}} \check{\psi}^{\dagger}_{\mathbf{q}} \check{\psi}_{\mathbf{q}} \\ &+ \frac{1}{2}\sum_{\mathbf{p}} (\check{b}^{\dagger}_{e,\mathbf{p}} \check{b}_{e,\mathbf{p}} + \check{b}^{\dagger}_{h,\mathbf{p}} \check{b}_{h,\mathbf{p}}) + \sum_{\mathbf{p}} \check{\varphi}^{\dagger}_{\mathbf{p}} \check{\varphi}_{\mathbf{p}}\,, \end{aligned} \tag{10}$$

which is a constant of motion $[\check{\mathcal{H}}, \mathcal{N}] = 0$.

We maintain that the total system in a non-equilibrium situation evolves under Hamiltonian $\check{\mathcal{H}} = \mathcal{H}_{\rm S} + \check{\mathcal{H}}_{\rm R} + \check{\mathcal{H}}_{\rm SR}$. Thereby $\check{\mathcal{H}}_{\rm S}$ is "simple" in the sense that it can be diagonalized, even though many-body aspects due to the presence of $\check{\mathcal{H}}_{\rm el-el}$ and $\check{\mathcal{H}}_{\rm el-ph}$ require a special treatment. $\check{\mathcal{H}}_{\rm SR}$ is responsible for the non-equilibrium situation since it governs the pumping and damping of electrons and holes and the leakage of photons into the free space. Note that $\mathcal{H}_{\rm SR}$ is not translationally invariant.

We now assume that $\check{\mathcal{H}}_{\rm SR}$ vanishes for times $t < t_0$, where $t_0 \to -\infty$ might be used as a suitable starting point. That is, before at t_0 the interaction $\check{\mathcal{H}}_{\rm SR}$ is turned on, the reservoirs and the e-h-p system are in separate thermal equilibrium states. Then the state of the total system is described by a product of the e-h-p system density operator $\check{\rho}_{\rm S}$ and the reservoir density operator $\check{\rho}_{\rm R}$

$$\check{\rho}_0 = \check{\rho}_{t_0\to-\infty} = \check{\rho}_{\rm S}\, \check{\rho}_{\rm R}\,, \tag{11}$$

where $\check{\rho}_{\rm S}$ commutes with $\check{\mathcal{H}}_{\rm S}$. To simplify the considerations we suppose the electronic baths and the external photon reservoir to be huge compared to $\check{\mathcal{H}}_{\rm S}$. As a result, in the steady state the two electronic baths remain in thermal equilibrium, even when they are coupled to the e-h-p system. Similarly the free-space photons act as a reservoir for cavity photons escaped from the e-h-p system.

Below, the task is to evaluate time-dependent expectation values of observables $\check{\mathcal{A}}$ for times $t \gg t_0$,

$$\langle \check{\mathcal{A}}(t) \rangle = \text{Tr}\,[\check{\rho}_0\, \check{\mathcal{A}}(t)]\,, \tag{12}$$

when the system has approached a steady state. Therefore we use the Heisenberg picture, in which the time-dependence of $\check{\mathcal{A}}$ is governed by the full Hamiltonian $\check{\mathcal{H}}$, and $\check{\rho}_0$ is time independent. Note that $\check{\rho}_0$ and $\check{\mathcal{H}}$ do not commute. This property causes the genuine time dependence of expectation values (12). Being τ_R some internal relaxation time, for times $t \gg \tau_R$ the system is expected to merge into a periodically driven steady state and remembers no longer its initial state.

C. Steady-state description

Now let us consider a steady-state situation in which both loss and pump processes are spatially homogenous with a coherent photon field that is only formed for $\mathbf{q} = 0$. For large times, the steady state will evolve according to

$$\langle \check{\psi}^{\dagger}_{\mathbf{q}}(t) \rangle = \delta_{\mathbf{q}0}\, \langle \psi^{\dagger}_0 \rangle\, e^{i\mu t}\,, \tag{13}$$

$$\langle (\check{e}^{\dagger}_{\mathbf{k}} \check{h}^{\dagger}_{-\mathbf{k}})(t) \rangle = d^{*}_{\mathbf{k}}\, e^{i\mu t}\,, \tag{14}$$

$$\langle (\check{e}^{\dagger}_{\mathbf{k}} \check{e}_{\mathbf{k}})(t) \rangle = n^{e}_{\mathbf{k}}\,, \tag{15}$$

$$\langle (\check{h}^{\dagger}_{-\mathbf{k}} \check{h}_{-\mathbf{k}})(t) \rangle = n^{h}_{-\mathbf{k}}\,, \tag{16}$$

where the quantities $\langle \psi^{\dagger}_0 \rangle$, $d^{*}_{\mathbf{k}}$, $n^{e}_{\mathbf{k}}$ and $n^{h}_{-\mathbf{k}}$ are time-independent and—together with μ—are subject to the evaluation below. *Ansatz* (13)–(16) implies that the dynamics of certain variables is captured on a rotating frame with a frequency μ, where in the steady state $\langle \psi^{\dagger}_0 \rangle$, $d^{*}_{\mathbf{k}}$, $n^{e}_{\mathbf{k}}$ and $n^{h}_{-\mathbf{k}}$ become time-independent [16].

In the first evaluation step the explicit time dependence in $\langle \check{\psi}^{\dagger}_{\mathbf{k}}(t) \rangle$ and $\langle (\check{e}^{\dagger}_{\mathbf{k}} \check{h}^{\dagger}_{-\mathbf{k}})(t) \rangle$ will be eliminated. This is achieved by performing a time-dependent gauge transformation:

$$\begin{aligned}\left(e^{\dagger}_{\mathbf{k}}, h^{\dagger}_{-\mathbf{k}}, \psi^{\dagger}_{\mathbf{q}}\right) &= e^{-i\mu\mathcal{N}t} \left(\check{e}^{\dagger}_{\mathbf{k}}, \check{h}^{\dagger}_{-\mathbf{k}}, \check{\psi}^{\dagger}_{\mathbf{q}}\right) e^{i\mu\mathcal{N}t} \\ &= \left(e^{-i(\mu/2)t} \check{e}^{\dagger}_{\mathbf{k}}, e^{-i(\mu/2)t} \check{h}^{\dagger}_{-\mathbf{k}}, e^{-i\mu t} \check{\psi}^{\dagger}_{\mathbf{q}}\right), \end{aligned} \tag{17}$$

$$\begin{aligned}\left(b^{\dagger}_{e,\mathbf{p}}, b^{\dagger}_{h,\mathbf{p}}, \varphi^{\dagger}_{\mathbf{p}}\right) &= e^{-i\mu\mathcal{N}t} \left(\check{b}^{\dagger}_{e,\mathbf{p}}, \check{b}^{\dagger}_{h,\mathbf{p}}, \check{\varphi}^{\dagger}_{\mathbf{p}}\right) e^{i\mu\mathcal{N}t} \\ &= \left(e^{-i(\mu/2)t} \check{b}^{\dagger}_{e,\mathbf{p}}, e^{-i(\mu/2)t} \check{b}^{\dagger}_{h,\mathbf{p}}, e^{-i\mu t} \check{\varphi}^{\dagger}_{\mathbf{p}}\right). \end{aligned} \tag{18}$$

Let us look at an example: The equation of motion for the operator $\check{\psi}^{\dagger}_{\mathbf{k}}(t)$ reads $(\mathrm{d}/\mathrm{d}t)\check{\psi}^{\dagger}_{\mathbf{k}}(t) = (i/\hbar)[\check{\mathcal{H}}, \check{\psi}^{\dagger}_{\mathbf{k}}](t)$. Going over from $\check{\psi}^{\dagger}_{\mathbf{k}}$ to the new variable $\psi^{\dagger}_{\mathbf{k}} = \check{\psi}^{\dagger}_{\mathbf{k}}\, e^{-i\mu t}$,

the equation for $\psi^\dagger_{\mathbf{k}}(t)$ becomes $(\mathrm{d}/\mathrm{dt})\psi^\dagger_{\mathbf{k}}(t) = (i/\hbar)[\check{\mathcal{H}} - \mu\mathcal{N}, \psi^\dagger_{\mathbf{k}}](t)$. Thus, using the following replacements

$$\varepsilon^\alpha_{\mathbf{k}} = \check{\varepsilon}^\alpha_{\mathbf{k}} - \frac{1}{2}\mu\,, \quad \omega_{\mathbf{q}} = \check{\omega}_{\mathbf{q}} - \mu\,, \tag{19}$$

$$\omega^\alpha_{\mathbf{p}} = \check{\omega}^\alpha_{\mathbf{p}} - \frac{1}{2}\mu\,, \quad \omega^\varphi_{\mathbf{p}} = \check{\omega}^\varphi_{\mathbf{p}} - \mu\,, \tag{20}$$

($\alpha = e, h$), the explicit time dependences in Eqs. (13) and (14) disappears. Following the equations of motion of the new variables we therefore introduce a new Hamiltonian,

$$\mathcal{H} = \check{\mathcal{H}} - \mu\mathcal{N}\,, \tag{21}$$

where both parts on the right hand side keep their operator form, when written in the new variables $e^\dagger_{\mathbf{k}}, h^\dagger_{\mathbf{k}}, \psi^\dagger_{\mathbf{q}} \ldots$ Note that replacements (19) and (20) only apply to the time dependence of $\check{\mathcal{A}}(t)$ in Eq. (12) (Heisenberg picture) but not to the density operator $\check{\rho}_0$, which keeps its operator form in the new variables and will be called ρ_0. The total particle number $\mathcal{N}$, written in the new variables, has the same operator form as in Eq. (10) and obeys $[\mathcal{H}, \mathcal{N}] = 0$. Thus the total particle flux $\mathrm{d}\langle\mathcal{N}\rangle/\mathrm{dt} = 0$ disappears, which means that a change of the particle numbers of the e-h-p subsystem and the electronic reservoirs must be balanced by a change of the free space photons.

We wish to stress that only in thermal equilibrium the quantity μ will turn out to act as a chemical potential. For time-dependent problems, such as the considered open e-h-p system, the dynamics is captured on a rotating frame with the frequency μ. Thereby μ is a given parameter which has to be fixed in a steady state [16].

D. Total Hamiltonian

With the above transformations and replacements the total Hamiltonian $\mathcal{H}$ takes the form:

$$\mathcal{H} = \mathcal{H}_{\mathrm{S}} + \mathcal{H}_{\mathrm{R}} + \mathcal{H}_{\mathrm{SR}}\,, \tag{22}$$

where $\mathcal{H}_{\mathrm{S}}$ describes the interacting e-h-p subsystem

$$\mathcal{H}_{\mathrm{S}} = \mathcal{H}_0 + \mathcal{H}_c + \mathcal{H}_g + \mathcal{H}_U\,, \tag{23}$$

with

$$\mathcal{H}_0 = \sum_{\mathbf{k}} \varepsilon^e_{\mathbf{k}} e^\dagger_{\mathbf{k}} e_{\mathbf{k}} + \sum_{\mathbf{k}} \varepsilon^h_{\mathbf{k}} h^\dagger_{-\mathbf{k}} h_{-\mathbf{k}} + \sum_{\mathbf{q}} \omega_{\mathbf{q}} \psi^\dagger_{\mathbf{q}} \psi_{\mathbf{q}}\,, \tag{24}$$

$$\mathcal{H}_c = \sum_{\mathbf{k}} (\Delta\, e^\dagger_{\mathbf{k}} h^\dagger_{-\mathbf{k}} + \mathrm{H.c.}) + \sqrt{N}(\Gamma\psi^\dagger_0 + \mathrm{H.c.})\,, \tag{25}$$

$$\mathcal{H}_g = -\frac{g}{\sqrt{N}} \sum_{\mathbf{kq}} (e^\dagger_{\mathbf{q}+\mathbf{k}} h^\dagger_{-\mathbf{k}} \psi_{\mathbf{q}} + \mathrm{H.c.})\,, \tag{26}$$

$$\mathcal{H}_U = -\frac{U}{N} \sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}} e^\dagger_{\mathbf{k}_1+\mathbf{k}} e_{\mathbf{k}_1}\, h^\dagger_{\mathbf{k}_2-\mathbf{k}} h_{\mathbf{k}_2}\,. \tag{27}$$

Here, the first term $\mathcal{H}_0$ of $\mathcal{H}_{\mathrm{S}}$ is diagonal, whereas the second part $\mathcal{H}_c$ is non-diagonal and contains infinitesimally small external fields $\Delta = 0^+$ and $\Gamma = 0^+$, which are introduced to account for possible ground-state phases with broken gauge symmetry. As shown below, in the course of the renormalization procedure, the fields Δ and Γ take over the role of order parameters for the exciton and photon condensates. Finally, the terms $\mathcal{H}_g$ and $\mathcal{H}_U$ in Eqs. (26) and (27) stand for the interactions between excitons and photons and for the Coulomb attraction between electrons and holes.

The remaining terms in Eq. (22) are the reservoir Hamiltonian $\mathcal{H}_{\mathrm{R}}$ and the interaction Hamiltonian $\mathcal{H}_{\mathrm{SR}}$ between the reservoirs and the e-h-p system. Written in the new variables, they have the same operator structure as Eqs. (8) and (9):

$$\mathcal{H}_{\mathrm{R}} = \sum_{\mathbf{p}} \omega^e_{\mathbf{p}}\, b^\dagger_{e,\mathbf{p}} b_{e,\mathbf{p}} + \sum_{\mathbf{p}} \omega^h_{\mathbf{p}}\, b^\dagger_{h,\mathbf{p}} b_{h,\mathbf{p}} + \sum_{\mathbf{p}} \omega^\varphi_{\mathbf{p}}\, \varphi^\dagger_{\mathbf{p}} \varphi_{\mathbf{p}}\,, \tag{28}$$

$$\mathcal{H}_{\mathrm{SR}} = \sum_{\mathbf{kp}} (\Gamma^e_{\mathbf{kp}}\, e^\dagger_{\mathbf{k}} b_{e,\mathbf{p}} + \mathrm{H.c.}) + \sum_{\mathbf{kp}} (\Gamma^h_{\mathbf{kp}}\, h^\dagger_{-\mathbf{k}} b_{h,-\mathbf{p}} + \mathrm{H.c.}) + \sum_{\mathbf{qp}} (\Gamma^\psi_{\mathbf{qp}}\, \psi^\dagger_{\mathbf{q}} \varphi_{\mathbf{p}} + \mathrm{H.c.})\,. \tag{29}$$

In order to separate the mean-field contributions from $\mathcal{H}_g$ and $\mathcal{H}_U$, we introduce time ordered operators:

$$\begin{aligned} :e^\dagger_{\mathbf{k}+\mathbf{q}} h^\dagger_{-\mathbf{k}} \psi_{\mathbf{q}}: &=: e^\dagger_{\mathbf{k}+\mathbf{q}} h^\dagger_{-\mathbf{k}} :: \psi_{\mathbf{q}} := e^\dagger_{\mathbf{k}+\mathbf{q}} h^\dagger_{-\mathbf{k}} \psi_{\mathbf{q}} \\ &\quad - \delta_{\mathbf{q},0}\big(d^*_{\mathbf{k}} :\psi_0: + \langle\psi_0\rangle :e^\dagger_{\mathbf{k}} h^\dagger_{-\mathbf{k}}:\big)\,, \end{aligned} \tag{30}$$

$$\begin{aligned} :e^\dagger_{\mathbf{k}_1+\mathbf{k}} e_{\mathbf{k}_1}\, h^\dagger_{\mathbf{k}_2-\mathbf{k}} h_{\mathbf{k}_2}: &= e^\dagger_{\mathbf{k}_1+\mathbf{k}} e_{\mathbf{k}_1}\, h^\dagger_{\mathbf{k}_2-\mathbf{k}} h_{\mathbf{k}_2} \\ &\quad - \delta_{\mathbf{k},0}\left(n^e_{\mathbf{k}_1} h^\dagger_{\mathbf{k}_2} h_{\mathbf{k}_2} + n^h_{\mathbf{k}_2} e^\dagger_{\mathbf{k}_1} e_{\mathbf{k}_1} - n^e_{\mathbf{k}_1}\, n^h_{\mathbf{k}_2}\right) \\ &\quad - \delta_{\mathbf{k}_1,-\mathbf{k}_2}\big(d^*_{\mathbf{k}+\mathbf{k}_1} :h_{-\mathbf{k}_1} e_{\mathbf{k}_1}: + d_{\mathbf{k}_1} :e^\dagger_{\mathbf{k}+\mathbf{k}_1} h^\dagger_{-\mathbf{k}-\mathbf{k}_1}:\big)\,. \end{aligned} \tag{31}$$

Here, $:\mathcal{A}: = \mathcal{A} - \langle\mathcal{A}\rangle$, and $n^e_{\mathbf{k}_1}$ and $n^h_{\mathbf{k}_2}$ are occupation numbers evaluated with the density operator ρ_0:

$$n^e_{\mathbf{k}_1} = \langle e^\dagger_{\mathbf{k}_1} e_{\mathbf{k}_1}\rangle\,, \quad n^h_{\mathbf{k}_2} = \langle h^\dagger_{\mathbf{k}_2} h_{\mathbf{k}_2}\rangle\,. \tag{32}$$

Obviously, a finite $d^*_{\mathbf{k}}$ indicates a particle-hole (exciton) condensate:

$$d^*_{\mathbf{k}} = \langle e^\dagger_{\mathbf{k}} h^\dagger_{-\mathbf{k}}\rangle\,. \tag{33}$$

With Eqs. (30)–(31) Hamiltonian $\mathcal{H}_{\mathrm{S}}$ is rewritten as

$$\mathcal{H}_{\mathrm{S}} = \hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_c + \hat{\mathcal{H}}_g + \hat{\mathcal{H}}_U\,, \tag{34}$$

where $\hat{\mathcal{H}}_0$ and $\hat{\mathcal{H}}_c$ have acquired one-particle contributions from separations (30) and (31):

$$\hat{\mathcal{H}}_0 = \sum_{\mathbf{k}} \hat{\varepsilon}^e_{\mathbf{k}} e^\dagger_{\mathbf{k}} e_{\mathbf{k}} + \sum_{\mathbf{k}} \hat{\varepsilon}^h_{\mathbf{k}} h^\dagger_{-\mathbf{k}} h_{-\mathbf{k}} + \sum_{\mathbf{q}} \omega_{\mathbf{q}} \psi^\dagger_{\mathbf{q}} \psi_{\mathbf{q}}\,, \tag{35}$$

$$\hat{\mathcal{H}}_c = \sum_{\mathbf{k}} (\hat{\Delta}\, e^\dagger_{\mathbf{k}} h^\dagger_{-\mathbf{k}} + \mathrm{H.c.}) + \sqrt{N}(\hat{\Gamma}\psi^\dagger_0 + \mathrm{H.c.})\,. \tag{36}$$

Thereby, the field parameters Δ and Γ have changed into

$$\hat{\Delta} = \Delta - \frac{g}{\sqrt{N}}\langle\psi_0\rangle - \frac{U}{N}\sum_{\mathbf{k}} d_{\mathbf{k}}\,, \tag{37}$$

$$\hat{\Gamma} = \Gamma - \frac{g}{N}\sum_{\mathbf{k}} d_{\mathbf{k}}\,, \tag{38}$$

and the electronic one-particle energies contain the Hartree shifts:

$$\hat{\varepsilon}^e_{\mathbf{k}} = \varepsilon^e_{\mathbf{k}} - \frac{U}{N}\sum_{\mathbf{q}} n^h_{-\mathbf{q}}\,, \tag{39}$$

$$\hat{\varepsilon}^h_{\mathbf{k}} = \varepsilon^h_{\mathbf{k}} - \frac{U}{N}\sum_{\mathbf{q}} n^e_{\mathbf{q}}\,. \tag{40}$$

Finally, the former interactions (26) and (27) have changed into $\hat{\mathcal{H}}_g$ and $\hat{\mathcal{H}}_U$, which now consist of fluctuation operators only:

$$\hat{\mathcal{H}}_g = -\frac{g}{\sqrt{N}}\sum_{\mathbf{kq}}(:e^\dagger_{\mathbf{q}+\mathbf{k}}h^\dagger_{-\mathbf{k}}\,\psi_{\mathbf{q}}: + \text{H.c.})\,, \tag{41}$$

$$\hat{\mathcal{H}}_U = -\frac{U}{N}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}} :e^\dagger_{\mathbf{k}_1+\mathbf{k}}e_{\mathbf{k}_1}\,h^\dagger_{\mathbf{k}_2-\mathbf{k}}h_{\mathbf{k}_2}:\,. \tag{42}$$

III. PRM FOR AN OPEN ELECTRON-HOLE-PHOTON SYSTEM

Applying the projector-based renormalization approach [27, 30] to the open exciton-polariton system, one starts, as usual, from an appropriate separation of the total Hamiltonian $\mathcal{H}$ into an "unperturbed" part $\mathcal{H}_0$ and a "perturbation" $\mathcal{H}_1$. In a many-particle system, $\mathcal{H}_1$ is usually the interaction, which prevents a straightforward solution of $\mathcal{H}$ since it leads to transitions between the eigenstates of $\mathcal{H}_0$. However, integrating out the interaction by a sequence of small unitary transformations, the Hamiltonian can be transformed into a diagonal operator. Thereby, transitions from $\mathcal{H}_1$ between eigenstates of $\mathcal{H}_0$ will be stepwise eliminated. For the actual evaluation one starts from the largest transition energy of $\mathcal{H}_0$, called Λ, and proceeds in small steps $\Delta\lambda$ to lower transition energies λ. Suppose all transitions between Λ and λ have already been eliminated, the resulting Hamiltonian, which contains only transitions with energies smaller than λ, will be called $\mathcal{H}_\lambda$. An additional elimination step from $\mathcal{H}_\lambda$ to a new Hamiltonian $\mathcal{H}_{\lambda-\Delta\lambda}$ with a somewhat reduced maximum transition energy $\lambda-\Delta\lambda$ is performed by means of a small unitary transformation,

$$\mathcal{H}_{\lambda-\Delta\lambda} = e^{X_{\lambda,\Delta\lambda}}\,\mathcal{H}_\lambda\,e^{-X_{\lambda,\Delta\lambda}}\,, \tag{43}$$

by which all excitations in $\mathcal{H}_\lambda$ between λ and $\lambda-\Delta\lambda$ will be eliminated. Here, $X_{\lambda,\Delta\lambda} = -X^\dagger_{\lambda,\Delta\lambda}$ is the generator of the unitary transformation. Its lowest-order expression is given by[27]

$$X_{\lambda,\Delta\lambda} = \frac{1}{\mathbf{L}_{0,\lambda}}\mathbf{Q}_{\lambda-\Delta\lambda}\mathcal{H}_{1,\lambda}\,, \tag{44}$$

Here, the quantities $\mathbf{Q}_{\lambda-\Delta\lambda}$ and $\mathbf{L}_{0,\lambda}$ are so-called superoperators which act on usual operators of the unitary space. Thereby $\mathbf{Q}_{\lambda-\Delta\lambda} = 1 - \mathbf{P}_{\lambda-\Delta\lambda}$ is a generalized projector that projects on all transition operators (with respect to the unperturbed Hamiltonian $\mathcal{H}_0$) with energies larger than $\lambda-\Delta\lambda$, whereas $\mathbf{P}_{\lambda-\Delta\lambda}$ is the orthogonal projector, which project on all transition operators with energies smaller than $\lambda-\Delta\lambda$. Examples for the action of $\mathbf{P}_\lambda$ and $\mathbf{Q}_\lambda$ are found in the subsections III A and III B below. Moreover, $\mathbf{L}_{0,\lambda}$ is the Liouville operator, which is defined by the commutator with $\mathcal{H}_{0,\lambda}$ applied to any operator variable $\mathcal{A}$, i.e., $\mathbf{L}_{0,\lambda}\mathcal{A} = [\mathcal{H}_{0,\lambda},\mathcal{A}]$. The explicit form of the generator $X_{\lambda,\Delta\lambda}$ is given in Eqs. (65) to (69).

We note that after each elimination step the unperturbed Hamiltonian as well as the perturbation become renormalized and therefore depend on λ. Continuing the renormalization scheme stepwise up to zero transition energy $\lambda = 0$ all transitions with energies larger than zero will be integrated out: In this way one arrives at a fully renormalized Hamiltonian $\mathcal{H}_{\lambda=0}$, which is diagonal (or quasi-diagonal) and therefore solvable. We finally like to point out that for sufficiently small $\Delta\lambda$, the evaluation of the transformation step (43) can be restricted to low orders in $\mathcal{H}_1$ which, in general, limits the validity of the approach to parameter values of $\mathcal{H}_1$ of the same magnitude as those of $\mathcal{H}_0$.

A. Ansatz for the system Hamiltonian

As mentioned above, the reservoirs are considered to be very large. Thus $\mathcal{H}_{\rm R}$ and $\mathcal{H}_{\rm SR}$ will not be renormalized by the PRM procedure. We therefore may restrict the renormalization to the e-h-p system only, and employ the following λ-dependent *ansatz* for $\mathcal{H}_{{\rm S},\lambda}$,

$$\mathcal{H}_{\rm S} \to \mathcal{H}_{{\rm S},\lambda} = \mathcal{H}_{0,\lambda} + \hat{\mathcal{H}}_{c,\lambda} + \hat{\mathcal{H}}_{g,\lambda} + \hat{\mathcal{H}}_{U,\lambda}\,, \tag{45}$$

where the operator structure of (45) is found from Eq. (43) by an expansion around $\lambda = \Lambda$ for small interactions $\hat{\mathcal{H}}_g + \hat{\mathcal{H}}_U$. As above-mentioned the parameters in $\mathcal{H}_{0,\lambda}$ and $\hat{\mathcal{H}}_{c,\lambda}$ depend on λ:

$$\mathcal{H}_{0,\lambda} = \sum_{\mathbf{k}}\varepsilon^e_{\mathbf{k},\lambda}e^\dagger_{\mathbf{k}}e_{\mathbf{k}} + \sum_{\mathbf{k}}\varepsilon^h_{\mathbf{k},\lambda}h^\dagger_{-\mathbf{k}}h_{-\mathbf{k}} + \sum_{\mathbf{q}}\omega_{\mathbf{q},\lambda}\psi^\dagger_{\mathbf{q}}\psi_{\mathbf{q}}\,, \tag{46}$$

$$\hat{\mathcal{H}}_{c,\lambda} = \sum_{\mathbf{k}}(\hat{\Delta}_{\mathbf{k},\lambda}\,e^\dagger_{\mathbf{k}}h^\dagger_{-\mathbf{k}} + \text{H.c.}) + \sqrt{N}(\hat{\Gamma}_\lambda\psi^\dagger_0 + \text{H.c.})\,. \tag{47}$$

Moreover, the quantity $\hat{\Delta}_{\mathbf{k},\lambda}$ has acquired an additional $\mathbf{k}$ dependence. The interactions take the form

$$\hat{\mathcal{H}}_{g,\lambda} = -\frac{g}{\sqrt{N}}\sum_{\mathbf{kq}}\mathbf{P}_\lambda\big(:e^\dagger_{\mathbf{k}+\mathbf{q}}h^\dagger_{-\mathbf{k}}\,\psi_{\mathbf{q}}: + \text{H.c.}\big)\,, \tag{48}$$

$$\hat{\mathcal{H}}_{U,\lambda} = -\frac{U}{N}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\mathbf{P}_\lambda\big(:e^\dagger_{\mathbf{k}_1}e_{\mathbf{k}_2}\,h^\dagger_{\mathbf{k}_3}h_{\mathbf{k}_1+\mathbf{k}_3-\mathbf{k}_2}:\big)\,. \tag{49}$$

As aforementioned, $\mathbf{P}_\lambda = 1 - \mathbf{Q}_\lambda$ is a generalized projection operator, complementary to $\mathbf{Q}_\lambda$, which projects on all transition operators with energies smaller than λ. The coupling parameters g and U will remain λ-independent in the renormalization procedure if one restricts oneself to renormalization contributions up to order g^2 and U^2.

Obviously the Hamiltonian $\mathcal{H}_{\mathrm{S},\lambda=\Lambda}$ reduces to $\mathcal{H}_\mathrm{S}$ by construction, provided the parameter values at the initial cutoff $\lambda = \Lambda$ fulfill

$$\varepsilon^e_{\mathbf{k},\Lambda} = \hat{\varepsilon}^e_{\mathbf{k}}\,, \quad \varepsilon^h_{\mathbf{k},\Lambda} = \hat{\varepsilon}^h_{\mathbf{k}}\,, \quad \omega_{\mathbf{q},\Lambda} = \omega_{\mathbf{q}}\,, \tag{50}$$

$$\hat{\Delta}_{\mathbf{k},\Lambda} = \hat{\Delta}\,, \quad \hat{\Gamma}_\Lambda = \hat{\Gamma}\,. \tag{51}$$

In order to study the action of $\mathbf{P}_\lambda$ in Eqs. (48) and (49) we start from the decomposition of $\hat{\mathcal{H}}_{g,\lambda}$ into dynamical eigenmodes of $\mathcal{H}_{0,\lambda}$,

$$\begin{aligned}\hat{\mathcal{H}}_{g,\lambda} = &-\frac{g}{\sqrt{N}}\sum_{\mathbf{kq}}\Theta_{\mathbf{kq},\lambda}\big(e^\dagger_{\mathbf{k}+\mathbf{q}}h^\dagger_{-\mathbf{k}}\psi_{\mathbf{q}} + \mathrm{H.c.}\big)\\ &+\frac{g}{\sqrt{N}}\sum_{\mathbf{k}}\Theta_{\mathbf{k},\lambda}(\langle\psi_0\rangle\, e^\dagger_{\mathbf{k}}h^\dagger_{-\mathbf{k}} + \mathrm{H.c})\\ &+\frac{g}{\sqrt{N}}\Theta_\lambda\sum_{\mathbf{k}}(d^*_{\mathbf{k}}\psi_0 + \mathrm{H.c.})\,,\end{aligned} \tag{52}$$

where Eq. (41) was used. In Eq. (52), we have introduced the Θ-functions

$$\Theta_{\mathbf{kq},\lambda} = \Theta(\lambda - |\varepsilon^e_{\mathbf{k}+\mathbf{q},\lambda} + \varepsilon^h_{-\mathbf{k},\lambda} - \omega_{\mathbf{q},\lambda}|)\,, \tag{53}$$

$$\Theta_{\mathbf{k},\lambda} = \Theta(\lambda - |\varepsilon^e_{\mathbf{k},\lambda} + \varepsilon^h_{-\mathbf{k},\lambda}|)\,, \tag{54}$$

$$\Theta_\lambda = \Theta(\lambda - |\omega_{\mathbf{q}=0,\lambda}|)\,, \tag{55}$$

which restrict transitions to those with excitation energies smaller than λ. Similarly one finds for $\hat{\mathcal{H}}_{U,\lambda}$:

$$\begin{aligned}\hat{\mathcal{H}}_{U,\lambda} = &-\frac{U}{N}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\Theta_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3,\lambda} : e^\dagger_{\mathbf{k}_1}e_{\mathbf{k}_2} :: h^\dagger_{\mathbf{k}_3}h_{\mathbf{k}_1+\mathbf{k}_3-\mathbf{k}_2} :\\ &+\frac{U}{N}\sum_{\mathbf{k}}\Theta_{\mathbf{k},\lambda}\sum_{\mathbf{k}'}(d_{\mathbf{k}'}\, e^\dagger_{\mathbf{k}}h^\dagger_{-\mathbf{k}} + \mathrm{H.c.})\end{aligned} \tag{56}$$

with

$$\Theta_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3,\lambda} = \Theta(\lambda - |\varepsilon^e_{\mathbf{k}_1,\lambda} - \varepsilon^e_{\mathbf{k}_2,\lambda} + \varepsilon^h_{\mathbf{k}_3,\lambda} - \varepsilon^h_{\mathbf{k}_1+\mathbf{k}_3-\mathbf{k}_2,\lambda}|). \tag{57}$$

In principle, the operator part $\mathcal{H}_{0,\lambda}+\hat{\mathcal{H}}_{c,\lambda}$ of the *ansatz* (45) for $\mathcal{H}_\mathrm{S}$ should take over the role of the unperturbed Hamiltonian and $\hat{\mathcal{H}}_{g,\lambda} + \hat{\mathcal{H}}_{U,\lambda}$ the role of the perturbation. This however would require a diagonalization of $\mathcal{H}_{0,\lambda} + \hat{\mathcal{H}}_{c,\lambda}$ and an expansion of $\hat{\mathcal{H}}_{g,\lambda}$ and $\hat{\mathcal{H}}_{U,\lambda}$ into eigenmodes of this "unperturbed" Hamiltonian. Since this procedure is rather complex, we prefer to use instead $\mathcal{H}_{0,\lambda}$ in the Θ-functions of Eqs. (52) and (56). Then the generator $X_{\lambda,\Delta\lambda}$ of the unitary transformation (43) has to be changed appropriately (see below).

