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Bosonization of Fermi liquids in a weak magnetic field

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Novel controlled non-perturbative techniques are a must in the study of strongly correlated systems, especially near quantum criticality. One of these techniques, bosonization, has been extensively used to understand one-dimensional, as well as higher dimensional electronic systems at finite density. In this paper, we generalize the theory of two-dimensional bosonization of Fermi liquids, in the presence of a homogeneous weak magnetic field perpendicular to the plane. Here, we extend the formalism of bosonization to treat free spinless fermions at finite density in a uniform magnetic field. We show that particle-hole fluctuations of a Fermi surface satisfy a *covariant Schwinger algebra*, allowing to express a fermionic theory with forward scattering interactions as a quadratic bosonic theory representing the quantum fluctuations of the Fermi surface. By means of a coherent-state path integral formalism we compute the fermion propagator as well as particle-hole bosonic correlations functions. We analyze the presence of de Haas-van Alphen oscillations and show how the quantum oscillations of the orbital magnetization, the Lifshitz-Kosevich theory, are obtained by means of the bosonized theory. We also study the effects of forward scattering interactions. In particular, we obtain oscillatory corrections to the Landau zero sound collective mode.

I. INTRODUCTION

Consistent advances in our understanding of strongly correlated fermionic systems depend on the development of controlled non-perturbative computation techniques. Important questions, such as the fate of interacting systems of fermions at finite density in a quantum critical regime,^{1,2} are still waiting to be answered. This is particularly pressing given the “non-Fermi-liquid” behaviors seen in many systems, notably in high temperature superconductors in their “strange metal” regime, and many others systems, such as $\text{Sr}_3\text{Ru}_2\text{O}_7$, and heavy-fermion materials near quantum critical points.

At the theoretical level, bosonization has played an important role in developing accurate low-energy theories in one and quasi-dimensional systems.^{3–8} The applicability of its higher dimensional generalization of this approach to describe the quantum critical behaviour of fermionic systems at finite density (i.e. with a Fermi surface) is still under study. Recent significant progress has been made in the development of bosonization dualities for theories of relativistic fermions in 2+1 dimensions.^{9–11}

The first step in the development of higher dimensional bosonization for Fermi fluids was made by Luther in the eighties.¹² Intense activity in the 1990s in this area led to the description of the Fermi liquid fixed point.^{13–19} The essential idea is that most relevant fermionic properties at low energies can be described by the dynamics of particle-hole excitations near the Fermi surface. In this sense, the Fermi surface can be considered as a quantum mechanical extended object, a membrane in momentum space, with its own quantum dynamics. Renormalization group methods for fermionic systems at finite density²⁰ found that two-body forward scattering interactions, parametrized by Landau parameters, are marginally irrelevant at the Fermi liquid fixed point, con-

sistent with the description provided by the Landau theory of Fermi liquids.²¹

Higher dimensional bosonization turned out to be a powerful tool to study metallic systems away from the Fermi liquid regime. Different phase transitions, for instance driven by Pomeranchuk instabilities, were studied using this approach,²² in particular the isotropic-nematic quantum phase transition.²³ It was shown that the nematic quantum critical point and as the nematic phase itself are dominated by an overdamped low energy mode, a Goldstone mode with dynamical critical exponent $z = 3$.²⁴ In these regimes, although the correlators of the order parameter scale both in frequency and momentum (with a certain dynamic critical exponent z), the fermionic correlators scale only in their frequency dependence, i.e. exhibit a form of “local quantum criticality.”²⁵

A powerful experimental tool used to study the properties of Fermi systems, and particularly their Fermi surface, are quantum oscillation experiments, in particular the de Haas-van Alphen effect (dHvA). In this case, the system is placed in a strong enough magnetic field that overcomes temperature effects, however weak enough to be far away from the Landau level quantization regime. In these conditions, thermodynamic observables such as the magnetization and susceptibility, and the heat capacity, exhibit oscillations as the magnetic field is varied. The Lifshitz-Kosevich (LK) theory,^{26,27} a semiclassical theory of free fermions in a magnetic field, relates the oscillations with geometric properties of the Fermi surface. This procedure has been used for a long time to study normal metals which are well described by the Landau theory of the Fermi liquid.

However, more recently, quantum oscillations have been observed in strongly correlated materials. The Lifshitz-Kosevich theory is nowadays widely used to determine Fermi surface properties in strongly corre-

lated systems,^{28–34} even near a quantum critical point³⁵. However, the validity of the LK theory is questionable in such extreme regimes where Fermi liquid theory is known to fail. Thus, understanding quantum oscillations in a strong coupling regime and, particularly, near a quantum critical point is an important and open problem.^{36–38} Generally, a magnetic field opens a gap in the fermionic spectrum and is a relevant perturbation that moves the system away from the critical point. How the system scales when the magnetic field approach zero at a Pomeranchuk instability³⁹ is a key question to understand the more general problem of quantum oscillations.

The main purpose of this paper is to extend the bosonization procedure in the presence of a weak homogeneous magnetic field. Here we will focus on the case of two-dimensional systems. The extension to higher dimensions is straightforward. In this paper, we develop a bosonization approach to a two dimensional spinless fermionic system, in the presence of a magnetic field B perpendicular to the plane. Previous attempts to bosonize a two-dimensional Fermi gas in a weak magnetic field were done in the Landau level basis⁴⁰, *i. e.*, by first projecting the fermionic fields into Landau levels. This technique was successfully applied to solve some problems in which static localized interactions produce transitions between, almost non-interacting, Landau levels.⁴¹ Here, we follow a different approach, keeping the formalism as close as possible to the conventional bosonization of Fermi liquids. With this tool, we expect to be able to compute correlations in the dHvA regime in strongly coupled fermionic systems.

As usual in bosonization, we consider a high density regime in which the Fermi surface can be taken to be locally flat. Assuming that the low energy physics is driven by small fluctuations of the shape of the Fermi surface,¹³ we project the Hamiltonian into a narrow shell of states, divided into patches, around the Fermi surface. The patches are small enough to allow the linearization of the dispersion relation, but big enough to contain a large number of Landau levels.

In this regime, it is possible to define local Fermi surface fluctuations in terms of densities particle-hole excitations and show that the corresponding operators satisfy a *covariant Schwinger algebra*. This algebra is at the root of the bosonization approach, see Eq. (3.15) below. Within this scheme, the fermion operator is defined as an exponential of a coherent superposition of bosonic fields, and show that this construction is consistent with the basic symmetries of the low energy theory. Importantly, the fermion operator in each patch depends on a superposition of bosonic excitations all around the Fermi surface.

We also present an extension of the coherent-state path integral formalism¹³ to describe a system in the presence of a magnetic field, suitable for the computation of correlation functions. The semiclassical approximation of this theory leads to an equation of motion that is a particular case of the Landau-Silin²¹ equation in the Fermi liquid

theory. Using these methods we computed the bosonic correlation functions necessary to characterize the linear response of the system. This theory allowed us to study the emergence of the dHvA oscillations from a bosonized approach. We also discuss the effects of forward scattering interactions within this formalism. In particular we compute the collective mode spectrum, finding damped oscillatory corrections to Landau zero sound. This is a manifestation of quantum oscillations in a Fermi liquid non-equilibrium property.

The paper is organized as follows: in Section II we compute an asymptotic form of the two-dimensional propagator of a Fermi gas in the regime of weak magnetic fields. In §III we present the main results of the bosonization approach. Then, we study symmetries of the low energy Hamiltonian in Section IV. In Section V we construct the fermionic operators and compute the fermion propagator by using the bosonized theory. In §VI we present a path integral representation of Fermi surface fluctuations and compute bosonic correlation functions. In Section VII we show how the dHvA oscillations can be computed from the bosonized action. In §VIII we discuss forward scattering interactions, showing an explicit calculation of collective modes and their dependence on the magnetic field. Finally we discuss our results in Section IX. Details of the calculations are presented in several appendices.

II. FERMION GAS IN A WEAK HOMOGENEOUS MAGNETIC FIELD

In this section we present a convenient representation of the asymptotic fermion propagator of a Fermi gas at high density in a weak magnetic field. The Fermi gas in a magnetic field is a very well known system. The effect of the magnetic field is, essentially, to discretize the fermionic spectrum into highly degenerated equally spaced Landau Levels. The gaps and the degeneracy grow with the strength of the magnetic field. Then, for sufficiently high magnetic field, a finite density of electrons will occupy just the first Landau level. In the presence of interactions, this regime leads to the integer and fractional quantum Hall fluids. In this work, we are interested, instead, in the regime of low magnetic fields, where there is a huge number of filled Landau levels in a small energy interval around the Fermi energy. The motivation is two-fold. We will first use conventional methods to derive an explicit expression for the fermion propagator computed directly, in order to compare it with the results obtained by multidimensional bosonization. On the other hand, the expression for the occupation number, obtained from the propagator, will be later used to derive the bosonic algebra in the bosonization procedure. The exact propagator of a Fermi gas is well known and was obtained by several methods, see *e.g.* Refs. [42–44]. In this paper we present an asymptotic form of the fermion propagator, especially useful in the weak magnetic field regime we will be interested in.

We begin by considering a two-dimensional degenerated gas of spinless fermions in a perpendicular homogeneous magnetic field. The Hamiltonian is,

$$H = \int d^2x \frac{1}{2m} |(\nabla + ie\mathbf{A}(\mathbf{x}))\psi(\mathbf{x})|^2, \quad (2.1)$$

where $\psi^\dagger(\mathbf{x})$ and $\psi(\mathbf{x})$ are fermionic creation and annihilation operators at position \mathbf{x} , satisfying,

$$\{\psi^\dagger(\mathbf{x}), \psi(\mathbf{x}')\} = i\delta^2(\mathbf{x} - \mathbf{x}'). \quad (2.2)$$

m is the effective electron mass, e is the electron electric charge and the homogeneous magnetic field is given by $\nabla \times \mathbf{A} = B\hat{z}$. In two dimensions, the magnetic field B is a pseudo-scalar.

The fermion propagator is defined as

$$iG_F(\mathbf{x}, \mathbf{y}, t, t') = \langle FS | \mathcal{T} \psi(\mathbf{x}, t) \psi^\dagger(\mathbf{y}, t') | FS \rangle, \quad (2.3)$$

where \mathcal{T} means time ordered product, and the filled Fermi sea ground state $|FS\rangle$ is built by the applications of the fermion creation operators $c_{n,k}^\dagger$ on the vacuum state, filling up the single particle states up to the Fermi energy ϵ_F :

$$|FS\rangle = \prod_{n,k} c_{n,k}^\dagger |0\rangle.$$

Here, $\{n, k\}$ are the quantum numbers of the filled Landau levels. The energy eigenvalues are $E_{n,k} = \hbar\omega_c(n + 1/2)$ with the cyclotron frequency $\omega_c = eB/m$, in units in which $\hbar = 1$. At zero magnetic field, the ground state $|FS\rangle$ is a completely filled circular Fermi surface of radius k_F , and chemical potential $\mu = \epsilon_F = k_F^2/2m$. At zero or very low temperatures, the dynamics is governed by the states laying in the small momentum shell $k_F - \lambda/2 < |\mathbf{k}| < k_F + \lambda/2$. Equivalently, the energy shell can be written as $\epsilon_F - \Delta\epsilon < E < \epsilon_F + \Delta\epsilon$. Since $\lambda \ll k_F$, the single-particle energy dispersion can be linearized around k_F , and $\Delta\epsilon = v_F\lambda/2$, where v_F is the Fermi velocity. From now on, we use units in which $c = 1$, $e = 1$, $\hbar = 1$.

The number of filled Landau levels contained in the momentum shell below the Fermi surface is $N_L = \Delta\epsilon/\omega_c = \lambda\ell_B^2 k_F/2$, where we have introduced the magnetic length $\ell_B^2 = 1/B = v_F/(k_F\omega_c)$. Even in the limit $\lambda \ll k_F$, we choose the magnetic field in such a way that there is still a macroscopic number of filled Landau levels, $N_L \gg 1$, in the energy shell. This condition can be stated as $\omega_c \ll \lambda v_F$, or equivalently, $\ell_B^2 \gg 1/(\lambda k_F)$.

In Appendix A we compute the fermion propagator in this regime, and obtain the explicit expression

$$iG_F(\mathbf{x}, \mathbf{y}, r_0) = \frac{ik_F}{2\pi} e^{-i\epsilon_F r_0} e^{i\theta_B} \int_0^{2\pi} d\varphi \sum_{\ell=-\infty}^{+\infty} G_{\ell,\varphi}(r, r_0), \quad (2.4)$$

where $r = |\mathbf{x} - \mathbf{y}|$ is the spacial separation, and $r_0 = t_x - t_y$ is the time difference. This propagator depends

on the choice of gauge. This dependence is contained in the phase θ_B . For instance, in the Landau gauge, $A_1 = -Bx_2$, $A_2 = 0$, where we have chosen a cartesian coordinates system $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$, the phase θ_B is

$$\theta_B(\mathbf{x}, \mathbf{y}) = \frac{1}{2\ell_B^2} (x_1 - y_1)(x_2 + y_2). \quad (2.5)$$

Instead, in the symmetric gauge $A_1 = -Bx_2/2$, $A_2 = Bx_1/2$, the phase is

$$\theta_B(\mathbf{x}, \mathbf{y}) = \frac{1}{2\ell_B^2} (x_1 y_2 - y_1 x_2). \quad (2.6)$$

The rest of the expression for the propagator is gauge-invariant.

In Eq. (2.4) we used the following notation

$$G_{\ell,\varphi}(r, r_0) = \frac{e^{ik_F r \cos \varphi}}{r \cos \varphi - v_F r_0 + 2\pi\ell\ell_B^2 k_F + i\alpha \text{sgn}(r_0)}. \quad (2.7)$$

where φ denotes the direction normal to the Fermi surface, i.e. $\cos \varphi = \mathbf{r} \cdot \mathbf{k}_F / r k_F$, and ℓ is an integer. Eqs. (2.4) and (2.7) are the main results of this section.

In order to check the $B \rightarrow 0$ limit, it is convenient to sum over ℓ in order to have a closed form for the propagator. We obtain,

$$iG_F(\mathbf{x}, \mathbf{y}, r_0) = \frac{ik_F}{2\pi} e^{-i\epsilon_F r_0} e^{i\theta_B} \int d\varphi e^{ik_F r \cos \varphi} \times \frac{1}{2\ell_B^2 k_F} \cot \left(\frac{1}{2\ell_B^2 k_F} (r \cos \varphi - v_F r_0 + i\alpha \text{sgn}(r_0)) \right). \quad (2.8)$$

Expanding the cotangent in powers of $1/\ell_B k_F$ we obtain,

$$iG_F(\mathbf{x}, \mathbf{y}, r_0) = \frac{ik_F}{2\pi} e^{-i\epsilon_F r_0} e^{i\theta_B} \times \int d\varphi e^{ik_F r \cos \varphi} \left\{ \frac{1}{r \cos \varphi - v_F r_0 + i\alpha \text{sgn}(r_0)} - \frac{1}{12\ell_B^4 k_F^2} (r \cos \varphi - v_F r_0) \right\}, \quad (2.9)$$

where the first term is the asymptotic Fermi liquid propagator, first computed by Luther,¹² and the second term is the leading order correction in the small magnetic field expansion.

