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Propagation and attenuation of sound in one-dimensional quantum liquids

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At low temperatures, elementary excitations of a one-dimensional quantum liquid form a gas that can move as a whole with respect to the center of mass of the system. This internal motion attenuates at exponentially long time scales. As a result, in a broad range of frequencies the liquid is described by two-fluid hydrodynamics, and the system supports two sound modes. The physical nature of the two sounds depends on whether the particles forming the quantum liquid have a spin degree of freedom. For particles with spin, the modes are analogous to the first and second sound modes in superfluid ^{4}He, which are the waves of density and entropy, respectively. When dissipative processes are taken into account, we find that at low frequencies the second sound is transformed into heat diffusion, while the first sound mode remains weakly damped and becomes the ordinary sound. In a spinless liquid the entropy and density oscillations are strongly coupled, and the resulting sound modes are hybrids of the first and second sound. As the frequency is lowered and dissipation processes become important, the crossover to single-fluid regime occurs in two steps. First the hybrid modes transform into predominantly density and entropy waves, similar to the first and second sound, and then the density wave transforms to the ordinary sound and the entropy wave becomes a heat diffusion mode. Finally, we account for the dissipation due to viscosity and intrinsic thermal conductivity of the gas of excitations, which controls attenuation of the sound modes at high frequencies.

I. INTRODUCTION

At low frequencies vibrations of a fluid propagate through it in the form of sound waves. The latter are oscillations of density accompanied by oscillations of entropy density, such that the entropy per particle remains unchanged[1]. When liquid ^{4}He undergoes the superfluid transition, the physics of sound waves changes dramatically. Instead of a single sound wave, two kinds of sound propagate in the superfluid. The first sound is an oscillation of mostly the particle density, whereas the second sound is a wave of entropy[2]. The physical origin of this behavior is that the gas of elementary excitations can move with respect to the rest of the superfluid without friction. Thus superfluid ^{4}He can be treated as a system of two interpenetrating fluids and described theoretically in the framework of two-fluid hydrodynamics[2–4]. The existence of two sound modes is a generic property of superfluids; it was observed experimentally in superfluid ^{4}He[4] and also more recently in an ultracold Fermi gas with resonant interactions[5].

In this paper we study sound in one-dimensional quantum liquids, such as the electron liquid in long quantum wires[6,7] or cold atomic gases in narrow traps[8,9]. These systems are commonly described theoretically in the framework of the Tomonaga-Luttinger liquid theory[10–13]. In the simplest form of this theory the elementary excitations of a one-dimensional quantum liquid are noninteracting bosons with linear spectrum. To account for the relaxation of the system to equilibrium, the formally irrelevant perturbations must be added to the Luttinger liquid Hamiltonian.

Sound waves propagate only in systems that relax to local thermal equilibrium on time scales shorter than the period of oscillations of density. Relaxation of one-dimensional quantum liquids has been studied in a number of recent papers[14–20]. The rate of collisions $\tau_{\rm ex}^{-1}$ between the thermally excited quasiparticles scales as a power law of the temperature T. In particular, in spinless Luttinger liquids $\tau_{\rm ex}^{-1}\propto T^7$[19,20], whereas for weakly interacting fermions with spin $\tau_{\rm ex}^{-1}\propto T^{15}$.

At time scales longer than $\tau_{\rm ex}$ the excitations of the one-dimensional quantum liquid come to equilibrium with each other and form a gas, which in general moves at some velocity $u_{\rm ex}$. It is important to keep in mind that $u_{\rm ex}$ does not necessarily equal the velocity of the center of mass of the liquid. The two velocities do equilibrate, but this relaxation involves different types of scattering processes, whose rate scales exponentially with temperature, $\tau^{-1}\propto e^{-D/T}$[21–23], where D is the bandwidth of the model. Thus, at low temperatures there is a broad range of frequencies

$$\tau^{-1}\ll\omega\ll\tau_{\rm ex}^{-1}, \tag{1}$$

in which the gas of excitations moves with respect to the rest of the fluid with negligible friction. In this regime the behavior of a one-dimensional quantum liquid is analogous to that of a superfluid. In particular, the system is described by two-fluid hydrodynamics and supports two sound modes[24,25]. At $\omega\ll\tau^{-1}$ the two-fluid behavior is destroyed, and only one sound mode remains. This is a manifestation of the fact that one-dimensional quantum liquid is not a superfluid.

In this paper we expand on the earlier results[24,25] by accounting for the dissipation processes affecting the sound modes. This enables us to investigate the crossover from the two-fluid behavior in the frequency range (1) to ordinary hydrodynamics at $\omega\to 0$. The crossover at

$\omega \sim \tau^{-1}$ is especially interesting in the case of spinless quantum liquids, where the two sound modes have an unusual hybrid nature[25]. In addition to including the slow relaxation at the time scale τ, we consider the effects of finite $\tau_{\rm ex}$, by including the effects of viscosity and thermal conductivity in the hydrodynamic equations. This enables us to obtain the full picture of propagation and attenuation of the sound modes in one-dimensional quantum liquids at all frequencies $\omega \ll \tau_{\rm ex}^{-1}$.

The paper is organized as follows. In Sec. II we discuss the properties of bosonic elementary excitations of interacting one-dimensional systems in the Luttinger liquid approximation. We obtain the basic thermodynamic characteristics of the quantum liquid in the lowest order at $T \ll D$. In Sec. III we derive the hydrodynamic equations of the one-dimensional quantum liquid near equilibrium, while accounting for both the fast and slow relaxation processes described by the rates $\tau_{\rm ex}^{-1}$ and τ^{-1}. In Sec. IV we solve the hydrodynamics equations in the frequency range (1) and obtain the two sound modes discussed in Refs.[24,25]. The crossover from two sound modes to a single sound at $\omega \ll \tau^{-1}$ is studied in Sec. V. The effect of the fast relaxation processes on the attenuation of sound is studied in Sec. VI. We discuss our results in Sec. VII.

II. THERMODYNAMICS OF LUTTINGER LIQUIDS

We start by reviewing the general properties of one-dimensional quantum liquids within the Luttinger liquid theory. Let us consider a system of N spinless particles confined to a region of length L with periodic boundary conditions. The Luttinger liquid theory describes the low-energy properties of the system. The elementary excitations are bosons that occupy discrete states with momenta k that are multiples of $2\pi\hbar/L$, excluding $k=0$. The momentum of the system is given by[11,12]

$$P = \frac{\pi\hbar}{L} NJ + \sum_k k N_k. \tag{2}$$

Here N_k is the occupation number of the quasiparticle state k, while J is an integer that is even if the particles of the quantum liquid are bosons and has parity opposite to that of N in the case of fermions. The first term in Eq. (2) accounts for the momentum of the moving liquid in the ground state defined by the absence of excitations, $N_k = 0$. The second term in Eq. (2) is the momentum of the bosonic excitations in the Luttinger liquid.

The number J will play an important role in the hydrodynamic description of the Luttinger liquid. It describes the quantization of momentum of the liquid in the ground state, which can be understood as follows. The smallest change of the momentum of the system that is not accompanied by creation or destruction of quasiparticles is obtained by adding momentum quantum $2\pi\hbar/L$ to each of the N particles of the fluid. This corresponds to $J \to J+2$ in Eq. (2). For Galilean invariant systems considered in this paper it is convenient to express the first term in Eq. (2) as mNu_0, where m is the mass of the particles, and the velocity

$$u_0 = \frac{\pi\hbar J}{mL}. \tag{3}$$

One can think of u_0 as the velocity of the reference frame in which $J=0$. In the two-fluid hydrodynamics it will play the role of the velocity of the superfluid component.

