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Algebraic Approach to Fractional Quantum Hall Effect

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We construct an algebraic description for the ground state and for the static response of the quantum Hall plateaux with filling factor $\nu = N/(2N+1)$ in the large N limit. By analyzing the algebra of the fluctuations of the shape of the Fermi surface of the composite fermions, we find the explicit form of the projected static structure (SSF) factor at large N and fixed $z = (2N+1)q\ell_B \sim 1$, where q is the wavenumber at which the system is probed, and ℓ_B is the magnetic length. When $z < 3.8$, we obtain the exact universal formula for the projected SSF. The formula does not depend on the particular form of the Hamiltonian.

Introduction.—The most striking property of two-dimensional electron gas in a strong magnetic field is its ability to form nontrivial topologically ordered gapped states—the factional quantum Hall (FQH) states [1]. Despite the numerous approaches, starting from the Laughlin wave function [2], that have been brought to bear on the problem, many important properties of the quantum Hall systems are still beyond theoretical control of analytic approaches. These include the spectrum and dispersion of quasiparticle excitations, the structure factors [3], the electromagnetic and gravitational response at finite wave number and frequency.

Generally, quantities that are not topologically protected should depend on the details of the system under consideration, e.g., the form of the electron-electron potential. A few exceptions have recently been found, suggesting that under some additional assumptions (e.g., rotational invariance), some physical quantities are protected by a combination of topology and symmetry. For example, the q^2 correction to the Hall conductivity is related to the shift [4]. It has also been argued that for a class of chiral states and their particle-hole conjugates, the first two terms in the small-q expansion of the projected static structure factor are uniquely defined by the filling factor, the shift, and the central charge [5–7]. On the other hand, it seems that the higher terms (in q) of these quantities are not universal, i.e., not determined solely by topological characteristics of the quantum Hall state.

In this paper, we consider a regime in quantum Hall physics where the full momentum dependence of the projected static structure factor, i.e., *all* terms in their momentum expansion, can be determined in a reliable fashion. The projected static structure factor (SSF) is an important property of the quantum Hall ground state [3]. The SSF, which can be calculated directly from projected SSF [3], plays a central role in the theory of FQH. Using the general form of the LLL wave function, one can calculate the Hall conductivity, Hall viscosity from the SSF [6]. We consider here quantum Hall plateaux at filling factors $\nu = N/(2N+1)$ and their particle-hole conjugates $\nu = (N+1)/(2N+1)$ (the Jain sequences). The limit being considered is that of large N, $N \gg 1$, and finite $z = (2N+1)q\ell_B \sim 1$, where q is the wavenumber at which the system is probed, and ℓ_B is the magnetic length. The latter condition $z \sim 1$ corresponds to wavelengths comparable to the cyclotron radius of the composite fermion, which diverges when the filling factor approaches $\nu = 1/2$. We will see that the momentum expansions of physical quantities are in fact series expansions over z. We will argue that equal-time structure factor [3] can be computed to all orders in the expansion over z. Our result for the projected structure factor can be written as

$$\bar{s}(q) = \frac{N+1}{8}(q\ell_B)^4 \frac{4J_2(z)}{zJ_1(z)}, \tag{1}$$

with

$$z = (2N+1)q\ell_B. \tag{2}$$

The result is valid as long as $z < z_1$, with $z_1 \approx 3.83$ being the first zero of the Bessel function J_1. For $z > z_1$, $\bar{s}(q)$ cannot be predicted without knowing precise form of the Hamiltonian. The divergence of the static structure factor at $z = z_1$ signals the approach of the first magnetoroton minimum [8], and at finite N this divergence should become a maximum of $\bar{s}(q)$.

We will argue that the result (1) is valid to the leading order in $1/N$ in the scaling regime $qN \sim 1$. We will also present the indication that, for short-range electron-electron interaction, the result is also valid to next-to-leading order in $1/N$, so the correction to Eq. (1) is suppressed by $1/N^2$.

Algebra of shapes.—Our understanding of the quantum Hall effect near half filling is based on the concept of the composite fermion [9–13]. In this paper, we use the modern, revised version of the composite fermion field theory compatible with particle-hole symmetry [14]. In this theory the composite fermion is a massless Dirac fermion with a π Berry phase around the composite Fermi surface, and the emergent field has no Chern-Simons term in its action. It has been suggested that the emergence of the Dirac composite fermion is a manifestation of a more general fermionic particle-vortex duality [15, 16].