Returning to Eq.(2.7), and after Fourier transforming its expression, we find the asymptotic propagator in momentum and frequency space,

$$\tilde{G}_\varphi(\mathbf{q}, \omega) = \frac{1}{\omega - \mathbf{v}_F \cdot \mathbf{q} + i v_F \lambda \text{sgn}(\omega)} \times \sum_{\ell=0}^{\infty} e^{-2\pi\ell N_L} \cos \left(2\pi\ell \frac{\mathbf{v}_F \cdot \mathbf{q}}{\omega_c} \right). \quad (2.10)$$

Thus, the propagator oscillates as a function of the energy of the particle-hole pair $\mathbf{v}_F \cdot \mathbf{q}$ with period ω_c/ℓ , damped by the number of filled Landau level N_L . In the limit of $\omega_c \rightarrow 0$, $N_L \rightarrow \infty$ and only $\ell = 0$ contributes, recovering the well known Fermi gas result.

A. Occupation number

We will now discuss the structure of the occupation number to check the consistency of the asymptotic propagator given by Eq. (2.4). It will be needed below to develop the bosonization technique. The occupation number, $n(\mathbf{k})$, is given by the equal-time expression

$$n(\mathbf{k}) = \int d^2x \, iG_F(\mathbf{x}, 0, 0) e^{i\mathbf{k} \cdot \mathbf{x}}. \quad (2.11)$$

Replacing Eq. (2.4) into Eq. (2.11), we write

$$n(\mathbf{k}) = \int_0^{2\pi} \frac{d\varphi}{2\pi} n_\varphi(\mathbf{k}), \quad (2.12)$$

with

$$n_\varphi(\mathbf{k}) = ik_F \sum_{\ell=-\infty}^{+\infty} \int d^2x \, e^{i\theta_B(\mathbf{x})} G_{\ell,\varphi}(\mathbf{x}, 0) e^{-i\mathbf{k} \cdot \mathbf{x}}. \quad (2.13)$$

To compute the integrals we define a local coordinate system for fixed direction φ . Since $\mathbf{k}_F = k_F(\cos \varphi, \sin \varphi)$, we define a local frame of two orthonormal vectors $\hat{\mathbf{n}}_\varphi$ and $\hat{\mathbf{t}}_\varphi$, respectively defining the local normal and tangential directions to the Fermi surface. In this basis, position and momentum are decomposed as $\mathbf{x} = x_n \hat{\mathbf{n}}_\varphi + x_t \hat{\mathbf{t}}_\varphi$, and $\mathbf{k} = k_n \hat{\mathbf{n}}_\varphi + k_t \hat{\mathbf{t}}_\varphi$. Thus, in the local frame, Eq. (2.13) reads,

$$n_\varphi(\mathbf{k}) = ik_F \exp\left(-\frac{i}{2\ell_B^2} \frac{\partial^2}{\partial k_t \partial k_n}\right) S_\varphi(k_n, k_t) \quad (2.14)$$

where we defined

$$S_\varphi(k_n, k_t) = \sum_{\ell=-\infty}^{+\infty} \int_{-2/\Lambda}^{2/\Lambda} dx_n \int_{-2/\Lambda}^{2/\Lambda} dx_t \, G_{\ell,\varphi}(x_n, 0) e^{-ik_n x_n - ik_t x_t} \quad (2.15)$$

with $\Lambda = 2\pi k_F/N$ is a momentum cut-off in the tangential direction to the Fermi surface, with $N \gg 1$ a large integer. Here, we have chosen the phase θ_B in the Landau gauge, Eq. (2.5).

These integrals can be easily done in the limit $\lambda \rightarrow 0$ and $\Lambda \rightarrow 0$. However, this limit should be taken carefully. Remembering that the number of filled Landau levels is $N_L = \lambda \ell_B^2 k_F / 2$, we need to take $\lambda \rightarrow 0$ and $\ell_B \rightarrow \infty$ keeping $N_L \gg 1$. Only in this way we recover the Fermi liquid behaviour as $B \rightarrow 0$. The result is,

$$n_\varphi(k_n, k_t) = k_F \delta(k_t) \Theta(k_F - k_n) \sum_{n=-\infty}^{+\infty} \delta_{n, \ell_B^2 k_F} (k_n - k_F) - ik_F \sin\left\{\frac{1}{\ell_B^2} \frac{\partial^2}{\partial k_t \partial k_n}\right\} \delta(k_t) \Theta(k_F - k_n). \quad (2.16)$$

This equation will be important when computing the algebra of Fermi surface fluctuations in the bosonization approach. Firstly, we observe that $n_\varphi(k_n, k_t)$ is given by very localized distributions. This is a consequence of the thermodynamic limit. Any technical difficulty that could eventually emerge for this reason, can be easily overcome by smoothing the distributions, keeping the cut-offs λ and Λ small, but finite. An important observation is that n_φ is gauge dependent. In particular, the second term in Eq. (2.16) comes from the phase θ_B in the Landau gauge. Then, it should give no contribution in the computation of any observable.

By replacing Eq. (2.16) into Eq. (2.12) we can compute the full occupation number of the fermions. As stated, the second line of Eq. (2.16) gives no contribution upon integration over φ . When written in terms of the angle φ , the first line reads, $n_\varphi(\mathbf{k}, \varphi) = k_F \delta(k \sin \varphi) \Theta(k_F - k \cos \varphi) \sum_n \delta_{n, \ell_B^2 k_F} (k \cos \varphi - k_F)$. Upon integration over φ we find,

$$n(\mathbf{k}) = \Theta(-q) \sum_{n=-\infty}^{+\infty} \delta_{n, \ell_B^2 k_F} + O\left(\frac{q}{k_F}\right), \quad (2.17)$$

where $q = k - k_F$ with $k = |\mathbf{k}|$. The Kronecker deltas simply indicate the location of the Landau levels,

$$q = -\frac{n}{\ell_B^2 k_F} = -\frac{n}{2} \left(\frac{\omega_c}{\epsilon_F}\right) k_F, \quad (2.18)$$

which are equally separated in intervals $\Delta q = 1/\ell_B^2 k_F$. The sum over n is actually limited by the cut-off λ to the range $-N_L \leq n \leq N_L$. The Heaviside function just locates the position of the Fermi surface. We depict this function in Fig. (1).

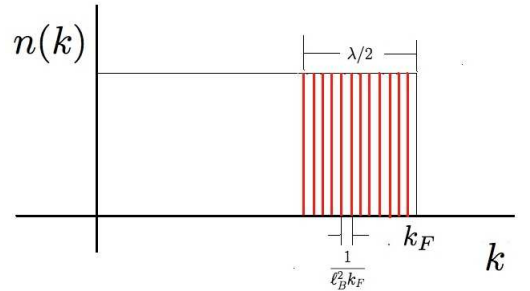


FIG. 1: Schematic representation of the fermionic occupation number in a weak magnetic field. The level spacing is $1/\ell_B^2 k_F$, while the number of filled levels near the Fermi surface is $N_L = \lambda \ell_B^2 k_F / 2$, where λ is a momentum cut-off. In the limit of $\ell_B \rightarrow \infty$ keeping λ fixed, the set of Landau level is dense and we have the usual Fermi gas expression $n(k) = \Theta(k_F - k)$.

In the limit $\ell_B \rightarrow \infty$, the filled Landau levels, for $q < \lambda/2$, diverges and become a dense set of points, such that

the following limit holds

$$\lim_{\ell_B \rightarrow \infty} \sum_{n=-\infty}^{+\infty} \delta_{n, \ell_B^2 k_F (k_F - k)} = \lim_{\ell_B \rightarrow \infty} \sum_{\ell=-\infty}^{+\infty} e^{i2\pi \ell \ell_B^2 k_F q} = 1, \quad (2.19)$$

where the last limit is taken in the sense of a distribution, *i. e.*, for huge values of ℓ_B , terms with $\ell \neq 0$ are strongly oscillating, giving no contribution when applied to smooth test functions. In this regime, the only relevant term is $\ell = 0$, obtaining the well known occupation number of a Fermi gas at zero temperature, $n(\mathbf{k}) = \Theta(k_F - k)$.

An additional consistency check of the asymptotic propagator deduced in this section is the computation of the density of states

$$N(\omega) = -\text{sgn}(\omega - \epsilon_F) \frac{1}{\pi} \text{Im} G_F(\mathbf{x}, \mathbf{x}, \omega). \quad (2.20)$$

Using Eq. (2.4), we obtain (see Appendix B),

$$N(\omega - \epsilon_F) = N(0) \sum_n \delta_{n, (\omega - \epsilon_F)/\omega_c}, \quad (2.21)$$

where $N(0) = k_F/v_F$ is the density of states at the Fermi surface of a two-dimensional Fermi gas.

III. BOSONIZATION

To proceed with bosonization, we follow closely the approach outlined in Refs. [13–15, and 22]. The main idea is to project the fermion operator into a restricted Hilbert space, built by restricting the momentum space to a small shell around a Fermi surface. We expect that this restriction correctly capture the low energy and long distance physics at high density, even in the presence of a small magnetic field, provided there is a very large number of Landau levels inside the shell.

For simplicity, we illustrate the procedure by considering a circular Fermi surface with Fermi momentum $|\mathbf{k}_F| = k_F$, which we coarse-grain by introducing N patches, each of width Λ and height λ , as shown in Fig. 2. The precise shape of each patch is not important. At the end of the calculations, the limit $\Lambda \rightarrow 0$, $N \rightarrow \infty$ should be taken, with the constraint that, in the continuum limit, they add up to the size of the Fermi surface, *i.e.* $N\Lambda = 2\pi k_F$. Each patch is labeled by an integer $S = 1, \dots, N$ or, equivalently, by an angle $\varphi_S = S\Lambda/k_F$, indicating the angular position of the patch on the Fermi surface with respect to some arbitrary chosen axes (see Fig. 2). Quite generally, we have that any set of functions f_S , label by N integer numbers, $S = 1, \dots, N$, are equivalent, in the continuum limit, to periodic functions $f(\varphi_S + 2\pi) = f(\varphi_S)$, in such a way that

$$\begin{aligned} \frac{\Lambda}{k_F} \sum_S f_S &\rightarrow \int_0^{2\pi} d\varphi_S f(\varphi_S) \\ \frac{k_F}{\Lambda} (f_{S+1} - f_S) &\rightarrow \frac{df(\varphi_S)}{d\varphi_S}. \end{aligned} \quad (3.1)$$

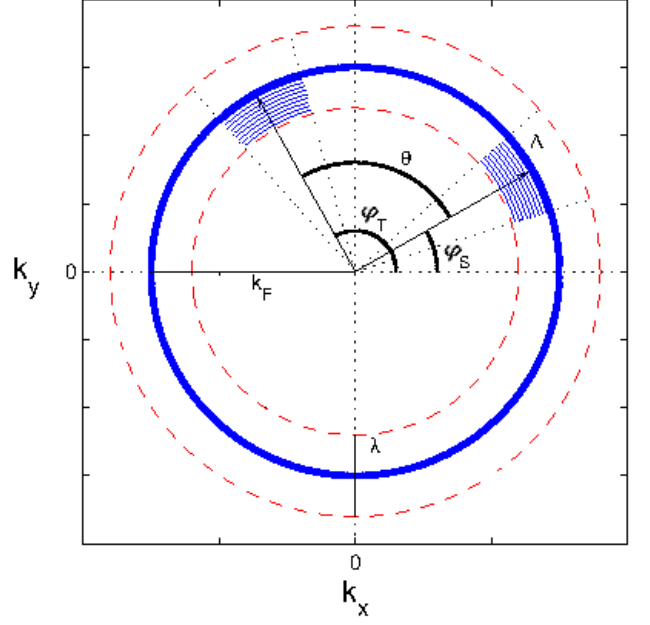


FIG. 2: Segmentation of a circular Fermi surface of radius k_F in patches of width Λ and height λ . Two patches label by the angles φ_S and φ_T , separated by an angle θ , are displayed. The shaded area represents a set of filled Landau levels inside the patch.

We introduce fermionic field operators $\psi_S(\mathbf{x}), \psi_S^\dagger(\mathbf{x})$ on each patch, S , defined as,

$$\psi_S(\mathbf{x}) = \frac{1}{L} e^{-i\mathbf{k}_S \cdot \mathbf{x}} \sum_{\mathbf{k}} \Omega_S(\mathbf{k}) c_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} \quad (3.2)$$

$$\psi_S^\dagger(\mathbf{x}) = \frac{1}{L} e^{i\mathbf{k}_S \cdot \mathbf{x}} \sum_{\mathbf{k}} \Omega_S(\mathbf{k}) c_{\mathbf{k}}^\dagger e^{-i\mathbf{k} \cdot \mathbf{x}}. \quad (3.3)$$

Here, the operators $c_{\mathbf{k}}^\dagger$ ($c_{\mathbf{k}}$) create (destroy) a fermion with definite momentum \mathbf{k} (for the moment, we are ignoring the spin degree of freedom). Here, \mathbf{k}_S is the Fermi momentum \mathbf{k}_F , at the center of the patch S , and $\Omega_S(\mathbf{k})$ is a compact support distribution that takes the values: one, for \mathbf{k} inside the patch S , and zero otherwise. We work in a large but finite volume, of linear size L , and area L^2 , in such a way that \mathbf{k} form a dense set of discrete values. Eventually, we may take the thermodynamic limit $L \rightarrow \infty$, for which $\sum_{\mathbf{k}} \rightarrow L^2 \int d^2k$. We are using the normalization factors in the Fourier transforms, Eqs. (3.2) and (3.3), in such a way to have the following bare field scaling dimensions: $[c_{\mathbf{k}}] = [c_{\mathbf{k}}^\dagger] = 1$ and $[\psi_S(\mathbf{x})] = [\psi_S^\dagger(\mathbf{x})] = L^{-1}$.

The usual fermionic anticommutation relations

$$\{c_{\mathbf{k}}^\dagger, c_{\mathbf{k}'}\} = \delta_{\mathbf{k}, \mathbf{k}'} \quad (3.4)$$

imply, for the coarse-grained fermions,

$$\{\psi_S^\dagger(\mathbf{x}), \psi_T(\mathbf{x}')\} = \delta_{S,T} \delta^2(\mathbf{x} - \mathbf{x}'). \quad (3.5)$$

The complete Fermi field can be reconstructed by summing over the entire Fermi surface,

$$\psi(\mathbf{x}) = \sum_{S=1}^N e^{-i\mathbf{k}_S \cdot \mathbf{x}} \psi_S(\mathbf{x}). \quad (3.6)$$

The patch fermions $\psi_S(\mathbf{x})$ belong to a mixed coordinate-momentum representation, where \mathbf{x} is a configurational space variable and S , or equivalently, φ_S is a momentum variable. Eq. (3.6), is the generalization of the well known one dimensional expression, $\psi(x) = e^{ik_F x} \psi_R(x) + e^{-ik_F x} \psi_L(x)$, that relates the microscopic field operator to the slowly varying right and left moving fields. Clearly, $\psi_S(x)$ generalizes the concept of left, ψ_L , and right, ψ_R , moving fermions. To be consistent with quantum mechanics, i.e. with the Heisenberg uncertainty principle, the fields $\psi_S(\mathbf{x})$ are slowly varying functions on a length scale $\sim k_F^{-1}$ (the mean distance between fermions in real space).