One sees that the last two terms in Eq. (52) and the last term in (56) represent one-particle contributions. They should best be included in the one-particle term $\hat{\mathcal{H}}_{c,\lambda}$ of $\mathcal{H}_{\mathrm{S},\lambda}$. That is only the first term in Eq. (52) and in Eq. (56) should be considered as "true" interactions. However, it has turned out that interactions formed by fluctuation operators should be preferred in the unitary transformation Eq. (43). Therefore, instead of Eqs. (52) and (56), we henceforth use modified interactions $\mathcal{H}_{g,\lambda}$ and $\mathcal{H}_{U,\lambda}$ based on fluctuation operators,

$$\mathcal{H}_{g,\lambda} = -\frac{g}{\sqrt{N}}\sum_{\mathbf{kq}}\Theta_{\mathbf{kq},\lambda}\big(:e^\dagger_{\mathbf{k}+\mathbf{q}}h^\dagger_{-\mathbf{k}}\psi_{\mathbf{q}}: + \mathrm{H.c.}\big)\,, \tag{58}$$

$$\mathcal{H}_{U,\lambda} = -\frac{U}{N}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\Theta_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3,\lambda} : e^\dagger_{\mathbf{k}_1}e_{\mathbf{k}_2}\, h^\dagger_{\mathbf{k}_3}h_{\mathbf{k}_1+\mathbf{k}_3-\mathbf{k}_2} :\,, \tag{59}$$

where the Θ-functions in front apply to all parts of the respective fluctuation operators. Of course, we have to repair this "mistake" by including the corresponding "counter-terms" in the one-particle part $\mathcal{H}_{c,\lambda}$ of $\mathcal{H}_{\mathrm{S},\lambda}$. Thus, we finally arrive at the following representation of $\mathcal{H}_{\mathrm{S},\lambda}$:

$$\mathcal{H}_{\mathrm{S},\lambda} = \mathcal{H}_{0,\lambda} + \mathcal{H}_{c,\lambda} + \mathcal{H}_{1,\lambda}\,, \tag{60}$$

$$\mathcal{H}_{1,\lambda} = \mathcal{H}_{g,\lambda} + \mathcal{H}_{U,\lambda}\,. \tag{61}$$

Here, $\mathcal{H}_{g,\lambda}$ and $\mathcal{H}_{U,\lambda}$ are given by Eqs. (58) and (59), whereas $\mathcal{H}_{c,\lambda}$ reads

$$\mathcal{H}_{c,\lambda} = \sum_{\mathbf{k}}(\Delta_{\mathbf{k},\lambda}\, e^\dagger_{\mathbf{k}}h^\dagger_{-\mathbf{k}} + \mathrm{H.c.}) + \sqrt{N}(\Gamma_\lambda\psi^\dagger_0 + \mathrm{H.c.})\,, \tag{62}$$

with

$$\begin{aligned}\Delta_{\mathbf{k},\lambda} = &\hat{\Delta}_{\mathbf{k},\lambda} + \frac{g}{\sqrt{N}}(\Theta_{\mathbf{k},\lambda} - \Theta_{\mathbf{k},\mathbf{q}=0,\lambda})\langle\psi_0\rangle\\ &+\frac{U}{N}\sum_{\mathbf{k}'}(\Theta_{\mathbf{k},\lambda} - \Theta_{\mathbf{k},\mathbf{k}',\mathbf{k};\lambda})\, d_{\mathbf{k}'}\,,\end{aligned} \tag{63}$$

$$\Gamma_\lambda = \hat{\Gamma}_\lambda + \frac{g}{N}\sum_{\mathbf{k}}(\Theta_\lambda - \Theta_{\mathbf{k},\mathbf{q}=0,\lambda})\, d_{\mathbf{k}}\,. \tag{64}$$

As before, at the initial cutoff $\lambda = \Lambda$ Hamiltonian $\mathcal{H}_{\mathrm{S},\lambda}$ must agree with $\mathcal{H}_\mathrm{S}$ (from Eq. (23)), which is fulfilled by ensuring Eqs. (50) and (51).

Let us add one remark: Carrying out the renormalization procedure the additional contributions in $\Delta_{\mathbf{k},\lambda}$ and Γ_λ in Eqs. (63) and (64) are expected to have very little influence on the results since they vanish both at the beginning (cutoff Λ) and at the end ($\lambda = 0$) of the PRM procedure.

B. Construction of the PRM generator

Next, we establish the generator $X_{\lambda,\Delta\lambda}$ of the unitary transformation (43). Following the lowest order expression (44), we look for an $X_{\lambda,\Delta\lambda}$ having the same operator structure as $\mathcal{H}_{1,\lambda}$. For this we make the ansatz

$$X_{\lambda,\Delta\lambda} = X^g_{\lambda,\Delta\lambda} + X^U_{\lambda,\Delta\lambda} = -X^\dagger_{\lambda,\Delta\lambda} \tag{65}$$

with

$$X^g_{\lambda,\Delta\lambda} = -\frac{g}{\sqrt{N}} \sum_{\mathbf{kq}} A_{\mathbf{kq}}(\lambda,\Delta\lambda) \big[: e^\dagger_{\mathbf{k+q}} h^\dagger_{-\mathbf{k}} \psi_{\mathbf{q}} : - \text{H.c.} \big] , \tag{66}$$

$$X^U_{\lambda,\Delta\lambda} = -\frac{U}{N} \sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} B_{\mathbf{k}_1\mathbf{k}_2;\mathbf{k}_3,\mathbf{k}_1+\mathbf{k}_3-\mathbf{k}_2}(\lambda,\Delta\lambda) \times : e^\dagger_{\mathbf{k}_1} e_{\mathbf{k}_2}\, h^\dagger_{\mathbf{k}_3} h_{\mathbf{k}_1+\mathbf{k}_3-\mathbf{k}_2} : , \tag{67}$$

and

$$A_{\mathbf{kq}}(\lambda,\Delta\lambda) = \frac{\Theta_{\mathbf{kq},\lambda}\big(1-\Theta_{\mathbf{kq},\lambda-\mathbf{\Delta}\lambda}\big)}{\varepsilon^e_{\mathbf{k+q},\lambda} + \varepsilon^h_{-\mathbf{k},\lambda} - \omega_{\mathbf{q},\lambda}} , \tag{68}$$

$$\begin{aligned} &B_{\mathbf{k}_1,\mathbf{k}_2;\mathbf{k}_3,\mathbf{k}_1+\mathbf{k}_3-\mathbf{k}_2}(\lambda,\Delta\lambda) \\ &\quad = \frac{\Theta_{\mathbf{k_1k_2k_3},\lambda}\big(1-\Theta_{\mathbf{k_1k_2k_3},\lambda-\mathbf{\Delta}\lambda}\big)}{\varepsilon^e_{\mathbf{k}_1,\lambda} - \varepsilon^e_{\mathbf{k_2},\lambda} + \varepsilon^h_{\mathbf{k_3},\lambda} - \varepsilon^h_{\mathbf{k}_1+\mathbf{k_3}-\mathbf{k_2},\lambda}} \\ &\quad = -B_{\mathbf{k_2},\mathbf{k}_1;\mathbf{k}_1+\mathbf{k_3}-\mathbf{k_2},\mathbf{k_3},}(\lambda,\Delta\lambda) . \end{aligned} \tag{69}$$

Here, the notation with four indices in $B_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_1+\mathbf{k}_3-\mathbf{k}_2}(\lambda,\Delta\lambda)$ emphasizes the momentum conservation. It can be recognized that the products of Θ-functions in Eqs. (68) and (69) assure that excitations between λ and $\lambda-\Delta\lambda$ are eliminated in each transformation step $\Delta\lambda$. For small $\Delta\lambda$, the transformation (43) can be restricted to an expansion up to second order in g and U, and to linear order in the order parameters $\Delta_{\mathbf{k},\lambda}$ and Γ_λ. Then $\mathcal{H}_{\mathrm{S},\lambda-\Delta\lambda}$ at the reduced cutoff $\lambda-\Delta\lambda$ reads

$$\begin{aligned} \mathcal{H}_{\mathrm{S},\lambda-\Delta\lambda} =& \mathcal{H}_{0,\lambda} + \mathcal{H}_{c,\lambda} + \mathcal{H}_{1,\lambda} \\ &+ [X_{\lambda,\Delta\lambda}, \mathcal{H}_{0,\lambda} + \mathcal{H}_{c,\lambda} + \mathcal{H}_{1,\lambda}] + \cdots \end{aligned} \tag{70}$$

Relation (70) connects the parameter values of $\mathcal{H}_{\mathrm{S},\lambda}$ at cutoff λ with those at the reduced cutoff $\lambda-\Delta\lambda$. That is, in order to find renormalization equations for the λ-dependent parameters one has to evaluate the commutators. For instance, from the first commutator $[X_{\lambda,\Delta\lambda}, \mathcal{H}_{0,\lambda}]$, one finds the following renormalization contributions to $\Delta_{\mathbf{k},\lambda}$ and Γ_λ:

$$\begin{aligned} \delta\Delta^{(0)}_{\mathbf{k},\lambda} =& -\frac{g}{\sqrt{N}} A_{\mathbf{k}0}(\lambda,\Delta\lambda)\, \omega_{0,\lambda} \langle\psi_0\rangle \\ &- \frac{U}{N} \sum_{\mathbf{k}_1} B_{\mathbf{k}_1\mathbf{k},-\mathbf{k}_1,-\mathbf{k}}(\lambda,\Delta\lambda)(\varepsilon^e_{\mathbf{k}_1,\lambda} + \varepsilon^h_{-\mathbf{k}_1,\lambda})\, d_{\mathbf{k}_1} , \end{aligned} \tag{71}$$

$$\delta\Gamma^{(0)}_\lambda = \frac{g}{N} \sum_{\mathbf{k}} A_{\mathbf{k}0}(\lambda,\Delta\lambda)(\varepsilon^e_{\mathbf{k},\lambda} + \varepsilon^h_{-\mathbf{k},\lambda})\, d_{\mathbf{k}} . \tag{72}$$

Combining these relations with the remaining renormalization contributions from the last two commutators in (70), one arrives at the following renormalization equations:

$$\Delta_{\mathbf{k},\lambda-\Delta\lambda} = \Delta_{\mathbf{k},\lambda} + \delta\Delta^{(0)}_{\mathbf{k},\lambda} + \delta\Delta^{(c)}_{\mathbf{k},\lambda} + \delta\Delta^{(U)}_{\mathbf{k},\lambda} , \tag{73}$$

$$\Gamma_{\lambda-\Delta\lambda} = \Gamma_\lambda + \delta\Gamma^{(0)}_\lambda + \delta\Gamma^{(c)}_\lambda + \delta\Gamma^{(g)}_\lambda . \tag{74}$$

Here, $\delta\Delta^{(c)}_{\mathbf{k},\lambda}$ and $\delta\Delta^{(U)}_{\mathbf{k},\lambda}$ are defined in Eqs. (A10) and (A22), whereas $\delta\Gamma^{(c)}_\lambda$ and $\delta\Gamma^{(g)}_\lambda$ are given in (A9) and (A16), respectively. The renormalization equations for the remaining parameters $\varepsilon^e_{\mathbf{k},\lambda}$, $\varepsilon^h_{\mathbf{k},\lambda}$, and $\omega_{\mathbf{q},\lambda}$ of $\mathcal{H}_\lambda$ are derived in Appendix A [Eqs. (A13)-(A15)] as well.

To solve the renormalization equations, one starts from the initial parameter values at cutoff Λ [Eqs. (50)-(51)] and proceeds in small steps $\Delta\lambda$ until $\lambda = 0$ is reached. In doing so, all transitions from $\mathcal{H}_{1,\lambda}$ between Λ and $\lambda = 0$ will be eliminated. We arrive at the fully renormalized Hamiltonian $\tilde{\mathcal{H}}_{\mathrm{S}} = \mathcal{H}_{\mathrm{S},\lambda=0} = \mathcal{H}_{0,\lambda=0} + \mathcal{H}_{c,\lambda=0}$:

$$\begin{aligned} \tilde{\mathcal{H}}_{\mathrm{S}} =& \sum_{\mathbf{k}} \tilde{\varepsilon}^e_{\mathbf{k}} e^\dagger_{\mathbf{k}} e_{\mathbf{k}} + \sum_{\mathbf{k}} \tilde{\varepsilon}^h_{\mathbf{k}} h^\dagger_{\mathbf{k}} h_{\mathbf{k}} + \sum_{\mathbf{q}} \tilde{\omega}_{\mathbf{q}} \psi^\dagger_{\mathbf{q}} \psi_{\mathbf{q}} \\ &+ \sum_{\mathbf{k}} (\tilde{\Delta}_{\mathbf{k}}\, e^\dagger_{\mathbf{k}} h^\dagger_{-\mathbf{k}} + \text{H.c.}) + \sqrt{N}(\tilde{\Gamma}\psi^\dagger_0 + \text{H.c.}) . \end{aligned} \tag{75}$$

Accordingly, $\tilde{\varepsilon}^e_{\mathbf{k}}$, $\tilde{\varepsilon}^h_{\mathbf{k}}$, $\tilde{\omega}_{\mathbf{q}}$, $\tilde{\Delta}_{\mathbf{k}}$, and $\tilde{\Gamma}$ are the fully renormalized energy parameters at $\lambda = 0$. They have to be determined self-consistently from the whole set of renormalization equations.

Since all transition operators from $\mathcal{H}_{1,\lambda}$ have been used up in the renormalization procedure, Hamiltonian $\tilde{\mathcal{H}}_{\mathrm{S}}$ is a one-particle operator which can be diagonalized. First, one defines "displaced" photon operators

$$\tilde{\Psi}^\dagger_{\mathbf{q}} = \psi^\dagger_{\mathbf{q}} + \frac{\sqrt{N}\tilde{\Gamma}^*}{\tilde{\omega}_{\mathbf{q}=0}} \delta_{\mathbf{q},0} , \tag{76}$$

which–up to a constant–leads to

$$\begin{aligned} \tilde{\mathcal{H}}_{\mathrm{S}} =& \sum_{\mathbf{k}} \tilde{\varepsilon}^e_{\mathbf{k}} e^\dagger_{\mathbf{k}} e_{\mathbf{k}} + \sum_{\mathbf{k}} \tilde{\varepsilon}^h_{\mathbf{k}} h^\dagger_{\mathbf{k}} h_{\mathbf{k}} + \sum_{\mathbf{q}} \tilde{\omega}_{\mathbf{q}} \tilde{\Psi}^\dagger_{\mathbf{q}} \tilde{\Psi}_{\mathbf{q}} \\ &+ \sum_{\mathbf{k}} (\tilde{\Delta}_{\mathbf{k}}\, e^\dagger_{\mathbf{k}} h^\dagger_{-\mathbf{k}} + \text{H.c.}) . \end{aligned} \tag{77}$$

The electronic part of $\tilde{\mathcal{H}}_{\mathrm{S}}$ is diagonalized by a subsequent Bogolyubov transformation

$$C^\dagger_{1\mathbf{k}} = \xi_{\mathbf{k}} e^\dagger_{\mathbf{k}} + \eta^*_{\mathbf{k}} h_{-\mathbf{k}} , \tag{78}$$

$$C^\dagger_{2\mathbf{k}} = -\eta_{\mathbf{k}} e^\dagger_{\mathbf{k}} + \xi^*_{\mathbf{k}} h_{-\mathbf{k}} \tag{79}$$

with coefficients

$$|\xi_{\mathbf{k}}|^2 = \frac{1}{2}\left[1 + \mathrm{sgn}(\tilde{\varepsilon}^e_{\mathbf{k}} + \tilde{\varepsilon}^h_{\mathbf{k}}) \frac{\tilde{\varepsilon}^e_{\mathbf{k}} + \tilde{\varepsilon}^h_{\mathbf{k}}}{W_{\mathbf{k}}}\right] , \tag{80}$$

$$|\eta_{\mathbf{k}}|^2 = \frac{1}{2}\left[1 - \mathrm{sgn}(\tilde{\varepsilon}^e_{\mathbf{k}} + \tilde{\varepsilon}^h_{\mathbf{k}}) \frac{\tilde{\varepsilon}^e_{\mathbf{k}} + \tilde{\varepsilon}^h_{\mathbf{k}}}{W_{\mathbf{k}}}\right] , \tag{81}$$

$$\eta_{\mathbf{k}}\xi_{\mathbf{k}} = \mathrm{sgn}(\tilde{\varepsilon}^e_{\mathbf{k}} + \tilde{\varepsilon}^h_{\mathbf{k}}) \frac{\tilde{\Delta}_{\mathbf{k}}}{W_{\mathbf{k}}} , \tag{82}$$

where

$$W_{\mathbf{k}} = \sqrt{(\tilde{\varepsilon}^e_{\mathbf{k}} + \tilde{\varepsilon}^h_{\mathbf{k}})^2 + 4|\tilde{\Delta}_{\mathbf{k}}|^2} . \tag{83}$$

In terms of the new quasiparticle operators $C_{1\mathbf{k}}^{(\dagger)}$ and $C_{2\mathbf{k}}^{(\dagger)}$ Hamiltonian $\tilde{\mathcal{H}}$ becomes diagonal:

$$\tilde{\mathcal{H}}_{\mathrm{S}} = \sum_{\mathbf{k}} \tilde{E}_{1\mathbf{k}} C_{1\mathbf{k}}^{\dagger} C_{1\mathbf{k}} + \sum_{\mathbf{k}} \tilde{E}_{2\mathbf{k}} C_{2\mathbf{k}}^{\dagger} C_{2\mathbf{k}} + \sum_{\mathbf{q}} \tilde{\omega}_{\mathbf{q}} \tilde{\Psi}_{\mathbf{q}}^{\dagger} \tilde{\Psi}_{\mathbf{q}} \tag{84}$$

with the quasiparticle energies

$$\tilde{E}_{(1,2)\mathbf{k}} = \frac{\tilde{\varepsilon}_{\mathbf{k}}^{e} - \tilde{\varepsilon}_{\mathbf{k}}^{h}}{2} \pm \mathrm{sgn}(\tilde{\varepsilon}_{\mathbf{k}}^{e} + \tilde{\varepsilon}_{\mathbf{k}}^{h}) \frac{W_{\mathbf{k}}}{2} \,. \tag{85}$$

As usual, the order parameter $\tilde{\Delta}_{\mathbf{k}}$ also acts as gap parameter for the quasiparticle bands.

IV. STEADY STATE

The fully transformed (renormalized) Hamiltonian $\tilde{\mathcal{H}}$ of the total system is

$$\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_{\mathrm{S}} + \mathcal{H}_{\mathrm{R}} + \mathcal{H}_{\mathrm{SR}} \,, \tag{86}$$

where $\tilde{\mathcal{H}}_{\mathrm{S}}$, $\mathcal{H}_{\mathrm{R}}$, and $\mathcal{H}_{\mathrm{SR}}$ are given by Eqs. (84), (28), and (29), respectively. As aforementioned, $\mathcal{H}_{\mathrm{R}}$ and $\mathcal{H}_{\mathrm{SR}}$ will not be affected by the unitary transformations (43). We are now going to calculate the steady-state expectation values $\langle \psi_{\mathbf{q}=0}^{\dagger} \rangle$, $\langle e_{\mathbf{k}}^{\dagger} h_{-\mathbf{k}}^{\dagger} \rangle = d_{\mathbf{k}}^{*}$, $n_{\mathbf{k}}^{e}$, and $n_{-\mathbf{k}}^{h}$.

A. Density operator for the initial state

First, the initial density operator ρ_0 must be specified. According to Eq. (12), ρ_0 is a product of the density operator ρ_{S} for the e-h-p subsystem and the density ρ_{R} for the reservoirs:

$$\rho_0 = \rho_{\mathrm{S}} \rho_{\mathrm{R}} \,. \tag{87}$$

Moreover, ρ_{R} factorizes into the density matrices $\rho_{\mathrm{R},e}$ and $\rho_{\mathrm{R},h}$ of the two electronic baths and into the density $\rho_{\mathrm{R},\psi}$ of the free-space photons,

$$\rho_{\mathrm{R}} = \rho_{\mathrm{R},e} \, \rho_{\mathrm{R},h} \, \rho_{\mathrm{R},\psi} \,, \tag{88}$$

where

$$\rho_{\mathrm{R},e} = \frac{e^{-\beta \sum_{\mathbf{p}} (\omega_{\mathbf{p}}^{e} - (\mu_e - \mu/2))\, b_{e,\mathbf{p}}^{\dagger} b_{e,\mathbf{p}}}}{Z_{\mathrm{R},e}} \,, \tag{89}$$

$$\rho_{\mathrm{R},h} = \frac{e^{-\beta \sum_{\mathbf{p}} (\omega_{\mathbf{p}}^{h} - (\mu_h - \mu/2))\, b_{h,\mathbf{p}}^{\dagger} b_{h,\mathbf{p}}}}{Z_{\mathrm{R},h}} \,. \tag{90}$$

Here, $Z_{\mathrm{R},e}$ and $Z_{\mathrm{R},h}$ are the partition functions for the electronic baths with

$$Z_{\mathrm{R},e/h} = \mathrm{Tr}_{e/h} \big(e^{-\beta \sum_{\mathbf{p}} (\omega_{\mathbf{p}}^{e/h} - (\mu_{e/h} - \mu/2))\, b_{e/h,\mathbf{p}}^{\dagger} b_{e/h,\mathbf{p}}} \big) \tag{91}$$

such that $\mathrm{Tr}_{e/h} \rho_{\mathrm{R},e/h} = 1$.

Note that both electronic bath energies $\omega_{\mathbf{p}}^{\alpha} (\alpha = e, h)$ in Eq. (20) include energy shifts $-(\mu/2)$ which however cancel in Eqs. (89) and (90). Therefore $\rho_{\mathrm{R},e}$ and $\rho_{\mathrm{R},h}$ describe thermal equilibrium situations for the electronic baths with temperature $1/\beta$ and chemical potentials μ_e and μ_h. As aforementioned both electronic baths were assumed to be huge, i.e., they always stay in thermal equilibrium, even in the steady state. In contrast, the quantity μ, introduced by Eqs. (13)–(16), will generally not act as a chemical potential since photons may "escape" from the e-h-p subsystem due to the leakage into the free-photon space. For vanishing coupling $\Gamma_{\mathbf{qp}}^{\psi}$ of cavity photons to free-space photons the e-h-p subsystem together with the electronic baths will reach a new thermal equilibrium with μ acting as the usual chemical potential again.

B. Electronic expectation values in the long-time limit

Let us consider the long-time behavior of a general expectation value,

$$\langle \mathcal{A}(t) \rangle = \mathrm{Tr}(\mathcal{A}(t)\, \rho_0) \,, \tag{92}$$

where–within the Heisenberg picture–the time dependence is governed by Hamiltonian $\mathcal{H}$ from Eq. (22). Since the total $\mathcal{H}$ does not commute with the initial density matrix ρ_0, $[\rho_0, \mathcal{H}] \neq 0$, the expectation value $\langle \mathcal{A}(t) \rangle$ is intrinsically time-dependent. The steady-state properties are found from the time-independent solutions of $\langle \mathcal{A}(t) \rangle$ for $t \to \infty$, which must obey

$$\lim_{t \to \infty} \frac{\mathrm{d}}{\mathrm{d}t} \langle \mathcal{A}(t) \rangle = 0 \,. \tag{93}$$

Remember, an explicit time-dependent factor $e^{i\mu t}$ was already extracted from Eqs. (13), (14) and (15). To evaluate $\langle \mathcal{A}(t) \rangle$, we use the invariance property of operator expressions against unitary transformations under a trace:

$$\langle \mathcal{A}(t) \rangle = \mathrm{Tr}(\mathcal{A}_{\lambda}(t)\, \rho_{0,\lambda}) = \mathrm{Tr}(\tilde{\mathcal{A}}(t)\, \tilde{\rho}_0) \,. \tag{94}$$

Here, $\mathcal{A}_{\lambda}$ and $\rho_{0,\lambda}$ are transformed operators at cutoff λ

$$\mathcal{A}_{\lambda} = e^{X_{\lambda}} \mathcal{A} e^{-X_{\lambda}} \quad \text{and} \quad \rho_{0,\lambda} = e^{X_{\lambda}} \rho_0 e^{-X_{\lambda}} \,. \tag{95}$$

The exponential function $e^{X_{\lambda}}$ stands for a compact notation of the unitary transformation operator between cutoffs Λ and λ. In the last equation of (94) the operators $\tilde{\mathcal{A}}$ and $\tilde{\rho}_0$ denote the fully renormalized operators at cutoff $\lambda = 0$, and the time dependence is now governed by $\tilde{\mathcal{H}}$. By contrast, the time dependence of $\mathcal{A}_{\lambda}(t)$ is given by the transformed Hamiltonian $\mathcal{H}_{\lambda} = \mathcal{H}_{\mathrm{S},\lambda} + \mathcal{H}_{\mathrm{R}} + \mathcal{H}_{\mathrm{SR}}$:

$$\mathcal{A}_{\lambda}(t) = e^{i\mathcal{H}_{\lambda} t} \mathcal{A}_{\lambda} e^{-i\mathcal{H}_{\lambda} t} \,. \tag{96}$$

We now derive the steady-state results for the electronic quantities

$$d_{\mathbf{k}}^{*} = d_{\mathbf{k}}^{*}(t \to \infty) \,, \tag{97}$$

$$n_{\mathbf{k}}^{e} = n_{\mathbf{k}}^{e}(t \to \infty) \,, \tag{98}$$

$$n_{-\mathbf{k}}^{h} = n_{-\mathbf{k}}^{h}(t \to \infty) \,. \tag{99}$$

Starting point are the time dependent expectation values,

$$d^*_{\mathbf{k}}(t) = \langle (e^\dagger_{\mathbf{k}} h^\dagger_{-\mathbf{k}})(t)\rangle = \langle (\tilde{e}^\dagger_{\mathbf{k}} \tilde{h}^\dagger_{-\mathbf{k}})(t)\rangle_{\tilde{\rho}_0}\,, \tag{100}$$

$$n^e_{\mathbf{k}}(t) = \langle (e^\dagger_{\mathbf{k}} e_{\mathbf{k}})(t)\rangle = \langle (\tilde{e}^\dagger_{\mathbf{k}} \tilde{e}_{\mathbf{k}})(t)\rangle_{\tilde{\rho}_0}\,, \tag{101}$$

$$n^h_{-\mathbf{k}}(t) = \langle (h^\dagger_{-\mathbf{k}} h_{-\mathbf{k}})(t)\rangle = \langle (\tilde{h}^\dagger_{-\mathbf{k}} \tilde{h}_{-\mathbf{k}})(t)\rangle_{\tilde{\rho}_0}\,, \tag{102}$$

where relation (94) was used on the right hand sides. The expectation values $\langle\cdots\rangle_{\tilde{\rho}_0}$ are formed with $\tilde{\rho}_0$, $\tilde{e}^\dagger_{\mathbf{k}}$ and $\tilde{h}^\dagger_{-\mathbf{k}}$ are the fully transformed one-particle operators, and the time dependence of the last expressions is formed with $\tilde{\mathcal{H}}$. According to Appendix C an appropriate *ansatz* for $\tilde{e}^\dagger_{\mathbf{k}}$ and $\tilde{h}^\dagger_{-\mathbf{k}}$ is

$$\begin{aligned}\tilde{e}^\dagger_{\mathbf{k}} &= \tilde{x}_{\mathbf{k}} e^\dagger_{\mathbf{k}} + \frac{1}{\sqrt{N}}\sum_{\mathbf{q}} \tilde{t}_{\mathbf{k}-\mathbf{q},\mathbf{q}} h_{\mathbf{q}-\mathbf{k}} :\psi^\dagger_{\mathbf{q}}: \\ &\quad + \frac{1}{N}\sum_{\mathbf{k}_1\mathbf{k}_2} \tilde{\alpha}_{\mathbf{k}_1\mathbf{k}\mathbf{k}_2} e^\dagger_{\mathbf{k}_1} :h^\dagger_{\mathbf{k}_2} h_{\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}}:\,,\end{aligned} \tag{103}$$

$$\begin{aligned}\tilde{h}^\dagger_{-\mathbf{k}} &= \tilde{y}_{\mathbf{k}} h^\dagger_{-\mathbf{k}} + \frac{1}{\sqrt{N}}\sum_{\mathbf{q}} \tilde{u}_{\mathbf{k},\mathbf{q}} e_{\mathbf{q}+\mathbf{k}} :\psi^\dagger_{\mathbf{q}}: \\ &\quad + \frac{1}{N}\sum_{\mathbf{k}_1\mathbf{k}_2} \tilde{\beta}_{\mathbf{k}_1\mathbf{k}_2,\mathbf{k}_2-\mathbf{k}_1-\mathbf{k}} :e^\dagger_{\mathbf{k}_1} e_{\mathbf{k}_2}: h^\dagger_{\mathbf{k}_2-\mathbf{k}_1-\mathbf{k}}\,.\end{aligned} \tag{104}$$

Here, the operator structure is caused by the electron-photon and the electron-electron interaction. Again, the parameters with tilde symbols are the fully renormalized quantities which result from the solution of the corresponding renormalization equations given in Appendix C. Inserting Eqs. (103) and (104) into Eqs. (100)-(102) one finds

$$\begin{aligned} d^*_{\mathbf{k}}(t) &= \tilde{x}_{\mathbf{k}}\tilde{y}_{\mathbf{k}}\,\hat{d}^*_{\mathbf{k}}(t) \\ &\quad + \frac{1}{N}\sum_{\mathbf{k}_1}\big[\tilde{x}_{\mathbf{k}}\tilde{\beta}_{\mathbf{k}_1,\mathbf{k},-\mathbf{k}_1}\hat{n}^e_{\mathbf{k}}(t) \\ &\quad + \tilde{y}_{\mathbf{k}}\tilde{\alpha}_{\mathbf{k}_1,\mathbf{k},-\mathbf{k}_1}\times(1-\hat{n}^h_{-\mathbf{k}}(t))\big]\,\hat{d}^*_{\mathbf{k}_1}(t) \\ &\quad - \frac{1}{N^2}\sum_{\mathbf{k}_1\mathbf{k}_2}\tilde{\alpha}_{\mathbf{k}_1\mathbf{k},-\mathbf{k}_2}\tilde{\beta}_{\mathbf{k}_2\mathbf{k}_1,\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}}\,\hat{n}^e_{\mathbf{k}_1}(t) \\ &\quad \times(1-\hat{n}^h_{\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}}(t))\,\hat{d}^*_{\mathbf{k}_2}(t)\,, \end{aligned} \tag{105}$$

$$\begin{aligned} n^e_{\mathbf{k}}(t) &= |\tilde{x}_{\mathbf{k}}|^2\hat{n}^e_{\mathbf{k}}(t) \\ &\quad + \frac{1}{N}\sum_{\mathbf{q}}|\tilde{t}_{\mathbf{k}-\mathbf{q},\mathbf{q}}|^2\,(1-\hat{n}^h_{\mathbf{k}-\mathbf{q}}(t))\,\hat{n}^\psi_{\mathbf{q}}(t) \\ &\quad + \frac{1}{N^2}\sum_{\mathbf{k}_1\mathbf{k}_2}|\tilde{\alpha}_{\mathbf{k}_1\mathbf{k}\mathbf{k}_2}|^2\hat{n}^e_{\mathbf{k}_1}(t)\hat{n}^h_{\mathbf{k}_2}(t)(1-\hat{n}^h_{\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}}(t))\,, \end{aligned} \tag{106}$$

and

$$\begin{aligned} n^h_{-\mathbf{k}}(t) &= |\tilde{y}_{\mathbf{k}}|^2\hat{n}^h_{-\mathbf{k}}(t) \\ &\quad + \frac{1}{N}\sum_{\mathbf{q}}|\tilde{u}_{\mathbf{k},\mathbf{q}}|^2\,\hat{n}^\psi_{\mathbf{q}}(t)(1-\hat{n}^e_{\mathbf{k}+\mathbf{q}}(t)) \\ &\quad + \frac{1}{N^2}\sum_{\mathbf{k}_1\mathbf{k}_2}|\tilde{\beta}_{\mathbf{k}_1\mathbf{k}_2,\mathbf{k}_2-\mathbf{k}_1-\mathbf{k}}|^2\hat{n}^e_{\mathbf{k}_1}(t)(1-\hat{n}^e_{\mathbf{k}_2}(t)) \\ &\quad \times\hat{n}^h_{\mathbf{k}_2-\mathbf{k}_1-\mathbf{k}}(t)\,, \end{aligned} \tag{107}$$

where an additional factorization approximation was used. Here $\hat{n}^e_{\mathbf{k}}(t)$, $\hat{n}^h_{-\mathbf{k}}(t)$, and $\hat{n}^\psi_{\mathbf{q}}(t)$ are time-dependent occupation numbers for electrons, holes, and photons, which are formed with $\tilde{\rho}_0$:

$$\hat{n}^e_{\mathbf{k}}(t) = \langle (e^\dagger_{\mathbf{k}} e_{\mathbf{k}})(t)\rangle_{\tilde{\rho}_0}\,, \tag{108}$$

$$\hat{n}^h_{-\mathbf{k}}(t) = \langle (h^\dagger_{-\mathbf{k}} h_{-\mathbf{k}})(t)\rangle_{\tilde{\rho}_0}\,, \tag{109}$$

$$\hat{n}^\psi_{\mathbf{q}}(t) = \langle (:\psi^\dagger_{\mathbf{q}}::\psi_{\mathbf{q}}:)(t)\rangle_{\tilde{\rho}_0}\,. \tag{110}$$

The quantity $\hat{n}^\psi_{\mathbf{q}}$ will be evaluated in Appendix C. Moreover, $\hat{d}^*_{\mathbf{k}}(t)$ accounts for the order parameter of exciton formation

$$\hat{d}^*_{\mathbf{k}}(t) = \langle (e^\dagger_{\mathbf{k}} h^\dagger_{-\mathbf{k}})(t)\rangle_{\tilde{\rho}_0}\,. \tag{111}$$

The time dependence in Eqs. (108)–(111) is determined by $\tilde{\mathcal{H}}$. Let us clarify the factorization approximations used in Eqs. (105)–(107) in more detail. As an example, we consider expression (106) for $n^e_{\mathbf{k}}(t)$. Starting point is Eq. (101). Inserting expression (103) for $\tilde{e}^\dagger_{\mathbf{k}}$ we find

$$\begin{aligned} &n^e_{\mathbf{k}}(t) = |\tilde{x}_{\mathbf{k}}|^2\langle (e^\dagger_{\mathbf{k}} e_{\mathbf{k}})(t)\rangle_{\tilde{\rho}_0} \\ &+\frac{1}{N}\sum_{\mathbf{q}\mathbf{q}'}\tilde{t}_{\mathbf{k}-\mathbf{q},\mathbf{q}}\tilde{t}^*_{\mathbf{k}-\mathbf{q}',\mathbf{q}'}\langle\big(h_{\mathbf{q}-\mathbf{k}}:\psi^\dagger_{\mathbf{q}}::\psi_{\mathbf{q}'}:h^\dagger_{\mathbf{q}'-\mathbf{k}}\big)(t)\rangle_{\tilde{\rho}_0} \\ &+\frac{1}{N^2}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}'_1\mathbf{k}'_2}\tilde{\alpha}^*_{\mathbf{k}_1\mathbf{k}\mathbf{k}_2}\tilde{\alpha}_{\mathbf{k}'_1\mathbf{k}\mathbf{k}'_2} \\ &\quad\times\langle\big(e^\dagger_{\mathbf{k}_1}:h^\dagger_{\mathbf{k}_2}h_{\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}}::h^\dagger_{\mathbf{k}'_1+\mathbf{k}'_2-\mathbf{k}}h_{\mathbf{k}'_2}e_{\mathbf{k}'_1}\big)(t)\rangle_{\tilde{\rho}_0}\,. \end{aligned} \tag{112}$$

Obviously, result (106) is obtained by factorizing corresponding operators in the expectation values of (112). For instance, in the second term the operator $h_{\mathbf{q}-\mathbf{k}}$ is factorized with $h^\dagger_{\mathbf{q}'-\mathbf{k}}$ and $:\psi_{\mathbf{q}'}:$ with $:\psi_{\mathbf{q}'}:$. This leads to the second term in expression (106). Note however that in Eq. (106) the following small contribution to second order in the order parameter $d_{\mathbf{k}}$ was neglected

$$\begin{aligned}(-1)\frac{1}{N^2}\sum_{\mathbf{k}_1\mathbf{k}_2}&\tilde{\alpha}^*_{\mathbf{k}_1\mathbf{k}\mathbf{k}_2}\tilde{\alpha}_{\mathbf{k}-(\mathbf{k}_1+\mathbf{k}_2),\mathbf{k}\mathbf{k}_2} \\ &\times\hat{n}^h_{\mathbf{k}_2}(t)\hat{d}_{\mathbf{k}_1}(t)\hat{d}_{\mathbf{k}-(\mathbf{k}_1+\mathbf{k}_2)}(t)\,.\end{aligned} \tag{113}$$

It results from an additional factorization of the last term in Eq. (112), where $e^\dagger_{\mathbf{k}_1}$ was factorized with $h^\dagger_{\mathbf{k}'_1+\mathbf{k}'_2-\mathbf{k}}$ and $h_{\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}}$ with $e_{\mathbf{k}'_1}$.