The energy of a Galilean invariant quantum liquid with the number of particles close to a reference value N_0 in the Luttinger liquid theory has the form[11,12]

$$E = \frac{mv^2}{2N_0}(N-N_0)^2 + \frac{\pi^2\hbar^2 NJ^2}{2mL^2} + \sum_k \epsilon(k) N_k. \tag{4}$$

Here v is the velocity of the bosonic excitations, determined by the compressibility of the quantum liquid in the ground state. The energy of the excitations is therefore usually written as $\epsilon(k) = v|k|$. This expression applies to the case of a stationary liquid, i.e., at $J=0$. At nonvanishing J, the excitation energy

$$\epsilon(k) = v|k| + u_0 k \tag{5}$$

is found by the Galilean transformation to the frame moving with velocity (3).

In addition to spinless quantum liquids, we will also consider a one-dimensional system of spin-$\frac{1}{2}$ fermions with repulsive interactions. This system can also be described by a Luttinger liquid theory with a few modifications to account for the spin degrees of freedom[13]. The main difference with the spinless case is that instead of one branch of bosonic excitations there are two such branches. The two types of bosons account for excitations in the charge and spin channels and propagate at two different speeds, v_ρ and v_σ, respectively. It is worth mentioning that in the case of spin-$\frac{1}{2}$ fermions in addition to the parameters N and J, the state of the Luttinger liquid depends on two similar quantum numbers in the spin channel. In the absence of magnetic field and other violations of spin rotation symmetry these variables take zero values and do not affect sound in the system.

We now turn to the discussion of relaxation of one-dimensional quantum liquids to thermal equilibrium. Weak interactions between the elementary excitations bring about their relaxation to an equilibrium state. These interactions belong to two classes. The strongest interactions are accounted for by the irrelevant perturbations to the Hamiltonian of the Luttinger liquid[13]. Their strength scales as a power law of temperature. An important feature of these interactions is that they conserve the total momentum of the excitations given by the second term in Eq. (2) and do not change J. As a result, at the time scale $\tau_{\rm ex}$ the occupation numbers N_k approach the equilibrium values

$$N_k = \left[\exp\left(\frac{\epsilon(k) - u_{\rm ex}k}{T}\right) - 1\right]^{-1}. \tag{6}$$

The parameter $u_{\rm ex}$ is the velocity of the gas of excitations. Conservation of momentum in the collisions of excitations means that in the thermodynamic equilibrium $u_{\rm ex}$ does not have to vanish.

Interactions of the second type conserve the total momentum of the system but not the two terms in the right hand side of Eq. (2) separately. The resulting scattering processes are analogous to the umklapp scattering of phonons in a crystal. At low temperatures the corresponding relaxation time is exponentially long, $\tau \propto e^{D/T}$[21–23]. These processes result in friction between the gas of excitations and the rest of the quantum liquid, and can be expressed as relaxation of $u_{\rm ex} - u_0$ to zero,

$$\frac{d}{dt}(u_{\rm ex} - u_0) = -\frac{u_{\rm ex} - u_0}{\tau}. \tag{7}$$

At long time scales $t \gg \tau$ both $u_{\rm ex}$ and u_0 approach the center of mass velocity of the liquid.

In Sec. III we apply the two-fluid hydrodynamic theory to the description of one-dimensional quantum liquids. This approach is applicable at time scales that are long compared with $\tau_{\rm ex}$, but not necessarily longer than τ. Under these conditions the motion of the quantum liquid is characterized by two velocities: u_0 and $u_{\rm ex}$. To obtain the hydrodynamic equations we will need expressions for a number of thermodynamic properties of the Luttinger liquid at given density of particles n, temperature, u_0, and $u_{\rm ex}$. Since we will only use linearized hydrodynamic equations, the expressions below are limited to terms up to first order in u_0, $u_{\rm ex}$, and the deviation of density n from its reference value n_0.

Substituting the expression (6) for the occupation numbers into Eq. (4) we obtain the energy density

$$\varepsilon = \frac{\pi T^2}{6\hbar\widetilde{v}}. \tag{8}$$

We wrote Eq. (8) in the form that applies to both the spinless quantum liquid and to the case of spin-$\frac{1}{2}$ fermions. In the former case $\widetilde{v} = v$, whereas in the latter one

$$\widetilde{v} = \left(\frac{1}{v_\rho} + \frac{1}{v_\sigma}\right)^{-1}. \tag{9}$$

To obtain the entropy density s we use the thermodynamic definition of temperature $T = (\partial\varepsilon/\partial s)_n$ and find

$$s = \frac{\pi T}{3\hbar\widetilde{v}}. \tag{10}$$

We then rewrite the result (8) in an alternative form

$$\varepsilon = \frac{3\hbar}{2\pi}\widetilde{v}s^2. \tag{11}$$

Using the thermodynamic expression for the pressure $\Pi = -\varepsilon + Ts + n(\partial\varepsilon/\partial n)_s$, we find

$$\Pi = \Pi^{(0)} + \varepsilon\frac{\partial_n(n\widetilde{v})}{\widetilde{v}}. \tag{12}$$

Here $\Pi^{(0)}$ is the pressure at zero temperature. It originates from the ground state energy density omitted in Eqs. (8) and (11) and cannot be determined within the Luttinger liquid theory.

Similarly, substitution of Eqs. (5) and (6) into the expression (2) gives the momentum density

$$p = mnu_0 + \frac{2\varepsilon}{\bar{v}^2}(u_{\rm ex} - u_0). \tag{13}$$

Here $\bar{v} = v$ in the case of spinless quantum liquid, while

$$\bar{v} = \left(\frac{v_\rho^{-1} + v_\sigma^{-1}}{v_\rho^{-3} + v_\sigma^{-3}}\right)^{1/2} \tag{14}$$

for spin-$\frac{1}{2}$ fermions.

III. HYDRODYNAMICS OF ONE-DIMENSIONAL QUANTUM LIQUIDS

As we saw in Sec. II, at frequencies $\omega \ll \tau_{\rm ex}^{-1}$ the motion of a one-dimensional quantum liquid is characterized by two velocities. Velocity u_0 is associated with the ground state motion of the liquid, whereas $u_{\rm ex}$ is the velocity of the gas of excitations. These two velocities are analogous to the velocities of the superfluid and normal components of superfluid ^{4}He[2,3]. In particular, the momentum density (13) of the liquid can be written in the form

$$p = \rho_0 u_0 + \rho_{\rm ex} u_{\rm ex}. \tag{15}$$

Here the mass densities of the two components are given by

$$\rho_0 = \rho - \rho_{\rm ex}, \quad \rho_{\rm ex} = \frac{\pi T^2}{3\hbar\widetilde{v}\bar{v}^2}, \tag{16}$$

with $\rho = mn$.