The low-energy effective Hamiltonian can be obtained by projecting the free fermion Hamiltonian of Eq. (2.1) to the restricted Hilbert space spanned by states in the small shell around the Fermi surface. We find,

$$H_F = i v_F \sum_S \int d^2 x \psi_S^\dagger(\mathbf{x}) (\hat{\mathbf{n}}_S \cdot \mathbf{D}) \psi_S(\mathbf{x}), \quad (3.7)$$

where v_F is the Fermi velocity, $\hat{\mathbf{n}}_S$ is a constant unit vector normal to the Fermi surface, pointing outward in the patch S , and we have represented the covariant derivative acting on the projected space spanned by $\psi_S(\mathbf{x}) \equiv \psi(\mathbf{k}_S, \mathbf{x})$, as

$$D_i = \nabla_i + i k_{S,i} - \frac{1}{\ell_B^2} \epsilon_{ij} \frac{\partial}{\partial k_{S,j}}. \quad (3.8)$$

Here, the covariant derivative has two parts. The first term acts on the slowly varying position variable \mathbf{x} , while the second part is in the momentum representation and acts on the Fermi surface. We will show that, since both parts commute with each other, this scale separation has important consequences on the symmetries of the system.

Eq. (3.8) was obtained by fixing the symmetric gauge $A_i = \frac{B}{2} \epsilon_{ij} x_j$. The term proportional to $i\mathbf{k}_S$ contributes to the energy of the ground state ($\epsilon_F = v_F k_F$) and can be ignored, since it cancels out upon normal ordering the Hamiltonian with respect to the filled Fermi sea. We decided to keep it in the definition of the covariant derivative, since it is important to study symmetries of the low energy Hamiltonian, as we will see in the next section.

To obtain the effective low energy Hamiltonian of Eq. (3.7), we linearized the dispersion relation in each patch, considering that the momentum excitations have $|\mathbf{k} - \mathbf{k}_F| < \lambda < \Lambda \ll k_F$. We have also ignored terms of the order $1/(\ell_B k_F)^2 \ll 1$. In other words, we have neglected sub-leading terms in $\omega_c/v_F k_F \ll 1$, meaning that the energy difference between Landau levels is much smaller than the Fermi energy. The Hamiltonian H_F is the generalization of the usual low-energy Hamiltonian

considered in bosonization of Fermi liquids.^{13–17,19,22} In the presence of a weak magnetic field the chiral fermionic components are coupled with the derivative term proportional to $1/(\ell_B^2 k_F)$.

Particle-hole excitations are created by acting with the operator $\hat{n}_{\mathbf{k}}(\mathbf{q}) = c_{\mathbf{k}+\mathbf{q}/2}^\dagger c_{\mathbf{k}-\mathbf{q}/2}$ on the reference state $|FS\rangle$. Deformations of the Fermi surface can be parametrized by a normal ordered particle-hole operator, smeared at each patch. We define

$$\delta n_S(\mathbf{q}) = \sum_{\mathbf{k}} \Omega_S(\mathbf{k} - \mathbf{q}/2) \Omega_S(\mathbf{k} + \mathbf{q}/2) \times \left\{ c_{\mathbf{k}+\mathbf{q}/2}^\dagger c_{\mathbf{k}-\mathbf{q}/2} - \langle FS | c_{\mathbf{k}+\mathbf{q}/2}^\dagger c_{\mathbf{k}-\mathbf{q}/2} | FS \rangle \right\}. \quad (3.9)$$

As before, $\Omega_S(\mathbf{k})$ is a distribution with support inside the patch S . By definition, in the ground state, with an undistorted Fermi surface, $\langle FS | \delta n_S | FS \rangle = 0$, since the operator is normal ordered with respect to this state.

The dynamics of the Fermi surface is governed by the Heisenberg equation

$$\frac{\partial \delta n_S(\mathbf{x})}{\partial t} = i [H_F, \delta n_S(\mathbf{x})]. \quad (3.10)$$

Using the fermionic anticommutator algebra, Eq. (3.4), we find that this equation of motion has the form of a collisionless Boltzmann type equation

$$\frac{\partial \delta n_S(\mathbf{x})}{\partial t} + \mathbf{v}_S \cdot \nabla \delta n_S(\mathbf{x}) - \omega_c \frac{\partial \delta n_S(\mathbf{x})}{\partial \varphi_S} = 0. \quad (3.11)$$

While the second term rules the excitations normal to the Fermi surface, the last term, coming from the Lorentz force, induces an evolution of the patch variable around the Fermi surface. This equation is a particular case of the Landau-Silin²¹ equation obtained by means of a phenomenological treatment of a Fermi liquid in an electromagnetic field. A similar quantum Boltzmann equation was deduced previously in the context of the quantum Hall effect at filling factor $\nu \sim 1/2$, by applying non-equilibrium Green function techniques to a fermion model coupled to a Chern-Simons field.⁴⁵ More recently, a related magneto transport equation with similar structure was deduced in a context of disorder strange metals.⁴⁶

The question is whether we can get this equation of motion from a pure Bosonic approach. To answer it, we firstly compute the commutation relations of $\delta n_S(\mathbf{x})$. Using Eq. (3.9) and (3.4) we find,

$$[\delta n_S(\mathbf{q}), \delta n_T(-\mathbf{q}')] = \sum_{\mathbf{k}} \Omega_S(\mathbf{k} - \mathbf{q}/2) \Omega_T(\mathbf{k} + \mathbf{q}'/2) (n_{\varphi_S}(\mathbf{q}) - n_{\varphi_T}(-\mathbf{q}')) \quad (3.12)$$

where $n_{\varphi_S}(\mathbf{q})$ and $n_{\varphi_T}(-\mathbf{q})$ are the occupation numbers at the patches S and T respectively, computed in the previous section (Eq. (2.16)). Expanding these quantities for small values of \mathbf{q} , we find,

$$[\delta n_S(\mathbf{q}), \delta n_T(-\mathbf{q}')] = \left(\frac{L^2 \Lambda}{2\pi} \right) \tilde{D}_S(\mathbf{q}) \delta_{S,T} \delta_{\mathbf{q},\mathbf{q}'} \quad (3.13)$$

where $\tilde{D}_S(\mathbf{q})$ is given by

$$\tilde{D}_S(\mathbf{q}) = \hat{\mathbf{n}}_S \cdot \mathbf{q} - \frac{i}{\ell_B^2 k_F} \frac{\partial}{\partial \varphi_S} \quad (3.14)$$

Fourier transforming Eq. (3.13), we have the commutation relation in real space,

$$[\delta n_S(\mathbf{x}), \delta n_T(\mathbf{y})] = - \left(\frac{\Lambda}{2\pi} \right) i D_S [\delta^2(\mathbf{x} - \mathbf{y}) \delta_{S,T}] , \quad (3.15)$$

where

$$D_S = \hat{\mathbf{n}}_S \cdot \nabla + \frac{1}{\ell_B^2 k_F} \frac{\partial}{\partial \varphi_S} . \quad (3.16)$$

This equation is the key of the bosonization procedure, and, for this reason, it is important to comment on its meaning and its range of applicability.

In Eq.(3.16) D_S is essentially the normal component of the covariant derivative at each patch S . At zero magnetic field, the expression $D_S = \hat{\mathbf{n}}_S \cdot \nabla$ reduces to the usual Schwinger term analogous to the one found in one-dimensional bosonization (where $S = R, L$) and in bosonization of Fermi liquids.^{13,16,18} The effect of the magnetic field is to produce a new term tangent to the Fermi surface. In this way, in the presence of a magnetic field, the commutation relations are no longer diagonal in the patch basis, and hence mix states on different patches of the Fermi surface. The algebra of Eq. (3.15) can be regarded as a *covariant Schwinger algebra*. Essentially the same commutation relations have been recently proposed, computed by means of semiclassical arguments, in the (quite different) context of fractional Hall states.⁴⁷

As usual in the treatment of multidimensional bosonization, in order to close the algebra we have discarded terms proportional to the ratio $\lambda/\Lambda \ll 1$. Then, the limit of a large number of patches N should be taken satisfying $\lambda < \Lambda \ll k_F$. While this is well established in the bosonization of a Fermi liquid, special care should be taken in the presence of a magnetic field. Indeed, we need to guarantee that, even in the limit of very small λ , there should be a huge number of Landau levels $N_L = \lambda \ell_B^2 k_F / 2 \gg 1$ inside the momentum shell around k_F . This requirement thus relates the number of patches N with the number of Landau levels N_L . In fact, the limits $N \rightarrow \infty, \ell_B \rightarrow \infty$ do not commute. To clarify this point, suppose that we may (wrongly) take the limit $N \rightarrow \infty$, fixing the magnetic length ℓ_B . Then, the width of the energy shell around the Fermi energy gets smaller than the gap between Landau levels, $\Delta\epsilon < \omega_c$. In this case, the concept of a Fermi liquid with a well defined Fermi surface is lost. As a last interesting remark, notice that in Eq. (3.15), the magnetic field contribution is exact, *i. e.*, there are no higher order correction terms in a $1/\ell_B k_F$ expansion.

Having established the bosonic character of the field δn_S , we can introduce the local quadratic Hamiltonian,

$$H_B = \frac{\pi v_F}{L^2 \Lambda} \sum_S \int d^2x : \delta n_S^2(\mathbf{x}) : , \quad (3.17)$$

where the colons mean normal order with respect to the filled Fermi sea state, $|FS\rangle$. By computing the Heisenberg equation of motion,

$$\frac{\partial \delta n_S(\mathbf{x})}{\partial t} = i [H_B, \delta n_S(\mathbf{x})] , \quad (3.18)$$

using the bosonic algebra of Eq. (3.15), we find that it is exactly the same equation, Eq. (3.11), that we have found above by direct computation in terms of the fermionic fields.

Thus, we established that, the low energy fermionic Hamiltonian H_F , of Eq. (3.7), with the standard anti-commutation relations, leads to the same equation of motion for the particle-hole excitations that the one derived by using the quadratic bosonic Hamiltonian H_B , of Eq. (3.17), with the bosonic *covariant Schwinger algebra*, Eq. (3.15). This is the main result of this section.

It is convenient to have a representation of the Fermi surface deformations, and the low energy Hamiltonian, in terms of canonical Bose fields. Since $\delta n_S(\mathbf{q})$ does not annihilate the ground state, we define the following operators:

$$a_S(\mathbf{q}) = \sqrt{\frac{2\pi}{L^2 \Lambda}} \quad (3.19)$$

$$\times \sum_{S'} \left[\alpha_{S,S'}(\mathbf{q}) \delta n_{S'}(\mathbf{q}) \Theta(q_{n'}) + \alpha_{S,S'}^\dagger(\mathbf{q}) \delta n_{S'}(-\mathbf{q}) \Theta(-q_{n'}) \right]$$

$$a_S^\dagger(\mathbf{q}) = \sqrt{\frac{2\pi}{L^2 \Lambda}} \quad (3.20)$$

$$\times \sum_{S'} \left[\alpha_{S,S'}^\dagger(\mathbf{q}) \delta n_{S'}(-\mathbf{q}) \Theta(q_{n'}) + \alpha_{S,S'}(\mathbf{q}) \delta n_{S'}(\mathbf{q}) \Theta(-q_{n'}) \right]$$

where $\alpha_{S,S'}(\mathbf{q})$ is a set of matrices to be determined, $q_{n'} = \hat{\mathbf{n}}_{S'} \cdot \mathbf{q}$, and Θ is the usual Heaviside distribution. It is immediate to verify that

$$a_S(\mathbf{q}) |FS\rangle = 0 . \quad (3.21)$$

In addition, the requirement that the bosonic operators a_S and a_S^\dagger satisfy canonical commutation relations

$$[a_S(\mathbf{q}), a_T^\dagger(\mathbf{q}')] = \delta_{S,T} (\delta_{\mathbf{q},\mathbf{q}'} + \delta_{\mathbf{q},-\mathbf{q}'}) , \quad (3.22)$$

fixes the matrix $\alpha_{S,T}(\mathbf{q})$ to satisfy,

$$\sum_{T'} \alpha_{T,T'}^\dagger(\mathbf{q}) \alpha_{T',S}(\mathbf{q}) = \tilde{D}_{S,T}^{-1}(\mathbf{q}) , \quad (3.23)$$

where $\tilde{D}_{S,S'}^{-1}(\mathbf{q})$ is the Green function

$$\left\{ \hat{\mathbf{n}}_S \cdot \mathbf{q} - \frac{i}{\ell_B^2 k_F} \frac{\partial}{\partial \varphi_S} \right\} \tilde{D}_{S,S'}^{-1}(\mathbf{q}) = \delta_{S,S'} . \quad (3.24)$$

We will see that this Green function plays a central role of the bosonization theory. It will recurrently appear in the structure of the bosonization construction, as well as in the computation of all correlation functions. In Appendix C, we study in detail its mathematical properties.

Inverting Eqs. (3.19) and (3.20), we obtain the desired relation that describes the Fermi surface deformation in terms of a set of canonical harmonic oscillators,

$$\delta n_S(\mathbf{q}) = \frac{L}{2\pi} \Lambda^{1/2} \quad (3.25)$$

$$\times \sum_{S'} \left[\beta_{S,S'} a_{S'}(\mathbf{q}) \Theta(q_{n'}) + \beta_{S,S'}^\dagger a_{S'}^\dagger(\mathbf{q}) \Theta(-q_{n'}) \right]$$

where we introduced the matrix $\beta = \alpha^{-1}$. We see that, a local deformation of the Fermi surface $\delta n_S(\mathbf{q})$ is given by a linear superposition of canonical harmonic oscillators around the Fermi surface.

Replacing Eq. (3.25) into Eq. (3.17), we obtain the bosonic Hamiltonian in terms of canonical harmonic oscillator fields,

$$H_B = v_F \sum_S \sum_{\mathbf{q}} a_S^\dagger(\mathbf{q}) \tilde{D}_S(\mathbf{q}) a_S(\mathbf{q}). \quad (3.26)$$

As expected, the quadratic bosonic Hamiltonian is not diagonal in the patch basis, when written in terms of canonical fields.

IV. SYMMETRIES OF THE LOW-ENERGY EFFECTIVE HAMILTONIAN

Symmetries of a microscopic model should be checked in an effective low-energy model. On the other hand, low energy Hamiltonians often tend to exhibit larger symmetries, that are broken at a more microscopic level. A standard example is rotational symmetry which is broken down to the (discrete) point group symmetry of a lattice but becomes continuous at a critical point. Another example more relevant here is the case of a Fermi liquid. A microscopic interacting fermionic Hamiltonian should have global gauge $U(1)$ invariance, related with the conservation of particle number. Of course, any low energy effective theory, such as Landau theory of Fermi liquids, or higher dimensional bosonization, should at least have this same symmetry (provided the total particle number is still conserved.)