Furthermore, following Refs. [8, 17], we will interpret the low-energy excitations of the Dirac composite Fermi liquid as fluctuations of the shape of the composite Fermi surface. At each point in spacetime, the latter can be parametrized, in polar coordinates in momentum space, as

$$p_F(\theta) = p_F^0 + \sum_n e^{-in\theta} u_n. \tag{3}$$

The scalar fields u_n satisfy nontrivial commutation relations [18],

$$[u_n(\mathbf{x}), u_{n'}(\mathbf{x}')] = 2\pi\big(nb\delta_{n+n',0} - i\delta_{n+n',1}\partial_{\bar{z}} - i\delta_{n+n',-1}\partial_z\big)\delta(\mathbf{x}-\mathbf{x}'). \tag{4}$$

Here we have taken the external magnetic field $B = 1$, so the magnetic length $\ell_B = 1$ [19]. The average effective magnetic field felt by the composite fermion is $b = 1/(2N+1)$ [20]. The kinetic equation of Landau's Fermi liquid theory at zero temperature can be reproduced by commuting u_n with a Lagrangian local in u_n, using the commutation relations (4).

In the case of the composite fermion of the fractional quantum Hall states, the coupling of the composite fermion to a gauge field leads to the freezing of u_0, u_1, and u_{-1}, corresponding to the density and currents of the composite fermions. This can be seen most clearly if one assumes that the gauge field a_μ has no kinetic term. This case extremization of the action with respect to a_μ the vanishing of the composite fermion density fluctuations and current, and so of u_0 and $u_{\pm1}$. We will argue later on that the presence of a kinetic term for a_μ only affects our calculations to subleading orders in $1/N$.

Eliminating u_0 and $u_{\pm1}$, we see that in Eq. (4), the only nonzero commutators are between u_n and $u_{n'}$ with indices n and n' having opposite signs. We can rewrite this equation as

$$[u_n(\mathbf{q}),\, u_{-n'}(-\mathbf{q}')] = C_{nn'}(\mathbf{q})\delta_{\mathbf{q}\mathbf{q}'} \quad n, n' \geq 2, \tag{5}$$

where $C_{nn'}$ is a tridiagonal matrix given by

$$C_{nn'}(\mathbf{q}) = 2\pi\left(nb\delta_{n,n'} + q_{\bar{z}}\delta_{n,n'+1} + q_z\delta_{n,n'-1}\right), \tag{6}$$

or

$$C = 2\pi\begin{pmatrix} 2b & q_z & 0 & 0 & \dots \\ q_{\bar{z}} & 3b & q_z & 0 & \dots \\ 0 & q_{\bar{z}} & 4b & q_z & \dots \\ 0 & 0 & q_{\bar{z}} & 5b & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \tag{7}$$

Canonical pairs.—At $q \neq 0$, the commutation relation between u_n does not have the canonical form. To bring it into the canonical form, one performs a unitary transformation

$$v_n(\mathbf{q}) = \sum_{m=2}^{\infty} U_{nm}(\mathbf{q}) u_m(\mathbf{q}), \tag{8}$$

so that

$$[v_n(\mathbf{q}),\, v_{-n'}(-\mathbf{q}')] = \tilde{C}_{nn'}(\mathbf{q})\delta_{\mathbf{q}\mathbf{q}'}, \tag{9}$$

where $\tilde{C}_{nn'}$ is now a diagonal matrix with coefficient

$$\begin{cases} \frac{2\pi}{2N+1}\lambda_n, & n = n', \\ 0, & n \neq n'. \end{cases} \tag{10}$$

Here λ_n [21] are the solutions to the characteristic equation

$$(m-\lambda)bx_m + q_{\bar{z}}x_{m-1} + q_z x_{m+1} = 0, \quad m \geq 2. \tag{11}$$