To fully describe the state of the one-dimensional fluid, in addition to the velocities u_0 and $u_{\rm ex}$ one should specify the particle density n and temperature T. Thus the motion of the liquid is described by four evolution equations. The first three of these equations express the usual conservation laws for the mass, energy, and momentum. In this paper we limit ourselves to the effects that are linear in small deviations from equilibrium. In this approximation energy dissipation is neglected, and energy conservation is equivalent to conservation of entropy. This yields continuity equations for the mass, entropy, and momentum densities

$$\partial_t\rho + \partial_x j_\rho = 0, \tag{17a}$$

$$\partial_t s + \partial_x j_s = 0, \tag{17b}$$

$$\partial_t p + \partial_x j_p = 0. \tag{17c}$$

Here j, j_s, and j_p are the particle, entropy, and momentum currents.

Let us now show that the fourth evolution equation has the form

$$\partial_t u_0 + \partial_x j_{u_0} = -\frac{\rho_{\rm ex}}{\rho}\,\frac{u_0 - u_{\rm ex}}{\tau}. \tag{17d}$$

At $\tau \to \infty$ there is a fourth conserved quantity, namely the quantum number J of the given state of the Luttinger liquid. Accordingly, Eq. (17d) takes the form of the continuity equation for the density J/L, see Eq. (3). We note that at $\tau \to \infty$ this equation is equivalent to the equation describing the time evolution of superfluid velocity in two-fluid hydrodynamic theory of liquid ^{4}He[2,3].

At finite τ the quantum number J is no longer conserved, resulting in the friction between the two components of the fluid, see Eq. (7). In this case the rate of change of velocity u_0 cannot be fully expressed as a gradient of the current j_{u_0} in Eq. (17d). Since the dissipative processes that result in friction conserve the momentum (15), the rate du_0/dt in the right-hand side of Eq. (17d) can be found by combining Eq. (7) with $\rho_0(du_0/dt) + \rho_{\rm ex}(du_{\rm ex}/dt) = 0$.

Because of the Galilean invariance, the mass current j_ρ in Eq. (17a) equals the momentum density,

$$j_\rho = \rho_0 u_0 + \rho_{\rm ex} u_{\rm ex}. \tag{18a}$$

The expressions for the remaining three currents are well understood in the theory of superfluidity of liquid ^{4}He[3]. Adapting them to one dimension, we obtain

$$j_s = s u_{\rm ex} - \frac{\kappa_{\rm ex}}{T}\partial_x T, \tag{18b}$$

$$j_p = \Pi - \zeta_1 \partial_x(p - \rho u_{\rm ex}) - \zeta_2 \partial_x u_{\rm ex}, \tag{18c}$$

$$j_{u_0} = \frac{\mu}{m} - \zeta_3 \partial_x(p - \rho u_{\rm ex}) - \zeta_4 \partial_x u_{\rm ex}, \tag{18d}$$

where μ is the chemical potential of the quantum liquid. Expressions (18b)–(18d) include the dissipative contributions to the respective currents, which are parametrized by four coefficients of bulk viscosity $\zeta_1, \ldots, \zeta_4$ and the thermal conductivity of the gas of excitations $\kappa_{\rm ex}$. We note that these parameters are proportional to $\tau_{\rm ex}$ and that the viscosity coefficients satisfy an Onsager relation $\zeta_1 = \zeta_4$.

IV. TWO SOUND MODES

Let us now consider the solutions of the hydrodynamic equations (17) and (18) for frequencies in the range (1), i.e., assuming $\tau \to \infty$ and $\tau_{\rm ex} = 0$. The former limit enables us to set the right-hand side of Eq. (17d) to zero. In addition, since the bulk viscosities and the thermal conductivity of the gas of excitations are proportional to $\tau_{\rm ex}$, we will set $\zeta_i = 0$ for all i and $\kappa_{\rm ex} = 0$ in Eqs. (18b)–(18d). Following Ref.[3], instead of the entropy density s we will use the entropy per unit mass $\sigma = s/\rho$. The resulting four equations are

$$\partial_t \rho + \rho_0\,\partial_x u_0 + \rho_{\rm ex}\,\partial_x u_{\rm ex} = 0, \tag{19a}$$

$$\partial_t \sigma + \frac{\rho_0}{\rho}\sigma\,\partial_x(u_{\rm ex} - u_0) = 0, \tag{19b}$$

$$\rho_0\,\partial_t u_0 + \rho_{\rm ex}\,\partial_t u_{\rm ex} + \partial_x \Pi = 0, \tag{19c}$$

$$\partial_t u_0 + \frac{1}{m}\partial_x \mu = 0. \tag{19d}$$

The four first order differential equations (19) can be easily reduced to two second order equations for ρ and σ. First, combining Eqs. (19a) and (19c), we obtain

$$\partial_t^2 \rho = \partial_x^2 \Pi. \tag{20}$$

Then, using the thermodynamic relation $d\,\Pi - n\,d\mu = s\,dT$ and excluding u_0 and $u_{\rm ex}$ from Eqs. (19b)–(19d), we find

$$\partial_t^2 \sigma = \frac{\rho_0}{\rho_{\rm ex}}\sigma^2 \partial_x^2 T. \tag{21}$$

In Eqs. (20) and (21) we will choose ρ and σ as two variables describing the state of the fluid, and treat the pressure and temperature as their functions $\Pi(\rho, \sigma)$ and $T(\rho, \sigma)$.

To study the sound modes in the system, we assume that ρ and σ have small variations of the form $\delta\rho \cos[q(x - ct)]$ and $\delta\sigma \cos[q(x - ct)]$. Expanding Eq. (20) and (21) to linear order in $\delta\rho$ and $\delta\sigma$, we get

$$\big(c^2 - A_{11}\big)\delta\rho - A_{12}\,\delta\sigma = 0, \tag{22a}$$

$$-A_{21}\,\delta\rho + \big(c^2 - A_{22}\big)\delta\sigma = 0. \tag{22b}$$

Here the coefficients A_{ij} are given by

$$A_{11} = \left(\frac{\partial \Pi}{\partial \rho}\right)_\sigma, \tag{23a}$$

$$A_{12} = \left(\frac{\partial \Pi}{\partial \sigma}\right)_\rho, \tag{23b}$$

$$A_{21} = \frac{\rho_0}{\rho_{\rm ex}}\sigma^2 \left(\frac{\partial T}{\partial \rho}\right)_\sigma, \tag{23c}$$

$$A_{22} = \frac{\rho_0}{\rho_{\rm ex}}\sigma^2 \left(\frac{\partial T}{\partial \sigma}\right)_\rho. \tag{23d}$$

The system of equations (22) has nonvanishing solutions with velocities given by

$$c_\pm^2 = \frac{A_{11} + A_{22}}{2} \pm \frac{1}{2}\sqrt{(A_{11} - A_{22})^2 + 4A_{12}A_{21}}. \tag{24}$$

Thus, in the frequency range (1) the one dimensional quantum liquid supports two sound modes.