However, new symmetries can emerge at low energies. In fact, in the Landau theory of the Fermi liquid the low energy Hamiltonian is invariant under a global phase transformation *at each point (patch) of the Fermi surface*. Then, the system with N patches has a $U(1)^N$ symmetry, related with the particle number conservation at each patch. In the thermodynamic limit, the theory formally has a huge $U(1)^\infty$ invariance. In the language of the renormalization group, we say that the microscopic interactions that break $U(1)^\infty$ symmetry are irrelevant operators at the Fermi liquid fixed point.^{15,20,48} We will now see that something similar occurs in the Fermi gas in a magnetic field.

A. Global $U(1)$ symmetry

Let us begin with the symmetries of the projected low energy Hamiltonian. Since the particle number is globally conserved, the Hamiltonian should be invariant under global phase transformations, $\psi'_S(\mathbf{x}) = \exp(i\alpha)\psi_S(\mathbf{x})$, for $S = 1, \dots, N$ and α a constant. However, unlike the Fermi liquid fixed point which has a $U(1)$ symmetry *for each patch*, the Hamiltonian of a Fermi gas coupled to an external uniform magnetic field is not invariant under phase transformation on each patch, $\psi_S(\mathbf{x}) \rightarrow \exp(i\alpha_S)\psi_S(\mathbf{x})$, since the magnetic field breaks this symmetry, thus implying that the charge is not conserved separately at each patch.

This effect has a close relation with the chiral anomaly. If we take for instance a one-dimensional sub-system by considering two opposite patches, say S_1 and $-S_1$, the Hamiltonian is invariant under the chiral transformation $\psi'_{\pm S_1}(\mathbf{x}) = \exp(\pm i\alpha)\psi_{\pm S_1}(\mathbf{x})$. In the presence of an external electric field along \mathbf{n}_{S_1} the divergence of the chiral current is not zero, meaning that the charge is not conserved in each patch independently. In this case, the system develops a charge density wave, breaking translation invariance⁴⁹. This chiral anomaly is a direct consequence of the Schwinger term in the $U(1)$ current algebra. The magnetic field has no effect on each one-dimensional sub-system. However, it breaks translation symmetry to magnetic translations. To take this effect into account, the chiral anomaly (the Schwinger term) should be modified by the magnetic field, mixing different patches as we have already described in Eq. (3.15). This term should transfer charge from path to path along the Fermi surface.

At the quantum level, the generator of the global $U(1)$ symmetry is the particle number operator ΔN , normal ordered with the filled Fermi sea reference state $|FG\rangle$. In the bosonic representation, it can be written as

$$\Delta N = \sum_S \Delta N_S = \sum_S \int d^2x \delta n_S(\mathbf{x}), \quad (4.1)$$

where we have introduced the number operator at each patch ΔN_S . Using the bosonized Hamiltonian H_B , given by Eq. (3.17), and the covariant Schwinger algebra of Eq. (3.15), it is simple to derive the commutation relation $[\Delta N_S, H_B] = -i\omega_c \frac{d\Delta N_S}{d\varphi_S}$. Consequently, the equation of motion for the particle number at each patch is

$$\frac{\partial \Delta N_S}{\partial t} + \omega_c \frac{\partial \Delta N_S}{\partial \varphi_S} = 0. \quad (4.2)$$

A general solution, $\Delta N_S \equiv \Delta N_S(\varphi_S - \omega_c t)$, is a local charge perturbation wrapping around the Fermi surface with velocity ω_c/k_F .

In the absence of magnetic field, $\omega_c \rightarrow 0$, $[\Delta N_S, H_B] = 0$, and the system is $U(1)$ invariant at each patch, independently. However, even for a weak magnetic field, this huge symmetry is broken. Of course, by summing up over the patches of the hole Fermi surface, we obtain

$[\Delta N, H_B] = 0$, as it should be, since the total charge is conserved.

B. Magnetic translations

The full Hamiltonian, Eq. (2.1), is not invariant under translations, due to the presence of the uniform magnetic field. Instead, it is invariant under the transformation $\psi(\mathbf{x}) \rightarrow \exp\{-i(B/2)\mathbf{a} \times \mathbf{x}\}\psi(\mathbf{x} + \mathbf{a})$, that characterizes the *group of magnetic translations*. Its infinitesimal generators are $D_i^* = \partial_i - ieA_i$, which satisfy

$$[D_i^*, D_j^*] = iB\epsilon_{ij}, \quad (4.3)$$

$$[D_i^*, D_j] = 0. \quad (4.4)$$

Elements of the group, *i. e.*, finite magnetic translation labeled by a vector \mathbf{a} , $T_{\mathbf{a}} = \exp(\mathbf{a} \cdot \mathbf{D}^*)$, satisfy the algebra

$$[T_{\mathbf{a}}, T_{\mathbf{b}}] = 2i \sin\left(\frac{B}{2}\hat{\mathbf{z}} \cdot (\mathbf{a} \times \mathbf{b})\right) T_{\mathbf{a}+\mathbf{b}}. \quad (4.5)$$

Using Eqs. (2.1), (4.3) and (4.4), we easily show that $[T_{\mathbf{a}}, H] = 0$, confirming that this is indeed a symmetry. Evidently, this symmetry should also be obeyed by the (patched) low-energy Hamiltonian.

It is instructive to see how the infinitesimal generators of magnetic translations are represented on the fermionic basis for each patch, $\psi_S(\mathbf{x})$. Considering the projection of D_i^* on the small shell around the Fermi surface, we have

$$D_i^* = \nabla_i + ik_{S,i} + \frac{1}{\ell_B^2}\epsilon_{ij}\frac{\partial}{\partial k_{S,j}}. \quad (4.6)$$

It is very easy to check that these operators in fact satisfy the algebra of Eqs. (4.3) and (4.4). In fact, Eq. (4.4) guarantees that these generators commute with the Hamiltonian, $[D_i^*, H_F] = 0$, implying that the low energy Hamiltonian is invariant under the infinitesimal transformation

$$\delta\psi_S(\mathbf{x}) = \left(\mathbf{a} \cdot \nabla + i\mathbf{a} \cdot \mathbf{k}_S + \frac{1}{\ell_B^2}\mathbf{a} \times \frac{\partial}{\partial \mathbf{k}_S}\right)\psi_S(\mathbf{x}). \quad (4.7)$$

The action of a finite magnetic translation on a patch fermion is

$$\psi'_S(\mathbf{x}) = T_{\mathbf{a}}\psi_S(\mathbf{x}) = e^{i\mathbf{k}_S \cdot \mathbf{a}}\psi_{S+\Delta S}(\mathbf{x} + \mathbf{a}) \quad (4.8)$$

where, $\Delta S = \hat{\mathbf{n}}_S \cdot \mathbf{a}/\ell_B^2 k_F$. The net effect on ψ_S is, not only to translate the position $\mathbf{x} \rightarrow \mathbf{x} + \mathbf{a}$, but also to shift the patch parameter S by ΔS . The transformation also changes the phase by an amount $\mathbf{k}_S \cdot \mathbf{a}$.

To clarify that the transformation of Eq. (4.8) is in fact a magnetic translation, it is necessary to reconstruct the full fermion operator by adding up the contributions

over the patches of the whole Fermi surface. We then have,

$$\begin{aligned} \psi'(\mathbf{x}) &= \sum_S e^{i\mathbf{k}_S \cdot \mathbf{x}} e^{i\mathbf{k}_S \cdot \mathbf{a}} \psi_{S+\Delta S}(\mathbf{x} + \mathbf{a}) \\ &= \sum_S e^{i\mathbf{k}_{S-\Delta S} \cdot (\mathbf{x} + \mathbf{a})} \psi_S(\mathbf{x} + \mathbf{a}), \end{aligned} \quad (4.9)$$

where, in the last equality, we have shifted $S \rightarrow S - \Delta S$. In the limit of small magnetic fields, $1/\ell_B k_F \ll 1$, we can write $\mathbf{k}_{S-\Delta S} \cdot \mathbf{x} = \mathbf{k}_S \cdot \mathbf{x} - (\mathbf{x} \times \mathbf{a}) \cdot \hat{\mathbf{z}} B/2$, and then

$$\psi'(\mathbf{x}) = e^{i\frac{B}{2}\hat{\mathbf{z}} \cdot (\mathbf{x} \times \mathbf{a})} \psi(\mathbf{x} + \mathbf{a}), \quad (4.10)$$

which is precisely a magnetic translation of the full fermion operator.

Our next task is construct a bosonic representation of the magnetic translation generators. It is known that the elements of the group can be represented as non-trivial linear superposition of fermion bilinears.^{50,51} In our case, it is more convenient to use a non-linear representation in terms of the bosonic excitations δn_S or, equivalently, a_S, a_S^\dagger . The desired representation of the generators of magnetic translations is

$$\mathbf{\Pi} = \sqrt{\frac{\Lambda}{k_F}} \frac{1}{k_{FL}} \sum_S \int d^2x \left(i\mathbf{k}_S \delta n_S(\mathbf{x}) + \frac{1}{2} \hat{\mathbf{n}}_S : \delta n_S^2(\mathbf{x}) : \right) \quad (4.11)$$

which coincides with the canonical momentum of a Fermi liquid.¹⁸ The covariant Schwinger algebra, Eq. (3.15), guarantees that this expression satisfies the magnetic translation algebra, Eq. (4.3). It is possible to rewrite Eq. (4.11) in terms of the canonical bosonic fields a_S and a_S^\dagger . However, in that representation, $\mathbf{\Pi}$ is not diagonal in the patch variable S .

C. Translations

As anticipated, the low-energy Hamiltonian, H_F , has more symmetries than the full Hamiltonian H . In fact, H_F is also invariant under translations $\psi'_S(\mathbf{x}) = \psi_S(\mathbf{x} + \mathbf{a})$. This happens because the projection process brings about a splitting of scales, as was already observed in the structure of the covariant derivative, Eq. (3.8). Indeed, the rapidly oscillating part, proportional to $\mathbf{k}_F \cdot \mathbf{x}$, has been extracted from the definition of $\psi_S(\mathbf{x})$, leading to a slowly varying field on scales $|\mathbf{x}| \gg k_F^{-1}$. The infinitesimal generator of translations, $\mathbf{P} = -i\nabla$, commutes with all the other symmetry generators and with the Hamiltonian, *i. e.*, $[P_i, P_j] = [P_i, D_j^*] = [P_i, D_j] = [P_i, H_F] = 0$. The bosonic representation of \mathbf{P} is written in a simpler way in terms of the bosonic fields a_S and a_S^\dagger ,

$$\mathbf{P} = \sum_S \int d^2q \mathbf{q} a_S^\dagger(\mathbf{q}) a_S(\mathbf{q}). \quad (4.12)$$

Notice that, if rewritten in terms of the fields δn_S , this is a non-local function of the patch variable S .

V. THE FERMION OPERATOR

To push forward the bosonization program we need to build the fermion operator in terms of canonical bosonic fields. The fermion operator should satisfy the following commutation relations,

$$\begin{aligned} [\Delta N, \psi_S(\mathbf{x})] &= -\psi_S(\mathbf{x}) \\ [\Pi_i, \psi_S(\mathbf{x})] &= \left(\nabla_i + ik_{S,i} + \frac{1}{\ell_B^2} \epsilon_{ij} \frac{\partial}{\partial k_{S,j}} \right) \psi_S(\mathbf{x}) \\ [P_i, \psi_S(\mathbf{x})] &= -i \nabla_i \psi_S(\mathbf{x}), \end{aligned} \quad (5.1)$$

where ΔN , Π and P are the infinitesimal generators of the main symmetries of the low energy system, already described in the previous section. Their bosonic representation is given by Eqs. (4.1), (4.11) and (4.12), respectively.

Interestingly, these three commutation relations are simultaneously satisfied by simply using,

$$[\psi_S(\mathbf{x}), \delta n_T(\mathbf{y})] = \delta(\mathbf{x} - \mathbf{y}) \delta_{S,T} \psi_S(\mathbf{x}). \quad (5.2)$$

For consistency, Eq. (5.2) can be independently confirmed by means of the definition of δn_S and the usual fermionic algebra.

Similarly to the Fermi liquid case,¹⁵ we propose the following ansatz for the fermion operator,

$$\psi_S(\mathbf{x}) = \frac{1}{2\pi} \sqrt{\Lambda \lambda} K_S e^{i\phi_S(\mathbf{x})}, \quad (5.3)$$

where K_S are the Klein factors (that insure anticommutation relations of the fermion operators at different patches), and $\phi_S(\mathbf{x})$ is a bosonic phase field. Both operators will be determined below.

Upon replacing Eq. (5.3) into Eq.(5.2), and additionally assuming that $[K_S, \delta n_T] = 0$, and $[\phi_S(\mathbf{x}), \delta n_T(\mathbf{x}')] = c$, where c is a complex number, we find that the bosonic field $\phi_S(\mathbf{x})$ must obey

$$[\delta n_S(\mathbf{y}), \phi_T(\mathbf{x})] = -i \delta(\mathbf{x} - \mathbf{y}) \delta_{S,T}. \quad (5.4)$$

Evidently, the fermionic phase, $\phi_S(\mathbf{x})$ is the canonical conjugate field to $\delta n_S(\mathbf{x})$. This algebra is then exactly solved by defining

$$\delta n_S(\mathbf{x}) = \frac{\Lambda}{(2\pi)^2} D_S \phi_S(\mathbf{x}), \quad (5.5)$$

with the commutation relations,

$$[\phi_S(\mathbf{x}), \phi_T(\mathbf{y})] = \frac{(2\pi)^2}{\Lambda} D_{S,T}^{-1}(\mathbf{x} - \mathbf{y}). \quad (5.6)$$

Eqs. (5.5) and (5.6) are an elegant (and important) generalization of the usual expressions of bosonization of Fermi liquids. Indeed, in the limit $\omega_c \rightarrow 0$, the covariant derivative becomes $D_S \rightarrow \hat{\mathbf{n}}_S \cdot \nabla$ and the Green function approaches the limit $D_{S,T}^{-1}(\mathbf{x} - \mathbf{y}) \rightarrow \text{sgn}(\mathbf{x} - \mathbf{y}) \delta_{S,T}$, thus recovering previous existing results.¹³⁻¹⁷

By inverting Eq. (5.5) we obtain the desired relation between the phase $\phi_S(\mathbf{x})$ of the fermionic operator and the bosonic particle-hole excitations,

$$\phi_S(\mathbf{x}) = -i \left(\frac{2\pi}{L} \right)^2 \frac{1}{\Lambda} \sum_{\mathbf{q}} e^{-i\mathbf{q} \cdot \mathbf{x}} \sum_{S'} \tilde{D}_{S,S'}^{-1}(\mathbf{q}) \delta n_{S'}(-\mathbf{q}). \quad (5.7)$$

We can now see the main change between bosonization in the absence of a magnetic field and with the application of a weak enough field. In the former case, the phase of the fermionic patch operator is a local function of the density fluctuation δn_S . In the latter, instead, the phase at a single patch S is defined by the contribution of δn_S on the entire Fermi surface. The structure of the patch superposition is coded in the Green function of the normal covariant derivative, $\tilde{D}_{S,S'}^{-1}$.