In principle the factorization (105)-(107) implies two approximations: (i) According to Sec. II B the initial density ρ_0 is a product of the density $\rho_{\rm S}$ for the e-h-p subsystem and of $\rho_{\rm R}$ for the reservoirs. Thereby the capacity of the reservoirs was assumed to be infinitely large so that only the density $\rho_{\rm S}$ of the e-h-p system is changed under the influence of the unitary transformations. Therefore, the renormalized density $\tilde{\rho}_{\rm S}$ should differ from the initial $\rho_{\rm S}$. However, these errors are of higher order in the interaction parameters g and U and should in principle be negligible. Moreover, which is more important, it turns

out that the final results (138)–(140) in the steady state for $\hat{d}^*_{\mathbf{k}}$, $\hat{n}^e_{\mathbf{k}}$, and $\hat{n}^h_{\mathbf{k}}$ are independent of the initial density ρ_0, as expected. Therefore the renormalization of $\tilde{\rho}_{\rm S}$ seems not be important. (ii) The time dependence of $d^*_{\mathbf{k}}(t)$, $n^e_{\mathbf{k}}(t)$, and $n^h_{-\mathbf{k}}(t)$ is governed not only by the e-h-p Hamiltonian $\tilde{\mathcal{H}}_{\rm S}$ but also by the coupling $\mathcal{H}_{\rm SR}$ to the electronic and photonic reservoirs. Therefore the correct time dependence might be influenced by the factorization in Eqs. (105)–(107).

In the next step, following the steady-state condition (93), equations of motion for $\hat{d}^*_{\mathbf{k}}(t)$, $\hat{n}^e_{\mathbf{k}}(t)$, and $\hat{n}^h_{-\mathbf{k}}(t)$ have to be derived. This is best done by expressing the operators in Eqs. (108)–(111) by the Bogolyubov quasiparticles $C^{(\dagger)}_{1,\mathbf{k}}$ and $C^{(\dagger)}_{2,\mathbf{k}}$ from Eqs. (78) and (79). According to Appendix C one first finds

$$\begin{aligned}\hat{d}^*_{\mathbf{k}}(t) &= \xi^*_{\mathbf{k}}\eta^*_{\mathbf{k}}(A^{11}_{\mathbf{k}}(t) - A^{22}_{\mathbf{k}}(t)) \\ &\quad +{\xi^*_{\mathbf{k}}}^2 A^{12}_{\mathbf{k}}(t) - {\eta^*_{\mathbf{k}}}^2 A^{21}_{\mathbf{k}}(t)\,, \end{aligned} \tag{114}$$

$$\begin{aligned}\hat{n}^e_{\mathbf{k}}(t) &= |\xi_{\mathbf{k}}|^2 A^{11}_{\mathbf{k}}(t) + |\eta_{\mathbf{k}}|^2 A^{22}_{\mathbf{k}}(t) \\ &\quad -(\xi^*_{\mathbf{k}}\eta_{\mathbf{k}} A^{12}_{\mathbf{k}}(t) + \xi_{\mathbf{k}}\eta^*_{\mathbf{k}} A^{21}_{\mathbf{k}}(t))\,, \end{aligned} \tag{115}$$

$$\begin{aligned}\hat{n}^h_{-\mathbf{k}}(t) &= |\eta_{\mathbf{k}}|^2(1 - A^{11}_{\mathbf{k}}(t)) + |\xi_{\mathbf{k}}|^2(1 - A^{22}_{\mathbf{k}}(t)) \\ &\quad - (\xi^*_{\mathbf{k}}\eta_{\mathbf{k}} A^{12}_{\mathbf{k}}(t) + \xi_{\mathbf{k}}\eta^*_{\mathbf{k}} A^{21}_{\mathbf{k}}(t)) \end{aligned} \tag{116}$$

with $(n, m = 1, 2)$

$$A^{nm}_{\mathbf{k}}(t) = \langle (C^\dagger_{n\mathbf{k}} C_{m\mathbf{k}})(t)\rangle_{\tilde{\rho}_0}\,. \tag{117}$$

The equations of motion for $A^{nm}_{\mathbf{k}}(t)$ are found by applying the Mori-Zwanzig projection operator formalism [31, 32]. According to Appendix C they read

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}} A^{12}_{\mathbf{k}}(t) &= -\big[2\gamma - i(\tilde{E}_{1\mathbf{k}} - \tilde{E}_{2\mathbf{k}})\big]\, A^{12}_{\mathbf{k}}(t) \\ &\quad -\gamma\,\xi_{\mathbf{k}}\eta^*_{\mathbf{k}}\big(f_e(\tilde{E}_{1\mathbf{k}}) + f_e(\tilde{E}_{2\mathbf{k}})\big) \\ &\quad -\gamma\,\xi_{\mathbf{k}}\eta^*_{\mathbf{k}}\big(f_h(-\tilde{E}_{1\mathbf{k}}) + f_h(-\tilde{E}_{2\mathbf{k}}) - 2\big) \\ &= \Big(\frac{\mathrm{d}}{\mathrm{dt}} A^{21}_{\mathbf{k}}(t)\Big)^\dagger, \end{aligned} \tag{118}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}} A^{11}_{\mathbf{k}}(t) &= -2\gamma\, A^{11}_{\mathbf{k}}(t) + 2\gamma\,|\xi_{\mathbf{k}}|^2 f_e(\tilde{E}_{1\mathbf{k}}) \\ &\quad +2\gamma\,|\eta_{\mathbf{k}}|^2\big(1 - f_h(-\tilde{E}_{1\mathbf{k}})\big)\,, \end{aligned} \tag{119}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}} A^{22}_{\mathbf{k}}(t) &= -2\gamma\, A^{22}_{\mathbf{k}}(t) + 2\gamma\,|\eta_{\mathbf{k}}|^2 f_e(\tilde{E}_{2\mathbf{k}}) \\ &\quad +2\gamma\,|\xi_{\mathbf{k}}|^2\big(1 - f_h(-\tilde{E}_{2\mathbf{k}})\big)\,. \end{aligned} \tag{120}$$

The damping rate γ, appearing in Eqs. (118)–(120), results from the coupling to the electronic reservoirs and is assumed to be the same for electrons and holes [see App. C.1, (C15)]. The functions $f_e(\omega)$ and $f_h(\omega)$ give the occupation numbers of bath electrons and bath holes in thermal equilibrium:

$$f_e(\omega^e_{\mathbf{p}}) = \langle b^\dagger_{e\mathbf{p}} b_{e\mathbf{p}}\rangle_{\rho_{\rm R}} = \frac{1}{1 + e^{\beta[\omega^e_{\mathbf{p}} - (\mu_e - \mu/2)]}}\,, \tag{121}$$

$$f_h(\omega^h_{-\mathbf{p}}) = \langle b^\dagger_{h,-\mathbf{p}} b_{h,-\mathbf{p}}\rangle_{\rho_{\rm R}} = \frac{1}{1 + e^{\beta[\omega^h_{-\mathbf{p}} - (\mu_h - \mu/2)]}}\,, \tag{122}$$

[compare Eqs. (89) and (90)]. The first contribution in each of the Eqs. (118)–(120) is a relaxation term for e-h-p quasiparticle pairs, while the last two terms stand for the relaxation of quasiparticle pairs into the electronic baths. The damping rate γ for all contributions is caused by that part of the interaction $\mathcal{H}_{\rm SR}$ which couples electrons and holes of the e-h-p system with the respective electronic baths. As above mentioned, we have adapted the usual assumption that the rates for electrons and holes are equal [compare Eq. (C12)]. Furthermore, the second term in Eq. (118), being proportional to $i(\tilde{E}_{1\mathbf{k}} - \tilde{E}_{2\mathbf{k}})$, is a frequency term and enters from the dynamics of $\tilde{\mathcal{H}}_{\rm S}$.

We are now in the position to study the steady-state expressions for $A^{nm}_{\mathbf{k}}(t)$. Defining the steady-state values in analogy to Eqs. (97)–(99),

$$A^{nm}_{\mathbf{k}} = A^{nm}_{\mathbf{k}}(t \to \infty)\,, \tag{123}$$

we arrive, for $t \to \infty$, with condition (93) at

$$\begin{aligned}&\big[i(\tilde{E}_{1\mathbf{k}} - \tilde{E}_{2\mathbf{k}}) - 2\gamma\big]\, A^{12}_{\mathbf{k}} = \gamma\,\xi_{\mathbf{k}}\eta^*_{\mathbf{k}} \\ &\times\big[f_e(\tilde{E}_{1\mathbf{k}}) + f_e(\tilde{E}_{2\mathbf{k}}) + f_h(-\tilde{E}_{1\mathbf{k}}) + f_h(-\tilde{E}_{2\mathbf{k}}) - 2\big]\,, \end{aligned} \tag{124}$$

$A^{21}_{\mathbf{k}} = (A^{12}_{\mathbf{k}})^*$, and (for $\gamma \neq 0$)

$$A^{11}_{\mathbf{k}} = |\xi_{\mathbf{k}}|^2 f_e(\tilde{E}_{1\mathbf{k}}) + |\eta_{\mathbf{k}}|^2\big[1 - f_h(-\tilde{E}_{1\mathbf{k}})\big]\,, \tag{125}$$

$$A^{22}_{\mathbf{k}} = |\eta_{\mathbf{k}}|^2\, f_e(\tilde{E}_{2\mathbf{k}}) + |\xi_{\mathbf{k}}|^2\big[1 - f_h(-\tilde{E}_{2\mathbf{k}})\big]\,, \tag{126}$$

where the common prefactor γ on both sides of Eqs. (125) and (126) has dropped. Therefore, both equations are only valid for finite γ. If $\gamma = 0$, no term would drive the system into a steady state.

To sum up, the steady-state quantities $d^*_{\mathbf{k}}$, $n^e_{\mathbf{k}}$, and $n^h_{-\mathbf{k}}$ can be first expressed by means of Eqs. (105)–(107):

$$\begin{aligned}d^*_{\mathbf{k}} &= \tilde{x}_{\mathbf{k}}\tilde{y}_{\mathbf{k}}\,\hat{d}^*_{\mathbf{k}} + \frac{1}{N}\sum_{\mathbf{k}_1}\Big[\tilde{x}_{\mathbf{k}}\tilde{\beta}_{\mathbf{k}_1,\mathbf{k},-\mathbf{k}_1}\hat{n}^e_{\mathbf{k}} \\ &\quad +\tilde{y}_{\mathbf{k}}\tilde{\alpha}_{\mathbf{k}_1,\mathbf{k},-\mathbf{k}_1}\times(1 - \hat{n}^h_{-\mathbf{k}_1})\Big]\,\hat{d}^*_{\mathbf{k}_1} \\ &\quad -\frac{1}{N^2}\sum_{\mathbf{k}_1\mathbf{k}_2}\tilde{\alpha}_{\mathbf{k}_1\mathbf{k},-\mathbf{k}_2}\tilde{\beta}_{\mathbf{k}_2\mathbf{k}_1,\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}}\,\hat{n}^e_{\mathbf{k}_1} \\ &\quad \times(1 - \hat{n}^h_{\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}}(t))\,\hat{d}^*_{\mathbf{k}_2}\,, \end{aligned} \tag{127}$$

$$\begin{aligned}n^e_{\mathbf{k}} &= |\tilde{x}_{\mathbf{k}}|^2\hat{n}^e_{\mathbf{k}} \\ &\quad +\frac{1}{N}\sum_{\mathbf{q}}|\tilde{t}_{\mathbf{k}-\mathbf{q},\mathbf{q}}|^2\,(1 - \hat{n}^h_{\mathbf{k}-\mathbf{q}})\,\hat{n}^\psi_{\mathbf{q}} \\ &\quad +\frac{1}{N^2}\sum_{\mathbf{k}_1\mathbf{k}_2}|\tilde{\alpha}_{\mathbf{k}_1\mathbf{k}\mathbf{k}_2}|^2\hat{n}^e_{\mathbf{k}_1}\hat{n}^h_{\mathbf{k}_2}(1 - \hat{n}^h_{\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}})\,, \end{aligned} \tag{128}$$

$$n^h_{-\mathbf{k}} = |\tilde{y}_{\mathbf{k}}|^2 \hat{n}^h_{-\mathbf{k}} + \frac{1}{N}\sum_{\mathbf{q}} |\tilde{u}_{\mathbf{k},\mathbf{q}}|^2\, \hat{n}^{\psi}_{\mathbf{q}}(1-\hat{n}^e_{\mathbf{k}+\mathbf{q}}) + \frac{1}{N^2}\sum_{\mathbf{k}_1\mathbf{k}_2} |\tilde{\beta}_{\mathbf{k}_1\mathbf{k}_2,\mathbf{k}_2-\mathbf{k}_1-\mathbf{k}}|^2 \hat{n}^e_{\mathbf{k}_1}(1-\hat{n}^e_{\mathbf{k}_2}) \times \hat{n}^h_{\mathbf{k}_2-\mathbf{k}_1-\mathbf{k}}\,. \tag{129}$$

Thereby, the quantities with hat symbols $\hat{d}^*_{\mathbf{k}}, \hat{n}^e_{\mathbf{k}}$, and $\hat{n}^h_{-\mathbf{k}}$, are written in terms of $A^{nm}_{\mathbf{k}}$:

$$\hat{d}^*_{\mathbf{k}} = \xi^*_{\mathbf{k}}\eta^*_{\mathbf{k}}(A^{11}_{\mathbf{k}} - A^{22}_{\mathbf{k}}) + {\xi^*_{\mathbf{k}}}^2 A^{12}_{\mathbf{k}} - {\eta^*_{\mathbf{k}}}^2 A^{21}_{\mathbf{k}}\,, \tag{130}$$

$$\hat{n}^e_{\mathbf{k}} = |\xi_{\mathbf{k}}|^2 A^{11}_{\mathbf{k}} + |\eta_{\mathbf{k}}|^2 A^{22}_{\mathbf{k}} - (\xi^*_{\mathbf{k}}\eta_{\mathbf{k}} A^{12}_{\mathbf{k}} + \xi_{\mathbf{k}}\eta^*_{\mathbf{k}} A^{21}_{\mathbf{k}})\,, \tag{131}$$

$$\hat{n}^h_{-\mathbf{k}} = |\eta_{\mathbf{k}}|^2(1-A^{11}_{\mathbf{k}}) + |\xi_{\mathbf{k}}|^2(1-A^{22}_{\mathbf{k}}) - (\xi^*_{\mathbf{k}}\eta_{\mathbf{k}} A^{12}_{\mathbf{k}} + \xi_{\mathbf{k}}\eta^*_{\mathbf{k}} A^{21}_{\mathbf{k}})\,, \tag{132}$$

where the steady-state results for $A^{nm}_{\mathbf{k}}$ are given by Eqs. (124)–(126).

C. Reformulation of the system dynamics

It makes sense to express the equations of motion (118)–(120) in terms of the variables with hat symbols $\hat{d}^*_{\mathbf{k}}(t)$, $\hat{n}^e_{\mathbf{k}}(t)$, and $\hat{n}^h_{-\mathbf{k}}(t)$. Let us start from $\hat{d}^*_{\mathbf{k}}(t)$. Using Eqs. (114)–(116) we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{d}^*_{\mathbf{k}} = i(\tilde{E}_{1\mathbf{k}} - \tilde{E}_{2\mathbf{k}})({\xi^*_{\mathbf{k}}}^2 A^{12}_{\mathbf{k}} + {\eta^*_{\mathbf{k}}}^2 A^{21}_{\mathbf{k}}) - 2\gamma\left[\xi^*_{\mathbf{k}}\eta^*_{\mathbf{k}}(A^{11}_{\mathbf{k}} - A^{22}_{\mathbf{k}}) + {\xi^*_{\mathbf{k}}}^2 A^{12}_{\mathbf{k}} - {\eta^*_{\mathbf{k}}}^2 A^{21}_{\mathbf{k}}\right] + 2\gamma\,\hat{d}^{0*}_{\mathbf{k}}\,, \tag{133}$$

where on the right hand side we have defined

$$\hat{d}^{0*}_{\mathbf{k}} = \frac{1}{2}\xi^*_{\mathbf{k}}\eta^*_{\mathbf{k}}\Big\{f_e(\tilde{E}_{1\mathbf{k}}) - f_h(-\tilde{E}_{1\mathbf{k}}) - [f_e(\tilde{E}_{2\mathbf{k}}) - f_h(-\tilde{E}_{2\mathbf{k}})]\Big\}\,. \tag{134}$$

Moreover, using the Bogolyubov transformation (78)–(82), as well as Eqs. (108), (109), and (111), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{d}^*_{\mathbf{k}}(t) = i(\tilde{\varepsilon}^e_{\mathbf{k}} + \tilde{\varepsilon}^h_{\mathbf{k}})\,\hat{d}^*_{\mathbf{k}}(t) + i\tilde{\Delta}^*_{\mathbf{k}}(1-\hat{n}^e_{\mathbf{k}}(t) - \hat{n}^h_{\mathbf{k}}(t)) - 2\gamma\left(\hat{d}^*_{\mathbf{k}}(t) - \hat{d}^{0*}_{\mathbf{k}}\right). \tag{135}$$

Similarly we derive the equations of motions for $\hat{n}^e_{\mathbf{k}}(t)$ and $\hat{n}^h_{-\mathbf{k}}(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{n}^e_{\mathbf{k}}(t) = 2\Im[\tilde{\Delta}_{\mathbf{k}}\hat{d}^*_{\mathbf{k}}(t)] - 2\gamma\big[\hat{n}^e_{\mathbf{k}}(t) - |\xi_{\mathbf{k}}|^2 f_e(\tilde{E}_{1\mathbf{k}}) - |\eta_{\mathbf{k}}|^2 f_e(\tilde{E}_{2\mathbf{k}})\big]\,, \tag{136}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{n}^h_{-\mathbf{k}}(t) = 2\Im[\tilde{\Delta}_{\mathbf{k}}\hat{d}^*_{\mathbf{k}}(t)] - 2\gamma\big[\hat{n}^h_{-\mathbf{k}}(t) - |\eta_{\mathbf{k}}|^2 f_h(-\tilde{E}_{1\mathbf{k}}) - |\xi_{\mathbf{k}}|^2 f_h(-\tilde{E}_{2\mathbf{k}})\big]\,, \tag{137}$$

where $2\Im[\tilde{\Delta}_{\mathbf{k}}\hat{d}^*_{\mathbf{k}}] = -i(\tilde{\Delta}_{\mathbf{k}}\hat{d}^*_{\mathbf{k}} - \tilde{\Delta}^*_{\mathbf{k}}\hat{d}_{\mathbf{k}})$ was used. The steady-state expectation values of $\hat{d}^*_{\mathbf{k}}$, $\hat{n}^e_{\mathbf{k}}$, and $\hat{n}^h_{-\mathbf{k}}$ are obtained from Eqs. (135)–(137) by setting the left hand sides equal to zero

$$\hat{d}^*_{\mathbf{k}} = -\frac{1}{(\tilde{\varepsilon}^e_{\mathbf{k}} + \tilde{\varepsilon}^h_{\mathbf{k}}) + 2i\gamma}\big[\tilde{\Delta}^*_{\mathbf{k}}(1-\hat{n}^e_{\mathbf{k}} - \hat{n}^k_{\mathbf{k}}) - 2i\gamma\,\hat{d}^{0*}_{\mathbf{k}}\big]\,, \tag{138}$$

and

$$\hat{n}^e_{\mathbf{k}} = |\xi_{\mathbf{k}}|^2 f_e(\tilde{E}_{1\mathbf{k}}) + |\eta_{\mathbf{k}}|^2 f_e(\tilde{E}_{2\mathbf{k}}) + \frac{1}{\gamma}\Im[\tilde{\Delta}_{\mathbf{k}}\hat{d}^*_{\mathbf{k}}]\,, \tag{139}$$

$$\hat{n}^h_{\mathbf{k}} = |\eta_{\mathbf{k}}|^2 f_h(-\tilde{E}_{1\mathbf{k}}) + |\xi_{\mathbf{k}}|^2 f_h(-\tilde{E}_{2\mathbf{k}}) + \frac{1}{\gamma}\Im[\tilde{\Delta}_{\mathbf{k}}\hat{d}^*_{\mathbf{k}}]\,. \tag{140}$$

Of course, this result is equivalent to the former equations (130)–(132). The steady-state expressions (138)–(140) can further be simplified by using definition (134) and Eqs. (80)–(83). According to Appendix C.2 one finds:

$$\hat{d}^*_{\mathbf{k}} = \frac{\tilde{\Delta}^*_{\mathbf{k}}}{(\tilde{\varepsilon}^e_{\mathbf{k}} + \tilde{\varepsilon}^h_{\mathbf{k}}) + 2i\,\gamma}\Big[(\hat{n}^e_{\mathbf{k}} + \hat{n}^h_{\mathbf{k}} - 1) + i\gamma\,\mathrm{sgn}(\tilde{\varepsilon}^e_{\mathbf{k}} + \tilde{\varepsilon}^h_{\mathbf{k}})\frac{F^+_{1\mathbf{k}}}{W_{\mathbf{k}}}\Big] \tag{141}$$

and

$$\hat{n}^e_{\mathbf{k}} + \hat{n}^h_{\mathbf{k}} - 1 = \frac{|\tilde{\varepsilon}^e_{\mathbf{k}} + \tilde{\varepsilon}^h_{\mathbf{k}}|}{2W_{\mathbf{k}}}F^+_{1\mathbf{k}} + \frac{1}{2}\frac{F^+_{2\mathbf{k}} - 2}{1 + \dfrac{4|\tilde{\Delta}_{\mathbf{k}}|^2}{(\tilde{\varepsilon}^e_{\mathbf{k}} + \tilde{\varepsilon}^h_{\mathbf{k}})^2 + (2\gamma)^2}}\,, \tag{142}$$

$$\hat{n}^e_{\mathbf{k}} - \hat{n}^h_{\mathbf{k}} = \frac{1}{2}F^-_{1\mathbf{k}} + \frac{|\tilde{\varepsilon}^e_{\mathbf{k}} + \tilde{\varepsilon}^h_{\mathbf{k}}|}{2W_{\mathbf{k}}}F^-_{2\mathbf{k}}\,. \tag{143}$$

The quantities $F^{\pm}_{1\mathbf{k}}$ and $F^{\pm}_{2\mathbf{k}}$ are defined by

$$F^{\pm}_{1\mathbf{k}} = f_e(\tilde{E}_{1\mathbf{k}}) - f_h(-\tilde{E}_{1\mathbf{k}}) \mp [f_e(\tilde{E}_{2\mathbf{k}}) - f_h(-\tilde{E}_{2\mathbf{k}})]\,, \tag{144}$$

$$F^{\pm}_{2\mathbf{k}} = f_e(\tilde{E}_{1\mathbf{k}}) + f_h(-\tilde{E}_{1\mathbf{k}}) \pm [f_e(\tilde{E}_{2\mathbf{k}}) + f_h(-\tilde{E}_{2\mathbf{k}})]\,. \tag{145}$$

Let us look again at the symmetric case $\tilde{\varepsilon}^e_{\mathbf{k}} = \tilde{\varepsilon}^h_{\mathbf{k}}$ with charge neutrality $\mu_e = \mu_h$ (compare Sec. IV B). Here, the quasiparticle energies $\tilde{E}_{1,2\mathbf{k}}$ reduce to

$$\tilde{E}_{1\mathbf{k}} = -\tilde{E}_{2\mathbf{k}} = \mathrm{sgn}(\tilde{\varepsilon}^e_{\mathbf{k}})\frac{W_{\mathbf{k}}}{2}\,, \tag{146}$$

and $F^{\pm}_{1\mathbf{k}}$ and $F^{\pm}_{2\mathbf{k}}$ to

$$F^+_{(1,2)\mathbf{k}} = 2\big[f(\tilde{E}_{1\mathbf{k}}) \mp f(\tilde{E}_{2\mathbf{k}})\big] \tag{147}$$

$$F^-_{(1,2)\mathbf{k}} = 0\,. \tag{148}$$

with

$$f(E) = \frac{1}{1+e^{\beta[E-(\mu_B-\mu)/2]}} = f_e(E) = f_h(E) \quad (149)$$

and $\mu_B = \mu_e + \mu_h = 2\mu_e = 2\mu_h$. From the relations (148) and (143) immediately follows $\hat{n}^e_{\mathbf{k}} = \hat{n}^h_{\mathbf{k}}$, which is a natural property of the symmetric case with charge neutrality.

D. Photon condensation

Next, let us study the steady-state expression $\langle\psi^\dagger_{\mathbf{q}}\rangle = \langle\psi^\dagger_{\mathbf{q}}(t\to\infty)\rangle$ for the photonic expectation value $\langle\psi^\dagger_{\mathbf{q}}(t)\rangle$. Starting from Eq. (94), we first rewrite

$$\langle\psi^\dagger_{\mathbf{q}}(t)\rangle = \langle\tilde{\psi}^\dagger_{\mathbf{q}}(t)\rangle_{\tilde{\rho}_0}\,, \quad (150)$$

where $\tilde{\psi}^\dagger_{\mathbf{q}}$ is the renormalized photon operator [cf. Eq. (C44)]. The dynamics on the right hand side is governed by $\tilde{\mathcal{H}}$ and the expectation value is formed with $\tilde{\rho}_0$. According to Appendix C.3 an appropriate representation for $\tilde{\psi}^\dagger_{\mathbf{q}}$ is

$$\tilde{\psi}^\dagger_{\mathbf{q}} = \tilde{z}_{\mathbf{q}}\psi^\dagger_{\mathbf{q}} + \frac{1}{\sqrt{N}}\sum_{\mathbf{k}}\tilde{v}_{\mathbf{kq}}\,:e^\dagger_{\mathbf{k+q}}h^\dagger_{-\mathbf{k}}:\,, \quad (151)$$

leading, with Eq. (150), to

$$\langle\psi^\dagger_{\mathbf{q}}(t)\rangle = \tilde{z}_{\mathbf{q}}\langle\psi^\dagger_{\mathbf{q}}(t)\rangle_{\tilde{\rho}_0} + \frac{1}{\sqrt{N}}\sum_{\mathbf{k}}\tilde{v}_{\mathbf{kq}}\,\langle:e^\dagger_{\mathbf{k+q}}h^\dagger_{-\mathbf{k}}:(t)\rangle_{\tilde{\rho}_0}\,, \quad (152)$$

where $\tilde{z}_{\mathbf{q}}$ and $\tilde{v}_{\mathbf{kq}}$ are the renormalized coefficients. An equation of motion for the expectation value $\langle\psi^\dagger_{\mathbf{q}}(t)\rangle_{\tilde{\rho}_0}$ can be derived from the generalized Langevin equations (C5)

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi^\dagger_{\mathbf{q}}(t) = i\tilde{\omega}_{\mathbf{q}}\psi^\dagger_{\mathbf{q}}(t) + i\sqrt{N}\tilde{\Gamma}^*\delta_{\mathbf{q},0} - \kappa\psi^\dagger_{\mathbf{q}}(t) + \mathcal{F}^\psi_{\mathbf{q}}\,, \quad (153)$$

from which one finds for $\mathbf{q}=0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi^\dagger_0(t)\rangle_{\tilde{\rho}_0} = i\omega_0\Big(\langle\psi^\dagger_0(t)\rangle_{\tilde{\rho}_0} + \frac{\sqrt{N}\tilde{\Gamma}^*}{\omega_0}\Big) - \kappa\langle\psi^\dagger_0(t)\rangle_{\tilde{\rho}_0}\,. \quad (154)$$

Thereby $\kappa \sim \pi\sum_{\mathbf{p}}|\Gamma^\psi_{\mathbf{qp}}|^2\delta(\omega^\varphi_{\mathbf{p}})$ is the damping rate for cavity photons into the free space due to a nonvanishing leakage. Moreover, $\tilde{\Gamma}^*$ is the renormalized field parameter which accounts for a possible photon condensation [cf. Eq. (25)]. Using condition (93), Eq. (154) leads to the steady-state result, $\langle\psi^\dagger_0\rangle_{\tilde{\rho}_0} = \langle\psi^\dagger_0(t\to\infty)\rangle_{\tilde{\rho}_0}$,

$$\langle\psi^\dagger_{\mathbf{q}}\rangle_{\tilde{\rho}_0} = -\frac{\sqrt{N}\tilde{\Gamma}^*}{\tilde{\omega}_0 + i\kappa}\delta_{\mathbf{q}0}\,. \quad (155)$$

Finally, neglecting the fluctuation term being proportional to $\langle:\, e^\dagger_{\mathbf{q+k}}h^\dagger_{-\mathbf{k}}\,:\rangle_{\tilde{\rho}_0}$ on the right hand side of Eq. (152), the steady-state result for $\langle\psi^\dagger_{\mathbf{q}}\rangle$ becomes

$$\langle\psi^\dagger_{\mathbf{q}}\rangle = -\tilde{z}_0\frac{\sqrt{N}\tilde{\Gamma}^*}{\tilde{\omega}_0 + i\kappa}\delta_{\mathbf{q}0}\,. \quad (156)$$

A corresponding expression for $n^\psi_{\mathbf{q}} = \langle:\,\psi^\dagger_{\mathbf{q}}::\psi_{\mathbf{q}}\,:\rangle$ is found in Appendix C.3.

E. Comparison with previous results

It may be worthwhile to compare our results (135)–(137) with those obtained by the Yamamoto group [25, 26]. Using a generating functional approach, the following equations were derived by these authors:

$$\frac{\mathrm{d}}{\mathrm{d}t}d^*_{\mathbf{k}}(t) = i(\varepsilon^{\mathrm{HF},e}_{\mathbf{k}} + \varepsilon^{\mathrm{HF},h}_{\mathbf{k}})d^*_{\mathbf{k}}(t) - 2\gamma(d^*_{\mathbf{k}}(t) - d^{0*}_{\mathbf{k}}) - i\Delta^{\mathrm{HF}*}_{\mathbf{k}}(1 - n^e_{\mathbf{k}}(t) - n^h_{\mathbf{k}}(t))\,, \quad (157)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}n^e_{\mathbf{k}}(t) = 2\Im[\Delta^{\mathrm{HF}}_{\mathbf{k}}d^*_{\mathbf{k}}(t)] - 2\gamma\big(n^e_{\mathbf{k}}(t) - n^0_{h,\mathbf{k}}\big)\,, \quad (158)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}n^h_{\mathbf{k}}(t) = 2\Im[\Delta^{\mathrm{HF}}_{\mathbf{k}}d^*_{\mathbf{k}}(t)] - 2\gamma\big(n^h_{\mathbf{k}}(t) - n^0_{h,\mathbf{k}}\big)\,. \quad (159)$$

In Eqs. (157)–(159), the time-independent quantities $d^{0*}_{\mathbf{k}}$ and $n^0_{e,\mathbf{k}}, n^0_{h,\mathbf{k}}$ on the right hand side are given in an integral formulation. In principle, the time-dependent quantities $d^*_{\mathbf{k}}(t), n^e_{\mathbf{k}}(t)$ and $n^h_{\mathbf{k}}(t)$ in Eqs. (157)–(159) should agree with our previous quantities (100)–(102), however, there are differences. Calculating the PRM quantities $d^*_{\mathbf{k}}(t), n^e_{\mathbf{k}}(t)$ and $n^h_{\mathbf{k}}(t)$ via Eqs. (105)–(107), fluctuation processes from $\mathcal{H}_g$ and $\mathcal{H}_U$ will be included to infinite order, while in Eqs. (157)–(159) the interactions $\mathcal{H}_g$ and $\mathcal{H}_U$ enter only in mean-field approximation. Hence the latter result can not directly be compared with the true PRM dynamics of $d^*_{\mathbf{k}}(t), n^e_{\mathbf{k}}(t)$ and $n^h_{\mathbf{k}}(t)$. However, one might compare equations (157)–(159) with equations (135)–(137) for the PRM quantities $\hat{d}^*_{\mathbf{k}}(t), \hat{n}^e_{\mathbf{k}}(t)$ and $\hat{n}^h_{\mathbf{k}}(t)$ (with hat symbols):

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{d}^*_{\mathbf{k}}(t) = i(\tilde{\varepsilon}^e_{\mathbf{k}} + \tilde{\varepsilon}^h_{\mathbf{k}})\,\hat{d}^*_{\mathbf{k}}(t) + i\tilde{\Delta}^*_{\mathbf{k}}(1 - \hat{n}^e_{\mathbf{k}}(t) - \hat{n}^h_{\mathbf{k}}(t)) - 2\gamma\,\big(\hat{d}^*_{\mathbf{k}}(t) - \hat{d}^{0*}_{\mathbf{k}}\big)\,, \quad (160)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{n}^e_{\mathbf{k}}(t) = 2\Im[\tilde{\Delta}_{\mathbf{k}}\hat{d}^*_{\mathbf{k}}(t)] - 2\gamma\big[\hat{n}^e_{\mathbf{k}}(t) - \hat{n}^0_{e,\mathbf{k}}\big]\,, \quad (161)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{n}^h_{-\mathbf{k}}(t) = 2\Im[\tilde{\Delta}_{\mathbf{k}}\hat{d}^*_{\mathbf{k}}(t)] - 2\gamma\big[\hat{n}^h_{-\mathbf{k}}(t) - \hat{n}^0_{h,\mathbf{k}}\big]\,, \quad (162)$$

where

$$\hat{d}^{0*}_{\mathbf{k}} = \frac{1}{2}\xi^*_{\mathbf{k}}\eta^*_{\mathbf{k}}F^+_{1\mathbf{k}}\,, \quad (163)$$

$$\hat{n}^0_{e,\mathbf{k}} = |\xi_{\mathbf{k}}|^2 f_e(\tilde{E}_{1\mathbf{k}}) + |\eta_{\mathbf{k}}|^2 f_e(\tilde{E}_{2\mathbf{k}})\,, \quad (164)$$

$$\hat{n}^0_{h,\mathbf{k}} = |\eta_{\mathbf{k}}|^2 f_h(-\tilde{E}_{1\mathbf{k}}) + |\xi_{\mathbf{k}}|^2 f_h(-\tilde{E}_{2\mathbf{k}}) = \hat{n}^0_{e,\mathbf{k}}\,. \quad (165)$$

Here, the additional fluctuation terms following from Eqs. (127)–(129) are absent. However there are differences between Eqs. (157)–(159) and (160)–(162) which

still remain: The energies $\tilde{\varepsilon}^e_{\mathbf{k}}$, $\tilde{\varepsilon}^h_{\mathbf{k}}$, and $\tilde{\Delta}^*_{\mathbf{k}}$ in (160)–(165) are renormalized quantities, whereas the energies $\varepsilon^{\mathrm{HF},e}_{\mathbf{k}}$, $\varepsilon^{\mathrm{HF},h}_{\mathbf{k}}$ and $\Delta^{\mathrm{HF}}_{\mathbf{k}}$ from Eqs. (157)–(159) are not. It remains for us to compare the time-independent quantities $d^{0*}_{\mathbf{k}}$, $n^0_{e,\mathbf{k}}$, and $n^0_{h,\mathbf{k}}$ in Eqs. (157)–(159) with the corresponding quantities $\hat{d}^{0*}_{\mathbf{k}}$, $\hat{n}^0_{e,\mathbf{k}}$, and $\hat{n}^0_{h,\mathbf{k}}$ in Eqs. (163)–(165).