For a system of spin-$\frac{1}{2}$ fermions, using Eqs. (10)–(12) we obtain the following results for A_{ij}

$$A_{11} = v_\rho^2 + \frac{\pi T^2}{6\hbar\rho\widetilde{v}^2}\partial_\rho^2\big(\rho^2\widetilde{v}\big), \tag{25a}$$

$$A_{12} = \frac{\rho T}{\widetilde{v}}\partial_\rho(\rho\widetilde{v}), \tag{25b}$$

$$A_{21} = \frac{\pi\bar{v}^2 T}{3\hbar\rho^2\widetilde{v}^2}\partial_\rho(\rho\widetilde{v}), \tag{25c}$$

$$A_{22} = \bar{v}^2. \tag{25d}$$

In Eq. (25a) we used the definition of velocity v_ρ in terms of the zero temperature compressibility, $v_\rho^2 = \partial_\rho \Pi^{(0)}$. Note that within the simple Luttinger liquid theory based on Eqs. (2) and (4) only A_{11} can be obtained with accuracy beyond the leading order at $T \to 0$.

Assuming the interactions between the fermions are repulsive, we have $v_\rho > v_\sigma$, and thus $v_\rho > \bar{v}$, see Eq. (14). When the temperature approaches zero Eq. (25) yields $A_{11} = v_\rho^2$, $A_{22} = \bar{v}^2$, $A_{12} = A_{21} = 0$, and the two sound velocities (24) are

$$c_+ = v_\rho, \quad c_- = \bar{v}. \tag{26}$$

In this limit, the faster and slower sound modes are pure waves of density ρ and entropy σ, respectively. They are completely analogous to the first and second sound in superfluid ^{4}He[2,3]. At small but finite temperature the nature of the sound modes remains largely the same, but there is a small mixing of the oscillations of ρ and σ. For example, the density oscillation $\delta\rho$ of the first sound mode is accompanied by a weak oscillation of entropy, $\delta\sigma \simeq [A_{21}/(v_\rho^2 - \bar{v}^2)]\delta\rho$. This is in contrast with the usual adiabatic sound in classical fluids, for which $\delta\sigma = 0$ at any temperature.

In a spinless Luttinger liquid the nature of the two sound modes is qualitatively different. To obtain A_{ij} in this case, one should substitute $v_\rho = \widetilde{v} = \bar{v} = v$ into Eq. (25). At $T \to 0$ this yields $A_{11} = A_{22} = v^2$ and $A_{12} = A_{21} = 0$. As a result, the two sound velocities (24) are equal, $c_\pm = v$. At a small but finite temperature, the leading order correction to A_{22} is quadratic in T[25], as is the correction to A_{11} in Eq. (25a). Thus the term $4A_{12}A_{21} \propto T^2$ dominates the square root in Eq. (24), resulting in a linear in temperature splitting of the sound velocities:

$$c_\pm = v \pm \frac{\sqrt{A_{12}A_{21}}}{2v} = v \pm \sqrt{\frac{\pi T^2}{12\hbar\rho v^3}}\, \partial_\rho(\rho v). \tag{27}$$

The sound mode propagating at the higher speed c_+ corresponds to in-phase oscillations of density and entropy. The relative magnitude of the oscillations $\delta\rho/\delta\sigma = \sqrt{A_{12}/A_{21}}$ approaches a finite limit at $T \to 0$. The mode propagating at the slower speed c_- is characterized by oscillations of ρ and σ that are of opposite signs. The two sound modes are therefore different from the usual first and second sounds, but instead combine the features of both.

This hybrid nature of the sound modes in a spinless quantum liquid was discussed in Ref.[25]. We note that in a three-dimensional weakly interacting Bose gas the speeds of the first and second sounds become equal at a certain temperature T_1[26]. This results in hybridization of these sound modes at temperatures in a narrow vicinity of T_1[27]. In contrast, in a one-dimensional spinless quantum liquid strong hybridization of sound modes occurs at any interaction strength, as long as $T \ll D$.

V. CROSSOVER BETWEEN THE REGIMES OF ONE AND TWO SOUND MODES

The two sound modes discussed in Sec. IV exist in the frequency range (1). Under these conditions the one-dimensional fluid behaves similarly to a superfluid and can be thought of as consisting of two components moving with two different velocities u_0 and $u_{\rm ex}$. In contrast to superfluids, there is friction between the two components of the fluid, but it is negligible at $\omega \gg \tau^{-1}$. On the other hand, at low frequencies $\omega \ll \tau^{-1}$ the friction between the two components is important, and one should expect equilibration of the two velocities, $u_0 = u_{\rm ex}$. In this limit the one-dimensional quantum liquid loses its superfluid properties and behaves as an ordinary fluid. In particular, only a single sound mode is present at $\omega \ll \tau^{-1}$. In this section we discuss the crossover between the regimes with one and two sound modes at $\omega \sim \tau^{-1}$.

The hydrodynamic equations (17) and (18) are applicable in this regime, with the only limitation on the frequency being $\omega \ll \tau_{\rm ex}^{-1}$. In contrast to our discussion in Sec. IV, one can no longer assume $\tau \to \infty$. On the other hand, one can still consider the limit $\tau_{\rm ex} \to 0$ and neglect the bulk viscosities ζ_i and $\kappa_{\rm ex}$ in Eq. (18). Under these circumstances the hydrodynamic equations (17) can again be rewritten in the form (19), but with Eq. (19d) replaced by

$$\partial_t u_0 + \frac{1}{m}\partial_x \mu = -\frac{\rho_{\rm ex}}{\rho}\,\frac{u_0 - u_{\rm ex}}{\tau}. \tag{28}$$

This change does not affect the derivation of Eq. (20), but Eq. (21) is replaced by

$$\partial_t^2 \sigma + \frac{1}{\tau}\partial_t \sigma = \frac{\rho_0}{\rho_{\rm ex}}\sigma^2 \partial_x^2 T. \tag{29}$$

Assuming that the small variations of ρ and σ in Eqs. (20) and (29) have the forms $\delta\rho\, e^{i(qx-\omega t)}$ and $\delta\sigma\, e^{i(qx-\omega t)}$, respectively, we find

$$\left(\omega^2 - A_{11}q^2\right)\delta\rho - A_{12}q^2\,\delta\sigma = 0, \tag{30a}$$

$$-A_{21}q^2\,\delta\rho + \left(\omega^2 + \frac{i\omega}{\tau} - A_{22}q^2\right)\delta\sigma = 0. \tag{30b}$$

This system of linear equations has nontrivial solutions if

$$\left(\omega^2 - A_{11}q^2\right)\left(\omega^2 + \frac{i\omega}{\tau} - A_{22}q^2\right) - A_{12}A_{21}q^4 = 0. \tag{31}$$

Equation (31) enables one to study propagation of perturbations of both density and entropy for arbitrary $\omega\tau$, provided $\omega\tau_{\rm ex} \to 0$.

A. Low frequency regime

We first consider the low-frequency limit $\omega\tau \to 0$. In this case Eq. (31) takes the form

$$(\omega^2 - A_{11}q^2)\left(\frac{i\omega}{\tau} - A_{22}q^2\right) - A_{12}A_{21}q^4 = 0. \quad (32)$$

This equation for $\omega(q)$ has two types of solutions. The first one corresponds to the ordinary sound in the fluid. For the right-moving wave we get

$$\omega = cq - i\tau\frac{A_{12}A_{21}}{2A_{11}}q^2. \quad (33)$$

Here

$$c = \sqrt{A_{11}}, \quad (34)$$

which in combination with Eq. (23a) gives the usual expression for the speed of sound in terms of the adiabatic compressibility[1]. The presence of the imaginary part of ω in Eq. (33) means that the sound wave $\delta\rho\, e^{i(qx-\omega t)}$ gradually decays over time.