It is useful to rewrite the phase of the fermionic field in terms of canonical fields operators. By replacing Eq. (3.25) into Eq. (5.7), we obtain

$$\begin{aligned} \phi_S(\mathbf{x}) &= -i \frac{2\pi}{L \Lambda^{1/2}} \sum_{\mathbf{q}} e^{-i\mathbf{q} \cdot \mathbf{x}} \\ &\times \sum_{S'} \left\{ \alpha_{S,S'}(\mathbf{q}) a_{S'}^\dagger(\mathbf{q}) \Theta(\mathbf{q}_{n'}) - \alpha_{S,S'}^\dagger(-\mathbf{q}) a_{S'}(\mathbf{q}) \Theta(-\mathbf{q}_{n'}) \right\}. \end{aligned} \quad (5.8)$$

Here, the phase operator is written in terms of a coherent superposition of harmonic oscillators, defined on the whole Fermi surface. The corresponding weight is given by the functions $\alpha_{S,S'}$, that can be roughly thought as the square root of the Green function, $\alpha \sim \tilde{D}_S^{-1/2}$. The precise definition of $\alpha_{S,S'}$ is given in Eq. (3.23). In the limit of zero magnetic field, the Green function becomes local in the patch basis, and Eq. (5.8) reduces to the usual local expression.^{15,19,22}

The Klein factors K_S should be built in order to insure anticommutation relations between fermions defined on different patches. Consider for instance,

$$K_S = \lim_{\mathbf{q} \rightarrow 0} e^{i\pi \sum_{S'=1}^{S-1} \delta n_{S'}(\mathbf{q})} \quad (5.9)$$

where we have chosen an arbitrary reference $S = 1$ and we have ordered the patches counterclockwise. It is then straightforward to show that the Klein factors commute with each other, $[K_S, K_T] = 0$, and that they are their own inverse, i.e. $K_S K_S^\dagger = 1$. Moreover, by using Eq. (5.4), we have,

$$K_T^\dagger \psi_S K_T = \begin{cases} -\psi_S & \text{if } T > S \\ +\psi_S & \text{if } T \leq S \end{cases} \quad (5.10)$$

This construction completes the definition of the fermion operator.

A. The fermion propagator at equal times

As an example, and to check the consistency of the construction, let us compute the fermionic equal time

expectation value on the same patch. Using the definition of the fermion operator, Eq. (5.3), we have

$$\langle \psi_S^\dagger(\mathbf{x}) \psi_S(0) \rangle = \frac{\lambda \Lambda}{(2\pi)^2} \langle e^{-i\phi_S(\mathbf{x})} e^{i\phi_S(0)} \rangle. \quad (5.11)$$

Using the Baker-Hausdorff formula,

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{A}+\hat{B}} : e^{\langle \hat{A}\hat{B} + (1/2)(\hat{A}^2 + \hat{B}^2) \rangle} \quad (5.12)$$

we can write,

$$\langle \psi_S^\dagger(\mathbf{x}) \psi_S(0) \rangle = \frac{\lambda \Lambda}{(2\pi)^2} e^{G_\phi(S, \mathbf{x})} \quad (5.13)$$

where we used the notation

$$\begin{aligned} G_\phi(S, \mathbf{x}) &= \langle FS | \phi_S(\mathbf{x}) \phi_S(0) - \phi_S^2(0) | FS \rangle \\ &= -\frac{1}{2} \langle FS | (\phi_S(\mathbf{x}) - \phi_S(0))^2 | FS \rangle. \end{aligned} \quad (5.14)$$

Using the representation of the bosonic field ϕ_S in terms of harmonic oscillators, Eqs. (5.8), and the fact that $a_S |FS\rangle = 0$, it is simple to obtain,

$$G_\phi(S, \mathbf{x}) = \left(\frac{2\pi}{L} \right)^2 \frac{1}{\Lambda} \sum_{\mathbf{q}, q_n > 0} (e^{i\mathbf{q} \cdot \mathbf{x}} - 1) D_{S,S}^{-1}(\mathbf{q}), \quad (5.15)$$

where $D_{S,S}^{-1}(\mathbf{q})$ is the diagonal sector of the covariant derivative Green function given by (see Appendix C for its computation),

$$D_{S,S}^{-1}(\mathbf{q}) = \sum_{\ell} \frac{e^{i2\pi\ell(\mathbf{v}_S \cdot \mathbf{q}/\omega_c)}}{\hat{\mathbf{n}}_S \cdot \mathbf{q}}. \quad (5.16)$$

Replacing Eq. (5.16) into Eq. (5.15), and integrating over q_n and q_t (in the local frame) we find,

$$G_\phi(S, \mathbf{x}) = \begin{cases} \sum_{\ell} \ln \left[\frac{i/\lambda + 2\pi\ell\ell_B^2 k_F}{\hat{\mathbf{n}} \cdot \mathbf{x} + i/\lambda + 2\pi\ell\ell_B^2 k_F} \right], & \text{for } |\hat{\mathbf{n}} \times \mathbf{x}| \Lambda \ll 1 \\ -\infty, & \text{for } |\hat{\mathbf{n}} \times \mathbf{x}| \Lambda \gg 1 \end{cases} \quad (5.17)$$

Plugging this result into the equal-time propagator of the patch fermion, Eq.(5.13), we find,

$$\begin{aligned} \langle \psi_S^\dagger(\mathbf{x}) \psi_S(0) \rangle &= \frac{i\Lambda}{(2\pi)^2} \prod_{\ell} \frac{1 + i2\pi\ell\ell_B^2 k_F \lambda}{\hat{\mathbf{n}} \cdot \mathbf{x} + i/\lambda + 2\pi\ell\ell_B^2 k_F} \\ &\sim \frac{i\Lambda}{(2\pi)^2} \sum_{\ell} \frac{1}{\hat{\mathbf{n}} \cdot \mathbf{x} + 2\pi\ell\ell_B^2 k_F + i/\lambda} + \\ &\quad + O(1/N_L) \end{aligned} \quad (5.18)$$

for $|\hat{\mathbf{n}}_S \times \mathbf{x}| \Lambda \ll 1$ and zero otherwise. The last expression coincides with the result for the fermion propagator, Eq. (2.7), at equal times $r_0 = 0$, with corrections at least of order N_L^{-1} .

VI. QUANTUM DYNAMICS OF THE FERMI SURFACE IN A MAGNETIC FIELD

To describe the Fermi surface dynamics, we write the generating functional in a coherent state path integral representation,¹⁵

$$Z = \int \left(\prod_S \mathcal{D}a_S^\dagger \mathcal{D}a_S \right) e^{iS[a_S, a_S^\dagger]}, \quad (6.1)$$

where the action is

$$S[a_S, a_S^\dagger] = \sum_S \int dt d^2x \left\{ a_S^\dagger i \partial_t a_S - H_B[a_S, a_S^\dagger] \right\}. \quad (6.2)$$

Using the explicit form of H_B , Eq. (3.26), the action takes de form

$$S[a] = \int d^2q d\omega \sum_{S,T} a_S^\dagger(\mathbf{q}, \omega) M_{S,T}^{-1}(\mathbf{q}, \omega) a_T(\mathbf{q}, \omega), \quad (6.3)$$

with

$$M_{S,T}^{-1}(\mathbf{q}, \omega) = \left\{ \omega - v_F \tilde{D}_S(\mathbf{q}) \right\} \delta_{S,T} \quad (6.4)$$

and the covariant derivative, $\tilde{D}_S(\mathbf{q})$, was given in Eq. (3.14).

The Bosonic correlation function,

$$\langle a_S^\dagger(\omega, \mathbf{q}), a_T(\omega, \mathbf{q}) \rangle = M(\omega, \mathbf{q}, \varphi_S, \varphi_T), \quad (6.5)$$

is found by solving the linear differential equation,

$$\left(\omega - \mathbf{v}_S \cdot \mathbf{q} + i\omega_c \frac{\partial}{\partial \varphi_S} \right) M_{S,T} = \delta_p(\varphi_S - \varphi_T), \quad (6.6)$$

where $\delta_p(\varphi_S) = \sum_n \delta(\varphi_S + 2\pi n)$ is the periodic Dirac delta function.

We can express the Green function in terms of eigenvalues, λ_n , and eigenfunctions, $\psi_n(\omega, \mathbf{q}, \varphi_S)$, of the operator $M_{S,T}^{-1}$. We have,

$$M_{S,T} = \sum_n \frac{1}{\lambda_n} \psi_n^*(\varphi_S) \psi_n(\varphi_T) \quad (6.7)$$

provided $\lambda_n \neq 0$. The eigenvalue equation reads,

$$\left(\omega - \mathbf{v}_S \cdot \mathbf{q} + i\omega_c \frac{\partial}{\partial \varphi_S} \right) \psi_n = \lambda_n \psi_n, \quad (6.8)$$

with periodic boundary conditions $\psi_n(\varphi_S) = \psi_n(\varphi_S + 2\pi)$. The solutions of this equation are the orthonormal eigenfunctions,

$$\psi_n(\varphi_S) = \frac{1}{2\pi} e^{in\varphi_S - i\frac{v_F \mathbf{q}}{\omega_c} \sin \varphi_S}, \quad (6.9)$$

and the eigenvalues are $\lambda_n = \omega - n\omega_c$, with n integer. By replacing Eq. (6.9) into Eq. (6.7), we obtain

$$\begin{aligned} M_{S,T}(\omega, \mathbf{q}) &= \frac{1}{(2\pi)^2} e^{-i\frac{v_F \mathbf{q}}{\omega_c} (\sin \varphi_S - \sin \varphi_T)} \\ &\quad \times \sum_{n=-\infty}^{\infty} \frac{e^{in(\varphi_S - \varphi_T)}}{\omega - n\omega_c}. \end{aligned} \quad (6.10)$$

with the desired property, $M_{S,T} = M_{T,S}^*$. We observe that $M_{S,T}$ has poles at frequencies corresponding to the energies of the Landau levels, $\omega = n\omega_c$. To the best of our knowledge, this is the first time that the Landau level quantization appears in a bosonization procedure. An equivalent phenomenon has been described in the context of a hydrodynamic description of composite fermions in a half-filled Landau level.⁴⁵ Here, the Landau levels appear as zero modes, $\lambda_n = 0$, of the operator $M_S^{-1} = \omega - \bar{D}_S(\mathbf{q})$, upon imposing periodic boundary conditions around the Fermi surface. Thus, in our context, Landau quantization is a global property of the Fermi surface, and cannot be seen in a single patch expansion.

The limit of small frequencies, $\omega \ll v_F q$, and in particular the limit $\omega \rightarrow 0$, is highly singular and needs to be handled with care. To this end, it is useful to rewrite the Green function in terms of the dual Fourier series of the Landau levels. Using the Poisson summation formula, we have

$$\sum_{n=-\infty}^{\infty} \frac{e^{in\theta}}{\omega - n\omega_c} = \sum_{\ell=-\infty}^{\infty} \int_{-\infty}^{\infty} dz \frac{e^{iz(\theta-2\pi\ell)}}{\omega - z\omega_c}. \quad (6.11)$$

Defining $\theta = \varphi_S - \varphi_T$ as the angle subtended by two patches (see Fig. 2) and performing the z integration (analytically continued to the complex plane), we find

$$M(\omega, \mathbf{q}, \varphi_S, \theta) = \frac{i}{2\omega_c} \sum_{\ell} \text{sgn}(\theta - 2\pi\ell) \exp(i\alpha_{\ell}), \quad (6.12)$$

where we defined the phase

$$\alpha_{\ell}(\omega, \mathbf{q}, \varphi_S, \theta) = \frac{\omega}{\omega_c} (\theta - 2\pi\ell) - \frac{\mathbf{v}_S \cdot \mathbf{q}}{\omega_c} \sin \theta - \frac{\mathbf{v}_S \times \mathbf{q}}{\omega_c} (1 - \cos \theta). \quad (6.13)$$

Eq. (6.12) is completely equivalent to Eq. (6.10).

In order to compute the intra-patch (diagonal sector) Green function, we coarse-grain the bosonic Green function on each patch, by integrating over the variable θ on one patch. We define

$$G_{SS}(\omega, \mathbf{q}) = \int_{-\Lambda/2k_F}^{+\Lambda/2k_F} d\theta M(\omega, \mathbf{q}, \varphi_S, \theta). \quad (6.14)$$

In the limit, $|\theta| \ll 1$, $\sin \theta \sim \theta$ and $1 - \cos \theta \sim \theta^2$. Thus, at leading order, the phase α_{ℓ} becomes,

$$\alpha_{\ell}(\omega, \mathbf{q}, \varphi_S, \theta) = \frac{\omega}{\omega_c} \theta - \frac{\mathbf{v}_S \cdot \mathbf{q}}{\omega_c} (\theta + 2\pi\ell)$$

where we have shifted $\theta \rightarrow \theta + 2\pi\ell$ in order to have a well-defined limit for small frequencies $\omega \ll \mathbf{v}_S \cdot \mathbf{q}$. The result of the integration in Eq. (6.14) is, in the limit $N_L/N \gg 1$,

$$G_{SS}(\omega, \mathbf{q}) = \sum_{\ell} \frac{e^{-i2\pi\ell(\frac{\mathbf{v}_S \cdot \mathbf{q}}{\omega_c})}}{\omega - \mathbf{v}_S \cdot \mathbf{q} + i\Delta\epsilon \text{sgn}(\omega)}, \quad (6.15)$$

where we have performed the analytic continuation $\omega \rightarrow \omega + i\Delta\epsilon \text{sgn}(\omega)$.

The $\ell = 0$ term in Eq.(6.15) is the well-known result for the particle-hole correlation function of a Fermi liquid. The effect of the magnetic field is to introduce oscillatory terms, which lead to the discretization of the spectrum. This is depicted schematically by the shaded area in Fig. 2. Finally, we note that this expression has the correct low frequency behavior since

$$\lim_{\omega \rightarrow 0} G_{SS}(\omega, \mathbf{q}) = -D_{SS}^{-1}(\mathbf{q}), \quad (6.16)$$

used above in Eq. (5.16). An independent detailed computation is presented in Appendix C.

In the same way we can compute the inter-patch (off-diagonal) bosonic correlation functions. Consider $\theta = \bar{\theta} + \delta\theta$, where $\bar{\theta}$ is the finite angle subtended by the patches S and T , and integrate over $|\delta\theta| < \Lambda/k_F$. The result is

$$G_{ST}(\omega, \mathbf{q}) = \sum_{\ell} \frac{\sin \left[\frac{\Lambda}{2k_F\omega_c} (\omega - \mathbf{v}_S \cdot \mathbf{q} \cos \bar{\theta} + \mathbf{v}_S \times \mathbf{q} \sin \bar{\theta}) \right]}{\omega - \mathbf{v}_S \cdot \mathbf{q} \cos \bar{\theta} + \mathbf{v}_S \times \mathbf{q} \sin \bar{\theta} + i\Delta\epsilon \text{sgn}(\omega)} e^{-i\alpha_{\ell}(\bar{\theta})}, \quad (6.17)$$

with $|\bar{\theta}| > \Lambda/k_F$. For weak magnetic fields, this is a strongly oscillating function of ω . Due to the analytic continuation $\omega \rightarrow \omega + i\Delta\epsilon \text{sgn}(\omega)$, the Fourier transform yields an exponential factor of the form $G_{S,T}(t, \mathbf{x}) \sim \exp(-N_L/N)$, that enforces the condition that the limit $\lim_{\omega_c \rightarrow 0} G_{S,T}(t, \mathbf{x}) = 0$, as it should be.