For this reason let us consider two limiting cases from Refs. [15, 25, 26]. Thereby, we use slightly modified conditions and restrict ourselves again to the symmetric case and charge neutrality.

1. $\min|2\tilde{E}_{(1,2)\mathbf{k}}| \geq \mu_B - \mu$

Here $\min|2\tilde{E}_{(1,2)\mathbf{k}}|$ is the minimal excitation energy of electron-hole pairs and the difference $\mu_B - \mu$ can be considered as being responsible for the particle supply from the pumping baths to the e-h-p system. According to Eq. (149) the Fermi function $f(E)$ can then be approximated by

$$f(E) \simeq \frac{1}{1+e^{-\beta E}}\,. \tag{166}$$

Using $\tilde{E}_{2\mathbf{k}} = -\tilde{E}_{1\mathbf{k}}$, $f(\tilde{E}_{2\mathbf{k}}) = f(-\tilde{E}_{1\mathbf{k}}) = 1 - f(\tilde{E}_{1\mathbf{k}})$ and Eq. (147) one has:

$$F^+_{1\mathbf{k}} \simeq 2\big[2f(\tilde{E}_{1k}) - 1\big] = -2\tanh\frac{\beta\tilde{E}_{1\mathbf{k}}}{2}\,, \tag{167}$$

$$F^+_{2\mathbf{k}} \simeq 2\,. \tag{168}$$

Hence, with $\tilde{E}_{(1,2)\mathbf{k}} = \pm\mathrm{sgn}(\tilde{\varepsilon}^e_{\mathbf{k}} + \tilde{\varepsilon}^h_{\mathbf{k}})W_{\mathbf{k}}/2$ and relation (82), one obtains for Eq. (163)

$$\hat{d}^{0*}_{\mathbf{k}} = -\frac{\tilde{\Delta}^*_{\mathbf{k}}}{2(W_{\mathbf{k}}/2)}\tanh\frac{\beta(W_{\mathbf{k}}/2)}{2}\,, \tag{169}$$

and similarly

$$\hat{n}^0_{e,\mathbf{k}} = \hat{n}^0_{h,\mathbf{k}} = \frac{1}{2} - \frac{\tilde{\varepsilon}^e_{\mathbf{k}} + \tilde{\varepsilon}^h_{\mathbf{k}}}{2W_{\mathbf{k}}}\tanh\frac{\beta(W_{\mathbf{k}}/2)}{2}\,. \tag{170}$$

These results agree with the corresponding expressions from the Japanese group [15, 25, 26].

The same results are also obtained with Eqs. (141)–(143) for the steady-state expressions of $\hat{d}^*_{\mathbf{k}}$ and $\hat{n}^{e,h}_{\mathbf{k}}$:

$$\hat{d}^*_{\mathbf{k}} = -\frac{\tilde{\Delta}^*_{\mathbf{k}}}{2(W_{\mathbf{k}}/2)}\tanh\frac{\beta(W_{\mathbf{k}}/2)}{2}\,, \tag{171}$$

$$\hat{n}^e_{\mathbf{k}} = \hat{n}^h_{\mathbf{k}} = \frac{1}{2} - \frac{\tilde{\varepsilon}^e_{\mathbf{k}} + \tilde{\varepsilon}^h_{\mathbf{k}}}{2W_{\mathbf{k}}}\tanh\frac{\tilde{\beta(W_{\mathbf{k}}/2)}}{2}\,, \tag{172}$$

and moreover (see App. C.3)

$$\langle\psi^\dagger_{\mathbf{q}}\psi_{\mathbf{q}}\rangle = |\tilde{z}_{\mathbf{q}}|^2\frac{N|\tilde{\Gamma}|^2}{\tilde{\omega}_0^2+\kappa^2}\delta_{\mathbf{q},0} + \frac{1}{N}\sum_{\mathbf{k}}|\tilde{v}_{\mathbf{kq}}|^2\hat{n}^e_{\mathbf{k}+\mathbf{q}}\hat{n}^h_{-\mathbf{k}}\,. \tag{173}$$

Note that the damping rate γ does not enter the equations for $\hat{d}^*_{\mathbf{k}}$ and $\hat{n}^{e,h}_{\mathbf{k}}$. Using a mean-field approximation, in Refs. [15, 25, 26] also a gap equation for the order parameter $\Delta_{\mathbf{k}}$ was derived, which was formally equivalent to a BCS gap equation. Therefore, $\beta = 1/k_BT$ and μ can be regarded as the inverse temperature and the chemical potential of the e-h-p system, even though β and μ were originally introduced as the inverse temperature of the pumping baths and the oscillation frequency of the photon and polarization fields. In other words, in case of vanishing damping κ [see Eq. (173)] the system can be considered as being in a quasi-equilibrium, because thermodynamic variables are defined. Thus, for $\kappa = 0$ the region with $\min|2\tilde{E}_{(1,2)\mathbf{k}}| \geq \mu_B - \mu$ is equivalent to the thermodynamic equilibrium theory of Ref. [30] for the isolated e-h-p system, apart from the explicit factor $e^{i\mu t}$ in Eqs. (13) and (14). For non-vanishing damping κ the number of cavity photons is only slightly reduced as long as κ is small compared to the cavity photon frequency $\tilde{\omega}_{\mathbf{q}=0}$.

2. $\mu_B - \mu \geq \min|2\tilde{E}_{(1,2)\mathbf{k}}|$

In this case the second term in the exponential of Eq. (149) dominates, i.e.:

$$f(E) \simeq \frac{1}{1+e^{-(\beta/2)(\mu_B-\mu)}} =: f_0\,, \tag{174}$$

and

$$F^+_{1\mathbf{k}} \simeq 0\,, \qquad F^+_{2\mathbf{k}} \simeq 4f_0\,. \tag{175}$$

With Eqs. (163)–(165) one finds for the time-independent quantities in Eqs. (160)–(162),

$$\hat{d}^{0*}_{\mathbf{k}} = 0\,, \quad \hat{n}^0_{e,\mathbf{k}} = \hat{n}^0_{h,\mathbf{k}} = f_0 \simeq 1\,, \tag{176}$$

where additionally in the last relation $(\beta/2)(\mu_B-\mu) \gg 1$ was used (low-temperature approximation).

The steady-state results for $\hat{d}^*_{\mathbf{k}}$ and $\hat{n}^e_{\mathbf{k}} = \hat{n}^h_{\mathbf{k}}$ are found from Eqs. (174)–(175) and (141)–(143):

$$\hat{d}^*_{\mathbf{k}} = \frac{\tilde{\Delta}^*_{\mathbf{k}}}{(\tilde{\varepsilon}^e_{\mathbf{k}} + \tilde{\varepsilon}^h_{\mathbf{k}}) + 2i\gamma}\,(\hat{n}^e_{\mathbf{k}} + \hat{n}^h_{\mathbf{k}} - 1)\,, \tag{177}$$

$$\hat{n}^e_{\mathbf{k}} + \hat{n}^h_{\mathbf{k}} - 1 = 2f_0 - 1 + \frac{2}{\gamma}\Im[\tilde{\Delta}_{\mathbf{k}}\hat{d}^*_{\mathbf{k}}]\,, \tag{178}$$

from which also follows

$$\hat{n}^e_{\mathbf{k}} + \hat{n}^h_{\mathbf{k}} - 1 = \frac{2f_0 - 1}{1 + \dfrac{4|\tilde{\Delta}_{\mathbf{k}}|^2}{(\tilde{\varepsilon}^e_{\mathbf{k}} + \tilde{\varepsilon}^h_{\mathbf{k}})^2 + 4\gamma^2}}\,. \tag{179}$$

In the considered regime $\mu_B - \mu \geq \min|2\tilde{E}_{(1,2)\mathbf{k}}|$ the e-h-p system can no longer be perceived as being in a quasi-equilibrium, solely formed by the isolated e-h-p system. This can be concluded from relation (179), assuming a small influence of the numerator ($|\tilde{\Delta}|^2$). Then for low temperatures the right hand side of (179) indicates that electrons

and holes are strongly excited and are in the high-density regime. Thus, increasing further the concentration of the total particle number of the e-h-p system,

$$n_{exc} = \frac{1}{N}\Big[\frac{1}{2}\sum_{\mathbf{k}}(n^e_{\mathbf{k}} + n^h_{\mathbf{k}}) + \sum_{\mathbf{q}}\langle\psi^\dagger_{\mathbf{q}}\psi_{\mathbf{q}}\rangle\Big]\,, \qquad (180)$$

only the number of cavity photons will mainly increase since possible electron-hole excitations tend already to be used up.

Cavity photons are also affected by a non-vanishing leakage ($\kappa \neq 0$) to the external photonic free-space. Then, the e-h-p system is no longer in an equilibrium situation with the electronic pumping baths. Therefore, one of the main differences between the two regimes $\mu_B - \mu \leq \min|2\tilde{E}_{(1,2)\mathbf{k}}|$ and $\mu_B - \mu \geq \min|2\tilde{E}_{(1,2)\mathbf{k}}|$ is the relative importance of electron-hole and photonic excitations. Whereas in the first regime particle-hole excitations are dominant this is not the case for the second regime.

As said before, photon excitations are less pronounced in regime $\mu_B - \mu \leq \min|2\tilde{E}_{(1,2)\mathbf{k}}|$. This means that the system is less affected by the photon leakage. In contrast, for $\mu_B - \mu \geq \min|2\tilde{E}_{(1,2)\mathbf{k}}|$ the system is in a high-density regime and is strongly affected by the photon leakage, which suggests that a large degree of non-equilibrium is achieved.

F. Self-consistency of the steady-state solution

Above we have derived the renormalization equations for the order parameters $\Delta_{\mathbf{k},\lambda}$ and Γ_λ and found a compact representation for the exciton-condensation parameter $d^*_{\mathbf{k}}$. The equations can be numerically solved, provided $\langle\psi_0\rangle$ and μ are known. However, these quantities are not yet determined since $\tilde{\Gamma}$ in Eq. (156) depends implicitly on $\langle\psi_0\rangle$ and μ as well as on the sets of quantities $d_{\mathbf{k}}$ and $\Delta_{\mathbf{k},\lambda}$. In particular, μ is not a chemical potential, since the total number of particles of the e-h-p system together with the particle number of the electronic baths is not fixed due to the leakage of cavity photons into the free space. To determine μ and $\langle\psi_0\rangle$ a "way out" has been discussed in the literature [13]. The starting point is Eq. (156),

$$(\tilde{\omega}_0 - i\kappa)\,\langle\psi_0\rangle = -\tilde{z}_0\sqrt{N}\tilde{\Gamma}\,, \qquad (181)$$

which is a complex equation due to presence of the damping rate κ. The final solution for $\tilde{\Gamma}$ results from the renormalization equation (A27):

$$\Gamma_{\lambda-\Delta\lambda} = \Gamma_\lambda + \delta\Gamma^{(0)}_\lambda + \delta\Gamma^{(c)}_\lambda + \delta\Gamma^{(g)}_\lambda\,, \qquad (182)$$

where the $\delta\Gamma^{(0)}_\lambda$, $\delta\Gamma^{(c)}_\lambda$, and $\delta\Gamma^{(g)}_\lambda$ defined in Eqs. (A3), (A9) and (A16) become:

$$\delta\Gamma^{(0)}_\lambda = \frac{g}{N}\sum_{\mathbf{k}} A_{\mathbf{k}0}(\lambda,\Delta\lambda)(\varepsilon^e_{\mathbf{k},\lambda} + \varepsilon^h_{-\mathbf{k},\lambda})\,d_{\mathbf{k}}\,, \qquad (183)$$

$$\delta\Gamma^{(c)}_\lambda = \frac{g}{N}\sum_{\mathbf{k}} A_{\mathbf{k}0}(\lambda,\Delta\lambda)(1 - n^e_{\mathbf{k}} - n^h_{-\mathbf{k}})\Delta_{\mathbf{k},\lambda}\,, \qquad (184)$$

$$\delta\Gamma^{(g)}_\lambda = \frac{2g^2}{N\sqrt{N}}\sum_{\mathbf{k}} A_{\mathbf{k}0}(\lambda,\Delta\lambda)(1 - n^e_{\mathbf{k}} - n^h_{-\mathbf{k}})\langle\psi_0\rangle\,. \qquad (185)$$

The initial value of Γ_λ is

$$\Gamma_\Lambda = \hat{\Gamma} = \Gamma - (g/N)\sum_{\mathbf{k}} d_{\mathbf{k}} \qquad (186)$$

($\Gamma = 0^+$). Note that the contribution $\delta\Gamma^{(g)}_\lambda$ is proportional to $\langle\psi_0\rangle$ as expected, whereas $\delta\Gamma^{(0)}_\lambda$ and $\delta\Gamma^{(c)}_\lambda$ depend on the order parameters $d_{\mathbf{k}}$ and $\Delta_{\mathbf{k},\lambda}$. Similarly, from Eqs. (127) and (130) one concludes that $d_{\mathbf{k}}$ is fixed if the order parameters $\tilde{\Delta}_{\mathbf{k}}$ are known. What remains to be shown is that $\Delta_{\mathbf{k},\lambda}$ is fixed for given $\langle\psi_0\rangle$ and $d_{\mathbf{k}}$, which follows from renormalization equation (73). Thus, putting everything together, $\tilde{\Gamma}$ can be considered as an implicit function of $\langle\psi_0\rangle$ and μ, i.e., $\tilde{\Gamma} = \tilde{\Gamma}[\langle\psi_0\rangle,\mu]$:

$$\frac{\langle\psi_0\rangle}{\sqrt{N}} = -\tilde{z}_0\frac{\tilde{\Gamma}[\langle\psi_0\rangle,\mu]}{\tilde{\omega}_0 - i\kappa}\,. \qquad (187)$$

However, the number of coupled equations by (187) is one less than the number of unknown variables, since $\langle\psi_0\rangle$ and $\tilde{\Gamma}$ are in general complex quantities. This can be seen from equation (124) for $A^{12}_{\mathbf{k}}$ and $A^{21}_{\mathbf{k}}$. Since the denominator in Eq. (124) is complex also $A^{12}_{\mathbf{k}}$ and $A^{21}_{\mathbf{k}}$ will be complex. Assuming $\langle\psi_0\rangle$ is complex, Eq. (187) would contain three unknown quantities, the real and the imaginary parts of $\langle\psi_0\rangle$ as well as the energy parameter μ, whereas the complex equation only fixes two of them. However the number of unknown variables can be reduced by fixing the phase of $\langle\psi_0\rangle$. Taking a phase for which the imaginary part of $\langle\psi_0\rangle$ vanishes, the number of coupled equations becomes equal to the number of unknown variables and the complex equation (187) only represents two independent equations for $\langle\psi_0\rangle$ and μ:

$$\frac{\langle\psi_0\rangle}{\sqrt{N}} = -\frac{\tilde{z}_0}{\tilde{\omega}_0^2 + \kappa^2}\big(\tilde{\omega}_0\,\Re\tilde{\Gamma} - \kappa\,\Im\tilde{\Gamma}\big)\,, \qquad (188)$$

$$0 = \frac{\tilde{z}_0}{\tilde{\omega}_0^2 + \kappa^2}\big(\kappa\,\Re\tilde{\Gamma} + \tilde{\omega}_0\,\Im\tilde{\Gamma}\big)\,, \qquad (189)$$

where $\tilde{\Gamma} = \Re\tilde{\Gamma} + i\Im\tilde{\Gamma}$ is complex. From equation (189) one obtains

$$\Im\tilde{\Gamma} = -(\kappa/\tilde{\omega}_0)\Re\tilde{\Gamma}, \qquad (190)$$

which leads for the first equation to

$$\frac{\langle\psi_0\rangle}{\sqrt{N}} = -\frac{\tilde{z}_0}{\tilde{\omega}_0}\Re\tilde{\Gamma}\,. \qquad (191)$$

Note that the last relation agrees with what is known for a closed system in thermal equilibrium, though it is now valid also for the general case of an open system. Eqs. (190) and (191) have to be solved self-consistently for μ and $\langle\psi_0\rangle$.

G. Limit of vanishing damping rate κ

In this subsection, we study the limit of a vanishing damping rate κ between the cavity photons and the free space photons. As stressed before, a finite leakage to external photons implies that the quantity μ does not act as a common chemical potential of the total system. The reason is, that photons can escape from the e-h-p system into the free-photon space. On the other hand, for vanishing κ thermal equilibrium should develop. Then μ should become the usual chemical potential for the remaining system, which is composed of the e-h-p subsystem and the two electronic baths. In this context, we are mostly interested in the case of strong damping rate γ for the coupling rate to the electronic baths.

Analyzing the limit $\kappa \to 0$, we start from Eq. (190) which states that the imaginary part of $\tilde{\Gamma}$ must vanish:

$$\Im\tilde{\Gamma} = 0\,. \qquad (192)$$

Thereby $\tilde{\Gamma}$ results from the solution of the renormalization equation (182) for Γ_λ,

$$\Gamma_{\lambda-\Delta\lambda}-\Gamma_\lambda=\delta\Gamma_\lambda^{(0)}+\delta\Gamma_\lambda^{(c)}+\delta\Gamma_\lambda^{(g)}\,, \tag{193}$$

with renormalization contributions $\delta\Gamma_\lambda^{(0)}$, $\delta\Gamma_\lambda^{(g)}$, and $\delta\Gamma_\lambda^{(g)}$, given by Eqs. (183)–(185). One finds for the imaginary parts:

$$\begin{aligned}\Im\Gamma_{\lambda-\Delta\lambda}-\Im\Gamma_\lambda=&\frac{g}{N}\sum_{\mathbf{k}}A_{\mathbf{k}0}(\lambda,\Delta\lambda) \\ &\times\left[(\varepsilon^e_{\mathbf{k},\lambda}+\varepsilon^h_{\mathbf{k},\lambda})\Im d_{\mathbf{k}}-(n^e_{\mathbf{k}}+n^h_{\mathbf{k}}-1)\Im\Delta_{\mathbf{k},\lambda}\right]\end{aligned} \tag{194}$$

with initial value $\Gamma_{\lambda=\Lambda}=\hat{\Gamma}=\Gamma-\frac{g}{N}\sum_{\mathbf{k}}d_{\mathbf{k}}$, $(\Gamma=0^+)$. Here we have already exploited that $\delta\Gamma_\lambda^{(g)}$ is real. In the following we again neglect all renormalization contributions to $d_{\mathbf{k}}$ and $n^{e,h}_{\mathbf{k}}$ from Eqs. (127)–(129), thereby replacing $d_{\mathbf{k}}$ and $n^{e,h}_{\mathbf{k}}$ by $\hat{d}_{\mathbf{k}}$ and $\hat{n}^{e,h}_{\mathbf{k}}$. Then, according to Eqs. (141) and (142) we obtain

$$\begin{aligned}\hat{d}_{\mathbf{k}}=&\frac{\tilde{\Delta}_{\mathbf{k}}\mathrm{sgn}(\tilde{\varepsilon}^e_{\mathbf{k}}+\tilde{\varepsilon}^h_{\mathbf{k}})}{2W_{\mathbf{k}}}F^+_{1\mathbf{k}} \\ &+\frac{1}{2}\frac{\tilde{\Delta}_{\mathbf{k}}}{(\tilde{\varepsilon}^e_{\mathbf{k}}+\tilde{\varepsilon}^h_{\mathbf{k}})-2i\gamma}\,\frac{F^+_{2\mathbf{k}}-2}{1+\dfrac{4|\tilde{\Delta}_{\mathbf{k}}|^2}{(\tilde{\varepsilon}^e_{\mathbf{k}}+\tilde{\varepsilon}^h_{\mathbf{k}})^2+(2\gamma)^2}}\end{aligned} \tag{195}$$

and

$$\hat{n}^e_{\mathbf{k}}+\hat{n}^h_{\mathbf{k}}-1=\frac{|\tilde{\varepsilon}^e_{\mathbf{k}}+\tilde{\varepsilon}^h_{\mathbf{k}}|}{2W_{\mathbf{k}}}F^+_{1\mathbf{k}}+\frac{1}{2}\frac{F^+_{2\mathbf{k}}-2}{1+\dfrac{4|\tilde{\Delta}_{\mathbf{k}}|^2}{(\tilde{\varepsilon}^e_{\mathbf{k}}+\tilde{\varepsilon}^h_{\mathbf{k}})^2+(2\gamma)^2}}. \tag{196}$$

Inserting Eqs. (195) and (196) into Eq. (194) we find

$$\begin{aligned}&\Im\Gamma_{\lambda-\Delta\lambda}-\Im\Gamma_\lambda=\frac{g}{N}\sum_{\mathbf{k}}A_{\mathbf{k}0}(\lambda,\Delta\lambda) \\ &\times\left[(\varepsilon^e_{\mathbf{k},\lambda}+\varepsilon^h_{-\mathbf{k},\lambda})\Im\tilde{\Delta}_{\mathbf{k}}-(\tilde{\varepsilon}^e_{\mathbf{k}}+\tilde{\varepsilon}^h_{-\mathbf{k}})\Im\Delta_{\mathbf{k},\lambda}\right]\frac{\mathrm{sgn}(\tilde{\varepsilon}^e_{\mathbf{k}}+\tilde{\varepsilon}^h_{-\mathbf{k}})}{2W_{\mathbf{k}}}F^+_{1\mathbf{k}} \\ &+\frac{g}{2N}\sum_{\mathbf{k}}A_{\mathbf{k}0}(\lambda,\Delta\lambda)\Big[(\varepsilon^e_{\mathbf{k},\lambda}+\varepsilon^h_{-\mathbf{k},\lambda})\Im\Big(\frac{\tilde{\Delta}_{\mathbf{k}}}{(\tilde{\varepsilon}^e_{\mathbf{k}}+\tilde{\varepsilon}^h_{-\mathbf{k}})-2i\gamma}\Big) \\ &\qquad-\Im\Delta_{\mathbf{k},\lambda}\Big]\frac{F^+_{2\mathbf{k}}-2}{1+\dfrac{4|\tilde{\Delta}_{\mathbf{k}}|^2}{(\varepsilon^e_{\mathbf{k},\lambda}+\varepsilon^h_{-\mathbf{k},\lambda})^2+(2\gamma)^2}}\,.\end{aligned} \tag{197}$$

This result can further be simplified. First of all, we neglect the first term in Eq. (197), which is small. It consists of the difference of two contributions which are of quite similar character. In particular, for small λ (almost full renormalization) the cancellation of the two terms is exact, and for $\lambda=\Lambda$ (initial point) contributions from the renormalization are small. Thus

$$\begin{aligned}&\Im\Gamma_{\lambda-\Delta\lambda}-\Im\Gamma_\lambda\approx\frac{g}{2N}\sum_{\mathbf{k}}A_{\mathbf{k}0}(\lambda,\Delta\lambda) \\ &\times\left[(\varepsilon^e_{\mathbf{k},\lambda}+\varepsilon^h_{-\mathbf{k},\lambda})\Im\Big(\frac{\tilde{\Delta}_{\mathbf{k}}}{(\tilde{\varepsilon}^e_{\mathbf{k}}+\tilde{\varepsilon}^h_{-\mathbf{k}})-2i\gamma}\Big)-\Im\Delta_{\mathbf{k},\lambda}\right] \\ &\times\frac{F^+_{2\mathbf{k}}-2}{1+\dfrac{4|\tilde{\Delta}_{\mathbf{k}}|^2}{(\varepsilon^e_{\mathbf{k},\lambda}+\varepsilon^h_{-\mathbf{k},\lambda})^2+(2\gamma)^2}}\,.\end{aligned} \tag{198}$$

Next, let us consider the limit of large damping γ, thereby assuming that the following conditions are fulfilled:

$$2\gamma\gg[\varepsilon^e_{\mathbf{k},\lambda}+\varepsilon^h_{-\mathbf{k},\lambda}|\quad\text{and}\quad 2\gamma\gg 2|\tilde{\Delta}_{\mathbf{k}}| \tag{199}$$

for most values of $\mathbf{k}$. The first condition is met easier for a semimetal than for a semiconductor. As a consequence of the conditions (199), an expansion of Eq. (198) for large γ gives to leading order γ^{-1}:

$$\begin{aligned}&\Im\Gamma_{\lambda-\Delta\lambda}-\Im\Gamma_\lambda\approx-\frac{g}{2N}\sum_{\mathbf{k}}A_{\mathbf{k}0}(\lambda,\Delta\lambda) \\ &\times\big(\Im\Delta_{\mathbf{k},\lambda}-\frac{\varepsilon^e_{\mathbf{k},\lambda}+\varepsilon^h_{-\mathbf{k},\lambda}}{2\gamma}\Re\tilde{\Delta}_{\mathbf{k}}\big)(F^+_{2\mathbf{k}}-2)\,.\end{aligned} \tag{200}$$

Here the term $\sim(1/\gamma)\Re\tilde{\Delta}_{\mathbf{k}}$ followed from the first contribution in the squared brackets of Eq. (198) and the denominator of the common factor behind the brackets was replaced by one. Expanding Eq. (195) to the same order as Eq. (200), the imaginary part of the initial condition, $\Gamma_\Lambda=\Gamma-\frac{g}{N}\sum_{\mathbf{k}}d_{\mathbf{k}}$ $(\Gamma=0^+)$, becomes for large γ

$$\Im\Gamma_\Lambda=\Im\Gamma-\frac{g}{N}\sum_{\mathbf{k}}\frac{(\Im\tilde{\Delta}_{\mathbf{k}})\mathrm{sgn}(\tilde{\varepsilon}^e_{\mathbf{k}}+\tilde{\varepsilon}^h_{\mathbf{k}})}{2W_{\mathbf{k}}}F^+_{1\mathbf{k}}+\frac{\Re\tilde{\Delta}_{\mathbf{k}}}{2\gamma}\,\frac{F^+_{2\mathbf{k}}-2}{2}\,. \tag{201}$$

Our aim is to study under which conditions the renormalized quantity $\tilde{\Gamma}$ is real so that $\Im\tilde{\Gamma}=0$ is valid. For this, according to Eq. (200), one also has to study the renormalization of $\Im\Delta_{\mathbf{k},\lambda}$. As is easily seen, $\Im\Delta_{\mathbf{k},\lambda}$ renormalizes to zero in dominant order, whereas $\Re\tilde{\Delta}_{\mathbf{k}}$ stays finite. Therefore, in order to arrive at the desired result $\Im\tilde{\Gamma}=0$, including the less dominant contribution on the right hand side of Eq. (200), the common factor $(F^+_{2\mathbf{k}}-2)$ must vanish. This condition can only be met by fixing the value of μ to the chemical potential μ_B of the electronic baths, $\mu=\mu_B$, so that $F^+_{2\mathbf{k}}-2=0$.

The remaining equations for $\Re\Gamma_\lambda$ and $\Re\Delta_{\mathbf{k},\lambda}$ are easily found from Eqs. (183) and (73), and completely agree with those of the equilibrium case. Also the quantities $\hat{d}_{\mathbf{k}}$ and $\hat{n}^e_{\mathbf{k}}+\hat{n}^h_{\mathbf{k}}-1$ for large γ agree with the corresponding equilibrium expressions:

$$\hat{d}_{\mathbf{k}}=\frac{\tilde{\Delta}_{\mathbf{k}}\mathrm{sgn}(\tilde{\varepsilon}^e_{\mathbf{k}}+\tilde{\varepsilon}^h_{\mathbf{k}})}{2W_{\mathbf{k}}}F^+_{1\mathbf{k}}\,, \tag{202}$$

$$\hat{n}^e_{\mathbf{k}}+\hat{n}^h_{\mathbf{k}}-1=\frac{|\tilde{\varepsilon}^e_{\mathbf{k}}+\tilde{\varepsilon}^h_{\mathbf{k}}|}{2W_{\mathbf{k}}}F^+_{1\mathbf{k}}\,. \tag{203}$$

To sum up, we have shown that the present extension of the PRM leads back to the usual thermodynamic equilibrium approach of Ref. [30]. The equilibrium is mainly of electronic nature with $\mu=\mu_B$ for the case that the following two conditions are fulfilled: (i) the damping κ of cavity photons to free space photons is zero and (ii) the coupling γ of the e-h-p subsystem to the electronic baths is sufficiently large in accordance with Eq. (199). In particular, the second condition is only fulfilled, when $\tilde{\Delta}_{\mathbf{k}}$ is sufficiently small. However, as shown in Fig. 4, its photonic part Δ_{ph} may tremendously increase at larger values of n_{exc} in the case of large detuning $d=3.5$, whereas at small detuning $d=-0.5$ the quantity Δ_{ph} already starts to increase at comparatively small values of n_{exc}. As discussed in more detail below, this behavior of Δ_{ph} can be understood as a phase space filling and Pauli blocking effect. At small n_{exc} additional excitations are either of excitonic or polaritonic nature until the electronic bands are completely filled. Then, for even larger n_{exc} photonic excitations dominate. Note, however, that even for large n_{exc}, when Δ_{ph} and $\tilde{\Delta}_{\mathbf{k}}$ become large, the total e-h-p subsystem, together with the electronic reservoirs, has to realize a thermodynamic equilibrium state for the case that κ is zero or sufficiently small.