Without losing generality we can take $q_x = q$, $q_y = 0$. The solution to the recursion relation (11), for $m \geq 3$ is

$$x_m = (-1)^m[\alpha J_{m-\lambda}(z) + \beta Y_{m-\lambda}(z)], \quad z = \frac{q}{b}. \tag{12}$$

In order for x_m to decrease at $m \to \infty$ we have to set $\beta = 0$. Then Eq. (11) is satisfied for $m = 2$ only when

$$J_{1-\lambda}(z) = 0, \qquad z = (2N+1)q. \tag{13}$$

This equation determines the eigenvalues of the matrix C. When z is smaller than the first zero of the Bessel function J_1, $z_1 \approx 3.8$, all solutions to Eq. (13) (as an equation for λ) are positive. In this case, one can define the creation and annihilation operators a_n and $a_n^\dagger$ as follows

$$a_n = \sqrt{\frac{2N+1}{2\pi\lambda_n}}v_n\,, \quad a_n^\dagger = \sqrt{\frac{2N+1}{2\pi\lambda_n}}v_{-n}\,. \tag{14}$$

The commutator of a_n and $a_n^\dagger$ now has the canonical form,

$$[a_n(\mathbf{q}),\, a_{n'}^\dagger(\mathbf{q}')] = \delta_{nn'}\delta_{\mathbf{q}\mathbf{q}'}. \tag{15}$$

In contrast, when $z > z_1$, some of λ_n are negative. For these eigenvalues, v_{-n} should be interpreted as an annihilation operator ($v_{-n} \sim a_n$), while v_n is a creation operator ($v_n \sim a_n^\dagger$).

Hamiltonian.—In order to find the ground state one needs a Hamiltonian. We will argue later it is sufficient to consider a Hamiltonian quadratic in u_n,

$$H = \sum_{\mathbf{q}} \sum_{m,n=-\infty}^{\infty} h_{mn}(\mathbf{q}) u_{-m}(-\mathbf{q}) u_n(\mathbf{q}). \tag{16}$$

Due to rotational invariance $h_{mn} \sim q^{|m-n|}$. Since $q \sim 1/N$. This means that, to leading and next-to-leading orders in $1/N$, the only nonzero elements of h_{mn} are those with $m - n = 0$ or ±1. In particular $h_{mn} = 0$ when m and n are of opposite sign: the first term in the Hamiltonian of this type is $h_{-2,2}u_2u_2$, where $h_{-2,2} \sim q^4$ and is suppressed by a high power of $1/N$. We thus will assume that h_{mn} are nonzero only when m and n are of the same

sign. This turns out to be the only information about the Hamiltonian that we will need.

One can transform H to be a quadratic form of a_n,

$$H = \sum_{\mathbf{q}} \sum_{m,n=2}^{\infty} H_{mn} a_m^\dagger(\mathbf{q}) a_n(\mathbf{q}). \tag{17}$$

The spectrum of excitations then can be obtained by diagonalizing the matrix H_{mn}. The ground state is the state that is annihilated by the a_n,

$$a_n|0\rangle = 0. \tag{18}$$

For $(2N+1)q < z_1$, this means $v_n|0\rangle = 0$, which also means that

$$u_n|0\rangle = 0. \tag{19}$$

Charge density.—Now we continue with our discussion of the density operator. In the composite fermion field theory, it is identified with the magnetic field of the emergent gauge field: $\delta\rho = -\delta b/(4\pi)$ [14, 22, 23]. We need to relate b to u_n to make further progress. To do that, first we note that the equation of motion for u_1 is [23, 24]

$$\dot{u}_1 = e_{\bar{z}} + i[u_1,\, H] = e_{\bar{z}} - 2\pi\partial_z \frac{\delta H}{\delta u_2}\,, \tag{20}$$

with $e_i = \partial_i a_0 - \partial_0 a_i$. Physically the term $e_{\bar{z}}$ comes from the acceleration of the composite fermion moving in an emergent electric field. Setting $u_1 = 0$ we obtain

$$e_{\bar{z}} = 2\pi\partial_z \frac{\delta H}{\delta u_2}\,. \tag{21}$$

Analogously

$$e_z = 2\pi\partial_{\bar{z}} \frac{\delta H}{\delta u_{-2}}\,. \tag{22}$$

We will combine Eqs. (21) and (22) with the Bianchi identity,

$$\delta\dot{b} + 2(\partial_z e_{\bar{z}} - \partial_{\bar{z}} e_z) = 0, \tag{23}$$

to find δb. This seems to still require the knowledge of the Hamiltonian. We now show that this is not true, and we can express b in terms of u_n without knowing the explicit form of the Hamiltonian. We now note that the equation of motion for u_n, with $n \geq 2$, is

$$\dot{u}_n = -i[u_n,\, H] = -i[u_n,\, u_{-m}] \frac{\delta H}{\delta u_{-m}}, \tag{24}$$