VII. DE HAAS-VAN ALPHEN OSCILLATIONS

A distinct characteristic of metals in weak magnetic fields is the presence of quantum oscillations.^{52,53} They encode the main features and existence of a Fermi surface. The de Haas-van Alphen effect (dHvA),^{27,53} consists of the periodic oscillation of the magnetization as a function of an external magnetic field. It is widely used as a tool to investigate the structure and topology of the Fermi surface. In this section we show how the dHvA can be computed by using the bosonization approach. In particular, we show that the structure of the oscillations is contained in the bosonized action, and depends on the dynamics of the collective modes of the Fermi surface. This is important because, on the one hand, confirms that the bosonization procedure we present here correctly captures the structural properties of the Fermi surface. On the other hand, it opens the possibility of computing quantum oscillations in extensions of the present work to strongly correlated systems, e.g. the nematic Fermi fluid.

We begin with the bosonic path integral representation of Eq. (6.1). Integrating over the bosonic fields a_S and a_S^{\dagger} we find the zero temperature limit of the thermodynamic potential, $F = -\ln Z$. We find

$$F(\omega_c) = -\text{Tr} \ln (M^{-1}), \quad (7.1)$$

where the operator M^{-1} is the kernel given by Eq. (6.4).

The orbital magnetic moment, \mathcal{M} , can be computed as

$$\begin{aligned}\mathcal{M} &= -\left(\frac{\mu_B}{L^2}\right) \frac{\partial F}{\partial B} = -\mu_B \left(\frac{v_F}{k_F L^2}\right) \frac{\partial F}{\partial \omega_c} \\ &= \mu_B \left(\frac{v_F}{k_F L^2}\right) \text{Tr} \left\{ M \frac{\partial M^{-1}}{\partial \omega_c} \right\}.\end{aligned}\quad (7.2)$$

Here $\mu_B = 1/2m = v_F/2k_F$ is the Bohr magneton in the natural units for our system. The operators M and M^{-1} are diagonal in both frequency and momentum variables (ω, \mathbf{q}) . However, they are not diagonal in the patch index S . Then, the trace can be computed as

$$\begin{aligned}\text{Tr} \left\{ M \times \frac{\partial M^{-1}}{\partial \omega_c} \right\} &= \\ \frac{L^2}{v_F \lambda} \int d\omega d^2 q \sum_{S,T} \langle S | M | T \rangle \langle T | \frac{\partial M^{-1}}{\partial \omega_c} | S \rangle.\end{aligned}\quad (7.3)$$

Here, for simplicity, we use the notation $|S\rangle \equiv |S, \omega, \mathbf{q}\rangle$.

On the other hand,

$$\langle T | \frac{\partial M^{-1}}{\partial \omega_c} | S \rangle = i \frac{\partial}{\partial \varphi_S} \delta(\varphi_S - \varphi_T). \quad (7.4)$$

Then the magnetization takes the form,

$$\mathcal{M} = i \left(\frac{\mu_B}{k_F \lambda}\right) \int d\omega d^2 q \sum_S \left. \frac{\partial M_{S,T}}{\partial \varphi_T} \right|_{T=S}. \quad (7.5)$$

Here, the condition $T = S$ should be understood as $\varphi_T = \varphi_S$ in the sense of a coarse-grained expression on a patch, *i. e.*, $-\Lambda/2k_F < \varphi_S - \varphi_T < \Lambda/2k_F$. Thus, using $\theta = \varphi_S - \varphi_T$ we have,

$$\left. \frac{\partial M_{S,T}}{\partial \varphi_T} \right|_{T=S} = - \lim_{\Lambda/k_F \rightarrow 0} \int_{-\Lambda/2k_F}^{\Lambda/2k_F} d\theta \frac{dM(\varphi_S, \theta)}{d\theta}, \quad (7.6)$$

where $M(\varphi_S, \theta)$ is given by Eq. (6.12).

Now, we choose a local coordinate system in each patch, $d^2 q = dq_n dq_t$, and integrate over q_t and ω . After taking the continuum limit $(\Lambda/k_F) \sum_S \rightarrow \int_0^{2\pi} d\varphi_S$, we find the following expression for the orbital magnetization \mathcal{M}

$$\begin{aligned}\mathcal{M} &= -\frac{1}{2} \left(\frac{v_F^2}{k_F \omega_c}\right) \sum_{\ell} \int_0^{\lambda} dq \int_0^{2\pi} d\varphi_S e^{i2\pi\ell \frac{v_F q}{\omega_c} \cos \varphi_S} \\ &= -\left(\frac{\pi v_F^2}{k_F \omega_c}\right) \sum_{\ell} \int_0^{\lambda} dq J_0(2\pi\ell v_F q / \omega_c),\end{aligned}\quad (7.7)$$

where $J_0(z)$ is the Bessel Function of the first kind. Finally, we compute the q integration, remembering that we are working in the asymptotic limit $v_F \lambda / \omega_c \gg 1$. Disregarding constant terms, we find for the oscillatory part of the magnetization the result

$$\mathcal{M}_{\text{osc}} = -2\mu_B \sqrt{\frac{\omega_c}{2\pi v_F \lambda}} \sum_{\ell=1} \frac{1}{\ell^{3/2}} \sin \left\{ \frac{2\pi\ell v_F \lambda}{\omega_c} + \frac{\pi}{4} \right\}. \quad (7.8)$$

We see that, the magnetization oscillates with the variable $1/\omega_c$, with period $1/(\ell v_F \lambda)$. Each harmonic is weakly damped by the power law $\ell^{-3/2}$. The overall oscillating function is damped at low fields with an envelope function $(\omega_c/2\pi v_F \lambda)^{1/2}$. This result coincides with the zero-temperature limit of the well-known Lifshitz-Kosevich formula.²⁷ From a geometrical point of view, the envelope function is the flux of the magnetic field through the area $1/2\pi k_F \lambda$, corresponding (in momentum space) to the area limited by the shell of width λ around the Fermi surface. It is interesting to note that the argument of the sin function, $v_F \lambda / \omega_c$, is the number of Landau levels contained in each patch. Thus, the magnetization completes one period every time an additional Landau level enters the patch, as the magnetic field is lowered.

In figure 3 we depict the orbital magnetization, Eq. (7.8), where we have summed the first twenty harmonics. Higher harmonics make no difference within the graphic precision. In fig. 4 we present the magnetic

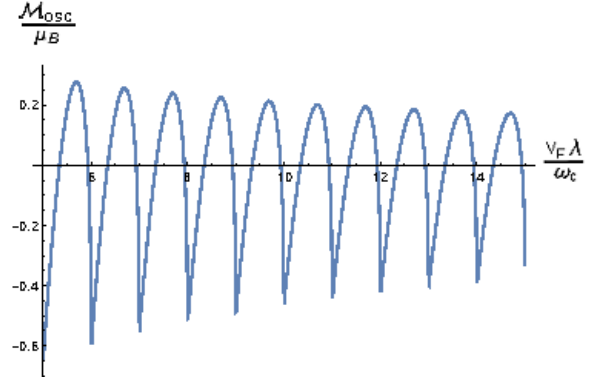


FIG. 3: Orbital magnetization $\mathcal{M}_{\text{osc}}/\mu_B$, Eq. (7.8), as a function of $v_F \lambda / \omega_c$. We have considered the first twenty terms in the sum. Higher harmonics make no difference within the graphic precision. Each time the variable $v_F \lambda / \omega_c$ is an integer, the magnetization has a cusp, corresponding with an additional complete filled Landau level inside the patch.

susceptibility $\chi = \partial \mathcal{M} / \partial B$. We observe divergencies, each time a new Landau level enters the region bounded by $|\mathbf{q}| \leq \lambda/2$. The divergence arises because higher harmonics are damped with the extremely low power $\ell^{-1/2}$, producing a divergent series at each Landau level. Even though, this is a zero temperature computation, temperature fluctuations put another energy scale in the problem. For relatively strong fields, $\omega_c > k_B T$ (where k_B is the Boltzmann constant), the results essentially coincide with the zero temperature expression. For weak fields, $\omega_c < k_B T$, temperature exponentially damps out higher harmonics, and only the leading term, $\ell = 1$, survives. In this case, the divergencies of the susceptibility are rounded by temperature.

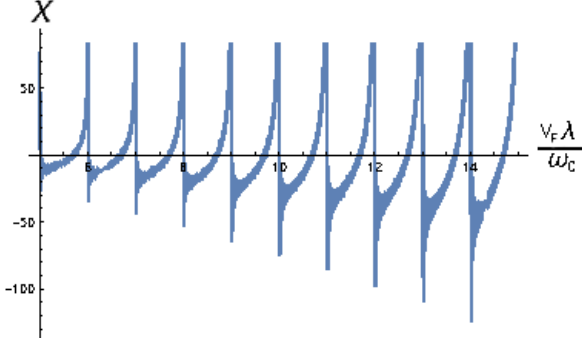


FIG. 4: Magnetic susceptibility $\partial \mathcal{M}_{\text{osc}} / \partial B$, as a function of $v_F \lambda / \omega_c$. We have considered the first twenty terms in the sum. For $v_F \lambda / \omega_c = \text{integer}$, the susceptibility diverges, corresponding with the inclusion on an additional complete filled Landau level inside the patch.

VIII. FORWARD SCATTERING INTERACTIONS

The power of bosonization as a non-perturbative technique is due in part to the fact that two-body forward scattering interactions can be written as quadratic operators in the bosonic basis. The more general two-body interaction term in the low-energy Hamiltonian can be written in terms of products of four fermionic operators, $\psi_{S_1}^\dagger(\mathbf{x})\psi_{S_2}^\dagger(\mathbf{y})\psi_{S_3}(\mathbf{x})\psi_{S_4}(\mathbf{y})$, where $\{S_1, S_2, S_3, S_4\}$ is a set of patches over the Fermi surface. Symmetries strongly constrain the possible terms that should be considered. Momentum conservation, $\mathbf{k}_{S_1} + \mathbf{k}_{S_2} + \mathbf{k}_{S_3} + \mathbf{k}_{S_4} = 0$, selects two kinds of processes: forward scattering interactions for which, $S_1 = S_3$ and $S_2 = S_4$, and BCS type interactions for which $S_1 = -S_2$ and $S_3 = -S_4$. Moreover, exchange processes ($S_1 = S_4$, $S_2 = S_3$) are absent in the spinless model with reasonably local interactions. In the absence of magnetic fields, these types of low-energy Fermi surface interactions were extensively studied both in the fermionic basis,^{20,54} as well as with bosonization techniques.^{13–19,22,24,25}

Considering a simple model, based on a circular Fermi surface, couplings between different patches can be classified according with irreducible representations of the rotation group. Each representation is labeled with an angular momentum quantum number ℓ . Then, it is possible to consider forward-scattering interactions between charge densities ($\ell = 0$), dipole moment density ($\ell = 1$), quadrupole moment density ($\ell = 2$), and so on. The simplest term is a density-density interaction of the form,

$$H_{\text{int}} = \sum_{S,T} \int d^2x d^2y f_0(\mathbf{x} - \mathbf{y}) \psi_S^\dagger(\mathbf{x}) \psi_S(\mathbf{x}) \psi_T^\dagger(\mathbf{y}) \psi_T(\mathbf{y}) \quad (8.1)$$

where, in the Fermi liquid phase, the interaction potential can be considered as being essentially local, $f_0(\mathbf{x} - \mathbf{y}) = f_0 \delta^2(\mathbf{x} - \mathbf{y})$.

By carefully computing the point-splitting product of fermion operators, $\lim_{\epsilon \rightarrow 0} : \psi_S^\dagger(\mathbf{x} + \epsilon \mathbf{n}) \psi_S(\mathbf{x} - \epsilon \mathbf{n}) :$, using the fermion operator defined in Eq. (5.3), we find the usual bosonization rule

$$\psi_S^\dagger(\mathbf{x}) \psi_S(\mathbf{x}) \rightarrow (1/L) \delta n_S(\mathbf{x}). \quad (8.2)$$

Notice, however, that the bosonization of the *local charge density operator* $\rho(\mathbf{x}) = \psi^\dagger(\mathbf{x})\psi(\mathbf{x})$ contains, in addition to this contribution, other operators involving fermion operators on different patches, e.g. the charge-density-wave operator that involves patches on opposite sides of the Fermi surface. We will not consider these interesting effects in this work.

With this identification, the interaction Hamiltonian, Eq. (8.1) is bosonized to a quadratic operator

$$H_{\text{int}} = \frac{f_0}{L^2} \int d^2x \sum_{S,T} : \delta n_S(\mathbf{x}) \delta n_T(\mathbf{x}) : \quad (8.3)$$

Thus, the interacting fermionic model is mapped into a system of non-trivially coupled harmonic oscillators.

The bosonization of forward scattering interactions with higher angular momentum, such as, for instance, quadrupolar interactions, is more involved. The reason is that the quadrupole density contains second order covariant derivatives that do not commute. Thus, in order to bosonize them, it is necessary to carefully implement a point-splitting computation on the Fermi surface variables. We postpone this study for a separate future presentation. We finish this section by showing a calculation of collective modes in the simpler, but non-trivial, case of density-density interactions.