V. SINGLE-PARTICLE SPECTRAL FUNCTION

The one-particle spectral function $A(\mathbf{k},\omega)$ for the steady state is defined by the Laplace transform of the time-dependent electron anti-commutator correlation function in the limit $t\to\infty$:

$$A_e(\mathbf{k},\omega) = \frac{1}{\pi}\lim_{t\to\infty}\Re\int_0^\infty d\tau e^{i\omega\tau}\langle[e^\dagger_{\mathbf{k}}(t),e_{\mathbf{k}}(t+\tau)]_+\rangle \tag{204}$$
$$= \frac{1}{\pi}\lim_{t\to\infty}\Re\int_0^\infty d\tau e^{i\omega\tau}\langle[\tilde{e}^\dagger_{\mathbf{k}}(t),\tilde{e}_{\mathbf{k}}(t+\tau)]_+\rangle_{\tilde{\rho}_0}.$$

In the first equation the time dependence is governed by the original Hamiltonian $\mathcal{H}$, whereas in the second line the dynamics is again given by $\tilde{\mathcal{H}}$, and also the expectation value is formed with the transformed density operator $\tilde{\rho}_0$. Note that in Eq. (204) the lower integration limit $\tau=0$ is a time much larger than τ_0. For that time a steady state has already been reached, with properties that do not depend on the details of the initial state. Likewise the two-time correlation function $\langle\tilde{e}^\dagger_{\mathbf{q}}(t)\tilde{e}_{\mathbf{q}}(t+\tau)\rangle_{\tilde{\rho}_0}$ depends only on the relative time difference τ and not on t. Thus, the stationary spectrum can be calculated at any fixed time t. At the end, t is shifted to infinity. Furthermore, $\tilde{e}^\dagger_{\mathbf{k}}$ is the transformed one-particle operator (103):

$$\tilde{e}^\dagger_{\mathbf{k}} = \tilde{x}_{\mathbf{k}}e^\dagger_{\mathbf{k}} + \frac{1}{\sqrt{N}}\sum_{\mathbf{q}}\tilde{t}_{\mathbf{k}-\mathbf{q},\mathbf{q}}h_{\mathbf{q}-\mathbf{k}}:\psi^\dagger_{\mathbf{q}}: \tag{205}$$
$$+\frac{1}{N}\sum_{\mathbf{k}_1\mathbf{k}_2}\tilde{\alpha}_{\mathbf{k}_1\mathbf{k}\mathbf{k}_2}e^\dagger_{\mathbf{k}_1}:h^\dagger_{\mathbf{k}_2}h_{\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}}:.$$

Let us consider the coherent part of the spectrum, which results from the first term on the right hand side of Eq. (205):

$$A_{e,\mathrm{coh}}(\mathbf{k},\omega) = \frac{|\tilde{x}_{\mathbf{k}}|^2}{\pi} \tag{206}$$
$$\times\lim_{t\to\infty}\Re\int_0^\infty d\tau e^{i\omega\tau}\langle[e^\dagger_{\mathbf{k}}(t),e_{\mathbf{k}}(t+\tau)]_+\rangle_{\rho_0}.$$

Decomposing $e^\dagger_{\mathbf{k}}$ and $e_{\mathbf{k}}$ into eigenmodes $C^{(\dagger)}_{1\mathbf{k}}$ and $C^{(\dagger)}_{2\mathbf{k}}$ of $\tilde{\mathcal{H}}_S$, according to Eqs. (78) and (79), $A_{\mathrm{coh}}(\mathbf{k},\omega)$ transforms to

$$A_{e,\mathrm{coh}}(\mathbf{k},\omega) = \frac{|\tilde{x}_{\mathbf{k}}|^2}{\pi}\lim_{t\to\infty}\Re\int_0^\infty d\tau e^{i\omega\tau}$$
$$\times\Big\{|\xi_{\mathbf{k}}|^2\langle[C^\dagger_{1\mathbf{k}}(t),C_{1\mathbf{k}}(t+\tau)]_+\rangle_{\rho_0}$$
$$+|\eta_{\mathbf{k}}|^2\langle[C^\dagger_{2\mathbf{k}}(t),C_{2\mathbf{k}}(t+\tau)]_+\rangle_{\rho_0}$$
$$-\xi^*_{\mathbf{k}}\eta_{\mathbf{k}}\langle[C^\dagger_{1\mathbf{k}}(t),C_{2\mathbf{k}}(t+\tau)]_+\rangle_{\rho_0}$$
$$-\xi_{\mathbf{k}}\eta^*_{\mathbf{k}}\langle[C^\dagger_{2\mathbf{k}}(t),C_{1\mathbf{k}}(t+\tau)]_+\rangle_{\rho_0}\Big\}. \tag{207}$$

The τ-dependence will be treated by employing the Mori-Zwanzig projection formalism described in Appendix C. Using the fermionic anti-commutator relations we find

$$\langle[C^\dagger_{1\mathbf{k}}(t),C_{1\mathbf{k}}(t+\tau)]_+\rangle_{\rho_0} = e^{(-i\tilde{E}_{1\mathbf{k}}-\gamma)\tau}, \tag{208}$$
$$\langle[C^\dagger_{1\mathbf{k}}(t),C_{2\mathbf{k}}(t+\tau)]_+\rangle_{\rho_0} = 0. \tag{209}$$

This leads for $A_{\mathrm{coh}}(\mathbf{k},\omega)$ to

$$A_{e,\mathrm{coh}}(\mathbf{k},\omega) = \frac{|\tilde{x}_{\mathbf{k}}|^2}{\pi}\lim_{t\to\infty}\Re\int_0^\infty d\tau e^{i\omega\tau} \tag{210}$$
$$\times\Big\{|\xi_{\mathbf{k}}|^2e^{(-i\tilde{E}_{1\mathbf{k}}-\gamma)\tau}+|\eta_{\mathbf{k}}|^2e^{(-i\tilde{E}_{2\mathbf{k}}-\gamma)\tau}\Big\}.$$

Finally, by integrating over τ and taking into account only the dissipative part of the integral, one finds

$$A_{e,\mathrm{coh}}(\mathbf{k},\omega) = \frac{|\tilde{x}_{\mathbf{k}}|^2}{\pi} \tag{211}$$
$$\times\Big\{|\xi_{\mathbf{k}}|^2\frac{\gamma}{(\tilde{E}_{1\mathbf{k}}-\omega)^2+\gamma^2}+|\eta_{\mathbf{k}}|^2\frac{\gamma}{(\tilde{E}_{2\mathbf{k}}-\omega)^2+\gamma^2}\Big\},.$$

Thus, the spectrum $A_{e,\mathrm{coh}}(\mathbf{k},\omega)$ consists of resonances at the quasiparticle energies $\tilde{E}_{1\mathbf{k}}$ and $\tilde{E}_{2\mathbf{k}}$ with damping γ and weights which are determined by $|\xi_{\mathbf{k}}|^2$ and $|\eta_{\mathbf{k}}|^2$, respectively. The spectral function $A_{h,\mathrm{coh}}(\mathbf{k},\omega)$ for holes can be written in the form (211) as well, however, with the weights $|\xi_{\mathbf{k}}|^2$ and $|\eta_{\mathbf{k}}|^2$ interchanged. The incoherent part of $A_e(\mathbf{k},\omega)$ can be obtained by help of the second and third term in Eq. (205), and is expected to lead to a background spectrum for the coherent part.

VI. STEADY-STATE LUMINESCENCE

The steady-state emission spectrum is obtained from the Laplace transform of the photon correlation function [33]:

$$S(\mathbf{q},\omega) = \frac{1}{\pi}\lim_{t\to\infty}\Re\int_0^\infty d\tau\, e^{i\omega\tau}\langle\psi^\dagger_{\mathbf{q}}(t)\psi_{\mathbf{q}}(t+\tau)\rangle \tag{212}$$

or—with the help of relation (94)—by

$$S(\mathbf{q},\omega) = \frac{1}{\pi}\lim_{t\to\infty}\Re\int_0^\infty d\tau\, e^{i\omega\tau}\langle\tilde{\psi}^\dagger_{\mathbf{q}}(t)\tilde{\psi}_{\mathbf{q}}(t+\tau)\rangle_{\tilde{\rho}_0}. \tag{213}$$

Again, in Eq. (212), the time dependence is governed by the original Hamiltonian $\mathcal{H}$, whereas in Eq. (213) the dynamics is given by the transformed Hamiltonian $\tilde{\mathcal{H}}$. Moreover, $\tilde{\psi}^\dagger_{\mathbf{q}}$ is the transformed photon operator (151), and the expectation value in Eq. (155) is formed with the transformed initial density operator $\tilde{\rho}_0$. We note that a quite similar photon correlation function $B(\mathbf{q},\omega)$ was studied in [30] for thermal equilibrium. However, in contrast to $S(\mathbf{q},\omega)$ the function $B(\mathbf{q},\omega)$ was a response function, that is a photon commutator correlation function. We would like to point out here that the renormalization equations (B41) and (B36) in Ref. [30] are not completely correct. The correct equations are given by the present Eqs. (C41) and (C42).

A. Coherent part

Also the luminescence spectrum consists of two parts. The coherent part results from the first term on the right hand side of Eq. (151):

$$S_{\mathrm{coh}}(\mathbf{q},\omega) = \frac{|\tilde{z}_{\mathbf{q}}|^2}{\pi}\lim_{t\to\infty}\Re\int_0^\infty d\tau\, e^{i\omega\tau}\langle\psi^\dagger_{\mathbf{q}}(t)\psi_{\mathbf{q}}(t+\tau)\rangle_{\tilde{\rho}_0}. \tag{214}$$

The time dependence of $\psi^\dagger_{\mathbf{q}}(t)$ on the right hand side of Eq. (214) is found from the solution of the equation of motion (153):

$$\psi^\dagger_{\mathbf{q}}(t) = -\frac{i\sqrt{N}\tilde{\Gamma}^*}{i\tilde{\omega}_0-\kappa}\delta_{\mathbf{q},0}+\Big(\psi^\dagger_{\mathbf{q}}+\frac{i\sqrt{N}\tilde{\Gamma}^*}{i\tilde{\omega}_0-\kappa}\delta_{\mathbf{q},0}\Big)e^{(i\tilde{\omega}_{\mathbf{q}}-\kappa)t}. \tag{215}$$

Substituting (215) into Eq. (214) leads to

$$S_{\rm coh}(\mathbf{q},\omega) = \frac{N|\tilde{z}_0|^2|\tilde{\Gamma}|^2}{\hat{\omega}_0^2+\kappa^2}\delta_{\mathbf{q},0}\,\delta(\omega)\,, \tag{216}$$

which shows that in the condensed phase a delta-function peak appears at $\omega = 0$.

B. Incoherent part

The incoherent part of $S(\mathbf{q},\omega)$ is given by

$$S_{\rm inc}(\mathbf{q},\omega) = \frac{1}{\pi}\lim_{t\to\infty}\Re\int_0^\infty d\tau\, e^{i\omega\tau}\langle :\, b_{\mathbf{q}}^\dagger(t)\,::\, b_{\mathbf{q}}(t+\tau)\,:\rangle_{\tilde{\rho}_0}\,, \tag{217}$$

where $b_{\mathbf{q}}^\dagger$ creates an exciton with wave vector $\mathbf{q}$ which is modified by the coefficients $\tilde{v}_{\mathbf{kq}}$:

$$b_{\mathbf{q}}^\dagger(t) = \frac{1}{\sqrt{N}}\sum_{\mathbf{k}}\tilde{v}_{\mathbf{kq}}\,(e_{\mathbf{k}+\mathbf{q}}^\dagger h_{-\mathbf{k}}^\dagger)(t)\,. \tag{218}$$

Thus

$$S_{\rm inc}(\mathbf{q},\omega) = \frac{1}{N\pi}\lim_{t\to\infty}\sum_{\mathbf{k}\mathbf{k}'}\tilde{v}_{\mathbf{kq}}\tilde{v}_{\mathbf{k}'\mathbf{q}}^*\Re\int_0^\infty d\tau\, e^{i\omega\tau} \times\langle :\,(e_{\mathbf{k}+\mathbf{q}}^\dagger h_{-\mathbf{k}}^\dagger)(t)\,::\,(h_{-\mathbf{k}'}e_{\mathbf{k}'+\mathbf{q}})(t+\tau)\,:\rangle_{\tilde{\rho}_0}\,. \tag{219}$$

In a factorization approximation this simplifies to

$$S_{\rm inc}(\mathbf{q},\omega) = \frac{1}{N\pi}\lim_{t\to\infty}\sum_{\mathbf{k}}|\tilde{v}_{\mathbf{kq}}|^2\Re\int_0^\infty d\tau\, e^{i\omega\tau} \times\langle e_{\mathbf{k}+\mathbf{q}}^\dagger(t)e_{\mathbf{k}+\mathbf{q}}(t+\tau)\rangle_{\tilde{\rho}_0}\langle h_{-\mathbf{k}}^\dagger(t)h_{-\mathbf{k}}(t+\tau)\rangle_{\tilde{\rho}_0}\,. \tag{220}$$

Note that expectation values $\langle :\,(e_{\mathbf{k}+\mathbf{q}}^\dagger h_{-\mathbf{k}}^\dagger)(t)\,:\rangle_{\tilde{\rho}_0}$ and $\langle :\,(h_{-\mathbf{k}'}^\dagger e_{\mathbf{k}'+\mathbf{q}})(t+\tau)\,:\rangle_{\tilde{\rho}_0}$ drop out in Eq. (219) so that only the pairwise factorization of Eq. (220) survives.

What remains to be done is the τ integration in Eq. (220). Again, this can best be achieved by using Bogolyubov quasiparticles in accordance with Eqs. (78) and (79). With

$$h_{-\mathbf{k}}^\dagger = \eta_{\mathbf{k}}^* C_{1\mathbf{k}} + \xi_{\mathbf{k}}^* C_{2\mathbf{k}}\,, \tag{221}$$

$$e_{\mathbf{k}}^\dagger = \xi_{\mathbf{k}}^* C_{1\mathbf{k}}^\dagger - \eta_{\mathbf{k}}^* C_{2\mathbf{k}}^\dagger\,, \tag{222}$$

one finds

$$\begin{aligned}\langle e_{\mathbf{k}}^\dagger(t)e_{\mathbf{k}}(t+\tau)\rangle_{\tilde{\rho}_0} &= |\xi_{\mathbf{k}}|^2\langle C_{1\mathbf{k}}^\dagger(t)C_{1\mathbf{k}}(t+\tau)\rangle_{\tilde{\rho}_0}\\ &+|\eta_{\mathbf{k}}|^2\langle C_{2\mathbf{k}}^\dagger(t)C_{2\mathbf{k}}(t+\tau)\rangle_{\tilde{\rho}_0}\\ &-\eta_{\mathbf{k}}\xi_{\mathbf{k}}^*\langle C_{1\mathbf{k}}^\dagger(t)C_{2\mathbf{k}}(t+\tau)\rangle_{\tilde{\rho}_0}\\ &-\eta_{\mathbf{k}}^*\xi_{\mathbf{k}}\langle C_{2\mathbf{k}}^\dagger(t)C_{1\mathbf{k}}(t+\tau)\rangle_{\tilde{\rho}_0}\end{aligned} \tag{223}$$

and

$$\begin{aligned}\langle h_{-\mathbf{k}}^\dagger(t)h_{-\mathbf{k}}(t+\tau)\rangle_{\tilde{\rho}_0} &= |\eta_{\mathbf{k}}|^2\langle C_{1\mathbf{k}}(t)C_{1\mathbf{k}}^\dagger(t+\tau)\rangle_{\tilde{\rho}_0}\\ &+|\xi_{\mathbf{k}}|^2\langle C_{2\mathbf{k}}(t)C_{2\mathbf{k}}^\dagger(t+\tau)\rangle_{\tilde{\rho}_0}\\ &+\eta_{\mathbf{k}}^*\xi_{\mathbf{k}}\langle C_{1\mathbf{k}}(t)C_{2\mathbf{k}}^\dagger(t+\tau)\rangle_{\tilde{\rho}_0}\\ &+\eta_{\mathbf{k}}\xi_{\mathbf{k}}^*\langle C_{2\mathbf{k}}(t)C_{1\mathbf{k}}^\dagger(t+\tau)\rangle_{\tilde{\rho}_0}\,.\end{aligned} \tag{224}$$

As before, the τ-dependence in Eqs. (223) and (224) is treated by employing the Mori-Zwanzig projection formalism. From the corresponding equations of motion one finds

$$\begin{aligned}\langle e_{\mathbf{k}}^\dagger(t)e_{\mathbf{k}}(t+\tau)\rangle_{\tilde{\rho}_0} &= |\xi_{\mathbf{k}}|^2 e^{(-i\tilde{E}_{1\mathbf{k}}-\gamma)\tau}\langle C_{1\mathbf{k}}^\dagger(t)C_{1\mathbf{k}}(t)\rangle_{\tilde{\rho}_0}\\ &+|\eta_{\mathbf{k}}|^2 e^{(-i\tilde{E}_{2\mathbf{k}}-\gamma)\tau}\langle C_{2\mathbf{k}}^\dagger(t)C_{2\mathbf{k}}(t)\rangle_{\tilde{\rho}_0}\\ &-\eta_{\mathbf{k}}\xi_{\mathbf{k}}^* e^{(-i\tilde{E}_{2\mathbf{k}}-\gamma)\tau}\langle C_{1\mathbf{k}}^\dagger(t)C_{2\mathbf{k}}(t)\rangle_{\tilde{\rho}_0}\\ &-\eta_{\mathbf{k}}^*\xi_{\mathbf{k}} e^{(-i\tilde{E}_{1\mathbf{k}}-\gamma)\tau}\langle C_{2\mathbf{k}}^\dagger(t)C_{1\mathbf{k}}(t)\rangle_{\tilde{\rho}_0}\end{aligned} \tag{225}$$

and

$$\begin{aligned}\langle h_{-\mathbf{k}}^\dagger(t)h_{-\mathbf{k}}(t+\tau)\rangle_{\tilde{\rho}_0} &= |\eta_{\mathbf{k}}|^2 e^{(i\tilde{E}_{1\mathbf{k}}-\gamma)\tau}\langle C_{1\mathbf{k}}(t)C_{1\mathbf{k}}^\dagger(t)\rangle_{\tilde{\rho}_0}\\ &+|\xi_{\mathbf{k}}|^2 e^{(i\tilde{E}_{2\mathbf{k}}-\gamma)\tau}\langle C_{2\mathbf{k}}(t)C_{2\mathbf{k}}^\dagger(t)\rangle_{\tilde{\rho}_0}\\ &+\eta_{\mathbf{k}}^*\xi_{\mathbf{k}} e^{(i\tilde{E}_{2\mathbf{k}}-\gamma)\tau}\langle C_{1\mathbf{k}}(t)C_{2\mathbf{k}}^\dagger(t)\rangle_{\tilde{\rho}_0}\\ &+\eta_{\mathbf{k}}\xi_{\mathbf{k}}^* e^{(i\tilde{E}_{1\mathbf{k}}-\gamma)\tau}\langle C_{2\mathbf{k}}(t)C_{1\mathbf{k}}^\dagger(t)\rangle_{\tilde{\rho}_0}\end{aligned} \tag{226}$$

with γ being the damping rate of the electrons and holes of the e-h-p system due to the coupling to the fermionic baths. Combining all parts of the correlation functions with the same τ-dependence one obtains:

$$\langle e_{\mathbf{k}}^\dagger(t)e_{\mathbf{k}}(t+\tau)\rangle_{\tilde{\rho}_0} = a_{1\mathbf{k}}^e(t)\,e^{(-i\tilde{E}_{1\mathbf{k}}-\gamma)\tau} + a_{2\mathbf{k}}^e(t)\,e^{(-i\tilde{E}_{2\mathbf{k}}-\gamma)\tau} \tag{227}$$

and

$$\langle h_{-\mathbf{k}}^\dagger(t)h_{-\mathbf{k}}(t+\tau)\rangle_{\tilde{\rho}_0} = a_{1\mathbf{k}}^h(t)e^{(i\tilde{E}_{1\mathbf{k}}-\gamma)\tau} + a_{2\mathbf{k}}^h(t)e^{(i\tilde{E}_{2\mathbf{k}}-\gamma)\tau}\,. \tag{228}$$

Here, we have introduced coefficients

$$a_{1\mathbf{k}}^e(t) = |\xi_{\mathbf{k}}|^2 A_{\mathbf{k}}^{11}(t) - \eta_{\mathbf{k}}^*\xi_{\mathbf{k}}A_{\mathbf{k}}^{21}(t)\,, \tag{229}$$

$$a_{2\mathbf{k}}^e(t) = |\eta_{\mathbf{k}}|^2 A_{\mathbf{k}}^{22}(t) - \eta_{\mathbf{k}}\xi_{\mathbf{k}}^*A_{\mathbf{k}}^{12}(t)\,, \tag{230}$$

$$a_{1\mathbf{k}}^h(t) = |\eta_{\mathbf{k}}|^2(1-A_{\mathbf{k}}^{11}(t)) - \eta_{\mathbf{k}}\xi_{\mathbf{k}}^*A_{\mathbf{k}}^{12}(t)\,, \tag{231}$$

$$a_{2\mathbf{k}}^h(t) = |\xi_{\mathbf{k}}|^2(1-A_{\mathbf{k}}^{22}(t)) - \eta_{\mathbf{k}}^*\xi_{\mathbf{k}}A_{\mathbf{k}}^{21}(t)\,, \tag{232}$$

and $A_{\mathbf{k}}^{nm}(t) = \langle (C_{n\mathbf{k}}^\dagger C_{m\mathbf{k}})(t)\rangle_{\tilde{\rho}_0}$ [compare Eq. (116)]. Finally, inserting the relations (227) and (228) into Eq. (220) and performing the integration over τ one finds

$$\begin{aligned}S_{\rm inc}(\mathbf{q},\omega) = \frac{1}{N\pi}\Re\sum_{\mathbf{k}}|\hat{v}_{\mathbf{kq}}|^2 \\ \times\Big[\frac{2\gamma}{(\tilde{E}_{1\mathbf{k}+\mathbf{q}}-\tilde{E}_{1\mathbf{k}}-\omega)^2+(2\gamma)^2}\,a_{1\mathbf{k}+\mathbf{q}}^e(t)\,a_{1\mathbf{k}}^h(t)\\ +\frac{2\gamma}{(\tilde{E}_{2\mathbf{k}+\mathbf{q}}-\tilde{E}_{2\mathbf{k}}-\omega)^2+(2\gamma)^2}\,a_{2\mathbf{k}+\mathbf{q}}^e(t)\,a_{2\mathbf{k}}^h(t)\\ +\frac{2\gamma}{(\tilde{E}_{1\mathbf{k}+\mathbf{q}}-\tilde{E}_{2\mathbf{k}}-\omega)^2+(2\gamma)^2}\,a_{1\mathbf{k}+\mathbf{q}}^e(t)\,a_{2\mathbf{k}}^h(t)\\ +\frac{2\gamma}{(\tilde{E}_{2\mathbf{k}+\mathbf{q}}-\tilde{E}_{1\mathbf{k}}-\omega)^2+(2\gamma)^2}\,a_{2\mathbf{k}+\mathbf{q}}^e(t)\,a_{1\mathbf{k}}^h(t)\Big]\,,\end{aligned} \tag{233}$$

where again only the dissipative part of the integral was considered. In Eq. (233) the coefficients $a_{1\mathbf{k}}^{e,h}(t)$ and $a_{2\mathbf{k}}^{e,h}(t)$ still depend on time t. The result for the steady state is obtained

in the limit $t \to \infty$. Thus,

$$S_{\rm inc}(\mathbf{q},\omega) = \frac{1}{N\pi}\sum_{\mathbf{k}} |\tilde{v}_{\mathbf{kq}}|^2 \tag{234}$$
$$\times\Big[\frac{2\gamma}{(\tilde{E}_{1\mathbf{k}+\mathbf{q}}-\tilde{E}_{1\mathbf{k}}-\omega)^2+(2\gamma)^2}\, a^e_{1\mathbf{k}+\mathbf{q}}\, a^h_{1\mathbf{k}}$$
$$+\frac{2\gamma}{(\tilde{E}_{2\mathbf{k}+\mathbf{q}}-\tilde{E}_{2\mathbf{k}}-\omega)^2+(2\gamma)^2}\, a^e_{2\mathbf{k}+\mathbf{q}}\, a^h_{2\mathbf{k}}$$
$$+\frac{2\gamma}{(\tilde{E}_{1\mathbf{k}+\mathbf{q}}-\tilde{E}_{2\mathbf{k}}-\omega)^2+(2\gamma)^2}\, a^e_{1\mathbf{k}+\mathbf{q}}\, a^h_{2\mathbf{k}}$$
$$+\frac{2\gamma}{(\tilde{E}_{2\mathbf{k}+\mathbf{q}}-\tilde{E}_{1\mathbf{k}}-\omega)^2+(2\gamma)^2}\, a^e_{2\mathbf{k}+\mathbf{q}}\, a^h_{1\mathbf{k}}\Big]\,.$$

Choosing the coefficients $a^e_{(1,2)\mathbf{k}}$ and $a^h_{(1,2)\mathbf{k}}$ to be real is compatible with Eqs. (227) and (228). We obtain:

$$a^e_{1\mathbf{k}+\mathbf{q}} = |\xi_{\mathbf{k}+\mathbf{q}}|^2 A^{11}_{\mathbf{k}+\mathbf{q}} - \Re\big(\eta^*_{\mathbf{k}+\mathbf{q}}\xi_{\mathbf{k}+\mathbf{q}}A^{21}_{\mathbf{k}+\mathbf{q}}\big)\,, \tag{235}$$
$$a^e_{2\mathbf{k}+\mathbf{q}} = |\eta_{\mathbf{k}+\mathbf{q}}|^2 A^{22}_{\mathbf{k}+\mathbf{q}} - \Re\big(\eta_{\mathbf{k}+\mathbf{q}}\xi^*_{\mathbf{k}+\mathbf{q}}A^{12}_{\mathbf{k}+\mathbf{q}}\big)\,, \tag{236}$$
$$a^h_{1\mathbf{k}} = |\eta_{\mathbf{k}}|^2(1-A^{11}_{\mathbf{k}}) - \Re\big(\eta_{\mathbf{k}}\xi^*_{\mathbf{k}}A^{12}_{\mathbf{k}}\big)\,, \tag{237}$$
$$a^h_{2\mathbf{k}} = |\xi_{\mathbf{k}}|^2(1-A^{22}_{\mathbf{k}}) - \Re\big(\eta^*_{\mathbf{k}}\xi_{\mathbf{k}}A^{21}_{\mathbf{k}}\big) \tag{238}$$

with $A^{nm}_{\mathbf{k}} = A^{nm}_{\mathbf{k}}(t\to\infty)$

$$A^{11}_{\mathbf{k}} = |\xi_{\mathbf{k}}|^2 f_e(\tilde{E}_{1\mathbf{k}}) + |\eta_{\mathbf{k}}|^2\big(1-f_h(-\tilde{E}_{1\mathbf{k}})\big)\,, \tag{239}$$
$$A^{22}_{\mathbf{k}} = |\eta_{\mathbf{k}}|^2\, f_e(\tilde{E}_{2\mathbf{k}}) + |\xi_{\mathbf{k}}|^2\big(1-f_h(-\tilde{E}_{2\mathbf{k}})\big)\,, \tag{240}$$

and

$$\Re\big(\eta_{\mathbf{k}}\xi^*_{\mathbf{k}}A^{12}_{\mathbf{k}}\big) = \frac{-2\gamma^2\,|\xi_{\mathbf{k}}|^2|\eta_{\mathbf{k}}|^2}{(\tilde{E}_{1\mathbf{k}}-\tilde{E}_{2\mathbf{k}})^2+(2\gamma)^2} \tag{241}$$
$$\times\big(f_e(\tilde{E}_{1\mathbf{k}}) + f_e(\tilde{E}_{2\mathbf{k}}) + f_h(-\tilde{E}_{1\mathbf{k}}) + f_h(-\tilde{E}_{2\mathbf{k}}) - 2\big)$$

with $A^{21}_{\mathbf{k}} = (A^{12}_{\mathbf{k}})^*$. Obviously, the denominators of Eq. (234) describe the frequency dependence of $S_{\rm inc}(\mathbf{q},\omega)$. It is caused by transitions between energy levels of the quasiparticle Hamiltonian $\hat{\mathcal{H}}$. Whereas the first two excitations in (234) are due to transitions within the same quasiparticle bands, $\tilde{E}_{1\mathbf{k}+\mathbf{q}} \to \tilde{E}_{1\mathbf{k}}$ and $\tilde{E}_{2\mathbf{k}+\mathbf{q}} \to \tilde{E}_{2\mathbf{k}}$, the last two excitations result from transitions between the two bands. The factors $a^e_{(1,2)\mathbf{k}+\mathbf{q}}$ and $a^h_{(1,2)\mathbf{k}}$ in (234) determine the weight of the transitions. Note that all transitions are broadened by 2γ, i.e., twice the damping rate γ of single electrons or holes into their respective baths. In particular, for the case $\mathbf{q}=0$ one finds two quasi-elastic excitations around $\omega=0$ with a broadening of 2γ as well.

VII. NUMERICAL RESULTS

Evaluating the theory developed so far, we assume, for simplicity, $\varepsilon^e_{\mathbf{k}} = \varepsilon^h_{\mathbf{k}}$ and charge neutrality $\mu_e = \mu_h$. We then self-consistently solve the set of renormalization equations (A23)–(A27), (B3)–(B8), and (C41)–(C42), together with Eqs. (127)–(129), and (156) for the expectation values, in momentum space (on a grid with $N = 160$ lattice sites), for a one-dimensional system. Convergence is assumed to be achieved when the relative error of all quantities is less than 10^{-10}.

In the numerical work, we fix the interaction parameters $g = 0.2$, $U = 2.0$, the zero-point cavity photon frequency $\omega_c = 0.5$, and consider a finite but very low temperature $T = 0.001$. All energies will be measured in units of the particle transfer amplitude t and the wave vectors in units of the lattice constant a, where we take as typical values $t \simeq 2eV$ and $a \simeq 5$Å, yielding $c \simeq 0.4\,c_0$ for the speed of light of the microcavity (c_0 is the speed of light in vacuum). We found that the physical properties only slightly depend on the actual value of c [30].

Since the coupling between electrons, holes and photons is most effective in case the excitation energy of an electron-hole pair (exciton) matches a photonic excitation, we introduce, for the following discussion, the so-called detuning

$$d = \omega_c - E_g\,, \tag{242}$$

where E_g denotes the minimum distance (gap) between the bare electron and hole bands [30]. A positive (negative) E_g indicates a semiconducting (semimetallic) setting.

A. Expectation values

We will start by examining the relation between μ and μ_B. Remember that $\mu_e = \mu_h$ is the common chemical potential of both electronic baths, a parameter that is fixed from outside. The quantity μ, on the other side, gets a physical meaning in (quasi-) equilibrium only, where it becomes the chemical potential of the system. Therefore a difference between μ and μ_B can be taken as a measure for an increased importance of cavity photons (compare Sec. IV E). Figure 1 gives μ as a function of μ_B at fixed damping rate κ (left panels) and γ (right panels), describing the coupling of the system to the photonic and electronic baths, respectively. The upper and lower panels of Fig. 1 reflect large and small detuning, where $d = 3.5$, $E_g = -3$ and $d = -0.5$, $E_g = 1$, respectively. In the former case, we observe a linear dependence of μ on μ_B over almost the whole energy range of the electron band (bare bandwidth $4t$); the saturation when μ approaches the upper band edge originates from electron phase space filling. In the latter case, μ_B has to overcome the band gap E_g first, thereafter μ grows monotonously. If the self-consistently calculated μ reaches ω_c, any further excitation is photonic in nature in both cases. This is the range where μ notably deviates from μ_B and non-equilibrium effects become important. These are more prominent for small detuning and less photon leakage.

Figure 2 directly relates μ to the total number n_{exc} of excitations in the electron-hole-photon system, which is given by Eq. (180). At small-to-moderate excitation densities and large (small) detuning, the excitations are excitons (polaritons) for the most part [30]. Here, the system is close to (quasi-) equlibrium and μ takes over the role of a true chemical potential. When n_{exc} increases, the photons play a major role, and the system moves away from the former equilibrium configuration, which was described by $\mu = \mu_B$. This is why the curves $\mu(n_{exc})$ flatten for large n_{exc}. Of course, above $n_{exc} = 1$ any further excitation has to be photonic. Again, for large detuning, the overall behavior of $\mu(n_{exc})$ only weakly depends on the damping/coupling parameters κ and γ.

We now aim at a characterization of the possible condensed phases of our e-h-p system. In Fig. 3 we show the excitonic order parameter,

$$\Delta_X = -(U/N)\sum_{\mathbf{k}} d_{\mathbf{k}}\,, \tag{243}$$

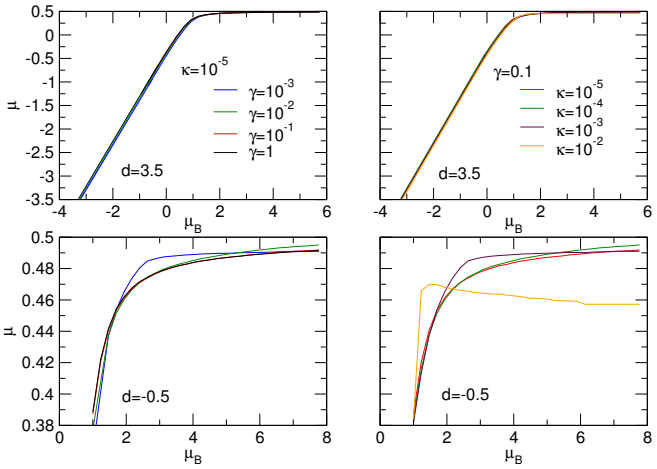

FIG. 1. (Color online) Parameter μ characterizing the steady state description (13)–(14), where μ becomes the chemical potential in equlibrium. Pictured here is μ as a function of μ_B for different values of γ at fixed $\kappa = 10^{-5}$ (left panels) and, likewise, for different values of κ at fixed $\gamma = 0.1$ (right panels). The two upper panels refer to detuning $d = 3.5$, the lower panels refer to $d = -0.5$; note the different scales of the ordinates.

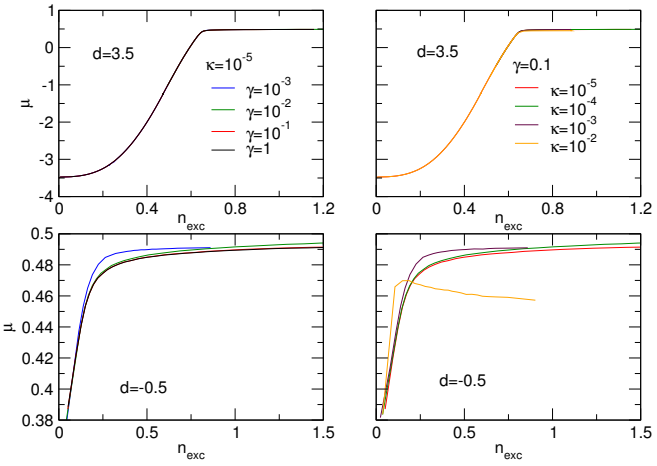

FIG. 2. (Color online) Steady-state parameter μ as a function of the total density of excitations in the e-h-p microcavity system, n_{exc} from (180). Shown are results for $d = 3.5$ (top) and $d = -0.5$ (bottom) at a fixed value of κ (left) and γ (right).

in dependence on the density of excitations. Figure 4 gives the corresponding photonic order parameter

$$\Delta_{ph} = -(g/\sqrt{N})\langle\psi_0\rangle \tag{244}$$

[compare Eq. (37)]. At large detuning ($d = 3.5$; upper panels), valence and conduction bands will penetrate each other and—for the considered values of the Coulomb interaction between electrons and holes ($U = 2$) and exciton-photon coupling ($g = 0.2$)—a gapful renormalized band structure develops, just as for a BCS-type excitonic insulator state [28]. Here, the condensate formed at low and intermediate excitation densities is mainly triggered by the Coulomb attraction

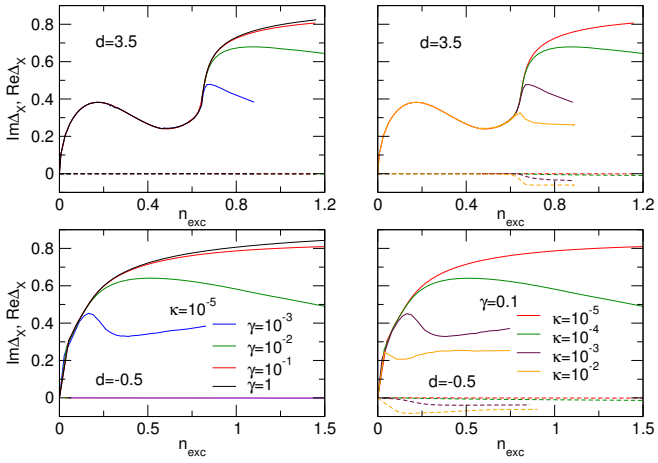

FIG. 3. (Color online) Excitonic order parameter Δ_X [see Eq. (243)], reflecting the exciton contribution to the condensate in the steady state. Depicted are the real parts (solid lines) and imaginary parts (dashed lines) of Δ_X as a function of n_{exc} for large (top) and small (bottom) detuning at fixed κ (left) and γ (right).