In this and subsequent equations, the indices n, m runs from 2 to ∞ and summation over repeating indices is implied. In momentum space,

$$\dot{u}_n(\mathbf{q}) = -iC_{nm}(\mathbf{q}) \frac{\delta H}{\delta u_{-m}(-\mathbf{q})}. \tag{25}$$

Inverting this equation, we can write

$$\frac{\delta H}{\delta u_{-n}(-\mathbf{q})} = iC^{-1}_{nm}(\mathbf{q})\dot{u}_m(\mathbf{q}), \tag{26}$$

where C^{-1}_{nm} are the elements of the matrix inverse of C. Analogously we find

$$\frac{\delta H}{\delta u_n(-\mathbf{q})} = -iu_{-m}(\mathbf{q})C^{-1}_{mn}(-\mathbf{q}). \tag{27}$$

This allows us to write $e_{\bar{z}}$ and e_z as

$$e_{\bar{z}}(\mathbf{q}) = -2\pi q_z C^{-1}_{2m}(\mathbf{q})\dot{u}_m(\mathbf{q}), \tag{28}$$

$$e_z(\mathbf{q}) = 2\pi q_{\bar{z}} \dot{u}_{-m}(\mathbf{q})C^{-1}_{m2}(-\mathbf{q}). \tag{29}$$

Substituting this equation into the Bianchi identity (23) and integrating over time, we then find the density of electrons in terms of the shape fluctuations

$$\delta\rho(\mathbf{q}) = -q_z^2 C^{-1}_{2m}(\mathbf{q})u_m(\mathbf{q}) - q_{\bar{z}}^2 u_{-m}(\mathbf{q})C^{-1}_{m2}(-\mathbf{q}). \tag{30}$$

As an example, consider the regime of very small q, $z \ll 1$. In this regime the only nonzero component of C^{-1} is $C^{-1}_{22} = (4\pi b)^{-1}$, and Eq. (30) becomes

$$\delta\rho = \frac{2N+1}{4\pi}(\partial_z^2 u_2 + \partial_{\bar{z}}^2 u_{-2}). \tag{31}$$

This matches with the result found in in the bimetric theory of the nematic phase transition [24, 25], where $u_{\pm 2}$ are identified with two independent components of a unimodular dynamical metric [26, 27] and $\delta\rho$ is proportional to the Gaussian curvature constructed form this metric.

We can now use the equation just derived to find the equal-time correlation of the density, $\langle\delta\rho\delta\rho\rangle$. Note that if q is such that $(2N+1)q$ is smaller than the first zero of the Bessel function J_1, the ground state is annihilated by u_n with $n > 0$: $u_n|0\rangle = \langle 0|u_{-n} = 0$. From that we find

$$\begin{aligned}\langle\delta\rho(\mathbf{q})\delta\rho(-\mathbf{q})\rangle &= \frac{q^4}{16} C^{-1}_{2m}(\mathbf{q})[u_m(\mathbf{q}),\, u_{-l}(-\mathbf{q})]C^{-1}_{l2}(\mathbf{q}) \\ &= \frac{q^4}{16} C^{-1}_{22}(\mathbf{q}).\end{aligned} \tag{32}$$

To find C^{-1}_{22}, one needs to solve the equation $C_{nm}x_m = \delta_{n2}$ and read out x_2. Using the same technique which we used to find the eigenvalues of C, we find

$$C^{-1}_{22}(\mathbf{q}) = \frac{1}{4\pi b}\frac{4J_2(z)}{zJ_1(z)}\,. \tag{33}$$

The result is then

$$\langle\delta\rho(\mathbf{q})\delta\rho(-\mathbf{q})\rangle = \frac{(2N+1)q^4}{64\pi}\frac{4J_2(z)}{zJ_1(z)}\,. \tag{34}$$

Finally, after dividing by the density of electrons, we find the projected structure factor (neglecting sub-subleading terms in $1/N$)

$$\bar{s}(q) = \frac{(N+1)q^4}{8}\,\frac{4J_2(z)}{zJ_1(z)}\,. \tag{35}$$

Note that this form of $\bar{s}(q)$ has been found previously near a nematic phase transition [24]; here we have demonstrated that this formula is valid in general.