A. Collective modes

An instructive application of the bosonization technique in a magnetic field is the computation of Fermi surface collective modes. Let us consider the interacting Hamiltonian

$$H = H_0 + H_{\text{int}} \quad (8.4)$$

where H_0 and H_{int} are given by Eqs. (3.17) and (8.3) respectively. Using the covariant Schwinger algebra, Eq. (3.15), we can write the Heisenberg equation of motion, $\partial_t \delta n_S = -i[H, \delta n_S]$, as

$$\begin{aligned} \frac{\partial \delta n_S(\mathbf{x})}{\partial t} + \mathbf{v}_S \cdot \nabla \delta n_S(\mathbf{x}) - \omega_c \frac{\partial \delta n_S(\mathbf{x})}{\partial \varphi_S} \\ + F_0 \mathbf{v}_S \cdot \nabla \sum_T \delta n_T(\mathbf{x}) = 0, \end{aligned} \quad (8.5)$$

where we defined the dimensionless parameter $F_0 = N(0)f_0$, where $N(0) = k_F/v_F$ is the one-particle density of states at the Fermi surface. Fourier transforming in \mathbf{x} and t we find,

$$\left(\omega - \mathbf{v}_S \cdot \mathbf{q} + i\omega_c \frac{\partial}{\partial \varphi_S} \right) \delta n_S = F_0 (\mathbf{v}_S \cdot \mathbf{q}) \rho(\mathbf{q}, \omega) \quad (8.6)$$

where we have introduced the (long wavelength) density $\rho(\mathbf{q}, \omega) = \sum_S \delta n_S(\mathbf{q}, \omega)$. This equation can be inverted as,

$$\delta n_S(\mathbf{q}, \omega) = \rho(\mathbf{q}, \omega) F_0 \sum_T M_{S,T} \mathbf{v}_T \cdot \mathbf{q}, \quad (8.7)$$

where the Green function $M_{S,T}$ is defined in Eq. (6.10), or equivalently, Eq. (6.12). This is a formal solution, since δn_S is present on both sides of the equation, thorough $\rho(\mathbf{q}, \omega)$. In order to find the self-consistent condition for the collective modes, we sum up Eq. (8.7) over the patch variable S , and we take the continuum limit. Thus, we find

$$\frac{v_F q}{(2\pi)^2} \int_0^{2\pi} d\varphi_S d\varphi_T M(\varphi_S, \varphi_T) \cos \varphi_T = \frac{1}{F_0}. \quad (8.8)$$

For weak magnetic fields, the Green function $M(\varphi_S, \varphi_T)$ is strongly peaked at $\varphi_S \sim \varphi_T$. Then, in that limit, the integral over φ_T can be easily done. At leading order in the magnetic field we obtain,

$$\sum_{\ell=-\infty}^{\infty} \int_0^{2\pi} \frac{d\varphi_S}{2\pi} \frac{\cos \varphi_S}{s - \cos \varphi_S} e^{i2\pi\ell \frac{v_F q}{\omega_c} \cos \varphi_S} = \frac{1}{F_0} \quad (8.9)$$

where we have introduced the usual Fermi liquid parameter $s = \omega/(v_F q)$. Solutions of Eq. (8.9) provide the dispersion relation $\omega(\mathbf{q})$ of the collective modes of the system. The term with $\ell = 0$ is the zero magnetic field condition, found in usual Fermi liquids. The presence of a weak magnetic field produces highly oscillatory components for $\ell \neq 0$. In order to compute corrections to the Fermi liquid result we need to integrate over φ_S . This calculation can be done exactly using the expansion⁵⁵

$$e^{iz \cos \varphi} = \sum_{k=-\infty}^{+\infty} i^k J_k(z) e^{ik\varphi} \quad (8.10)$$

where $J_k(z)$ is the k^{th} order Bessel function of the first kind. By replacing Eq. (8.10) into Eq. (8.9), calculating the angular integrals over φ_S , and taking advantage of the symmetry properties of the Bessel functions, $J_{2k}(z) = J_{-2k}(z)$, $J_{2k+1}(z) = -J_{-(2k+1)}(z)$, we find that Eq.(8.9) takes the form

$$\chi_0(s) \left\{ 1 + 2 \sum_{\ell=1}^{\infty} J_0 \left(2\pi\ell \frac{v_F q}{\omega_c} \right) \right\} + 4 \sum_{n=1}^{\infty} \chi_{2n}(s) \sum_{\ell=1}^{\infty} J_{2n} \left(2\pi\ell \frac{v_F q}{\omega_c} \right) = \frac{1}{F_0} \quad (8.11)$$

where. $\chi_n(s)$ are multipole Fermi liquid susceptibilities

$$\chi_n(s) = \int_0^{2\pi} \frac{d\varphi}{(2\pi)} \frac{\cos \varphi}{s - \cos \varphi} e^{in\varphi} \quad (8.12)$$

When needed, we have analytically continued the variable $s \rightarrow s + i\epsilon \text{sgn}(s)$. For $s > 1$,

$$\chi_n(s) = -\delta_{k,0} + \frac{s}{\sqrt{s^2 - 1}} \left(s - \sqrt{s^2 - 1} \right)^n. \quad (8.13)$$

In the second line of Eq. (8.11), the sum over n can be exactly performed in the asymptotic limit $v_F q/\omega_c \gg 1$. We have found the following expression,

$$\begin{aligned} \sum_{n=1}^{\infty} \chi_{2n}(s) J_{2n} \left(2\pi\ell \frac{v_F q}{\omega_c} \right) = \\ - (1 + \chi_0(s)) \frac{(s - \sqrt{s^2 - 1})^2}{1 + (s - \sqrt{s^2 - 1})^2} J_0 \left(2\pi\ell \frac{v_F q}{\omega_c} \right) \\ + O \left[\left(\frac{\omega_c}{v_F q} \right)^{3/2} \right] \end{aligned} \quad (8.14)$$

Replacing this result into Eq. (8.11), we finally get

$$\chi_0(s) \left\{ 1 + 4 \frac{\sqrt{s^2 - 1}}{s} \sum_{\ell=1}^{\infty} J_0 \left(2\pi\ell \frac{v_F q}{\omega_c} \right) \right\} = \frac{1}{F_0} \quad (8.15)$$

This is a non-trivial algebraic equation for $\omega(\mathbf{q})$. In fact it is a non-analytic equation in terms of the two dimensionless variables, $s = \omega/v_F q$ and $\omega_c/v_F q$. We have solved Eq.(8.15) perturbatively in the parameter $(\omega_c/v_F q)^{1/2}$, considering $s > 1$, and obtained,

$$\begin{aligned} s = \frac{1 + F_0}{\sqrt{(1 + F_0)^2 - F_0^2}} \\ + \Delta(F_0) \left(\frac{\omega_c}{v_F q} \right)^{1/2} \sum_{\ell=1}^{\infty} \frac{e^{-2\pi\ell\epsilon/\omega_c}}{\sqrt{\ell}} \cos \left(\frac{2\pi\ell v_F q}{\omega_c} - \frac{\pi}{4} \right) \\ + O \left[\left(\frac{\omega_c}{v_F q} \right)^{3/2} \right], \end{aligned} \quad (8.16)$$

where we have define the quantity

$$\Delta(F_0) = \frac{4}{\pi} \frac{F_0^3}{(F_0 + 1)[(1 + F_0)^2 - F_0^2]^{3/2}}. \quad (8.17)$$

Equation (8.16) is the main result of this section. It represent the corrections to the zero sound dispersion relation at leading order in $(\omega_c/v_F q)^{1/2}$. Taking the limit $\omega_c \rightarrow 0$ in Eq. (8.16), only survives the first line of the equation, *i. e.*,

$$s = \frac{1 + F_0}{\sqrt{(1 + F_0)^2 - F_0^2}}, \quad (8.18)$$

which is the well known linear dispersion relation of the underdamped Landau zero sound in two dimensions.¹⁴ The second line in Eq. (8.16), is the leading order correction due to the presence of the homogeneous magnetic field. The corrections are highly oscillatory terms whose periods are integer multiples of ω_c/ℓ . The damping factor in the sum over ℓ appears due to the analytic continuation $\omega \rightarrow \omega + i\epsilon \text{sgn}(\omega)$, necessary to properly define the Green function. In the presence of a magnetic field, it is subtle to take the limits $\omega_c \rightarrow 0$ and $\epsilon \rightarrow 0$. If we take $\epsilon \rightarrow 0$ at fixed magnetic field, all harmonics contribute to the sum, producing singularities each time the energy

of the particle-hole pair, $v_F q$, which coincides with the energy of a Landau level. Conversely, if we take the limit $\omega_c \rightarrow 0$ first, all harmonics are damped to zero. In practice, the scale of ϵ is given by temperature, that blurs the finely discretized dense spectrum. Thus, if $\omega_c \lesssim k_B T$, only very few harmonics contribute to the sum, producing a nicely oscillatory behavior. On the other hand, if $\omega_c \gtrsim k_B T$, more and more harmonics contribute to the sum, producing peaks whenever a Landau level coincides with the energy of the particle-hole pair. In the extreme case of $\omega_c \gg k_B T$, equivalent to zero temperature, the result turns out to be discontinuous peaks. However, in this regime, the gaps are important and the Fermi liquid type description breaks down. We depict this behavior in figure 5. Using Eq. (8.16), where we have plotted ω/ω_c , as a function of the dimensionless variable $v_F q/\omega_c$. The straight line is the classical result of Landau, for a typical value of $F_0 = 1$. In 5a, we fixed the damping term $\epsilon/\omega_c = 0.15$. In this case, only the very few first harmonics appreciably contribute to the sum, producing the oscillatory behavior shown in the figure. In Fig. 5b, we fixed the damping term one order of magnitude weaker, $\epsilon/\omega_c = 0.05$, corresponding to a regime with stronger magnetic fields. We observe pronounced peaks for integer values of $v_F q/\omega_c$, *i. e.*, when the energy of the particle hole excitation is commensurate with the energy of a Landau level.

In Fig. 6 we have fixed the value of the particle-hole excitation energy $v_F q$, and have depicted the frequency of the collective mode in terms of the inverse magnetic field ω_c^{-1} , in the same scale, using Eq. (8.16). To draw the picture we chose $F_0 = 1$ and $\epsilon/v_F q = 0.002$. We observe a damped oscillation as we decrease the magnetic field. This behavior is another manifestation of quantum oscillations. However, different from the dHvA effect, it is proper of a Fermi liquid and it is not present in the Fermi gas. Indeed, the amplitude of the oscillations is governed by $\Delta(F_0)$. As we can see from Eq. (8.17), $\lim_{F_0 \rightarrow 0} \Delta(F_0) = 0$ and, in this case, there is no correction to the free dispersion, $\omega = v_F q$.

IX. DISCUSSION AND CONCLUSIONS

In this paper we have presented a bosonization technique that describes a 2D system of fermions at fixed density in a weak magnetic field. We focused on a regime in which there is a large number of Landau levels in a small energy band around the chemical potential. In this way, the system is, in some sense, near a Fermi liquid regime, since the concept of Fermi surface still make sense. The main idea of the bosonization approach is to project the states of the system onto a restricted subspace, spanned by states very near the Fermi surface. At very low temperature, it is assumed that fluctuations of the Fermi surface are responsible for the main properties of a fermionic system. Processes originating from transitions deep inside the Fermi sea are completely irrelevant in this low en-

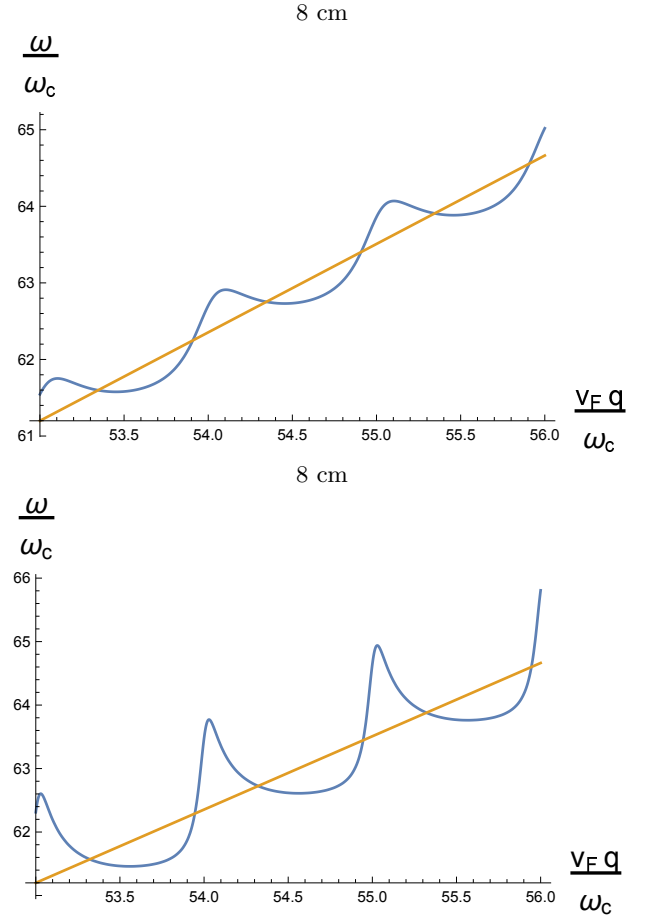


FIG. 5: Dispersion relation for the collective mode described by Eq. (8.16). In both figures we have fixed $F_0 = 1$. The straight line is the classical Landau result for the zero sound mode. In (a), we have fixed $\epsilon/\omega_c = 0.15$, corresponding to a weak magnetic field regime ($\omega_c < k_B T_c$). In (b), we fixed $\epsilon/\omega_c = 0.05$, corresponding with stronger values of the magnetic field ($\omega_c \gtrsim k_B T_c$). The peaks coincide with values of $v_F q/\omega_c$ integers.

ergy regime. Within this perspective, we coarse-grained the fermion operator in small patches of the Fermi surface, in which we can linearize the dispersion relation. Fermi surface deformations can be parametrized by a set of bosonic operators describing the creation of particle-hole excitations in each patch.

We showed that these operators satisfy a magnetic-field-dependent *covariant Schwinger algebra*. In addition to the usual Schwinger term, proportional to the normal derivative of the delta function, there is now a tangential term, proportional to the magnetic field, that mixes neighbor patches. This algebra is one of the important results of the paper, and it is the cornerstone of bosonization construction. The bosonization program is completed by constructing the full fermion operator in terms of bosonic excitations. Unlike the case of bosoniza-

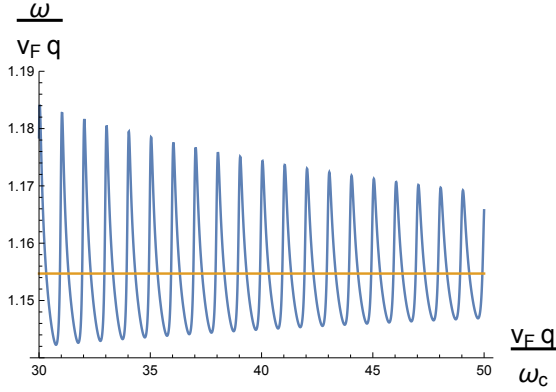


FIG. 6: Energy of the collective mode excitation in units of $v_F q$, for fixed q , as a function of the inverse magnetic field in the same scale, $v_F q/\omega_c$. The curve was depicted with Eq. (8.16) where we have fixed $F_0 = 1$ and $\epsilon/v_F q = 0.002$. The horizontal line is the value of the zero sound energy in the absence of magnetic field, given by Eq. (8.18).

tion of the Fermi liquid, the phase of a fermionic operator on each patch depends on a coherent superposition of bosonic fields all around the Fermi surface. With this construction at hand, we computed the fermionic equal time propagator, showing that it coincides with the one computed with standard procedures. Moreover, we have shown that the low-energy fermionic Hamiltonian can be exactly mapped to a local quadratic bosonic Hamiltonian.

The dynamics of the Fermi surface can be represented by a path integral coherent state formalism. The effective action is written in terms of a set of couple canonical bosonic fields. We showed how to use this formalism to compute bosonic correlation functions. As an example, we have computed the orbital magnetization and showed that the bosonized action correctly captures quantum oscillations responsible for the de Haas-van Alphen effect. The fact that quantum oscillations can be obtained from the bosonized action opens the possibility of computing them at strong coupling. We have also studied the bosonization of forward scattering interactions, encoded in the Landau parameter F_0 . We computed the spectrum of collective modes, in particular, corrections to the Landau zero sound mode. We have shown that the corrections are damped oscillations, in terms of the momentum, and either in term of the inverse magnetic field. These results resemble the problem of quantum oscillations and, in fact, have the same origin. However, this oscillation, in a non-equilibrium property, is completely due to interactions. In fact, the correction vanishes in the limit of $F_0 \rightarrow 0$.