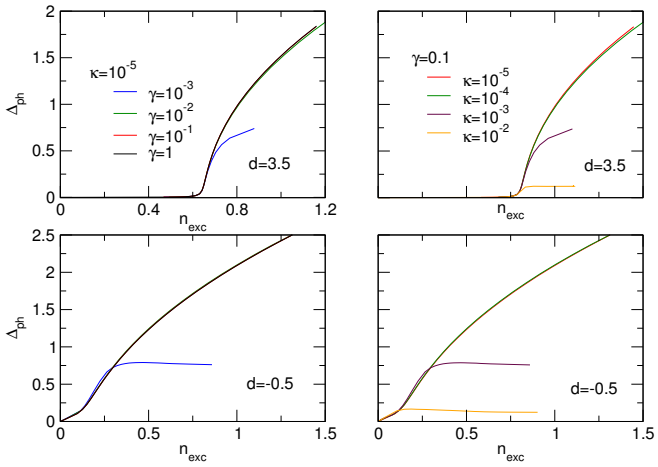

FIG. 4. (Color online) Photonic order parameter Δ_{ph} [see Eq. (243)], reflecting the photon contribution to the condensate in the steady state. Here, Δ_{ph} is given as a function of n_{exc} for $d = 3.5$ (top) and $d = -0.5$ (bottom) at fixed $\kappa = 10^{-5}$ (left) and $\gamma = 0.1$ (right).

between electrons and holes and therefore is predominantly an excitonic one; cf. the vanishing value of Δ_{ph} in Fig. 4. If we would have strengthened the Coulomb interaction at fixed n_{exc}, we would be able to observe a BCS-BEC crossover in the excitonic condendate [9, 34]. Increasing the density of excitation n_{exc} the location of the correlation-induced gap is shifted to larger k values, and phase-space and Fermi-surface effects become increasingly important. This is indicated by the downturn of $\Re\Delta_X$. At still larger values of n_{exc}, photonic excitations come into play more and more. As a consequence, the condensate turns from excitonic to polaritonic, and finally to a purely photonic one (lasing regime [26]). For small detuning ($d = -0.5$; lower panels), where the system is in the semiconducting regime from the very beginning, both exci-

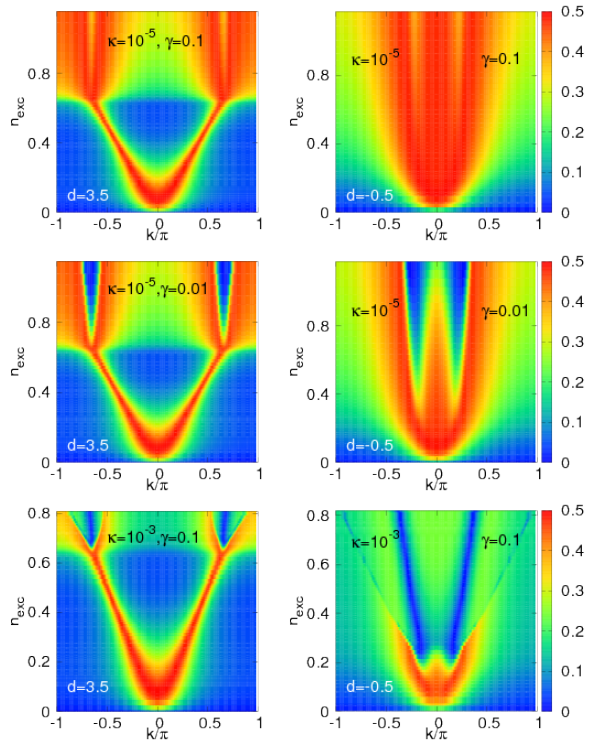

FIG. 5. (Color online) Electron-hole pairing amplitude $d_{\mathbf{k}}$ [see Eq.(33)], indicating exciton condensation. Shown is an intensity plot of its real part in the momentum-density plane for semimetallic ($d = 3.5$; left panels) and semiconducting ($d = -0.5$; right panels) situations.

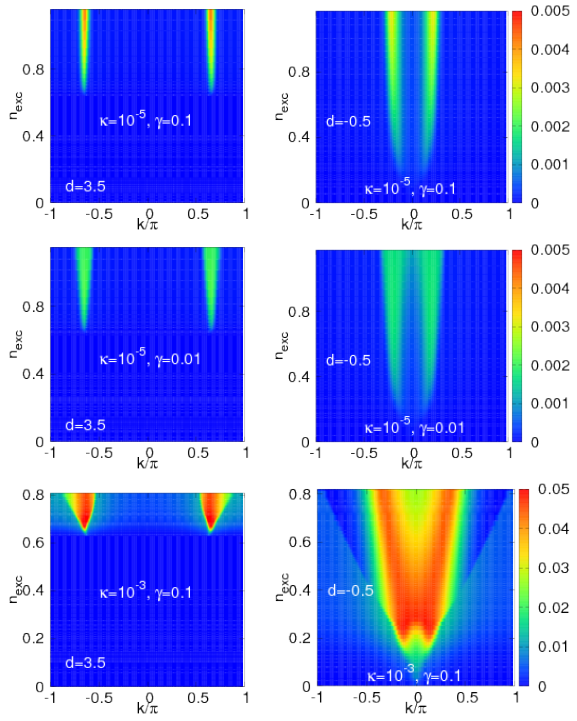

FIG. 6. (Color online) Exciton pairing amplitude $d_{\mathbf{k}}$. Shown is the intensity plot of its imaginary part in the momentum-density plane for $d = 3.5$ (left) and $d = -0.5$ (right).

tonic and photonic order parameters are finite, even at small excitation densities, which can be taken as a clear signature of a strong coupling between the light and matter degrees of freedom. As a result, a BEC of polaritons forms. Again the photons are dominant at large n_{exc} (especially in the lasing regime). Obviously the influence of the bath degrees of freedom on the results is more pronounced for smaller (larger) values of γ (κ). This is in accord with the analytical results of Sec. IV G, indicating that an equilibrium description is appropriate in the limit of large (vanishing) γ (κ). When γ gets smaller, we found self-consistent solutions of the renormalization equations in a smaller range of n_{exc} only. Note that the excitonic order parameter receives a finite imaginary part only for sufficiently large values of κ, almost irrespective of γ.

Figures 5, 6, and 7 show the wave-vector resolved, excitation-density dependent intensity of the real and imaginary parts of the electron-hole pairing amplitude $d_{\mathbf{k}}$ and the photon density expectation value $\langle \psi^\dagger_{\mathbf{q}} \psi_{\mathbf{q}} \rangle$. Not surprisingly, the results for small photon leakage κ and relatively large coupling to the electronic baths (upper panels) are more or less the same as in equilibrium [30]. In both the semimetallic (left panels) and semiconducting (right panels) regimes the amplitude for electron-hole pairing is largest at $k = 0$ if $n_{exc} \to 0$. Increasing the excitation density at large detuning, the maximum is shifted to larger k-values in the course of exciton formation, respecting the band structure, phase space filling and Pauli blocking, until, when μ approaches $\omega_c = 0.5$ near $n_{exc} \simeq 2/3$, the photon field severely interferes. From this

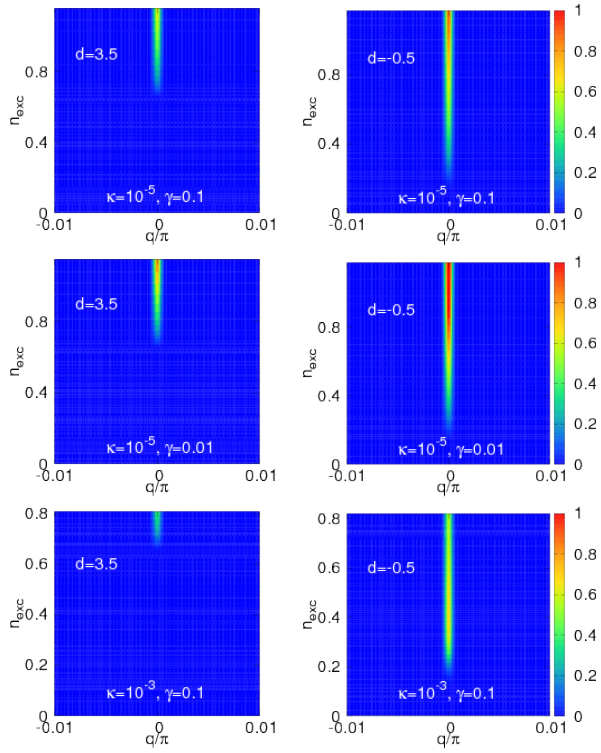

FIG. 7. (Color online) Intensity of the photon field $\langle \psi^\dagger_{\mathbf{q}} \psi_{\mathbf{q}} \rangle$ in the momentum-density plane for detunings $d = 3.5$ (left) and $d = -0.5$ (right).

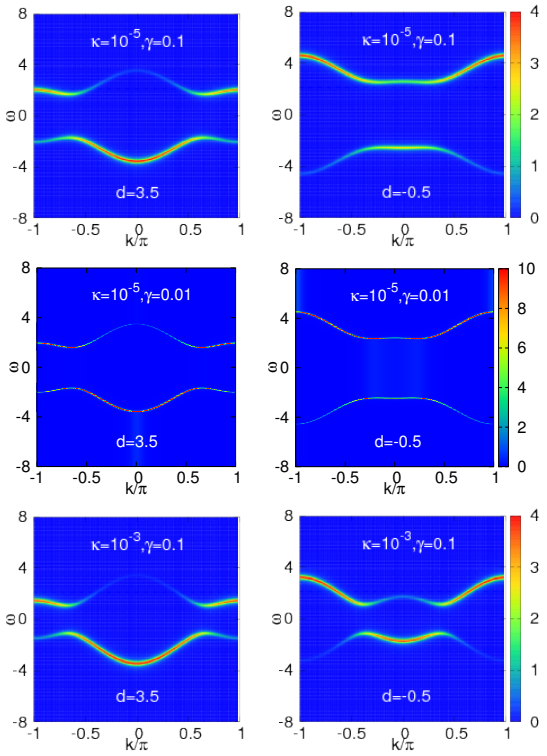

FIG. 8. (Color online) Electron single-particle spectrum in the steady state of the considered e-h-p microcavity system. The quasiparticle band dispersion clearly appears in the intensity plot of the coherent part of the fully renormalized spectral function, $A_{e,\mathrm{coh}}(\mathbf{k},\omega)$ given by Eq. (211). Results are given for typical semimetallic ($d = 3.5$, left) and semiconducting ($d = -0.5$, right) situations. Here, the excitation density $n_{exc} = 0.8$. Note that the frequency is measured from μ.

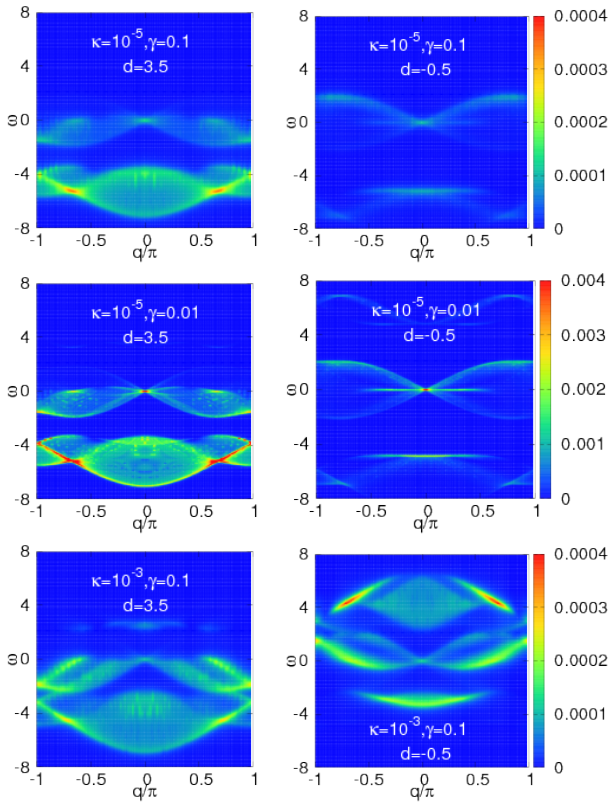

FIG. 9. (Color online) Steady-state luminescence of the e-h-p microcavity system under consideration. Shown is an intensity plot of its incoherent part, $S_{\mathrm{inc}}(\mathbf{q},\omega)$ given by Eq. (234), at $n_{exc} = 0.8$, for $d = 3.5$ (left) and $d = -0.5$ (right).

moment, the real (imaginary) part of the pairing amplitude is substantially reduced (enhanced) and the photon density becomes finite. Clearly, in view of the above, this effect gets stronger the smaller γ (see middle panels) or the larger κ will be (see lower panels). For $n_{exc} \gtrsim 2/3$ the pairing amplitude $d_{\mathbf{k}}$ is enhanced for almost all k- values, with the exception of the momenta (energies) where the photons interfere. Here, the system is more or less characterized by its large photon loss in the environment, whereby the leakage strengthens at larger κ (see lower left panels). At even greater n_{exc} one expects to enter the lasing regime [26]. For small detuning, exciton formation is intimately related to electron-hole excitation across the bare band gap, i.e., the coupling to the photons affects the properties of the system from the very beginning and, as a consequence, a broad maximum in $d_{\mathbf{k}}$ develops when n_{exc} increases. The strong signatures emerging in the imaginary part of $d_{\mathbf{k}}$ can be attributed to polariton formation. As a matter of course the maximum intensity of the photon field is always at $q = 0$, but the abrupt increase of the photon density changes to larger excitation densities for larger detuning.

B. Spectral properties

We now consider selected spectral quantities characterizing the physical properties of the e-h-p system if it is coupled to electronic and photonic baths. Thereby, we first examine how the correlations and fluctuations resulting from the Coulomb and light-matter interactions will renormalize the band structure. Of course, this band structure has to be calculated in a self-consistent way for a given excitation density since the electron and hole contributions to the spectral function are interrelated in the PRM scheme. Hereafter we consider $n_{exc} = 0.8$. The quasiparticle band dispersion shows up in the coherent part of the single-particle spectrum, $A_{e,\mathrm{coh}}(\mathbf{k},\omega)$ in Fig. 8, which probes both the occupied and unoccupied states as it is defined via the anticommutator (Green) function in Eq. (207). As briefly mentioned already above, at large detuning the bare bands interpenetrate and the electron-hole Coulomb attraction favors the formation of a macroscopic quantum-coherent excitonic insulator state, in formal analogy to the occurrence of the BCS-type superconducting phase. This becomes evident by looking at the quasiparticle bands shown in Fig. 8 left panels): Here the (correlation induced) band gaps open at finite momenta (around k_F), where an almost complete backfolding of the bands is observed. At the considered large n_{exc} the gap appears at the momenta where the real (imaginary) parts of the electron-hole pairing amplitude is substantially sup-

pressed (enhanced), cf. Figs. 5 and 6, which indicates the importance of photons in both the polartion BEC and lasing phases. A so-called "lasing gap", where (light-induced) electron-hole pairs will be formed around the laser frequency (momentum of the kinetic hole burning), recently has been predicted theoretically [26], but has not been observed experimentally so far. For small detuning the renormalized band structure is different in nature (right panels). In principle, the quasiparticle bands retain their (bare) semiconductor-like arrangement, but the particle-photon coupling causes a noticeable flattening (plateau structure) of the conduction band bottom and valence band top, thereby enlarging the single-particle spectral gap. The plateau structure can be attributed to a polariton (photonic) BEC. For both detunings, a smaller value of γ will reduce the spreading of the coherent signal while it enhances its intensity (see middle panels). A larger value of κ, keeping γ at fixed, will reduce the magnitude of the gap in the renormalized band structure. In this case the leakage to the external photon vacuum is enlarged, leading to a weakening of the excitonic order parameter $\tilde{\Delta}_{\mathbf{k}}$ and thus to a weakening of the quasiparticle band gap.

Finally, we will look at the steady-sate luminescence of the e-h-p system, considering the same parameters as for the single-particle spectra. Clearly the coherent part of the luminescence spectrum is the dominant one, however, $S_{\rm coh}(\mathbf{q},\omega)$ has neither a nontrivial $\mathbf{q}$- nor a nontrivial ω-dependence [see Eq. (216)]. Therefore, in Fig. 9, we only display the behavior of the incoherent part of the luminescence, $S_{\rm inc}(\mathbf{q},\omega)$, which is characterized by particle-hole excitations according to Eq. (218). The results shown in Fig. 9 include all possible annihilation and creation processes of electron-hole pairs inside and in-between the fully renormalized quasiparticle bands $\tilde{E}_{(1,2)\mathbf{k}}$ without any additional photons involved. From Eq. (235) it is evident that the interband contributions between the two bands $\tilde{E}_{1\mathbf{k}}$ and $\tilde{E}_{2\mathbf{k}}$ are the dominant ones, caused by terms being proportional to $|\xi_{\mathbf{k}}|^2|\xi_{\mathbf{k}+\mathbf{q}}|^2$ and $|\xi_{\mathbf{k}+\mathbf{q}}|^2|\xi_{\mathbf{k}}|^2$ in the prefactors of the last two contributions. Special attention deserves the significant flattening of the excitonic response at small momentum transfer for small detuning and $\kappa = 10^{-5}$, which is due to a strong light-matter interaction and indicates the formation of an exciton-polariton condensate.

VIII. SUMMARY AND CONCLUSIONS

The projector-based renormalization method (PRM) is a reliable and powerful analytical technique that has already been successfully applied to a wide range of *equilibrium* solid-state physics problems in the past; examples are magnetism, superconductivity, charge density wave formation, phonon-softening, or valence and metal-insulator transitions. The main purpose of this work was to provide a consistent extension of the PRM for dealing with more general *non-equilibrium* situations in open systems, as they appear when quantum systems are coupled to external reservoirs. A prime example for this is the light-matter coupling in semiconductor microcavities, where electrons and holes—for example, after being excited with light—can form excitonic bound states due to their Coulomb interaction, or can recombine into photons, when cavity photons can escape into the vacuum (e.g., because of mirrors with imperfect reflectivity). Furthermore, in such systems coherent quantum condensates may arise, realized as BCS or BEC equilibrium states, but also manifest nonequilibrium (lasing-like) phases. The PRM framework, we developed can treat, if combined with the Mori-Zwanzig projection technique, these equilibrium and nonequilibrium situations in a rather unified way. The steady-state properties of the system are thereby obtained from the long-term behavior of appropriate expectation values and, equally important, the many-body correlations and fluctuations processes are taken into account beyond mean-field in the whole range of excitation densities. Besides expectation values also spectral properties can be evaluated in the steady state. From a theoretical point of view, this ensures diverse future application possibilities of the proposed approach.

Other examples, where the newly developed PRM approach might be applied, coming from the very topical and promising field of ultracold atomic physics [35]. In these systems particles (atoms, molecules) or even BECs are loaded into optical lattices created by dynamic cavity fields and are studied in connection with different ordering phenomena, quantum phase transitions, superradiance phase transitions, driving and dissipation [36–41]. Here the particles in a quantum many-body correlated phase of matter strongly influence the properties of light and vice versa, whereby the tunable interplay between rather short-ranged direct particle-particle interaction and long-range interaction mediated by the coupling to the optical cavity mode is of particular importance. Then in particular the quantum properties of light scattered from the emergent structured cold-atom phases will require a non-equilibrium or at least steady-state description [42]. The Hamiltonians being normally discussed in this context are extended Bose-Hubbard-type models supplemented by an atom-field interaction part or Dicke-type models, and take into account dissipation due to photon leakage (coupling to reservoirs). The proposed PRM, adjusted correspondingly, is definitely suitable for treating such models.

In this paper we considered a rather generic open model system consisting of interacting electrons, holes and cavity photons and their corresponding reservoirs. The focus was on exciton and polariton formation, and their possible condensation in the course of spontaneous breaking demonstrated by nonvanishing excitonic and photonic order parameters. In the steady state, the nature of the condensate changes from an exciton to a polariton and finally to a photon dominated ground-state when the density of excitations increases. Thereby a finite expectation value of the photonic field operator is intrinsically connected with a finite imaginary part of the excitonic order parameter function and, from a physical perspective, with photon loss. Having assumed an electron/hole band symmetric case and charge neutrality, the difference between the self-consistently determined quantity μ (which takes over the role of a true chemical potential of the system in thermal equilibrium only) and the sum of the chemical potentials of the electron and hole baths μ_B can be used in order to quantify nonequilibrium effects. For small-to-intermediate excitation densities and large detuning (semimetallic situation) Fermi-surface and Pauli- blocking effects are important and the condensate is reminiscent of the BCS-type excitonic insulator phase, whereas for small detuning (semiconducting situation) the condensate typifies a Bose-Einstein condensate of preformed electron-hole pairs (Excitons). Note that if we would have increased the Coulomb interaction at fixed excitation density we could realize a BCS-BEC crossover in the excitonic condensate due to the growing Hartree shift between valence and conduction bands. In any case the fully renormal-

ized band dispersions were obtained from the coherent part of the single-particle spectral function and show the opening of the band gaps and significant differences between large and small detuning situations, such as a strong band backfolding and a pronounced band flattening of the valence (conduction) band top (bottom) in the former and latter case, respectively. As soon as we enter the regime where the photons and therefore nonequilibrium effects play an important role, our results will noticeably depend on the parameters γ and κ parametrizing respectively the couplings to the electron/hole and photon reservoirs. In this context we have shown that the present steady-state approach cannot be reduced to the case of a closed electron-hole-photon system simply by setting γ and κ to zero; instead one gets a description of thermal equilibrium in the limit of large γ. On the other hand, the photon leakage/loss strengthens at larger values of κ.

It might make sense to emphasize once more the key findings of the nonequilibrium effects of the steady state and to compare our results with previous results for the thermal equilibrium situation from Ref. [30]: (i) At small-to-moderate excitation densities n_{exc} the e-h-p subsystem is close to thermal equilibrium. In particular, for the largest used value $\gamma = 1$ of the coupling of cavity electrons and holes to their respective baths, the results of Fig. 1 - Fig. 4 agree very well with those from Ref. [30]. But also for smaller γ (and the largest detuning value $d = 3.5$) the e-h-p subsystem and the electronic baths stay in a common (quasi-) equilibrium. Note that the linear slope with $\mu = \mu_B$ in Fig. 1 stands alone for an increase of electrons and holes, whereas the following flattening of $\mu(\mu_B)$ is a Pauli-blocking effect after all quasiparticle states are already occupied by electrons and holes. Any further increase of n_{exc} or μ_B is solely governed by an increase of cavity photons. (ii) Increasing n_{exc} further the number of cavity photons increases. Thereby, for a sufficient large number of cavity photons they become affected by their coupling κ to the external free photons, which leads to a loss of cavity photons. This loss is intrinsically connected to the appearance of finite imaginary parts of the excitonic order parameters in Fig. 3 or Fig. 6, in particular for larger values of κ. For the case of small detuning $d = -0.5$ in Fig. 3 or Fig. 6 this effect is more pronounced already at small n_{exc} since in the semiconducting case photons are also present already at smaller excitation densities n_{exc}. The coupling of cavity photons to free space photons affects the properties of the system from the very beginning. To summarize, nonlinear effects become important whenever a sufficiently large number of cavity photons is present. Thereby, the coupling κ to external photons plays an important role but also the detuning of the system and less important the value of the coupling γ.

The limitations of the present theoretical approach are: (i) The initial density matrix ρ_0 was assumed to be factorizable into a part $\rho_{\rm S}$ for the subsystem $\mathcal{H}_{\rm S}$ and into a reservoir density $\rho_{\rm R}$, $\rho_0 = \rho_{\rm S}\rho_R$. Thereby $\rho_{\rm S}$ was assumed to describe thermal equilibrium for $\mathcal{H}_{\rm S}$, and $\rho_{\rm R}$ should be infinitely large so that it is not changed by renormalization effects. (ii) The interactions $\mathcal{H}_1$ of the subsystem $\mathcal{H}_{\rm S}$ was assumed to be 'small' and was treated in the renormalization equations in perturbation theory. The renormalization was only done in small steps $\Delta\lambda$, so that extreme high renormalization processes are taken into account in the fully renormalized quantities. Therefore, the renormalization method is usually valid for parameters of $\mathcal{H}_1$ which are of the same order as those of $\mathcal{H}_0$, i.e. far beyond usual perturbation theory. (iii) Finally, the influence of the reservoirs were taken into account in perturbation theory up to second order in the interaction $\mathcal{H}_{\rm SR}$ between the subsystem $\mathcal{H}_{\rm S}$ and the reservoirs $\mathcal{H}_{\rm R}$.

Although we exclusively focused on the exciton-polariton problem in this contribution, the extended PRM, bridging equilibrium and steady state descriptions, can be used to tackle other strongly open/driven quantum model systems with strong correlations, which opens opens a new avenue for exploring many-body effects in non-equilibrium situations, i.e., for ultracold atoms in coupled to radiation fields. Work along this line is in progress.

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Appendix A: Renormalization of $\mathcal{H}_\lambda$

The renormalization equations for the λ-dependent parameters of $\mathcal{H}_\lambda$ will be derived from Eq. (43) which transforms $\mathcal{H}_\lambda$ to $\mathcal{H}_{\lambda-\Delta\lambda}$. For sufficiently small renormalization steps $\Delta\lambda$, when the expansion in g and U can be limited to $\mathcal{O}(g^2)$ and $\mathcal{O}(U^2)$, we have

$$\begin{aligned}\mathcal{H}_{\mathrm{S},\lambda-\Delta\lambda} &= \mathcal{H}_{0,\lambda}+\mathcal{H}_{c,\lambda}+\mathcal{H}_{1,\lambda}\\ &\quad+[X_{\lambda,\Delta\lambda},\mathcal{H}_{0,\lambda}+\mathcal{H}_{c,\lambda}+\mathcal{H}_{1,\lambda}]\\ &\quad+\cdots,\end{aligned} \tag{A1}$$

where the representation (60) for $\mathcal{H}_{\mathrm{S},\lambda}$ has been used. Renormalization contributions arise from the three commutators on the right hand side which must be evaluated explicitly. Contributions of order $\mathcal{O}(X^2_{\lambda,\Delta\lambda})$ and higher will be neglected. From the first commutator $[X_{\lambda,\Delta\lambda},\mathcal{H}_{0,\lambda}]$ one finds renormalization contributions to $\Delta_{\mathbf{k},\lambda}$ and Γ_λ. They read according to Sec. III B:

$$\begin{aligned}\delta\Delta^{(0)}_{\mathbf{k},\lambda} &= -\frac{g}{\sqrt{N}}A_{\mathbf{k}0}(\lambda,\Delta\lambda)\,\omega_{0,\lambda}\langle\psi_0\rangle \\ &\quad-\frac{U}{N}\sum_{\mathbf{k}_1}B_{\mathbf{k}_1\mathbf{k},-\mathbf{k}_1,-\mathbf{k}}(\lambda,\Delta\lambda)(\varepsilon^e_{\mathbf{k}_1,\lambda}+\varepsilon^h_{-\mathbf{k}_1,\lambda})\,d_{\mathbf{k}_1}\,,\end{aligned} \tag{A2}$$

$$\delta\Gamma^{(0)}_\lambda = \frac{g}{N}\sum_{\mathbf{k}}A_{\mathbf{k}0}(\lambda,\Delta\lambda)(\varepsilon^e_{\mathbf{k},\lambda}+\varepsilon^h_{-\mathbf{k},\lambda})\,d_{\mathbf{k}}\,, \tag{A3}$$

where we have used expressions (65)-(67) for the generator $X_{\lambda,\Delta\lambda}$

$$X_{\lambda,\Delta\lambda} = X^g_{\lambda,\Delta\lambda}+X^U_{\lambda,\Delta\lambda} = -X^\dagger_{\lambda,\Delta\lambda} \tag{A4}$$

with

$$X^g_{\lambda,\Delta\lambda} = -\frac{g}{\sqrt{N}}\sum_{\mathbf{kq}}A_{\mathbf{kq}}(\lambda,\Delta\lambda)\big[:e^\dagger_{\mathbf{k}+\mathbf{q}}h^\dagger_{-\mathbf{k}}\psi_{\mathbf{q}}:-\mathrm{H.c.}\big]\,, \tag{A5}$$

$$\begin{aligned}X^U_{\lambda,\Delta\lambda} &= -\frac{U}{N}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}B_{\mathbf{k}_1\mathbf{k}_2;\mathbf{k}_3,\mathbf{k}_1+\mathbf{k}_3-\mathbf{k}_2}(\lambda,\Delta\lambda)\\ &\quad\times:e^\dagger_{\mathbf{k}_1}e_{\mathbf{k}_2}\,h^\dagger_{\mathbf{k}_3}h_{\mathbf{k}_1+\mathbf{k}_3-\mathbf{k}_2}:\,.\end{aligned} \tag{A6}$$

Note that both parts $X^g_{\lambda,\Delta\lambda}$ and $X^U_{\lambda,\Delta\lambda}$ contribute to $\delta\Delta^{(0)}_{\mathbf{k},\lambda}$, whereas only $X^g_{\lambda,\Delta\lambda}$ contributes to $\delta\Gamma^{(0)}_\lambda$. For the second commutator $[X_{\lambda,\Delta\lambda},\mathcal{H}_{c,\lambda}]$ one finds:

$$\begin{aligned}[X^g_{\lambda,\Delta\lambda},\mathcal{H}_{c,\lambda}] &= \frac{g}{\sqrt{N}}\sum_{\mathbf{k}}\big(A_{\mathbf{k}0}(\lambda,\Delta\lambda)\Delta_{\mathbf{k},\lambda}\\ &\qquad\times(1-n^e_{\mathbf{k}}-n^h_{-\mathbf{k}})\,\psi^\dagger_0+\mathrm{H.c.}\big)\\ &\quad+\frac{g}{\sqrt{N}}\sum_{\mathbf{k}}\big(A_{\mathbf{k}0}(\lambda,\Delta\lambda)\Delta_{\mathbf{k},\lambda}\langle\psi^\dagger_0\rangle\\ &\qquad\times(1-e^\dagger_{\mathbf{k}}e_{\mathbf{k}}-h^\dagger_{-\mathbf{k}}h_{-\mathbf{k}})+\mathrm{H.c.}\big)\\ &\quad-g\Gamma_\lambda\sum_{\mathbf{k}}\big(A_{\mathbf{k}0}(\lambda,\Delta\lambda)\,e^\dagger_{\mathbf{k}}h^\dagger_{-\mathbf{k}}+\mathrm{H.c.}\big)\,,\end{aligned} \tag{A7}$$

$$\begin{aligned}[X^U_{\lambda,\Delta\lambda},\mathcal{H}_{c,\lambda}] &= -\frac{U}{N}\sum_{\mathbf{k}\mathbf{k}_1}\Big[B_{\mathbf{k}_1,\mathbf{k},-\mathbf{k}_1,-\mathbf{k}}(\lambda,\Delta\lambda)\,\Delta_{\mathbf{k},\lambda}\\ &\qquad\times\big(d^*_{\mathbf{k}_1}(1-e^\dagger_{\mathbf{k}}e_{\mathbf{k}}-h^\dagger_{-\mathbf{k}}h_{-\mathbf{k}})\\ &\qquad+(1-n^e_{\mathbf{k}}-n^h_{-\mathbf{k}})\,e^\dagger_{\mathbf{k}_1}h^\dagger_{-\mathbf{k}_1}\big)+\mathrm{H.c.}\Big]\end{aligned} \tag{A8}$$

(omitting irrelevant constants), where again only renormalization contributions of the operator structure of $\mathcal{H}_{\mathrm{S},\lambda}$ are retained. We point out that the contributions in Eqs. (A7) and (A8) which renormalize $\varepsilon^e_{\mathbf{k}}$ and $\varepsilon^h_{-\mathbf{k}}$ are of second order in the order parameters; they should be small and will be neglected. The remaining contributions renormalize Γ_λ and $\Delta_{\mathbf{k},\lambda}$:

$$\delta\Gamma^{(c)}_\lambda = \frac{g}{N}\sum_{\mathbf{k}}A_{\mathbf{k}0}(\lambda,\Delta\lambda)\Delta_{\mathbf{k},\lambda}(1-n^e_{\mathbf{k}}-n^h_{-\mathbf{k}}), \tag{A9}$$

$$\begin{aligned}\delta\Delta^{(c)}_{\mathbf{k},\lambda} &= -g\Gamma_\lambda A_{\mathbf{k}0}(\lambda,\Delta\lambda)\\ &\quad-\frac{U}{N}\sum_{\mathbf{k}_1}B_{\mathbf{k},\mathbf{k}_1,-\mathbf{k},-\mathbf{k}_1}(\lambda,\Delta\lambda)\,\Delta_{\mathbf{k}_1,\lambda}\\ &\quad\times(1-n^e_{\mathbf{k}_1}-n^h_{-\mathbf{k}_1})\,.\end{aligned} \tag{A10}$$