The second solution of Eq. (32) is purely imaginary,

$$\omega = -i\tau\frac{A_{11}A_{22} - A_{12}A_{21}}{A_{11}}q^2. \quad (35)$$

It shows that the second sound cannot exist at $\omega \ll \tau^{-1}$. Instead, Eq. (35) describes diffusive propagation of heat in the system.

In a system with thermal conductivity κ and specific heat at constant pressure c_p the dependence $\omega(q)$ has the form $\omega = -i(\kappa/c_p)q^2$. This enables one to obtain the expression for the thermal conductivity of the quantum liquid

$$\kappa = \tau A_{22}c_v. \quad (36)$$

Here we used the thermodynamic relation

$$\frac{c_p}{c_v} = \frac{A_{11}A_{22}}{A_{11}A_{22} - A_{12}A_{21}} \quad (37)$$

with A_{ij} given by Eq. (23) to express κ in terms of the specific heat at constant density c_v. In a Luttinger liquid at low temperature, the entropy density (10) is a linear function of T, resulting in $c_v = T(\partial s/\partial T)_\rho = s$. We therefore conclude from Eqs. (36) and (25d) that the thermal conductivity of a one-dimensional quantum liquid is given by

$$\kappa = \frac{\pi T\bar{v}^2\tau}{3\hbar\widetilde{v}}, \quad (38)$$

where we used the result (25d) for A_{22} at $T \to 0$. The known result for the thermal conductivity of a spinless one-dimensional quantum liquid[28] is obtained from Eq. (38) by substitution $\bar{v} = \widetilde{v} = v$. It is worth mentioning that with the aid of Eqs. (36) and (37) the expression (33) for the frequency of sound can be brought to the form

$$\omega = cq - \frac{i\kappa}{2}\left(\frac{1}{c_v} - \frac{1}{c_p}\right)q^2. \quad (39)$$

The last term in Eq. (39) reproduces the well-known expression for the rate of sound absorption caused by the finite thermal conductivity of the fluid[1].

B. High frequency regime

In the limit of large frequencies, $\omega\tau \to \infty$, Eq. (31) has solutions of the form $\omega = c_\pm q$[29] with velocities given by Eq. (24), and thus reproduces the results of Sec. IV. We now obtain the attenuation rate of these sound modes in the first order in $(\omega\tau)^{-1}$. A straightforward calculation yields the solutions of Eq. (31) in the form

$$\omega = c_\pm q - \frac{i}{4\tau}\left[1 \mp \frac{|A_{11} - A_{22}|}{\sqrt{(A_{11} - A_{22})^2 + 4A_{12}A_{21}}}\right]. \quad (40)$$

This expression gives very different attenuation rates of the sound modes in the cases of liquids of spinless particles and spin-$\frac{1}{2}$ fermions.

As discussed in Sec. IV, in a system with spins we have $A_{12}A_{21} \ll (A_{11} - A_{22})^2$. In this regime we obtain the damping of the first and second sound modes in the form

$$\omega - v_\rho q = -\frac{i}{2\tau}\frac{A_{12}A_{21}}{(A_{11} - A_{22})^2} = -\frac{i}{\tau}\frac{\pi\bar{v}^2[\partial_\rho(\rho\widetilde{v})]^2T^2}{6\hbar\rho\widetilde{v}^3(v_\rho^2 - \bar{v}^2)^2}, \quad (41a)$$

$$\omega - \bar{v}q = -\frac{i}{2\tau}. \quad (41b)$$

The attenuation rate of the first sound is smaller than that of the second sound by a parameter of order $(T/D)^2$. In the case of a spinless quantum liquid we have $A_{12}A_{21} \gg (A_{11} - A_{22})^2$, and Eq. (40) yields

$$\omega - c_\pm q = -\frac{i}{4\tau}. \quad (42)$$

Thus the two hybrid sound modes decay at the same rate.

The qualitatively different behavior of the attenuation rates of the different sound modes can be interpreted as follows. Thermal conductivity gives rise to dissipation in systems with non-uniform temperature. As discussed in Sec. IV, the second sound mode in a system with spins is an almost pure wave of entropy and, therefore, temperature. Thus the second sound is attenuated quite effectively [Eq. (41b)]. In contrast, the first sound is primarily a wave of density, with only a weak disturbance in temperature. As a result the processes of thermal conductivity attenuate the first sound less effectively [Eq. (41a)]. In a spinless system, the two hybrid sounds carry equal in magnitude oscillations of entropy, resulting in equal dissipation, see Eq. (42).

C. Crossover regime in a quantum liquid of spin-$\frac{1}{2}$ fermions

We begin the consideration of the crossover regime $\omega\tau \sim 1$ with the first sound. From our results (33) and (41a) in the limiting cases or small and large $\omega\tau$, we expect the attenuation to remain small as $(T/D)^2$ at $\omega\tau \sim 1$. We therefore substitute $\omega = cq+\delta\omega$ into Eq. (31) and linearize it in small $\delta\omega$. This yields

$$\delta\omega = \frac{A_{12}A_{21}}{2\sqrt{A_{11}}}\frac{q^2}{(A_{11}-A_{22})q + \frac{i}{\tau}\sqrt{A_{11}}}. \tag{43}$$

Taking into account Eq. (25), it is easy to see that for $v_\rho > \bar{v}$ the correction (43) is indeed small, $|\delta\omega|/\omega \sim (T/D)^2$. The rate of decay of the first sound is given by the imaginary part of ω,

$$\operatorname{Im}\omega = -\frac{q^2\tau}{2}\frac{A_{12}A_{21}}{(q\tau)^2(A_{11}-A_{22})^2 + A_{11}}. \tag{44}$$

In the limits $q \to 0$ and $q \to \infty$ Eq. (44) reproduces our earlier results (33) and (41a), respectively. We note also that $\operatorname{Re}\delta\omega \propto q$ at large q. Thus the speed of the first sound c_+ is slightly different from the speed $c = \sqrt{A_{11}}$ of the ordinary sound,

$$c_+ = c + \frac{A_{12}A_{21}}{2\sqrt{A_{11}}(A_{11}-A_{22})}. \tag{45}$$

Using Eq. (25) it is easy to see that at low temperature $|c - v_\rho| \sim |c_+ - v_\rho| \sim v_\rho(T/D)^2$. More generally, the result (45) for the speed of the first sound holds at frequencies in the range (1) and can be obtained directly from Eq. (24).

To study the crossover from the second sound at $\omega\tau \gg 1$ to diffusive heat transport at $\omega\tau \ll 1$ we rewrite Eq. (31) in the form

$$\omega^2 + \frac{i\omega}{\tau} - A_{22}q^2\left(1 + \frac{A_{12}A_{21}q^2}{A_{22}(\omega^2 - c^2q^2)}\right) = 0. \tag{46}$$

At the crossover we have $|\omega - cq| \sim \tau^{-1}$. Then, using Eq. (25) we estimate the second term in parentheses of Eq. (46) to be of the order of $(T/D)^2 \ll 1$. Neglecting it and solving the resulting quadratic equation, we obtain

$$\omega = \pm\sqrt{(\bar{v}q)^2 - \frac{1}{(2\tau)^2}} - \frac{i}{2\tau}, \tag{47}$$

where we also used Eq. (25d). At $\bar{v}q \gg \tau^{-1}$ Eq. (47) describes the second sound with attenuation $\operatorname{Im}\omega = (2\tau)^{-1}$, in agreement with Eq. (41b). This solution exists at all $\bar{v}q > (2\tau)^{-1}$, with real part vanishing at $\bar{v}q = (2\tau)^{-1}$. As smaller q the frequencies are imaginary. At $q \to 0$ one of these damped modes becomes $\omega = -i\tau(\bar{v}q)^2$, thereby transforming into the heat diffusion mode (35).