Discussion of the result.—We now expand $s(q)$ at small q,

$$\bar{s}(q) = \frac{N+1}{8} q^4 + \frac{N^2(N+2)}{48} q^6 + \cdots \tag{36}$$

The q^4 coefficient of the static structure factor $\bar{s}(q) = s_4 q^4 + s_6 q^6$ saturates the Haldane bound [26]

$$s_4 \equiv \lim_{q\to 0} \frac{\bar{s}(q)}{(q\ell_B)^4} = \frac{|\mathcal{S}-1|}{8}, \tag{37}$$

where $\mathcal{S}$ is the shift [28, 29], which is equal to $N+2$ for the state under consideration. The saturation of Haldane bound was derived earlier for a large class of trial states [30, 31]. However, Jain's sequences do not belong to those states, so the saturation of the Haldane bound is a nontrivial fact. Note, however, that Jain's states share with the states considered in Ref. [30, 31] the property of chirality: all edge modes propagate in one direction.

We now consider s_6. Little is know about the constraints on s_6. In contrast to s_4, there is no constraint analogous to the Haldane bound for s_6. Nevertheless, in Refs. [5–7] it has been argued that for certain chiral states (including the Laughlin and Pfaffian states), s_6 is completely determined by topological characteristics of the quantum Hall state. Extending the result of Ref. [5–7] to states where the composite fermions fill multiple Landau levels, one finds [25]

$$\bar{s}_6 = \frac{1}{8\nu}\left(\nu \mathrm{var}(s) + \nu\varsigma^2 - \nu\varsigma + \frac{\nu - c}{12}\right), \tag{38}$$

where ς is the guiding center orbital spin of the composite fermion and $\mathrm{var}(s)$ is the orbital spin variance [32]. For the Jain's state with $\nu = N/(2N+1)$, only the first two terms contribute to leading and subleading orders in N: $\nu\mathrm{var}(s) = \frac{1}{12}N(N^2-1)$ and $\varsigma = \frac{1}{2}(N+1)$, so $\bar{s}_6 = \frac{1}{48}N^3(N+2)$, in agreement with Eq. (36).

Equation (35) can also be obtained from the dynamic structure factor in a direct but not very illuminating manner. We have performed a (rather cumbersome) calculation of the dynamic structure factor by using the equations of motions for u_n in a theory with H having the form (16) with the diagonal elements given by the Landau parameters in the presence of a space- and time-varying external scalar potential A_0. We confirm that the dynamic structure factor has a complicated structure with an infinite number of poles corresponding to an infinite number of neutral excitations. After integrating over frequency, we find [33] that, rather miraculously, the infinite sum over residues can be taken exactly for $z < z_1$, the Landau parameters drop out, and the result matches with Eq. (35) [34].

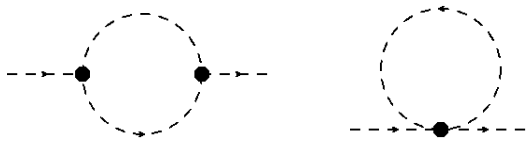

FIG. 1. One-loop contribution to u self-energy.