Next we look at the last commutator $[X_{\lambda,\Delta\lambda},\mathcal{H}_{1,\lambda}]$ in Eq. (A1). Neglecting off-diagonal commutators we first obtain,

$$\begin{aligned}[X^g_{\lambda,\Delta\lambda},\mathcal{H}_{g,\lambda}] &= \frac{2g^2}{N}\sum_{\mathbf{kq}}A_{\mathbf{k}-\mathbf{q},\mathbf{q}}(\lambda,\Delta\lambda)(n^\psi_{\mathbf{q}}+n^h_{\mathbf{q}-\mathbf{k}})\,e^\dagger_{\mathbf{k}}e_{\mathbf{k}}\\ &\quad+\frac{2g^2}{N}\sum_{\mathbf{kq}}A_{\mathbf{kq}}(\lambda,\Delta\lambda)(n^\psi_{\mathbf{q}}+n^e_{\mathbf{q}+\mathbf{k}})\,h^\dagger_{-\mathbf{k}}h_{-\mathbf{k}}\\ &\quad-\frac{2g^2}{N}\sum_{\mathbf{kq}}A_{\mathbf{kq}}(\lambda,\Delta\lambda)(1-n^e_{\mathbf{k}+\mathbf{q}}-n^h_{-\mathbf{k}})\\ &\quad\times\big(\psi^\dagger_{\mathbf{q}}\psi_{\mathbf{q}}-\delta_{\mathbf{q},0}(\langle\psi^\dagger_0\rangle\psi_0-\langle\psi_0\rangle\psi^\dagger_0)\big)\,,\end{aligned} \tag{A11}$$

where we have introduced the following expectation value for the photon fluctuations:

$$n^\psi_{\mathbf{q}} = \langle\psi^\dagger_{\mathbf{q}}\psi_{\mathbf{q}}\rangle-\delta_{\mathbf{q},0}\langle\psi^\dagger_0\rangle\langle\psi_0\rangle\,. \tag{A12}$$

Eq. (A11) leads to renormalization contributions of $\varepsilon^e_{\mathbf{k},\lambda}$, $\varepsilon^h_{-\mathbf{k},\lambda}$, $\omega_{\mathbf{q},\lambda}$, and Γ_λ:

$$\delta\varepsilon^{e(g)}_{\mathbf{k},\lambda} = \frac{2g^2}{N}\sum_{\mathbf{q}}A_{\mathbf{k}-\mathbf{q},\mathbf{q}}(\lambda,\Delta\lambda)(n^\psi_{\mathbf{q}}+n^h_{\mathbf{q}-\mathbf{k}})\,, \tag{A13}$$

$$\delta\varepsilon^{h(g)}_{\mathbf{k}\lambda} = \frac{2g^2}{N}\sum_{\mathbf{q}}A_{\mathbf{kq}}(\lambda,\Delta\lambda)(n^\psi_{\mathbf{q}}+n^e_{\mathbf{q}+\mathbf{k}})\,, \tag{A14}$$

$$\delta\omega^{(g)}_{\mathbf{q},\lambda} = -\frac{2g^2}{N}\sum_{\mathbf{k}}A_{\mathbf{kq}}(\lambda,\Delta\lambda)(1-n^e_{\mathbf{k}+\mathbf{q}}-n^h_{-\mathbf{k}})\,, \tag{A15}$$

$$\delta\Gamma^{(g)}_\lambda = \frac{2g^2}{N\sqrt{N}}\sum_{\mathbf{k}}A_{\mathbf{k}0}(\lambda,\Delta\lambda)(1-n^e_{\mathbf{k}}-n^h_{-\mathbf{k}})\langle\psi_0\rangle\,. \tag{A16}$$

The evaluation of the second commutator $[X^U_{\lambda,\Delta\lambda},\mathcal{H}_{U,\lambda}]$ to $[X_{\lambda,\Delta\lambda},\mathcal{H}_{1,\lambda}]$ is more evolved. Our starting point is

$$\begin{aligned}[X^U_{\lambda,\Delta\lambda},\mathcal{H}_{U,\lambda}] &= \frac{U^2}{N^2}\sum_{\substack{\mathbf{k}_1\mathbf{k}_2\mathbf{q}\\ \mathbf{k}'_1\mathbf{k}'_2\mathbf{q}'}}\Gamma^{\mathbf{k}_1,\mathbf{k}_1-\mathbf{q};\mathbf{k}_2,\mathbf{k}_2+\mathbf{q}}_{\mathbf{k}'_1,\mathbf{k}'_1-\mathbf{q}';\mathbf{k}'_2,\mathbf{k}'_2+\mathbf{q}'}(\lambda,\Delta\lambda)\\ &\quad\times\big[:e^\dagger_{\mathbf{k}_1}e_{\mathbf{k}_1-\mathbf{q}}h^\dagger_{\mathbf{k}_2}h_{\mathbf{k}_2+\mathbf{q}}:,h^\dagger_{\mathbf{k}'_2+\mathbf{q}'}h_{\mathbf{k}'_2}e^\dagger_{\mathbf{k}'_1-\mathbf{q}'}e_{\mathbf{k}'_1}:\big]\,,\end{aligned} \tag{A17}$$

where we have introduced

$$\Gamma^{\mathbf{k}_1,\mathbf{k}_1-\mathbf{q},\mathbf{k}_2,\mathbf{k}_2+\mathbf{q}}_{\mathbf{k}'_1,\mathbf{k}'_1-\mathbf{q}',\mathbf{k}'_2,\mathbf{k}'_2+\mathbf{q}'}(\lambda,\Delta\lambda) \tag{A18}$$

$$\begin{aligned}&= \frac{1}{2}\big[B_{\mathbf{k}'_1,\mathbf{k}'_1-\mathbf{q}',\mathbf{k}'_2,\mathbf{k}'_2+\mathbf{q}'}(\lambda,\Delta\lambda)\,\Theta_{\mathbf{k}_1,\mathbf{k}_1-\mathbf{q},\mathbf{k}_2,\mathbf{k}_2+\mathbf{q},\lambda}\\ &\quad+B_{\mathbf{k}_1,\mathbf{k}_1-\mathbf{q},\mathbf{k}_2,\mathbf{k}_2+\mathbf{q}}(\lambda,\Delta\lambda)\,\Theta_{\mathbf{k}'_1,\mathbf{k}'_1-\mathbf{q}',\mathbf{k}'_2,\mathbf{k}'_2+\mathbf{q}',\lambda}\big]\,.\end{aligned} \tag{A19}$$

We first extract the part of the commutator (A17) that renormalizes the electronic one-particle energies. It reads

$$\frac{U^2}{N^2}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{q}}\sum_{\mathbf{k}_1'\mathbf{k}_2'\mathbf{q}'}\Gamma^{\mathbf{k}_1,\mathbf{k}_1-\mathbf{q};\mathbf{k}_2,\mathbf{k}_2+\mathbf{q}}_{\mathbf{k}_1',\mathbf{k}_1'-\mathbf{q}';\mathbf{k}_2',\mathbf{k}_2'+\mathbf{q}'}(\lambda,\Delta\lambda)$$
$$\times\Big[\delta_{\mathbf{k}_2+\mathbf{q},\mathbf{k}_2'+\mathbf{q}}:e^\dagger_{\mathbf{k}_1}e_{\mathbf{k}_1-\mathbf{q}}:\ h^\dagger_{\mathbf{k}_2}h_{\mathbf{k}_2'}\ :e^\dagger_{\mathbf{k}_1'-\mathbf{q}'}e_{\mathbf{k}_1'}:$$
$$-\delta_{\mathbf{k}_2,\mathbf{k}_2'}:e^\dagger_{\mathbf{k}_1}e_{\mathbf{k}_1-\mathbf{q}}:\ h^\dagger_{\mathbf{k}_2'+\mathbf{q}'}h_{\mathbf{k}_2+\mathbf{q}}\ :e^\dagger_{\mathbf{k}_1'-\mathbf{q}'}e_{\mathbf{k}_1'}:$$
$$+\delta_{\mathbf{k}_1-\mathbf{q},\mathbf{k}_1'-\mathbf{q}'}:h^\dagger_{\mathbf{k}_2'+\mathbf{q}'}h_{\mathbf{k}_2'}:\ e^\dagger_{\mathbf{k}_1}e_{\mathbf{k}_1'}\ :h^\dagger_{\mathbf{k}_2}h_{\mathbf{k}_2+\mathbf{q}}:$$
$$-\delta_{\mathbf{k}_1,\mathbf{k}_1'}:h^\dagger_{\mathbf{k}_2'+\mathbf{q}'}h_{\mathbf{k}_2'}:\ e^\dagger_{\mathbf{k}_1'-\mathbf{q}'}e_{\mathbf{k}_1-\mathbf{q}}\ :h^\dagger_{\mathbf{k}_2}h_{\mathbf{k}_2+\mathbf{q}}:\Big].$$

From this, by truncation, we extract those contributions which are proportional to $e^\dagger_{\mathbf{k}}e_{\mathbf{k}}$ or $h^\dagger_{-\mathbf{k}}h_{-\mathbf{k}}$ and arrive at the renormalization contributions to $\varepsilon^e_{\mathbf{k}}$ and $\varepsilon^h_{\mathbf{k}}$:

$$\delta\varepsilon^{e(U)}_{\mathbf{k},\lambda}=\frac{U^2}{N^2}\sum_{\mathbf{k}_2\mathbf{q}}\Big(B_{\mathbf{k},\mathbf{k}-\mathbf{q};\mathbf{k}_2,\mathbf{k}_2+\mathbf{q}}(\lambda,\Delta\lambda)$$
$$\times\big[(n^h_{\mathbf{k}_2}-n^h_{\mathbf{k}_2+\mathbf{q}})(1-n^e_{\mathbf{k}-\mathbf{q}})+n^h_{\mathbf{k}+\mathbf{q}}(1-n^h_{\mathbf{k}_2})$$
$$-B_{\mathbf{k}+\mathbf{q},\mathbf{k};\mathbf{k}_2,\mathbf{k}_2+\mathbf{q}}(\lambda,\Delta\lambda)$$
$$\times\big[(n^h_{\mathbf{k}_2}-n^h_{\mathbf{k}_2+\mathbf{q}})\,n^e_{\mathbf{k}+\mathbf{q}}+n^h_{\mathbf{k}+\mathbf{q}}(1-n^h_{\mathbf{k}_2})\big]\Big),\qquad(A20)$$

$$\delta\varepsilon^{h(U)}_{\mathbf{k},\lambda}=\frac{U^2}{N^2}\sum_{\mathbf{k}_1\mathbf{q}}\Big(B_{\mathbf{k}_1,\mathbf{k}_1-\mathbf{q};\mathbf{k}-\mathbf{q},\mathbf{k}}(\lambda,\Delta\lambda)$$
$$\times\big[(n^e_{\mathbf{k}_1}-n^e_{\mathbf{k}_1-\mathbf{q}})(1-n^h_{\mathbf{k}-\mathbf{q}})-n^e_{\mathbf{k}_1}(1-n^e_{\mathbf{k}_1-\mathbf{q}})$$
$$-B_{\mathbf{k}_1,\mathbf{k}_1-\mathbf{q};\mathbf{k},\mathbf{k}+\mathbf{q}}(\lambda,\Delta\lambda)$$
$$\times\big[(n^e_{\mathbf{k}_1}-n^e_{\mathbf{k}_1-\mathbf{q}})\,n^h_{\mathbf{k}+\mathbf{q}}-n^e_{\mathbf{k}_1}(1-n^e_{\mathbf{k}_1-\mathbf{q}})\big]\Big).\qquad(A21)$$

In the same way, the renormalization contributions to $\Delta_{\mathbf{k},\lambda}$ can be extracted from (A17). Again, by truncation, we collect the parts being proportional to $e^\dagger_{\mathbf{k}}h^\dagger_{-\mathbf{k}}$ or $h_{-\mathbf{k}}e_{\mathbf{k}}$. Since $X^U(\lambda,\Delta\lambda)$ and $\mathcal{H}_{U,\lambda}$ are time-ordered expressions, a truncation within $X^U(\lambda,\Delta\lambda)$ and within $\mathcal{H}_{U,\lambda}$ is thereby forbidden. One finds

$$\delta\Delta^{(U)}_{\mathbf{k},\lambda}=-\frac{U^2}{N^2}\sum_{\mathbf{k}_1\mathbf{q}}\Big(\Gamma^{\mathbf{k}_1+\mathbf{q},\mathbf{k}_1;-\mathbf{k},-\mathbf{k}+\mathbf{q}}_{\mathbf{k}_1+\mathbf{q},\mathbf{k};-\mathbf{k}_1,-\mathbf{k}+\mathbf{q}}(\lambda,\Delta\lambda)$$
$$\times(n^e_{\mathbf{k}_1+\mathbf{q}}-n^h_{-\mathbf{k}+\mathbf{q}})d_{\mathbf{k}_1}$$
$$+\Gamma^{\mathbf{k},\mathbf{k}-\mathbf{q};-(\mathbf{k}_1+\mathbf{q}),-\mathbf{k}_1}_{\mathbf{k}_1,\mathbf{k}-\mathbf{q};-(\mathbf{k}_1+\mathbf{q}),-\mathbf{k}}(\lambda,\Delta\lambda)(n^e_{\mathbf{k}-\mathbf{q}}-n^h_{-(\mathbf{k}+\mathbf{q})})d_{\mathbf{k}_1}\Big).\qquad(A22)$$

Summing up, the following renormalization equations between the energy parameters of $\mathcal{H}_\lambda$ and $\mathcal{H}_{\lambda-\Delta\lambda}$ were found:

$$\varepsilon^e_{\mathbf{k},\lambda-\Delta\lambda}=\varepsilon^e_{\mathbf{k},\lambda}+\delta\varepsilon^{e(g)}_{\mathbf{k},\lambda}+\delta\varepsilon^{e(U)}_{\mathbf{k},\lambda},\qquad(A23)$$
$$\varepsilon^h_{\mathbf{k},\lambda-\Delta\lambda}=\varepsilon^h_{\mathbf{k},\lambda}+\delta\varepsilon^{h(g)}_{\mathbf{k},\lambda}+\delta\varepsilon^{h(U)}_{\mathbf{k},\lambda},\qquad(A24)$$
$$\omega_{\mathbf{q},\lambda-\Delta\lambda}=\omega_{\mathbf{q},\lambda}+\delta\omega^{(g)}_{\mathbf{q},\lambda},\qquad(A25)$$
$$\Delta_{\mathbf{k},\lambda-\Delta\lambda}=\Delta_{\mathbf{k},\lambda}+\delta\Delta^{(0)}_{\mathbf{k},\lambda}+\delta\Delta^{(c)}_{\mathbf{k},\lambda}+\delta\Delta^{(U)}_{\mathbf{k},\lambda},\qquad(A26)$$
$$\Gamma_{\lambda-\Delta\lambda}=\Gamma_\lambda+\delta\Gamma^{(0)}_\lambda+\delta\Gamma^{(c)}_\lambda+\delta\Gamma^{(g)}_\lambda.\qquad(A27)$$

Appendix B: Renormalization of electronic operators

Starting from an appropriate ansatz for the single-fermion operators $e^\dagger_{\mathbf{k},\lambda}$ and $h^\dagger_{-\mathbf{k},\lambda}$, according to Eqs. (103) and (104), we have

$$e^\dagger_{\mathbf{k},\lambda}=x_{\mathbf{k},\lambda}e^\dagger_{\mathbf{k}}+\frac{1}{\sqrt{N}}\sum_{\mathbf{q}}t_{\mathbf{k}-\mathbf{q},\mathbf{q},\lambda}h_{\mathbf{q}-\mathbf{k}}:\psi^\dagger_{\mathbf{q}}:$$
$$+\frac{1}{N}\sum_{\mathbf{k}_1\mathbf{k}_2}\alpha_{\mathbf{k}_1\mathbf{k}\mathbf{k}_2,\lambda}\,e^\dagger_{\mathbf{k}_1}:h^\dagger_{\mathbf{k}_2}h_{\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}}:,\qquad(B1)$$

$$h^\dagger_{-\mathbf{k},\lambda}=y_{\mathbf{k},\lambda}h^\dagger_{-\mathbf{k}}+\frac{1}{\sqrt{N}}\sum_{\mathbf{q}}u_{\mathbf{k},\mathbf{q},\lambda}\,e_{\mathbf{q}+\mathbf{k}}:\psi^\dagger_{\mathbf{q}}:$$
$$+\frac{1}{N}\sum_{\mathbf{k}_1\mathbf{k}_2}\beta_{\mathbf{k}_1\mathbf{k}_2,\mathbf{k}_2-\mathbf{k}_1-\mathbf{k},\lambda}:e^\dagger_{\mathbf{k}_1}e_{\mathbf{k}_2}:h^\dagger_{\mathbf{k}_2-\mathbf{k}_1-\mathbf{k}}.\qquad(B2)$$

In analogy to the renormalization equations for the parameters of $\mathcal{H}_\lambda$, one derives the following set of renormalization equations for the coefficients $t_{\mathbf{k}-\mathbf{q},\mathbf{q},\lambda}$, $\alpha_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3,\lambda}$, $u_{\mathbf{q},-\mathbf{k},\lambda}$, and $\beta_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3,\lambda}$:

$$t_{\mathbf{k}-\mathbf{q},\mathbf{q},\lambda-\Delta\lambda}=t_{\mathbf{k}-\mathbf{q},\mathbf{q},\lambda}+gx_{\mathbf{k},\lambda}A_{\mathbf{k}-\mathbf{q},\mathbf{q}}(\lambda,\Delta\lambda),\qquad(B3)$$
$$\alpha_{\mathbf{k}_1\mathbf{k}\mathbf{k}_2,\lambda-\Delta\lambda}=\alpha_{\mathbf{k}_1\mathbf{k}\mathbf{k}_2,\lambda}-Ux_{\mathbf{k},\lambda}B_{\mathbf{k}_1\mathbf{k}\mathbf{k}_2}(\lambda,\Delta\lambda),\qquad(B4)$$

$$u_{\mathbf{k}\mathbf{q},\lambda-\Delta\lambda}=u_{\mathbf{k}\mathbf{q},\lambda}-g\,y_{\mathbf{k},\lambda}A_{\mathbf{k},\mathbf{q}}(\lambda,\Delta\lambda),\qquad(B5)$$
$$\beta_{\mathbf{k}_1\mathbf{k}_2,\mathbf{k}-\mathbf{k}_1+\mathbf{k}_2,\lambda-\Delta\lambda}=\beta_{\mathbf{k}_1\mathbf{k}_2,\mathbf{k}-\mathbf{k}_1+\mathbf{k}_2,\lambda}$$
$$-Uy_{\mathbf{k},\lambda}B_{\mathbf{k}_1\mathbf{k}_2,\mathbf{k}-\mathbf{k}_1+\mathbf{k}_2}(\lambda,\Delta\lambda).\qquad(B6)$$

To obtain renormalization equations for $x_{\mathbf{k},\lambda}$ and $y_{\mathbf{k},\lambda}$ we use the anti-commutator relations for fermionic operators, $[e^\dagger_{\mathbf{k}}(\lambda),e_{\mathbf{k}}(\lambda)]_+=1$ and $[h^\dagger_{-\mathbf{k}}(\lambda),h_{-\mathbf{k}}(\lambda)]_+=1$, which are valid for any λ. We arrive at

$$|x_{\mathbf{k},\lambda}|^2=1-\frac{1}{N}\sum_{\mathbf{q}}|t_{\mathbf{k}-\mathbf{q},\mathbf{q},\lambda}|^2(n^\psi_{\mathbf{k}-\mathbf{q}}+n^h_{-\mathbf{q}})$$
$$-\frac{1}{N^2}\sum_{\mathbf{k}_1\mathbf{k}_2}|\alpha_{\mathbf{k}_1\mathbf{k}\mathbf{k}_2,\lambda}|^2\Big[n^h_{\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}}(1-n^h_{\mathbf{k}_2})$$
$$-n^e_{\mathbf{k}_1}(n^h_{\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}}-n^h_{\mathbf{k}_2})\Big],\qquad(B7)$$

$$|y_{\mathbf{k},\lambda}|^2=1-\frac{1}{N}\sum_{\mathbf{q}}|u_{\mathbf{k},\mathbf{q},\lambda}|^2(n^\psi_{\mathbf{q}}+n^e_{\mathbf{q}+\mathbf{k}})$$
$$-\frac{1}{N^2}\sum_{\mathbf{k}_1\mathbf{k}_2}|\beta_{\mathbf{k}_1\mathbf{k}_2,\mathbf{k}_2-\mathbf{k}_1-\mathbf{k}_2,\lambda}|^2\Big[n^e_{\mathbf{k}_1}(1-n^e_{\mathbf{k}_2})$$
$$+(1-n^h_{\mathbf{k}_2-\mathbf{k}_1-\mathbf{k}})(n^e_{\mathbf{k}_2}-n^e_{\mathbf{k}_1})\Big].\qquad(B8)$$

Here, a factorization approximation was used. The expectation values $n^e_{\mathbf{k}}$, $n^h_{-\mathbf{k}}$, and $n^\psi_{\mathbf{q}}$ on the right hand side are best chosen as steady-state expectation values and have been defined before in Eqs. (98) and (99). Moreover $n^\psi_{\mathbf{q}}=\langle:\psi^\dagger_{\mathbf{q}}::\psi_{\mathbf{q}}:\rangle$. Equations (B3)–(B6) together with Eqs. (B7), (B8), taken at $\lambda\to\lambda-\Delta\lambda$, represent a complete set of renormalization equations for the λ-dependent coefficients in Eqs. (B1) and (B2). They connect the parameter values at λ with those at $\lambda-\Delta\lambda$. The initial parameter values at cutoff $\lambda=\Lambda$ are:

$$\{x_{\mathbf{k},\Lambda},y_{\mathbf{k},\Lambda}\}=1,\qquad(B9)$$
$$\{t_{\mathbf{k}\mathbf{q},\Lambda},u_{\mathbf{k}\mathbf{q},\Lambda},\alpha_{\mathbf{k}_1\mathbf{k}\mathbf{k}_2,\lambda},\beta_{\mathbf{k}_1\mathbf{k}\mathbf{k}_2,\Lambda}\}=0.\qquad(B10)$$

By integrating the full set of renormalization equations between Λ and $\lambda=0$ one is led to the fully renormalized one-

particle operators:

$$\tilde{e}^\dagger_{\mathbf{k}} = \tilde{x}_{\mathbf{k}} e^\dagger_{\mathbf{k}} + \frac{1}{\sqrt{N}} \sum_{\mathbf{q}} \tilde{t}_{\mathbf{k}-\mathbf{q},\mathbf{q}} h_{\mathbf{q}-\mathbf{k}} \, : \psi^\dagger_{\mathbf{q}} : + \frac{1}{N} \sum_{\mathbf{k}_1\mathbf{k}_2} \tilde{\alpha}_{\mathbf{k}_1\mathbf{k}\mathbf{k}_2} e^\dagger_{\mathbf{k}_1} : h^\dagger_{\mathbf{k}_2} h_{\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}} : , \tag{B11}$$

$$\tilde{h}^\dagger_{-\mathbf{k}} = \tilde{y}_{\mathbf{k}} h^\dagger_{-\mathbf{k}} + \frac{1}{\sqrt{N}} \sum_{\mathbf{q}} \tilde{u}_{\mathbf{k},\mathbf{q}} e_{\mathbf{q}+\mathbf{k}} \, : \psi^\dagger_{\mathbf{q}} : + \frac{1}{N} \sum_{\mathbf{k}_1\mathbf{k}_2} \tilde{\beta}_{\mathbf{k}_1\mathbf{k}_2,\mathbf{k}_2-\mathbf{k}_1-\mathbf{k}} : e^\dagger_{\mathbf{k}_1} e_{\mathbf{k}_2} : h^\dagger_{\mathbf{k}_2-\mathbf{k}_1-\mathbf{k}} . \tag{B12}$$

With (B11) and (B12) one obtains in the limit $t \to \infty$:

$$n^e_{\mathbf{k}} = \langle (\tilde{e}^\dagger_{\mathbf{k}} \tilde{e}_{\mathbf{k}})(t\to\infty) \rangle_{\tilde{\rho}_0} = |\tilde{x}_{\mathbf{k}}|^2 \hat{n}^e_{\mathbf{k}} + \frac{1}{N} \sum_{\mathbf{q}} |\tilde{t}_{\mathbf{k}-\mathbf{q},\mathbf{q}}|^2 \, (1-\hat{n}^h_{\mathbf{k}-\mathbf{q}}) \, \hat{n}^\psi_{\mathbf{q}} + \frac{1}{N^2} \sum_{\mathbf{k}_1\mathbf{k}_2} |\tilde{\alpha}_{\mathbf{k}_1\mathbf{k}\mathbf{k}_2}|^2 \hat{n}^e_{\mathbf{k}_1} \hat{n}^h_{\mathbf{k}_2} (1-\hat{n}^h_{\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}}) , \tag{B13}$$

$$n^h_{-\mathbf{k}} = \langle (\tilde{h}^\dagger_{-\mathbf{k}} \tilde{h}_{-\mathbf{k}})(t\to\infty) \rangle_{\tilde{\rho}_0} = |\tilde{y}_{\mathbf{k}}|^2 \hat{n}^h_{-\mathbf{k}} + \frac{1}{N} \sum_{\mathbf{q}} |\tilde{u}_{\mathbf{k},\mathbf{q}}|^2 \, \hat{n}^\psi_{\mathbf{q}} (1-\hat{n}^e_{\mathbf{k}+\mathbf{q}}) + \frac{1}{N^2} \sum_{\mathbf{k}_1\mathbf{k}_2} |\tilde{\beta}_{\mathbf{k}_1\mathbf{k}_2,\mathbf{k}_2-\mathbf{k}_1-\mathbf{k}}|^2 \hat{n}^e_{\mathbf{k}_1} (1-\hat{n}^e_{\mathbf{k}_2}) \hat{n}^h_{\mathbf{k}_2-\mathbf{k}_1-\mathbf{k}} , \tag{B14}$$

and similarly

$$d^*_{\mathbf{k}} = \tilde{x}_{\mathbf{k}} \tilde{y}_{\mathbf{k}} \, \hat{d}^*_{\mathbf{k}} - \frac{1}{N} \sum_{\mathbf{k}_1} \big(\tilde{x}_{\mathbf{k}} \tilde{\beta}_{\mathbf{k}_1,\mathbf{k},-\mathbf{k}_1} \hat{n}^e_{\mathbf{k}} + \tilde{y}_{\mathbf{k}} \tilde{\alpha}_{\mathbf{k}_1,\mathbf{k},-\mathbf{k}_1} (\hat{n}^h_{-\mathbf{k}} - 1) \big) \, \hat{d}^*_{\mathbf{k}_1} , \tag{B15}$$

where a small term proportional to $\alpha_{\mathbf{k}_1,\mathbf{k},-\mathbf{k}_1} \beta_{\mathbf{k}_1,\mathbf{k},-\mathbf{k}_1}$ of $\mathcal{O}(U^2)$ was neglected. Another small contribution being proportional to $\langle (: \psi^\dagger_0 :)_{t\to\infty} \rangle_{\tilde{\rho}_0}$ was neglected as well. The quantities $\hat{n}^e_{\mathbf{k}}$, $\hat{n}^h_{-\mathbf{k}}$, and $\hat{n}^\psi_{\mathbf{q}}$ on the right hand side of Eqs. (B13)–(B15) are steady-state expectation values however formed with the renormalized density $\tilde{\rho}_0$ [also compare Eqs. (108)-(110)]:

$$\hat{n}^e_{\mathbf{k}} = \langle (e^\dagger_{\mathbf{k}} e_{\mathbf{k}})(t\to\infty) \rangle_{\tilde{\rho}_0} , \tag{B16}$$

$$\hat{n}^h_{-\mathbf{k}} = \langle (h^\dagger_{-\mathbf{k}} h_{-\mathbf{k}})(t\to\infty) \rangle_{\tilde{\rho}_0} , \tag{B17}$$

$$\hat{n}^\psi_{\mathbf{q}} = \langle (: \psi^\dagger_{\mathbf{q}} :: \psi_{\mathbf{q}} :)(t\to\infty) \rangle_{\tilde{\rho}_0} . \tag{B18}$$

The corresponding order parameter for the formation of excitons is

$$\hat{d}^*_{\mathbf{k}} = \langle (e^\dagger_{\mathbf{k}} h^\dagger_{-\mathbf{k}})(t\to\infty) \rangle_{\tilde{\rho}_0} . \tag{B19}$$

Appendix C: Steady state expectation values

Evaluating the expectation values $\langle \mathcal{A}(t) \rangle$ for $t \to \infty$, we use relation (94) and the steady-state condition (93):

$$\langle \mathcal{A}(t) \rangle = \langle \tilde{\mathcal{A}}(t) \rangle_{\tilde{\rho}_0} , \tag{C1}$$

$$\frac{\mathrm{d}}{\mathrm{dt}} \langle \mathcal{A}(t\to\infty) \rangle = \frac{\mathrm{d}}{\mathrm{dt}} \langle \tilde{\mathcal{A}}(t\to\infty) \rangle_{\tilde{\rho}_0} = 0 . \tag{C2}$$

On the right hand sides, the time dependence is governed by $\tilde{\mathcal{H}}$. $\tilde{\rho}_0$ denotes the fully transformed initial density, and $\tilde{\mathcal{A}}$ is the transformed operator of $\mathcal{A}$.

1. Electronic quantities

To analyze time-dependent expectation values for large times, the steady-state condition (93),

$$\frac{\mathrm{d}}{\mathrm{dt}} \langle (C^\dagger_{n\mathbf{k}} C_{m\mathbf{k}})(t) \rangle_{\tilde{\rho}_0} = 0 \quad \text{for } t \to \infty , \tag{C3}$$

must be fulfilled. Here, the time dependence is governed by the renormalized Hamiltonian

$$\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_{\mathrm{S}} + \mathcal{H}_{\mathrm{R}} + \mathcal{H}_{\mathrm{SR}} , \tag{C4}$$

and the expectation values are formed with the transformed initial density $\tilde{\rho}_0$. According to this "recipe", we first derive equations of motions using generalized Langevin equations. These dynamical equations are best found within the Mori-Zwanzig projection formalism [31, 32, 43] for a set of dynamical variables $\{\mathcal{A}_\nu = C^\dagger_{n\mathbf{k}} C_{m\mathbf{k}}, b^\dagger_{e,\mathbf{p}} b_{e,\mathbf{p}}, b^\dagger_{h,-\mathbf{p}} b_{h,-\mathbf{p}}\}$ $(n,m=1,2)$:

$$\frac{\mathrm{d}}{\mathrm{dt}} \mathcal{A}_\nu(t) = i \sum_\mu \mathcal{A}_\mu(t) \, \omega_{\mu\nu} - \int_0^t dt' \sum_\mu \mathcal{A}_\mu(t-t') \Sigma_{\mu\nu}(t') + \mathcal{F}_\nu(t) , \tag{C5}$$

where we have introduced a generalized scalar product $(\mathcal{A}|\mathcal{B}) = \langle \mathcal{A}^\dagger \mathcal{B} \rangle_{\tilde{\rho}_0}$ for operator variables $\mathcal{A}$, $\mathcal{B}$. The $\omega_{\mu\nu}$ and $\Sigma_{\mu\nu}(t)$ are generalized frequencies and self-energies, respectively, and $\mathcal{F}_\nu(t)$ is the random force:

$$\omega_{\mu\nu} = \sum_\eta \chi^{-1}_{\mu\eta} \left(\mathcal{A}_\eta | \tilde{\mathbf{L}} \mathcal{A}_\nu \right) , \tag{C6}$$

$$\Sigma_{\mu\nu}(t) = \sum_\eta \chi^{-1}_{\mu\eta} \left(\mathcal{A}_\eta | \tilde{\mathbf{L}}\mathbf{Q} \, e^{i\mathbf{Q}\tilde{\mathbf{L}}\mathbf{Q}\,t} \, \mathbf{Q}\tilde{\mathbf{L}} \mathcal{A}_\nu \right) , \tag{C7}$$

$$\mathcal{F}_\nu(t) = i \, e^{i\mathbf{Q}\tilde{\mathbf{L}}\mathbf{Q}\,t} \, \mathbf{Q}\tilde{\mathbf{L}} \mathcal{A}_\nu . \tag{C8}$$

The quantity $\tilde{\mathbf{L}}$ is the Liouville operator, defined by the commutator of $\tilde{\mathcal{H}}$ with any operator observable $\mathcal{A}$, i.e., $\tilde{\mathbf{L}}\mathcal{A} = [\tilde{\mathcal{H}}, \mathcal{A}]$, and $\mathbf{Q}$ is a generalized projector in the operator space which projects perpendicular to the subspace spanned by the set $\{A_\nu\}$. Moreover $\chi^{-1}_{\mu\eta}$ is the inverse of the generalized susceptibility matrix $\chi_{\eta'\mu} = (\mathcal{A}_{\eta'}|\mathcal{A}_\mu)$:

$$\sum_\mu \chi_{\eta'\mu} \chi^{-1}_{\mu\eta} = \delta_{\eta'\eta} . \tag{C9}$$

Since $\tilde{\mathcal{H}}$ does not commute with $\tilde{\rho}_0$ the expectation values $\langle \mathcal{A}_\nu(t) \rangle_{\tilde{\rho}_0}$ are intrinsically time dependent.