The evolution of the first and second sound through the crossover at $\omega\tau \sim 1$ can be summarized as follows. The first sound mode does not experience significant changes at the crossover and gradually evolves into the ordinary sound at $\omega\tau \ll 1$. Its attenuation is always small, $\operatorname{Im}\omega \ll \operatorname{Re}\omega$. In contrast, the second sound mode experiences a sharp transition when its wavevector q crosses $\bar{q} = 1/2\bar{v}\tau$. The wave-like propagation of heat, associated with the second sound, occurs at $q > \bar{q}$, with damping becoming strong at $q \sim \bar{q}$. At $q < \bar{q}$ wave-like behavior is absent and at $q \ll \bar{q}$ the heat propagation becomes diffusive.

D. Crossover regime in a spinless quantum liquid

The crossover from two hybrid sound modes in a spinless quantum liquid to a single sound at $\omega\tau \to 0$ is more complicated. We start by noticing that as discussed in Sec. IV, at low temperatures the difference between A_{11} and A_{22} in a spinless system is negligible, and one can set $A_{11} = A_{22} = v$. Then Eq. (31) can be rewritten in the form

$$\left(\omega^2 - v^2q^2\right)^2 + \frac{i\omega}{\tau}\left(\omega^2 - v^2q^2\right) - (v\delta c)^2q^4 = 0, \tag{48}$$

Here we have introduced the difference of velocities of the hybrid sound modes,

$$\delta c = c_+ - c_- = \frac{\sqrt{A_{12}A_{21}}}{v}, \tag{49}$$

cf. Eq (27). We treat Eq. (48) as a quadratic equation for $\omega^2 - v^2q^2$ and write the two solutions in the form

$$\omega^2 - v^2q^2 = -\frac{i\omega}{2\tau} \pm \sqrt{-\left(\frac{\omega}{2\tau}\right)^2 + (v\delta c)^2q^4}. \tag{50}$$

This expression is equivalent to Eq. (48), and can be further analyzed by an appropriate approximation of ω in the right-hand side.

At $vq \gg \tau^{-1}$ one can approximate $\omega = vq$ in the right-hand side of Eq. (50) and obtain

$$\omega - vq = -\frac{i}{4\tau} \pm \frac{1}{2}\sqrt{-\frac{1}{(2\tau)^2} + (\delta c)^2q^2}. \tag{51}$$

In the limit $q \to \infty$ Eq. (51) recovers our earlier result (42). This asymptotic behavior requires $q \gg q^*$, where

$$q^* = \frac{1}{2\tau\delta c}. \tag{52}$$

Note that because of the smallness of the difference of velocities $\delta c \ll v$ in the spinless case, at $q \sim q^*$ we have $\omega\tau \gg 1$. The attenuation rate remains constant, $\operatorname{Im}\omega = (4\tau)^{-1}$, for all $q > q^*$[30], while $\operatorname{Re}\omega$ has a small nonlinear correction. At $q < q^*$ the real part $\operatorname{Re}\omega = vq$, whereas the attenuation of the two sound modes is different,

$$\omega - vq = \frac{i}{4\tau}\left[-1 \pm \sqrt{1 - (2\tau\delta cq)^2}\right]. \tag{53}$$

At $q \ll q^*$ the mode corresponding to the plus sign in Eq. (53) becomes weakly damped,

$$\omega - vq = -\frac{i}{2}\tau(\delta c q)^2. \tag{54}$$

For this mode $|\omega - vq| \ll |vq|$ for all q. Thus the approximation $\omega = vq$ leading to Eq. (51) remains applicable even at $vq \lesssim \tau^{-1}$. Indeed, since the speed of sound $c = v$ in the spinless quantum liquid, at $q \to 0$ Eq. (54) recovers the attenuation (33) of the ordinary sound. Thus this mode behaves similarly to the first sound mode as discussed in Sec. V C.

The sound mode corresponding to the minus sign in Eq. (53) behaves very differently. Its attenuation $\operatorname{Im}\omega \sim \tau^{-1}$ is small compared to vq only at $vq \gg \tau^{-1}$. Thus at $vq \lesssim \tau^{-1}$ Eqs. (51) and (53) are inapplicable. Instead, in this regime one can neglect the last term in Eq. (48), resulting in $\omega^2 - v^2q^2 + i\omega/\tau = 0$. The solution of this quadratic equation has the form

$$\omega = \pm\sqrt{(vq)^2 - \frac{1}{(2\tau)^2}} - \frac{i}{2\tau}, \tag{55}$$

analogous to that of Eq. (47) for the second sound in a quantum liquid of spin-$\frac{1}{2}$ fermions. Again, at $q \ll (v\tau)^{-1}$ we recover the general expression (35) for the heat diffusion mode to leading order at $T \to 0$. For q in the parametrically broad region $(v\tau)^{-1} \ll q \ll q^*$ or, equivalently,

$$\tau^{-1} \ll \omega \ll \tau^{-1}\frac{v}{\delta c}, \tag{56}$$

Eqs. (53) and (55) are both applicable, and for the right-moving sound mode we have $\omega = vq - i/2\tau$.

We now summarize our results for the crossover in a spinless quantum liquid from the regime of two hybrid sound modes at $\omega \gg \tau^{-1}$ to the regime of ordinary sound at $\omega \to 0$. The simple description of the two sound modes with speeds (27) applies not in the whole frequency range (1), but at

$$\tau^{-1}\frac{v}{\delta c} \ll \omega \ll \tau_{\rm ex}^{-1}. \tag{57}$$

At frequencies near $\tau^{-1}(v/\delta c)$ or $q \sim q^*$ the nature of the sound modes changes. By combining Eqs. (30a) and (51) we obtain the ratio of the variations of density and entropy

$$\frac{\delta\rho}{\delta\sigma} = i\sqrt{\frac{A_{12}}{A_{21}}}\,\frac{q^*}{q}\left[1 \pm \sqrt{1 - \left(\frac{q}{q^*}\right)^2}\,\right]. \tag{58}$$

In the frequency range (57) we recover the ratio $\delta\rho/\delta\sigma = \pm\sqrt{A_{12}/A_{21}}$ for the two hybrid sound modes that approaches a finite value at $T \to 0$. At q below q^* the oscillations of ρ and σ acquire a phase shift $\pi/2$, while the ratio $|\delta\rho/\delta\sigma|$ depends on q. Near the lower end of the range (56) we obtain $|\delta\rho/\delta\sigma| \sim \sqrt{A_{12}/A_{21}}\,(T/D)^{\mp 1}$, i.e., the two modes are similar to the first and second sound in the liquid of spin-$\frac{1}{2}$ fermions. Finally, at $\omega \ll \tau^{-1}$ the first sound mode becomes the ordinary sound, whereas the second one is replaced by heat diffusion.