Sources of corrections.—In our treatment we have set u_0 and $u_{\pm 1}$ to zero, which means that we have assumed the absence of the kinetic term for a_μ in the Lagrangian. One source of corrections to the result (35) is the kinetic term of a_μ. In the Dirac composite fermion theory the Chern-Simons term for a_μ is forbidden by particle-hole symmetry, and for short-ranged electron-electron interaction the kinetic term appears first at order $(\partial_i a_j)^2 \sim N^{-2}a^2$ when $q \sim 1/N$. In contrast, the loop induces a term $\frac{\omega}{q}a^2$. The regime we are interested in is $q \sim 1/N$ and $\omega \sim 1/N^z$, where z is the dynamic critical exponent of a fermion coupled to a U(1) gauge field (up to three loops $z = 3/2$ [35], but the result may be changed in higher loops [36]), so the fermion-loop induced term is $\sim N^{1-z}a^2$. The bare term is suppressed by N^{3-z} relative to the fermion loop contribution. Assuming $z < 2$ the kinetic term for a_μ can be neglected without affecting the result to leading and next-to-leading orders in $1/N$.

We now try to estimate corrections due to interactions between the modes u_n. A treatment of this technically difficult problem is deferred to future work. We can, however, make the following observation. The action for u_n can be expected to have the form

$$\mathcal{L} \sim Nu\partial_t u + \frac{1}{N^{z-1}}(u^2 + u^3 + u^4 + \cdots). \tag{39}$$

Here the N dependence of the coefficient of the $u\partial_t u$ is dictated by the commutation relations (4), that of the u^2 term is determined by the energy gap, which is expected to be $O(N^{-z})$, and the coefficients of the interactions terms (u^3, u^4) come from the assumption that nonlinear effects in energy appear when $u \sim 1$. We need estimate the magnitude of the loop diagrams. There are two one-loop contributions to the u self-energy.

First, internal lines with energy much larger than the gap (N^{-z}) simply lead to renormalization of the low-energy effective theory for u_n. Thus, it is sufficient to limit oneself to $O(1)$ number of internal modes with the energy of order the gap. Second, we can restrict ourselves to internal momentum where non-Fermi liquid effects are expected to be important. At energy scale ω this occurs for $q \lesssim \omega^{1/z}$, in our case, $q \sim N^{-1/2}$. The first diagram in Fig. 1 then is of order

$$\left(\frac{1}{N^{z-1}}\right)^2 \int d\omega\, dq \left(N^{z-1}\right)^2 \sim \frac{1}{N^{z-1}}\frac{1}{N}. \tag{40}$$

This is by a factor of $1/N^2$ smaller than the inverse bare propagator for u. Analogously, the second diagram in

Fig. 1 can be estimated in the same way to be suppressed by $1/N^2$.

When the interaction of the electron is not short-ranged, but Coulomb, the the situation is different. We can estimate the influence of the Coulomb interaction by comparing the bare Coulomb interaction $|q|a_i^2$ with the one-loop contribution to the gauge self-energy, which is of order $\frac{\omega}{q}a^2 \sim a^2$ (ignoring powers of $\ln N$). The bare action is of order $1/N$ compared the term generated by the fermion loop. This means the results will be modified to order $1/N$. The result (35) remains valid at leading order in $1/N$ but one expects corrections to first order in $1/N$.

Conclusions.—In this paper we have shown that the projected structure factor can be computed reliably for states on Jain's sequences near $\nu = 1/2$, and the result depends on the kinematic structure of the Fermi surface of the composite fermion rather than detailed knowledge of the Hamiltonian. As the result, even when composite fermions at $\nu = 1/2$ form a non-Fermi liquid, we can still compute $\bar{s}(q)$ reliably, although in a regime of momentum which shrinks to zero as the filling factor approaches $1/2$. As far as we know, there are no proposed parent Hamiltonian for a general Jain state. However, from our results, we suspect that ground state wave function of $\nu = N/(2N+1)$ is robust under perturbation of the corresponding parent Hamiltonian by LLL two-body interactions [37]. It would be interesting to see if the algebraic method used here is useful in a wider range of problems related to the non-Fermi liquid.

The result (35) is amenable to numerical verification. One should keep in mind that in the more directly measurable *unprojected* static structure factor $s(q) = 1 - e^{-q^2/2} + \bar{s}(q)$, the contribution from the projected structure factor $\bar{s}(q)$ is subdominant in $1/N$. Recent numerical data [38] are consistent with Eq. (35) at small z, but also (and even a bit better) with a truncation of $\bar{s}(q)$ to the sum of the q^4 and q^6 terms. It may be that larger N is needed to clearly see the presence of a q^8 terms in $\bar{s}(q)$.

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