Let us first consider the equations for the electronic variables $\mathcal{A}^{nm}_{\mathbf{k}} := C^\dagger_{n\mathbf{k}} C_{m\mathbf{k}}$. Because they are dynamical eigenmodes of $\tilde{\mathcal{H}}_{\mathrm{S}}$ the frequencies $\omega_{\mu\nu}$ and self-energies $\Sigma_{\mu\nu}$ can be easily evaluated in lowest non-vanishing order perturbation theory in the interaction $\mathcal{H}_{\mathrm{SR}}$. One finds in Markov approxi-

mation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}^{12}_{\mathbf{k}}(t) &= i(\tilde{E}_{1\mathbf{k}}-\tilde{E}_{2\mathbf{k}})\mathcal{A}^{12}_{\mathbf{k}}\\ &-\mathcal{A}^{12}_{\mathbf{k}}\Big[|\xi_{\mathbf{k}}|^2\,\gamma^e_{\mathbf{k}}(\tilde{E}_{2\mathbf{k}})+|\eta_{\mathbf{k}}|^2\,\gamma^e_{\mathbf{k}}(\tilde{E}_{1\mathbf{k}})\\ &\quad+|\xi_{\mathbf{k}}|^2\,\gamma^h_{\mathbf{k}}(-\tilde{E}_{1\mathbf{k}})+|\eta_{\mathbf{k}}|^2\,\gamma^h_{\mathbf{k}}(-\tilde{E}_{2\mathbf{k}})\Big]\\ &-\xi_{\mathbf{k}}\eta^*_{\mathbf{k}}\Big[\,\mathcal{A}^{11}_{\mathbf{k}}\,\big(-\gamma^e_{\mathbf{k}}(\tilde{E}_{1\mathbf{k}})+\gamma^h_{\mathbf{k}}(-\tilde{E}_{1\mathbf{k}})\big)\\ &\quad+\mathcal{A}^{22}_{\mathbf{k}}\,\big(-\gamma^e_{\mathbf{k}}(\tilde{E}_{2\mathbf{k}})+\gamma^h_{\mathbf{k}}(-\tilde{E}_{2\mathbf{k}})\big)\Big]\\ &-\pi\xi_{\mathbf{k}}\eta^*_{\mathbf{k}}\sum_{\mathbf{p}}|\Gamma^e_{\mathbf{kp}}|^2\big(\delta(\omega^e_{\mathbf{p}}-\tilde{E}_{1\mathbf{k}})+\delta(\omega^e_{\mathbf{p}}-\tilde{E}_{2\mathbf{k}})\big)\,b^\dagger_{e\mathbf{p}}b_{e\mathbf{p}}\\ &+\pi\xi_{\mathbf{k}}\eta^*_{\mathbf{k}}\sum_{\mathbf{p}}|\Gamma^h_{\mathbf{kp}}|^2\big(\delta(\omega^h_{-\mathbf{p}}+\tilde{E}_{1\mathbf{k}})+\delta(\omega^h_{-\mathbf{p}}+\tilde{E}_{2\mathbf{k}})\big)\\ &\quad\times b_{h,-\mathbf{p}}b^\dagger_{h,-\mathbf{p}}+\mathcal{F}^{12}_{\mathbf{k}}\\ &=\Big(\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}^{21}_{\mathbf{k}}(t)\Big)^\dagger, \end{aligned}\tag{C10}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}^{11}_{\mathbf{k}}(t) &= -2\,\mathcal{A}^{11}_{\mathbf{k}}\big(|\xi_{\mathbf{k}}|^2\,\gamma^e_{\mathbf{k}}(\tilde{E}_{1\mathbf{k}})+|\eta_{\mathbf{k}}|^2\,\gamma^h_{\mathbf{k}}(-\tilde{E}_{1\mathbf{k}})\big)\\ &\quad+(\xi^*_{\mathbf{k}}\eta_{\mathbf{k}}\,\mathcal{A}^{12}_{\mathbf{k}}+\xi_{\mathbf{k}}\eta^*_{\mathbf{k}}\mathcal{A}^{21}_{\mathbf{k}})\,\big(\gamma^e_{\mathbf{k}}(\tilde{E}_{1\mathbf{k}})-\gamma^h_{\mathbf{k}}(-\tilde{E}_{1\mathbf{k}})\big)\\ &\quad+2\pi|\xi_{\mathbf{k}}|^2\sum_{\mathbf{p}}|\Gamma^e_{\mathbf{kp}}|^2\delta(\omega^e_{\mathbf{p}}-\tilde{E}_{1\mathbf{k}})\,b^\dagger_{e\mathbf{p}}b_{e\mathbf{p}}\\ &\quad+2\pi|\eta_{\mathbf{k}}|^2\sum_{\mathbf{p}}|\Gamma^h_{\mathbf{kp}}|^2\delta(\omega^h_{-\mathbf{p}}+\tilde{E}_{1\mathbf{k}})\,b_{h,-\mathbf{p}}b^\dagger_{h,-\mathbf{p}}\\ &\quad+\mathcal{F}^{11}_{\mathbf{k}}\,, \end{aligned}\tag{C11}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}^{22}_{\mathbf{k}}(t) &= -2\,\mathcal{A}^{22}_{\mathbf{k}}\big(|\eta_{\mathbf{k}}|^2\,\gamma^e_{\mathbf{k}}(\tilde{E}_{2\mathbf{k}})+|\xi_{\mathbf{k}}|^2\,\gamma^h_{\mathbf{k}}(-\tilde{E}_{2\mathbf{k}})\big)\\ &\quad+(\xi^*_{\mathbf{k}}\eta_{\mathbf{k}}\,\mathcal{A}^{12}_{\mathbf{k}}+\xi_{\mathbf{k}}\eta^*_{\mathbf{k}}\,\mathcal{A}^{21}_{\mathbf{k}})\,\big(\gamma^e_{\mathbf{k}}(\tilde{E}_{2\mathbf{k}})-\gamma^h_{\mathbf{k}}(-\tilde{E}_{2\mathbf{k}})\big)\\ &\quad+2\pi|\eta_{\mathbf{k}}|^2\sum_{\mathbf{p}}|\Gamma^e_{\mathbf{kp}}|^2\delta(\omega^e_{\mathbf{p}}-\tilde{E}_{2\mathbf{k}})\,b^\dagger_{e\mathbf{p}}b_{e\mathbf{p}}\\ &\quad-2\pi|\xi_{\mathbf{k}}|^2\sum_{\mathbf{p}}|\Gamma^h_{\mathbf{kp}}|^2\delta(\omega^h_{-\mathbf{p}}+\tilde{E}_{2\mathbf{k}})\,b_{h,-\mathbf{p}}b^\dagger_{h,-\mathbf{p}}\\ &\quad+\mathcal{F}^{22}_{\mathbf{k}}\,. \end{aligned}\tag{C12}$$

Note that the last two terms in equations (C10)–(C12) are proportional to the electron occupation number operators $b^\dagger_{e\mathbf{p}}b_{e\mathbf{p}}$ and $b_{h,-\mathbf{p}}b^\dagger_{h,-\mathbf{p}}$ of the electronic reservoirs. However, the equations of motion for $b^\dagger_{e,\mathbf{p}}b_{e,\mathbf{p}}$ and $b^\dagger_{h,-\mathbf{p}}b_{h,-\mathbf{p}}$ are not needed. The electronic baths are assumed to be large and stay in thermal equilibrium even when they are coupled to the e-h-p system.

Moreover, the imaginary parts of the self-energies $\Im\gamma^{e,h}(\omega)$ will be neglected, which would lead to frequency shifts. The remaining real parts $\Re\gamma^{e,h}_{\mathbf{k}}(\omega)$ lead to a damping of electrons and holes as a result of the coupling to the electronic reservoirs:

$$\Re\gamma^e_{\mathbf{k}}(\omega)=\pi\sum_{\mathbf{p}}|\Gamma^e_{\mathbf{kp}}|^2\delta(\omega^e_{\mathbf{p}}-\omega)\,,\tag{C13}$$

$$\Re\gamma^h_{\mathbf{k}}(\omega)=\pi\sum_{\mathbf{p}}|\Gamma^h_{\mathbf{kp}}|^2\delta(\omega^h_{-\mathbf{p}}-\omega)\,.\tag{C14}$$

To simplify the further evaluation we assume that electrons and holes possess the same damping rate, which is also supposed not to depend on $\mathbf{k}$ and ω, i.e.,

$$\Re\gamma^{e,h}_{\mathbf{k}}(\omega)=\Re\gamma_{\mathbf{k}}(\omega)\approx\gamma\,.\tag{C15}$$

Then Eqs. (C10)–(C12) reduce to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}^{12}_{\mathbf{k}}(t) &= i(\tilde{E}_{1\mathbf{k}}-\tilde{E}_{2\mathbf{k}})\mathcal{A}^{12}_{\mathbf{k}}-2\gamma\,\mathcal{A}^{12}_{\mathbf{k}}\\ &-\pi\xi_{\mathbf{k}}\eta^*_{\mathbf{k}}\sum_{\mathbf{p}}|\Gamma^e_{\mathbf{kp}}|^2\big(\delta(\omega^e_{\mathbf{p}}-\tilde{E}_{1\mathbf{k}})+\delta(\omega^e_{\mathbf{p}}-\tilde{E}_{2\mathbf{k}})\big)\,b^\dagger_{e\mathbf{p}}b_{e\mathbf{p}}\\ &+\pi\xi_{\mathbf{k}}\eta^*_{\mathbf{k}}\sum_{\mathbf{p}}|\Gamma^h_{\mathbf{kp}}|^2\big(\delta(\omega^h_{-\mathbf{p}}+\tilde{E}_{1\mathbf{k}})+\delta(\omega^h_{-\mathbf{p}}+\tilde{E}_{2\mathbf{k}})\big)\\ &\quad\times b_{h,-\mathbf{p}}b^\dagger_{h,-\mathbf{p}}+\mathcal{F}^{12}_{\mathbf{k}}\\ &=\Big(\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}^{21}_{\mathbf{k}}(t)\Big)^\dagger, \end{aligned}\tag{C16}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}^{11}_{\mathbf{k}}(t) &= -2\gamma\,\mathcal{A}^{11}_{\mathbf{k}}\\ &\quad+2\pi|\xi_{\mathbf{k}}|^2\sum_{\mathbf{p}}|\Gamma^e_{\mathbf{kp}}|^2\delta(\omega^e_{\mathbf{p}}-\tilde{E}_{1\mathbf{k}})\,b^\dagger_{e\mathbf{p}}b_{e\mathbf{p}}\\ &\quad+2\pi|\eta_{\mathbf{k}}|^2\sum_{\mathbf{p}}|\Gamma^h_{\mathbf{kp}}|^2\delta(\omega^h_{-\mathbf{p}}+\tilde{E}_{1\mathbf{k}})\,b_{h,-\mathbf{p}}b^\dagger_{h,-\mathbf{p}}\\ &\quad+\mathcal{F}^{11}_{\mathbf{k}}\,, \end{aligned}\tag{C17}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}^{22}_{\mathbf{k}}(t) &= -2\gamma\,\mathcal{A}^{22}_{\mathbf{k}}\\ &\quad+2\pi|\eta_{\mathbf{k}}|^2\sum_{\mathbf{p}}|\Gamma^e_{\mathbf{kp}}|^2\delta(\omega^e_{\mathbf{p}}-\tilde{E}_{2\mathbf{k}})\,b^\dagger_{e\mathbf{p}}b_{e\mathbf{p}}\\ &\quad-2\pi|\xi_{\mathbf{k}}|^2\sum_{\mathbf{p}}|\Gamma^h_{\mathbf{kp}}|^2\delta(\omega^h_{-\mathbf{p}}+\tilde{E}_{2\mathbf{k}})\,b_{h,-\mathbf{p}}b^\dagger_{h,-\mathbf{p}}\\ &\quad+\mathcal{F}^{22}_{\mathbf{k}}\,. \end{aligned}\tag{C18}$$

The equations of motions for the expectation values, formed with $\tilde{\rho}_0$,

$$A^{nm}_{\mathbf{k}}(t)=\langle\mathcal{A}^{nm}_{\mathbf{k}}(t)\rangle_{\tilde{\rho}_0}=\langle(C^\dagger_{n\mathbf{k}}C_{m\mathbf{k}})(t)\rangle_{\tilde{\rho}_0}\,,\tag{C19}$$

can immediately be found from Eqs. (C10)–(C12) :

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}A^{12}_{\mathbf{k}}(t) &= \big[i(\tilde{E}_{1\mathbf{k}}-\tilde{E}_{2\mathbf{k}})-2\gamma\big]\,A^{12}_{\mathbf{k}}(t)\\ &-\pi\xi_{\mathbf{k}}\eta^*_{\mathbf{k}}\sum_{\mathbf{p}}|\Gamma^e_{\mathbf{kp}}|^2\big(\delta(\omega^e_{\mathbf{p}}-\tilde{E}_{1\mathbf{k}})+\delta(\omega^e_{\mathbf{p}}-\tilde{E}_{2\mathbf{k}})\big)\,\langle b^\dagger_{e\mathbf{p}}b_{e\mathbf{p}}\rangle_{\tilde{\rho}_0}\\ &+\pi\xi_{\mathbf{k}}\eta^*_{\mathbf{k}}\sum_{\mathbf{p}}|\Gamma^h_{\mathbf{kp}}|^2\big(\delta(\omega^h_{-\mathbf{p}}+\tilde{E}_{1\mathbf{k}})+\delta(\omega^h_{-\mathbf{p}}+\tilde{E}_{2\mathbf{k}})\big)\\ &\quad\times\langle b_{h,-\mathbf{p}}b^\dagger_{h,-\mathbf{p}}\rangle_{\tilde{\rho}_0}\\ &=\Big(\frac{\mathrm{d}}{\mathrm{d}t}A^{21}_{\mathbf{k}}(t)\Big)^\dagger, \end{aligned}\tag{C20}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}A^{11}_{\mathbf{k}}(t) &= -2\gamma\,A^{11}_{\mathbf{k}}(t)\\ &\quad+2\pi|\xi_{\mathbf{k}}|^2\sum_{\mathbf{p}}|\Gamma^e_{\mathbf{kp}}|^2\delta(\omega^e_{\mathbf{p}}-\tilde{E}_{1\mathbf{k}})\,\langle b^\dagger_{e\mathbf{p}}b_{e\mathbf{p}}\rangle_{\tilde{\rho}_0}\\ &\quad+2\pi|\eta_{\mathbf{k}}|^2\sum_{\mathbf{p}}|\Gamma^h_{\mathbf{kp}}|^2\delta(\omega^h_{-\mathbf{p}}+\tilde{E}_{1\mathbf{k}})\,\langle b_{h,-\mathbf{p}}b^\dagger_{h,-\mathbf{p}}\rangle_{\tilde{\rho}_0}\,, \end{aligned}\tag{C21}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}A^{22}_{\mathbf{k}}(t) &= -2\gamma\,A^{22}_{\mathbf{k}}(t)\\ &\quad+2\pi|\eta_{\mathbf{k}}|^2\sum_{\mathbf{p}}|\Gamma^e_{\mathbf{kp}}|^2\delta(\omega^e_{\mathbf{p}}-\tilde{E}_{2\mathbf{k}})\,\langle b^\dagger_{e\mathbf{p}}b_{e\mathbf{p}}\rangle_{\tilde{\rho}_0}\\ &\quad+2\pi|\xi_{\mathbf{k}}|^2\sum_{\mathbf{p}}|\Gamma^h_{\mathbf{kp}}|^2\delta(\omega^h_{-\mathbf{p}}+\tilde{E}_{2\mathbf{k}})\,\langle b_{h,-\mathbf{p}}b^\dagger_{h,-\mathbf{p}}\rangle_{\tilde{\rho}_0}\,, \end{aligned}\tag{C22}$$

where the random forces $\mathcal{F}^{nm}_{\mathbf{k}}$ do to contribute since the $\langle\mathcal{F}^{nm}_{\mathbf{k}}\rangle_{\tilde\rho_0}$ vanish at least up to second order in $\mathcal{H}_{\rm SR}$. Moreover, because the expectation values of the bath variables $\langle b^\dagger_{e\mathbf{p}}b_{e\mathbf{p}}\rangle_{\tilde\rho_0}$ and $\langle b^\dagger_{h,-\mathbf{p}}b_{h,-\mathbf{p}}\rangle_{\tilde\rho_0}$ do not depend on time, we may use Fermi functions for

$$\langle b^\dagger_{e\mathbf{p}}b_{e\mathbf{p}}\rangle_{\tilde\rho_0} = \frac{1}{1+e^{\beta[\omega^e_{\mathbf{p}}-(\mu_e-\mu/2)]}} = f_e(\omega^e_{\mathbf{p}})\,, \tag{C23}$$

$$\langle b^\dagger_{h,-\mathbf{p}}b_{h,-\mathbf{p}}\rangle_{\tilde\rho_0} = \frac{1}{1+e^{\beta[\omega^h_{-\mathbf{p}}-(\mu_h-\mu/2)]}} = f_h(\omega^h_{-\mathbf{p}}). \tag{C24}$$

Exploiting the presence of the δ-functions, Eqs. (C20)–(C22) can be simplified to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}A^{12}_{\mathbf{k}}(t) &= \big[i(\tilde E_{1\mathbf{k}}-\tilde E_{2\mathbf{k}})-2\gamma\big]\,A^{12}_{\mathbf{k}}(t)\\ &\quad-\gamma\,\xi_{\mathbf{k}}\eta^*_{\mathbf{k}}\big(f_e(\tilde E_{1\mathbf{k}})+f_e(\tilde E_{2\mathbf{k}})\big)\\ &\quad-\gamma\,\xi_{\mathbf{k}}\eta^*_{\mathbf{k}}\big(f_h(-\tilde E_{1\mathbf{k}})+f_h(-\tilde E_{2\mathbf{k}})-2\big)\\ &= \Big(\frac{\mathrm{d}}{\mathrm{dt}}A^{21}_{\mathbf{k}}(t)\Big)^\dagger\,,\end{aligned} \tag{C25}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}A^{11}_{\mathbf{k}}(t) &= -2\gamma\,A^{11}_{\mathbf{k}}(t)+2\gamma\,|\xi_{\mathbf{k}}|^2 f_e(\tilde E_{1\mathbf{k}})\\ &\quad+2\gamma\,|\eta_{\mathbf{k}}|^2\big(1-f_h(-\tilde E_{1\mathbf{k}})\big)\,,\end{aligned} \tag{C26}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}A^{22}_{\mathbf{k}}(t) &= -2\gamma\,A^{22}_{\mathbf{k}}(t)+2\gamma\,|\eta_{\mathbf{k}}|^2 f_e(\tilde E_{2\mathbf{k}})\\ &\quad+2\gamma\,|\xi_{\mathbf{k}}|^2\big(1-f_h(-\tilde E_{2\mathbf{k}})\big)\,,\end{aligned} \tag{C27}$$

where Eqs. (C13) and (C15) have been used.

We are now in the position to evaluate the limit $t\to\infty$ for the expectation values

$$A^{nm}_{\mathbf{k}} = \lim_{t\to\infty}\langle\mathcal{A}^{nm}_{\mathbf{k}}(t)\rangle_{\tilde\rho_0} = \lim_{t\to\infty}\langle(C^\dagger_{n\mathbf{k}}C_{m\mathbf{k}})(t)\rangle_{\tilde\rho_0}\,. \tag{C28}$$

Using the steady-state condition (C3) one finds

$$A^{12}_{\mathbf{k}} = \gamma\,\xi_{\mathbf{k}}\eta^*_{\mathbf{k}}\,\frac{1}{\big(i(\tilde E_{1\mathbf{k}}-\tilde E_{2\mathbf{k}})-2\gamma\big)}\times \tag{C29}$$

$$\times\Big[\big(f_e(\tilde E_{1\mathbf{k}})+f_e(\tilde E_{2\mathbf{k}})\big)+\big(f_h(-\tilde E_{1\mathbf{k}})+f_h(-\tilde E_{2\mathbf{k}})-2\big)\Big]\,, \tag{C30}$$

and (for $\gamma\neq 0$)

$$A^{11}_{\mathbf{k}} = |\xi_{\mathbf{k}}|^2 f_e(\tilde E_{1\mathbf{k}})+|\eta_{\mathbf{k}}|^2\big(1-f_h(-\tilde E_{1\mathbf{k}})\big)\,, \tag{C31}$$

$$A^{22}_{\mathbf{k}} = |\eta_{\mathbf{k}}|^2 f_e(\tilde E_{2\mathbf{k}})+|\xi_{\mathbf{k}}|^2\big(1-f_h(-\tilde E_{2\mathbf{k}})\big)\,. \tag{C32}$$

2. Derivation of equations (141)–(143)

Let us consider the steady-state result for $\hat d^*_{\mathbf{k}}$, $\hat n^e_{\mathbf{k}}$, and $\hat n^h_{\mathbf{k}}$, Eqs. (138), (139), and (140), respectively. Here, $\hat d^{0*}_{\mathbf{k}}$, defined by Eq. (134) with Eq. (144), takes the form

$$\hat d^{0*}_{\mathbf{k}} = \frac{1}{2}\xi^*_{\mathbf{k}}\eta^*_{\mathbf{k}}F^+_{1\mathbf{k}}\,. \tag{C33}$$

Transforming first the last terms (imaginary parts) in Eqs. (139) and (140), one gets

$$\begin{aligned}\frac{1}{\gamma}\Im[\tilde\Delta_{\mathbf{k}}\hat d^*_{\mathbf{k}}] = -\frac{\Im}{\gamma}\Big\{&\frac{1}{\tilde\varepsilon^e_{\mathbf{k}}+\tilde\varepsilon^h_{\mathbf{k}}+2i\gamma}\big[|\tilde\Delta_{\mathbf{k}}|^2(1-\hat n^e_{\mathbf{k}}-\hat n^h_{\mathbf{k}})\\ &-2i\gamma\tilde\Delta_{\mathbf{k}}\hat d^{0*}_{\mathbf{k}}\big]\Big\}\end{aligned} \tag{C34}$$

or

$$\begin{aligned}\frac{1}{\gamma}\Im[\tilde\Delta_{\mathbf{k}}\hat d^*_{\mathbf{k}}] = -\frac{\Im}{\gamma}\Big\{&\frac{1}{\tilde\varepsilon^e_{\mathbf{k}}+\tilde\varepsilon^h_{\mathbf{k}}+2i\gamma}\big[|\tilde\Delta_{\mathbf{k}}|^2(1-\hat n^e_{\mathbf{k}}-\hat n^h_{\mathbf{k}})\\ &-\frac{i\,\gamma}{W_{\mathbf{k}}}\mathrm{sgn}(\tilde\varepsilon^e_{\mathbf{k}}+\tilde\varepsilon^h_{\mathbf{k}})F^+_{1\mathbf{k}}\big]\Big\}\,.\end{aligned} \tag{C35}$$

Thus

$$\begin{aligned}\frac{1}{\gamma}\Im[\tilde\Delta_{\mathbf{k}}\hat d^*_{\mathbf{k}}] = \frac{|\tilde\Delta_{\mathbf{k}}|^2}{(\tilde\varepsilon^e_{\mathbf{k}}+\tilde\varepsilon^h_{\mathbf{k}})^2+(2\gamma)^2}\Big\{&2(1-\hat n^e_{\mathbf{k}}-\hat n^h_{\mathbf{k}})\\ &+\frac{|\tilde\varepsilon^e_{\mathbf{k}}+\tilde\varepsilon^h_{\mathbf{k}}|}{W_{\mathbf{k}}}\,F^+_{1\mathbf{k}}\Big\}\,.\end{aligned} \tag{C36}$$

Here, we have used Eq. (138) for $\hat d^*_{\mathbf{k}}$ with $\hat d^{0*}_{\mathbf{k}}$ given by Eq. (C33) and (82):

$$\begin{aligned}\hat d^*_{\mathbf{k}} = \frac{\tilde\Delta^*_{\mathbf{k}}}{(\tilde\varepsilon^e_{\mathbf{k}}+\tilde\varepsilon^h_{\mathbf{k}})+2i\,\gamma}\Big[&(\hat n^e_{\mathbf{k}}+\hat n^h_{\mathbf{k}}-1)\\ &+i\gamma\,\mathrm{sgn}(\tilde\varepsilon^e_{\mathbf{k}}+\tilde\varepsilon^h_{\mathbf{k}})\frac{F^+_{1\mathbf{k}}}{W_{\mathbf{k}}}\Big]\end{aligned} \tag{C37}$$

Then, from (C36) together with Eqs. (139) and (140), one finds:

$$\begin{aligned}\hat n^e_{\mathbf{k}}+\hat n^h_{\mathbf{k}}-1 = &\frac{|\tilde\varepsilon^e_{\mathbf{k}}+\tilde\varepsilon^h_{\mathbf{k}}|}{2W_{\mathbf{k}}}F^+_{1\mathbf{k}}+\\ &+\frac{1}{2}\frac{F^+_{2\mathbf{k}}-2}{1+\dfrac{4|\tilde\Delta_{\mathbf{k}}|^2}{(\tilde\varepsilon^e_{\mathbf{k}}+\tilde\varepsilon^h_{\mathbf{k}})^2+(2\gamma)^2}}\,,\end{aligned} \tag{C38}$$

and

$$\hat n^e_{\mathbf{k}}-\hat n^h_{\mathbf{k}} = \frac{1}{2}F^-_{1\mathbf{k}}+\frac{|\tilde\varepsilon^e_{\mathbf{k}}+\tilde\varepsilon^h_{\mathbf{k}}|}{2W_{\mathbf{k}}}F^-_{2\mathbf{k}}\,. \tag{C39}$$

3. Photonic expectation values

To calculate the photon condensation parameter $\langle\psi^\dagger_{\mathbf{q}=0}\rangle$, we use the *ansatz* for the λ-dependent photon operator,

$$\psi^\dagger_{\mathbf{q},\lambda} = z_{\mathbf{q},\lambda}\psi^\dagger_{\mathbf{q}}+\frac{1}{\sqrt{N}}\sum_{\mathbf{k}}v_{\mathbf{kq},\lambda}\;:e^\dagger_{\mathbf{k+q}}h^\dagger_{-\mathbf{k}}:\,, \tag{C40}$$

where again the operator structure was taken over from a small $X_{\lambda,\Delta\lambda}$ expansion. Furthermore, $:e^\dagger_{\mathbf{k+q}}h^\dagger_{-\mathbf{k}}:=e^\dagger_{\mathbf{k+q}}h^\dagger_{-\mathbf{k}}-\langle e^\dagger_{\mathbf{k+q}}h^\dagger_{-\mathbf{k}}\rangle$. In analogy to the preceding section, one easily obtains renormalization equations for the λ-dependent coefficients $z_{\mathbf{q},\lambda}$ and $v_{\mathbf{kq},\lambda}$:

$$v_{\mathbf{kq},\lambda-\Delta\lambda} = v_{\mathbf{kq},\lambda}-g z_{\mathbf{q},\lambda}A_{\mathbf{kq}}(\lambda,\Delta\lambda)\,, \tag{C41}$$

$$|z_{\mathbf{q},\lambda}|^2 = 1-\frac{1}{N}\sum_{\mathbf{k}}|v_{\mathbf{kq},\lambda}|^2(1-n^e_{\mathbf{k+q}}-n^h_{-\mathbf{k}})\,. \tag{C42}$$

Deriving the last equation, the commutator relation $[\psi_{\mathbf{q},\lambda},\psi^\dagger_{\mathbf{q},\lambda}]=1$ was used. Eq. (C41) and Eq. (C42), both taken at $\lambda\to\lambda-\Delta\lambda$, represent a complete set of renormalization equations for the λ-dependent coefficient in (C40). Here, the initial parameter values are

$$z_{\mathbf{q},\Lambda} = 1\,,\quad v_{\mathbf{kq},\Lambda} = 0\,. \tag{C43}$$

The integration between $\lambda = \Lambda$ and $\lambda = 0$ leads to the fully renormalized photon operator

$$\tilde{\psi}^\dagger_{\mathbf{q}} = \tilde{z}_{\mathbf{q}} \psi^\dagger_{\mathbf{q}} + \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} \tilde{v}_{\mathbf{kq}} \, : e^\dagger_{\mathbf{k+q}} h^\dagger_{-\mathbf{k}} : \, . \tag{C44}$$

Using Eq. (94), one finds in the large-t limit:

$$\begin{aligned} \langle \psi^\dagger_{\mathbf{q}}(t \to \infty) \rangle &= \tilde{z}_{\mathbf{q}} \langle \psi^\dagger_{\mathbf{q}}(t \to \infty) \rangle_{\tilde{\rho}_0} \\ &+ \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} \tilde{v}_{\mathbf{kq}} \, \langle (: e^\dagger_{\mathbf{k+q}} h^\dagger_{-\mathbf{k}} :)(t \to \infty) \rangle_{\tilde{\rho}_0} \\ &\simeq \tilde{z}_{\mathbf{q}} \langle \psi^\dagger_{\mathbf{q}}(t \to \infty) \rangle_{\tilde{\rho}_0}, \end{aligned} \tag{C45}$$

where the second contribution, being proportional to fluctuation operators, was neglected. Similarly,

$$\begin{aligned} \langle (\psi^\dagger_{\mathbf{q}} \psi_{\mathbf{q}})(t \to \infty) \rangle &= |\tilde{z}_{\mathbf{q}}|^2 \langle (\psi^\dagger_{\mathbf{q}} \psi_{\mathbf{q}})(t \to \infty) \rangle_{\tilde{\rho}_0} \\ &+ \frac{1}{N} \sum_{\mathbf{k}} |\tilde{v}_{\mathbf{kq}}|^2 \, \hat{n}^e_{\mathbf{k+q}} \, \hat{n}^h_{-\mathbf{k}} \, . \end{aligned} \tag{C46}$$

We then evaluate the remaining quantities $\langle \psi^\dagger_{\mathbf{q}}(t \to \infty) \rangle_{\tilde{\rho}_0} := \langle \psi^\dagger_{\mathbf{q}} \rangle_{\tilde{\rho}_0}$ and $\langle (\psi^\dagger_{\mathbf{q}} \psi_{\mathbf{q}})(t \to \infty) \rangle_{\tilde{\rho}_0} := \langle \psi^\dagger_{\mathbf{q}} \psi_{\mathbf{q}} \rangle_{\tilde{\rho}_0}$. Our starting point is an equation of motion for the time-dependent photon creation operator $\psi^\dagger_{\mathbf{q}}(t)$. Using again the Mori-Zwanzig approach of Sec. C.1, one obtains with Eqs. (77) and (C5),

$$\frac{\mathrm{d}}{\mathrm{dt}} \psi^\dagger_{\mathbf{q}}(t) = i\tilde{\omega}_{\mathbf{q}} \psi^\dagger_{\mathbf{q}}(t) + i\sqrt{N}\tilde{\Gamma}^* \delta_{\mathbf{q},0} - \kappa \psi^\dagger_{\mathbf{q}}(t) + \mathcal{F}^\psi_{\mathbf{q}} \, , \tag{C47}$$

and

$$\frac{\mathrm{d}}{\mathrm{dt}} \langle \psi^\dagger_{\mathbf{q}}(t) \rangle_{\tilde{\rho}_0} = i\tilde{\omega}_{\mathbf{q}} \langle \psi^\dagger_{\mathbf{q}}(t) \rangle_{\tilde{\rho}_0} + i\sqrt{N}\tilde{\Gamma}^* \delta_{\mathbf{q},0} - \kappa \langle \psi^\dagger_{\mathbf{q}}(t) \rangle_{\tilde{\rho}_0}, \tag{C48}$$

where κ is the damping rate of cavity photons into the free space. For the steady state at $t \to \infty$ one finds from Eq. (C48)

$$\langle \psi^\dagger_{\mathbf{q}}(t \to \infty) \rangle_{\tilde{\rho}_0} = -\frac{\sqrt{N}\hat{\Gamma}^*}{\tilde{\omega}_0 + i\kappa} \, \delta_{\mathbf{q},0} \, , = \langle \psi^\dagger_{\mathbf{q}} \rangle_{\tilde{\rho}_0} \tag{C49}$$

and, with Eq. (C45),

$$\langle \psi^\dagger_{\mathbf{q}}(t \to \infty) \rangle = -\hat{z}_{\mathbf{q}=0} \, \frac{\sqrt{N}\tilde{\Gamma}^*}{\tilde{\omega}_0 + i\kappa} \, \delta_{\mathbf{q},0} = \langle \psi^\dagger_{\mathbf{q}} \rangle \, . \tag{C50}$$

To evaluate the expectation value $\langle (\psi^\dagger_{\mathbf{q}} \psi_{\mathbf{q}})(t \to \infty) \rangle_{\tilde{\rho}_0}$, one best starts from the solution (215) of the equation of motion (C47), thereby neglecting the fluctuation force $\mathcal{F}^\psi_{\mathbf{q}}$:

$$\psi^\dagger_{\mathbf{q}}(t) = -\frac{i\sqrt{N}\tilde{\Gamma}^*}{i\tilde{\omega}_0 - \kappa} \delta_{\mathbf{q},0} + \big(\psi^\dagger_{\mathbf{q}} + \frac{i\sqrt{N}\tilde{\Gamma}^*}{i\tilde{\omega}_0 - \kappa} \delta_{\mathbf{q},0} \big) e^{(i\tilde{\omega}_{\mathbf{q}} - \kappa)t} \, . \tag{C51}$$

For $t \to \infty$ one is led to

$$\langle (\psi^\dagger_{\mathbf{q}} \psi_{\mathbf{q}})(t \to \infty) \rangle_{\tilde{\rho}_0} = \frac{N|\tilde{\Gamma}|^2}{\tilde{\omega}_0^2 + \kappa^2} \delta_{\mathbf{q},0} = \langle \psi^\dagger_{\mathbf{q}} \psi_{\mathbf{q}} \rangle_{\tilde{\rho}_0} \, . \tag{C52}$$

For the fluctuation number $\hat{n}^\psi_{\mathbf{q}}$ of cavity photons one obtains with Eq. (C49):

$$\hat{n}^\psi_{\mathbf{q}} = \langle (: \psi^\dagger_{\mathbf{q}} :: \psi_{\mathbf{q}} :)(t \to \infty) \rangle_{\tilde{\rho}_0} = 0 \, . \tag{C53}$$

Thus, the fluctuation number, formed with $\tilde{\rho}_0$ vanishes. In contrast, for the full quantity $n^\psi_{\mathbf{q}} = \langle : \psi^\dagger_{\mathbf{q}} :: \psi_{\mathbf{q}} : \rangle$, which is formed by the initial density ρ_0, one finds from Eq. (C46) and (C50):

$$n^\psi_{\mathbf{q}} = \frac{1}{N} \sum_{\mathbf{k}} |\tilde{v}_{\mathbf{kq}}|^2 \, \hat{n}^e_{\mathbf{k+q}} \, \hat{n}^h_{-\mathbf{k}} \, . \tag{C54}$$

That is, the full fluctuation number $n^\psi_{\mathbf{q}}$ of cavity photons is determined by the coupling of cavity photons to electronic particle-hole excitations of the e-h-p system.

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