VI. ATTENUATION OF SOUND DUE TO THE FAST SCATTERING PROCESSES

The two sound modes in one-dimensional quantum liquids exist only in the frequency range (1). The limitation on the high frequencies was imposed in Eq. (1) in order to make sure that the gas of excitations is always near equilibrium, which is achieved at the time scale $\tau_{\rm ex}$. Because the relaxation processes are not instantaneous, they lead to attenuation of sound. In the two-fluid hydrodynamic description of the system this effect is accounted for by the dissipative corrections to the currents j_s, j_p, and j_{u_0} in Eq. (18). The strength of the dissipative processes in the quantum liquid is characterized by the bulk viscosities ζ_i and the thermal conductivity of the gas of excitations $\kappa_{\rm ex}$, which are all proportional to $\tau_{\rm ex}$. We now study the effect of these processes on the attenuation of the sound modes by assuming that $\tau_{\rm ex}$ is finite, while $\tau \to \infty$.

We start by rederiving Eqs. (20) and (21), while accounting for the nonvanishing ζ_i and $\kappa_{\rm ex}$ in Eq. (18). A straightforward calculation yields

$$\partial_t^2\rho = \partial_x^2\Pi - \zeta_1\rho_0\partial_x^3(u_0 - u_{\rm ex}) - \zeta_2\partial_x^3 u_{\rm ex}, \tag{59a}$$

$$\partial_t^2\sigma = \frac{\rho_0}{\rho_{\rm ex}}\sigma^2\partial_x^2 T - \frac{\rho_0^2\sigma}{\rho\rho_{\rm ex}}(\zeta_1 - \rho\zeta_3)\partial_x^3(u_0 - u_{\rm ex}) - \frac{\rho_0\sigma}{\rho\rho_{\rm ex}}(\zeta_2 - \rho\zeta_4)\partial_x^3 u_{\rm ex} + \frac{\kappa_{\rm ex}}{\rho T}\partial_t\partial_x^2 T. \tag{59b}$$

Our goal is to account for the dissipation caused by nonvanishing viscosity and thermal conductivity in the first order in ζ_i and $\kappa_{\rm ex}$. Therefore we replace the derivatives of velocities u_0 and $u_{\rm ex}$ in Eqs. (59a) and (59b) by the expressions obtained from dissipationless hydrodynamic equations (19a) and (19b),

$$\partial_x u_{\rm ex} = -\frac{\partial_t\rho}{\rho} - \frac{\partial_t\sigma}{\sigma}, \quad \partial_x(u_0 - u_{\rm ex}) = \frac{\rho}{\rho_0}\frac{\partial_t\sigma}{\sigma}. \tag{60}$$

This results in a system of two equations for two variables: ρ and σ,

$$\partial_t^2\rho = \partial_x^2\Pi + \frac{\zeta_2}{\rho}\partial_t\partial_x^2\rho + \frac{\zeta_2 - \rho\zeta_1}{\sigma}\partial_t\partial_x^2\sigma, \tag{61a}$$

$$\partial_t^2\sigma = \frac{\rho_0}{\rho_{\rm ex}}\sigma^2\partial_x^2 T + \frac{\rho_0\sigma}{\rho^2\rho_{\rm ex}}(\zeta_2 - \rho\zeta_1)\partial_t\partial_x^2\rho + \frac{\rho_0}{\rho\rho_{\rm ex}}\,\widetilde{\zeta}\,\partial_t\partial_x^2\sigma + \frac{\kappa_{\rm ex}}{\rho T}\,\partial_t\partial_x^2 T. \tag{61b}$$

Here we have introduced

$$\widetilde{\zeta} = \zeta_2 - 2\rho\zeta_1 + \rho^2\zeta_3 \tag{62}$$

and applied the Onsager relation $\zeta_4 = \zeta_1$.

We now substitute into Eqs (61) small variations of ρ and σ in the form $\delta\rho\, e^{i(qx-\omega t)}$ and $\delta\sigma\, e^{i(qx-\omega t)}$ and obtain the following system of two linear equations

$$\left(\omega^2 - q^2\hat{A} + i\omega q^2\hat{\alpha}\right)\begin{pmatrix}\delta\rho\\ \delta\sigma\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}, \tag{63}$$

where $\hat{A}$ and $\hat{\alpha}$ are 2×2 matrices, with the matrix elements A_{ij} given by Eq. (23) and

$$\alpha_{11} = \frac{\zeta_2}{\rho}, \tag{64a}$$

$$\alpha_{12} = \frac{\zeta_2 - \rho\zeta_1}{\sigma}, \tag{64b}$$

$$\alpha_{21} = \frac{\rho_0\sigma}{\rho^2\rho_{\rm ex}}(\zeta_2 - \rho\zeta_1) + \frac{\kappa_{\rm ex}}{\rho T}\left(\frac{\partial T}{\partial \rho}\right)_\sigma, \tag{64c}$$

$$\alpha_{22} = \frac{\rho_0}{\rho\rho_{\rm ex}}\widetilde{\zeta} + \frac{\kappa_{\rm ex}}{\rho T}\left(\frac{\partial T}{\partial \sigma}\right)_\rho. \tag{64d}$$

The frequencies of the sound modes as functions of q are obtained by equating the determinant of the matrix in the left-hand side of Eq. (63) with zero.

In the leading order at $q \to 0$ the term proportional to $\hat{\alpha}$ in Eq. (63) can be neglected, and the solutions take the form $\omega^2 = c_\pm^2 q^2$, with the speeds given by Eq. (24). Accounting for $\hat{\alpha}$ in the first order, we obtain a small imaginary part of ω, which for the right-moving waves takes the form

$$\omega = c_\pm q - i\frac{q^2}{2}\gamma_\pm, \tag{65}$$

where

$$\gamma_\pm = \frac{\alpha_{11}+\alpha_{22}}{2} \pm \frac{(A_{11}-A_{22})(\alpha_{11}-\alpha_{22})}{2\sqrt{(A_{11}-A_{22})^2 + 4A_{12}A_{21}}} \pm \frac{A_{12}\alpha_{21} + A_{21}\alpha_{12}}{\sqrt{(A_{11}-A_{22})^2 + 4A_{12}A_{21}}}. \tag{66}$$

We now analyze the sound attenuation given by Eqs. (65) and (66) for one-dimensional quantum liquids at low temperatures.

In the case of the liquid of spin-$\frac{1}{2}$ fermions at $T \to 0$ the matrix elements A_{12} and A_{21} scale linearly with T, whereas the difference $A_{11} - A_{22}$ approaches a finite positive value. As a result, for the attenuation of the first sound we get

$$\begin{aligned}\gamma_+ &= \alpha_{11} + \frac{A_{12}A_{21}\alpha_{22}}{(A_{11}-A_{22})^2} + \frac{A_{12}\alpha_{21} + A_{21}\alpha_{12}}{A_{11}-A_{22}}.\\ &= \frac{\zeta_2}{\rho} + 2\frac{\zeta_2 - \rho\zeta_1}{\rho}\frac{\bar{v}^2\partial_\rho(\rho\widetilde{v})}{\widetilde{v}(v_\rho^2 - \bar{v}^2)} + \frac{\widetilde{\zeta}}{\rho}\frac{\bar{v}^4[\partial_\rho(\rho\widetilde{v})]^2}{\widetilde{v}^2(v_\rho^2-\bar{v}^2)^2}\\ &\quad + T\kappa_{\rm ex}\frac{v_\rho^2[\partial_\rho(\rho\widetilde{v})]^2}{\rho\widetilde{v}^2(v_\rho^2-\bar{v}^2)^2}.\end{aligned} \tag{67a}$$

Here we have evaluated the coefficients next to ζ_1, ζ_2, $\widetilde{\zeta}$, and $\kappa_{\rm ex}$ to leading order at $T \to 0$. Unlike the result (67a), the attenuation of the second sound is dominated by just one of the four matrix elements of $\hat{\alpha}$,

$$\gamma_- = \alpha_{22} = \widetilde{\zeta}\,\frac{3\hbar\widetilde{v}\bar{v}^2}{\pi T^2} + \kappa_{\rm ex}\frac{3\hbar\widetilde{v}}{\pi T}. \tag{67b}$$

Note, that the decay rate of the first sound is much smaller than that of the second sound, $\gamma_+/\gamma_- \sim (T/D)^2$. This is a result of the fact that the dissipation occurs in the gas of excitations, which is much more strongly disturbed by the second sound than the first one.

For a spinless quantum liquid we have $(A_{11}-A_{22})^2 \ll A_{12}A_{21}$. Then using the expressions (64) we conclude that the two hybrid modes decay at the same rate, such that in Eq. (65) $\gamma_\pm = \alpha_{22}/2$. This result takes the form

$$\gamma_\pm = \widetilde{\zeta}\,\frac{3\hbar v^3}{2\pi T^2} + \kappa_{\rm ex}\frac{3\hbar v}{2\pi T} \tag{68}$$

in terms of the parameters of the quantum liquid.

VII. SUMMARY AND DISCUSSION OF THE RESULTS

Relaxation of one-dimensional quantum liquids is characterized by two very different time scales τ and $\tau_{\rm ex}$. As a consequence, in a broad range of frequencies (1) the quantum liquid behaves as a superfluid and supports two sound modes. The properties of these two modes depend on the microscopic nature of the quantum liquid. In a liquid of one-dimensional spin-$\frac{1}{2}$ fermions, charge and spin excitations propagate at two different velocities v_ρ and v_σ. As a result the two sound modes also propagate at different velocities, which in the low-temperature limit are v_ρ and $\bar{v}$ given by Eq. (14). Their nature is essentially the same as that of the first and second sounds in superfluid ^{4}He, with the former being predominantly a wave of density and the latter—a wave of entropy. In a spinless one-dimensional quantum liquid all low-energy elementary excitations propagate at the same velocity v. In this case both sound modes propagate with speed v in the zero temperature limit, but at finite T the speeds split according to Eq. (27). These are hybrid modes, which are fundamentally different from the first and second sounds. They are combined oscillations of density and entropy, either in phase or with the phase shift π.

To study the propagation and attenuation of sound in one-dimensional quantum liquids we have adapted the two-fluid hydrodynamic theory developed for superfluid ^{4}He[2,3] to one dimension. In addition, we accounted for the processes of slow relaxation of the system to thermodynamic equilibrium, which are absent in true superfluids. The resulting theory describes the properties of the fluid at frequencies $\omega \ll \tau_{\rm ex}^{-1}$ as long as the deviations from equilibrium are small. It enabled us to study attenuation of sound in the single-fluid regime $\omega \ll \tau^{-1}$, two-fluid regime (1), and in the crossover region between them. The crossover is described in detail in Secs. V C and V D. We point out that in the case of spinless quantum liquid the crossover splits into two: one at $\omega \sim \tau^{-1}$ and the other at $\omega \sim \tau^{-1}v/\delta c$. The intermediate frequency regime (56) arises due to the presence of a large parameter $v/\delta c \sim D/T$; it occupies a small part of the exponentially broad region (1). In the range (56) the dissipation processes significantly affect the nature of the sound modes, transforming hybrid sounds into modes similar to the first and second sound.

We then studied the effect on sound attenuation of the fast relaxation processes occurring on the time scale $\tau_{\rm ex}$. The resulting sound attenuation is described by the last term in Eq. (65). In the frequency range (1) this contribution may compete with the attenuation caused by the slow relaxation processes. The total attenuation rate is obtained by adding $-i(q^2/2)\gamma_\pm$ to Eqs. (41) and (42), with $\gamma_\pm$ given by Eqs. (67) and (68), respectively. It is worth noting that the different physical nature of the sound waves for systems with and without spins results in qualitatively different sound attenuation. Because the latter is enabled by the gas of excitations, the first sound, being predominantly a density wave, is weakly damped compared with the second sound in a liquid of spin-$\frac{1}{2}$ fermions. On the other hand, the gas of excitations is affected equally by both hybrid modes in a spinless system, resulting in equal attenuation rates.

The phenomenological parameters ζ_1, ζ_2, ζ_3, and $\kappa_{\rm ex}$ are expected to be proportional to the fast relaxation time $\tau_{\rm ex}$. However, at this time a microscopic expression has been obtained[31,32] only for ζ_2, which coincides with the bulk viscosity in the single-fluid regime $\omega \to 0$. In a single-channel liquid of one-dimensional spinless fermions, the temperature dependence of the bulk viscosity is given by $\zeta_2 \propto T^4\tau_{\rm ex}$[31]. Assuming that all ζ_i have the same temperature dependence, we expect the viscous contributions to $\gamma_\pm$ given by the first term in Eq. (68) to scale as $T^2\tau_{\rm ex}$. Although no calculations of $\kappa_{\rm ex}$ are available at this time, we expect the contribution of the thermal conductivity of the gas of excitations to sound attenuation to be of the same order as that of viscous dissipation.

One-dimensional quantum liquids of interacting electrons can be studied experimentally in long quantum wire devices, such as that of Ref.[6]. Alternatively, atoms confined in elongated traps[8,9] can also form a one-dimensional quantum liquid. The two sound modes can be identified by observing propagation of small disturbances of particle density, similarly to the experiment[5] performed in elongated three-dimensional atomic traps. To achieve the quantum regime, the system must be cooled to temperatures below the bandwidth D, which is usually of the order of the Fermi energy E_F for systems of fermions. The observation of sound modes requires temperatures that are low enough for the exponentially small rate τ^{-1} to become much smaller than the power-law rate $\tau_{\rm ex}^{-1}$. One of the more favorable systems in this respect is that of weakly interacting spin-$\frac{1}{2}$ fermions, for which $\tau_{\rm ex}^{-1} \propto T^{15}$, as opposed to $\tau_{\rm ex}^{-1} \propto T^7$ in spinless systems[19,20]. The result of Ref.[15] applies in the temperature range $nU_0 \ll T \ll E_F$, in which the collective charge and spin excitations are not formed. (Here U_0 is the zero-momentum Fourier component of the interaction potential.) In this regime the Fermi velocity is the only speed of low-energy excitations in the system, despite the presence of spins. As a result, we expect this system to have properties similar to those of spinless quantum liquids, including two hybrid sound modes.

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[30] Strictly speaking the square-root singularity in Eq. (51) at $q = q^*$ is smeared slightly by $\delta q^* \sim (v\tau)^{-1}$. This smearing can be obtained by substituting $\omega = vq - i/4\tau$ instead of $\omega = vq$ into the right-hand side of Eq. (50).
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