In this work we considered the case of a fluid of spinless fermions at finite density with the main goal to establish the bosonization rules for a system in a uniform magnetic field. We have also discussed some effects of Fermi liquid corrections due to forward scattering interactions. An

important and challenging extension is to apply these ideas and techniques to the more interesting case of 2D Fermi systems at and near quantum criticality. Higher dimensional bosonization was used before to study the quantum phase transition to a nematic state driven by a Pomeranchuk instability.²⁵ Such systems are the simplest examples of non-Fermi liquids. In a separate publication we will study the effects of a uniform magnetic field in these non-Fermi liquid phases of matter and quantum critical points. The great interest of such studies is how non-Fermi liquid behavior affects quantum oscillations and similar magnetic field driven phenomena.

Note: After this work was completed we became aware of a recent paper by Nguyen and Son⁵⁶ on an algebraic approach to the fractional Hall effect. Although the problem that these authors study is formally different than ours, their formalism is in spirit similar to the theory of higher dimensional bosonization of dense Fermi systems that we use here.

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Appendix A: The Fermion Propagator in a uniform magnetic field

The propagator for two-dimensional spinless free fermions in a uniform magnetic field, with p filled Landau levels, can be written as,⁵⁷

$$\begin{aligned}
 iG(\mathbf{x}, \mathbf{y}, t - t') = & \Theta(t - t') \sum_{m=p}^{\infty} \int \frac{dk}{2\pi} e^{-i\omega_m(t-t')} \varphi_{mk}(\mathbf{x}) \varphi_{mk}^*(\mathbf{y}) \\
 & - \Theta(t' - t) \sum_{m=0}^{p-1} \int \frac{dk}{2\pi} e^{-i\omega_m(t-t')} \varphi_{mk}(\mathbf{x}) \varphi_{mk}^*(\mathbf{y}),
 \end{aligned} \tag{A1}$$

where $\omega_m = \omega_c(m + 1/2)$ and the eigenfunctions of the Hamiltonian of Eq. (2.1), in the Landau gauge $A_1 =$

$-Bx_2, A_2 = 0$, are given by

$$\varphi_{mk}(\mathbf{x}) = \sqrt{\frac{1}{2^m m! \sqrt{\pi} \ell_B}} \times e^{ikx_1} e^{-\frac{1}{2} \left(\frac{x_2}{\ell_B} - k \ell_B \right)^2} H_m \left(\frac{x_2}{\ell_B} - k \ell_B \right), \quad (\text{A2})$$

where H_m are the Hermite polynomials of order m .

Integrating over k we find,

$$iG(\mathbf{x}, \mathbf{y}, t - t') = \frac{1}{2\pi \ell_B^2} e^{-\left(\frac{r}{2\ell_B}\right)^2} e^{i\theta_B} \times \left\{ \Theta(t - t') \sum_{m=p}^{\infty} e^{-i\omega_m(t-t')} L_m[(r/2\ell_B)^2] - \Theta(t' - t) \sum_{m=0}^{p-1} e^{-i\omega_m(t-t')} L_m[(r/2\ell_B)^2] \right\}, \quad (\text{A3})$$

where $r = |\mathbf{x} - \mathbf{y}|$, $r_0 = t_x - t_y$,

$$\theta_B(\mathbf{x}, \mathbf{y}) = \frac{1}{2} B(x_1 - y_1)(x_2 + y_2) \quad (\text{A4})$$

with $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$, and L_m are the Laguerre polynomials.

In the weak magnetic field regime, the number of filled Landau levels is huge, $p \gg 1$, and $\epsilon_F = k_F^2/2M \sim \omega_c p$. We are interested in the limit $\omega_c \rightarrow 0$ (or $\ell_B \rightarrow \infty$) and $p \rightarrow \infty$, while keeping the chemical potential constant, $\omega_c p = \epsilon_F = \mu$. To this end, we will use the following asymptotic property of Laguerre polynomials⁵⁸

$$L_n(x) = e^{\frac{1}{2}x} J_0(2\sqrt{nx}), \quad (\text{A5})$$

valid for $n \gg 1$ and $x \gg 1/(4n)$. Here, J_0 is the Bessel function of the first kind of order zero. Using this property we find,

$$iG(\mathbf{x}, \mathbf{y}, t - t') = \frac{1}{2\pi \ell_B^2} e^{i\theta_B} \times \left\{ \Theta(t - t') \sum_{m=p}^{p+\Delta} e^{-i\omega_m(t-t')} J_0[\sqrt{2m}(r/\ell_B)] - \Theta(t' - t) \sum_{m=p-\Delta}^p e^{-i\omega_m(t-t')} J_0[\sqrt{2m}(r/\ell_B)] \right\}, \quad (\text{A6})$$

with $p \gg 1$, and $k_F r \gg 1$. The cut-off $\Delta = (E - \omega_p)/\omega_c$ is the number of Landau levels near the Fermi energy.

To go further we use the Poisson summation formula,

$$\sum_{n=a}^b F(n) = \frac{F(a) + F(b)}{2} + \sum_{\ell=-\infty}^{+\infty} \int_a^b dx F(x) e^{i2\pi \ell x}. \quad (\text{A8})$$

Then, the propagator reads,

$$iG(\mathbf{x}, \mathbf{y}, t - t') = \frac{1}{2\pi \ell_B^2} e^{i\theta_B} \sum_{\ell=-\infty}^{+\infty} \left\{ \Theta(t - t') \int_{m=p}^{p+\Delta} dm e^{i2\pi \ell m} e^{-i\omega_m(t-t')} J_0[\sqrt{2m}(r/\ell_B)] - \Theta(t' - t) \int_{m=p-\Delta}^p dm e^{i2\pi \ell m} e^{-i\omega_m(t-t')} J_0[\sqrt{2m}(r/\ell_B)] \right\} + \frac{1}{2\pi \ell_B^2} e^{i\theta_B} e^{-i\epsilon_F(t-t')} J_0(k_F r) \text{sgn}(t - t'). \quad (\text{A9})$$

Upon the change of variables $m = (1/2)\ell_B^2 k^2$, we obtain

$$iG(\mathbf{x}, \mathbf{y}, t - t') = \frac{1}{2\pi} e^{i\theta_B} \times \sum_{\ell=-\infty}^{+\infty} \left\{ \Theta(t - t') \int_{k_F}^{k_F + \lambda/2} dk k e^{i\pi \ell \ell_B^2 k^2} e^{-i\omega_k(t-t')} J_0(kr) - \Theta(t' - t) \int_{k_F - \lambda/2}^{k_F} dk k e^{i\pi \ell \ell_B^2 k^2} e^{-i\omega_k(t-t')} J_0(kr) \right\} + \frac{1}{2\pi \ell_B^2} e^{i\theta_B} e^{-i\epsilon_F(t-t')} J_0(k_F r) \text{sgn}(t - t'). \quad (\text{A10})$$

We now define $q = k - k_F$ in the first integral and $q = k_F - k$ in the second one. We next use the integral representation of the Bessel function,

$$J_0(x) = \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{ix \cos \varphi} \quad (\text{A11})$$

to obtain

$$iG(\mathbf{x}, \mathbf{y}, t - t') = \frac{k_F}{2\pi} e^{-i\epsilon_F(t-t')} e^{i\theta_B} \sum_{\ell=-\infty}^{+\infty} \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{ik_F r \cos \varphi} \times \left\{ \Theta(t - t') \int_0^{\lambda/2} dq e^{iq(r \cos \varphi - v_F(t-t') + 2\pi \ell \ell_B^2 k_F)} - \Theta(t' - t) \int_0^{\lambda/2} dq e^{-iq(r \cos \varphi - v_F(t-t') + 2\pi \ell \ell_B^2 k_F)} \right\} + O(\ell_B^{-2}). \quad (\text{A12})$$

Implementing a smooth cut-off for the momentum integrals we finally find,

$$iG(\mathbf{x}, \mathbf{y}, t - t') = \frac{ik_F}{2\pi} e^{-i\epsilon_F(t-t')} e^{i\theta_B} \sum_{\ell=-\infty}^{+\infty} \int_0^{2\pi} \frac{d\varphi}{2\pi} \frac{e^{ik_F r \cos \varphi}}{r \cos \varphi - v_F(t - t) + 2\pi \ell \ell_B^2 k_F + i\alpha \text{sgn}(t - t')}, \quad (\text{A13})$$

which coincides with Eq. (2.4).

Appendix B: Density of States

The density of states can be computed from the imaginary part of the self-correlation function as,

$$N(\omega) = -\text{sgn}(\omega - \epsilon_F) \frac{1}{\pi} \text{Im} G_F(\mathbf{x}, \mathbf{x}, \omega), \quad (\text{B1})$$

where

$$\begin{aligned} G_F(\mathbf{x}, \mathbf{x}, \omega) &= \int_{-\infty}^{+\infty} dt G_F(\mathbf{x}, \mathbf{x}, t) e^{i\omega t} \\ &= \frac{k_F}{2\pi} \sum_{\ell=-\infty}^{+\infty} \int_0^{2\pi} d\varphi \int_{-\infty}^{+\infty} dt G_{\ell, \varphi}(0, t) e^{-i(\epsilon_F - \omega)t}. \end{aligned} \quad (\text{B2})$$

Notice that $N(\omega)$ is Gauge invariant since $\theta_B(\mathbf{x}, \mathbf{x}) = 0$ in any Gauge. By computing the time integral in the complex plane we find,

$$\begin{aligned} G_{\ell, \varphi}(\omega) &= \frac{-1}{v_F} \int_{-\infty}^{+\infty} \frac{e^{-i(\epsilon_F - \omega)t}}{t - 2\pi\ell \frac{\ell_B^2 k_F}{v_F} + i\alpha \text{sgn } t} dt \\ &= \frac{-2\pi i}{v_F} e^{-i2\pi\ell \frac{(\omega - \epsilon_F)}{\omega_c}} [\Theta(\omega - \epsilon_F)\Theta(-\ell) - \Theta(\epsilon_F - \omega)\Theta(\ell)]. \end{aligned} \quad (\text{B3})$$

Replacing this expression into Eq. (B1) we obtain

$$N(\omega - \epsilon_F) = N(0) \sum_n \delta_{n, \frac{\omega - \epsilon_F}{\omega_c}}, \quad (\text{B4})$$

where $N(0) = k_F/v_F$ is the density of states at the Fermi surface of a two-dimensional Fermi gas. Equation (B4) is just another consistency check of the asymptotic propagator, Eqs. (2.4) and (2.7).

Appendix C: Green function of the covariant derivative

The covariant derivative

$$\tilde{D}_S = \hat{\mathbf{n}}_S \cdot \mathbf{q} - \frac{i}{\ell_B^2 k_F} \frac{\partial}{\partial \varphi_S}, \quad (\text{C1})$$

as well as, its inverse $D_{S,T}^{-1}(\mathbf{q})$ play a central role in the bosonization procedure. For this reason, it is useful to have a deeper understanding of their properties. There are two simple, however instructive, limits: $\ell_B \rightarrow \infty$ (with $\mathbf{q} \neq 0$), and $\mathbf{q} \rightarrow 0$ (with $B \neq 0$). In the former case, $\tilde{D}_S = \hat{\mathbf{n}}_S \cdot \mathbf{q}$. The only solution of the homogeneous equation $\tilde{D}_S \psi = 0$, is the trivial one $\psi_0 = 0$. Then, given properly initial conditions, the Green function is uniquely determined and it is given by $D_{S,T}^{-1} = \delta_{S,T}/\hat{\mathbf{n}}_S \cdot \mathbf{q}$. The latter case is also quite simple. When $\mathbf{q} = 0$, the periodic solution of $\tilde{D}_S(0)\psi = 0$ is trivial, $\psi = 0$, and then, the Green function is well defined. It is given by

$$D_{ST}^{-1}(0) = i \frac{\ell_B^2 k_F}{2} \sum_n \text{sgn}(\varphi_S - \varphi_T - 2n\pi). \quad (\text{C2})$$

However, in the presence of a finite \mathbf{q} particle-hole pair excitation and a finite magnetic field $\ell_B \neq \infty$, the homogeneous equation $\tilde{D}_S(\mathbf{q})\psi = 0$ has a non-trivial periodic solution, $\psi_0(\mathbf{q}, \varphi_S)$. Therefore, the operator $\tilde{D}_S(q)$ has a zero mode and, rigorously, its inverse does not exist.

Then, we can look for a modified Green function by limiting the functional space to be orthogonal to the zero mode. To be concrete, consider one wants to solve the following inhomogeneous first order linear equation,

$$\tilde{D}_S(\mathbf{q})\psi(\mathbf{q}, \varphi_S) = f(\mathbf{q}, \varphi_S) \quad (\text{C3})$$

where $f(\mathbf{q}, \varphi_S)$ is a smooth function of \mathbf{q} and φ_S . Since the homogeneous equation

$$\tilde{D}_S(\mathbf{q})\psi_0(\mathbf{q}, \varphi_S) = 0, \quad (\text{C4})$$

has a non-trivial solution then, for any $f(\varphi_S)$, Eq. (C3) has no solution. However, it has an infinite number of solutions whether the function $f(\varphi_S)$ is orthogonal to the zero mode⁵⁹ ψ_0 , *i. e.*

$$\int_0^{2\pi} d\varphi_S f^*(\mathbf{q}, \varphi_S) \psi_0(\mathbf{q}, \varphi_S) = 0. \quad (\text{C5})$$

Restricting the domain to the orthogonal functional space, the solutions of Eq. (C3) can be written as

$$\psi(\varphi_S) = \int_0^{2\pi} d\varphi_T G_m(\varphi_S, \varphi_T) f(\varphi_T) \quad (\text{C6})$$

in which we have defined the *modified Green function* $G_m(\varphi_S, \varphi_T)$ satisfying

$$\begin{aligned} \left\{ \hat{\mathbf{n}}_S \cdot \mathbf{q} - \frac{i}{\ell_B^2 k_F} \frac{\partial}{\partial \varphi_S} \right\} G_m(\mathbf{q}, \varphi_S, \varphi_T) &= \delta_p(\varphi_S - \varphi_T) \\ &- \frac{1}{2\pi} \psi_0^*(\varphi_S) \psi_0(\varphi_T). \end{aligned} \quad (\text{C7})$$

It is immediate to show that the inhomogeneity of Eq. (C7) is orthogonal to the zero mode ψ_0 .

We can now solve Eq. (C7) for $\varphi_S \neq \varphi_T + 2n\pi$, obtaining

$$\begin{aligned} G_m(\varphi_S, \varphi_T) &= i\ell_B^2 k_F \psi_0(\varphi_S) \times \\ &\times \left\{ \psi_0^*(\varphi_T) - \frac{1}{2\pi} \psi_0(\varphi_T) \int_{\varphi_T}^{\varphi_S} [\psi_0^*(\varphi')]^2 d\varphi' \right\} \end{aligned} \quad (\text{C8})$$

(to simplify notation we are not displaying the momentum \mathbf{q} explicitly). We complete the calculation by imposing that

$$\lim_{\varphi_T \rightarrow \varphi_S^+} G_m(\varphi_S, \varphi_T) - \lim_{\varphi_T \rightarrow \varphi_S^-} G_m(\varphi_S, \varphi_T) = i\ell_B^2 k_F \quad (\text{C9})$$

The result is

$$G_m(\varphi_S, \varphi_T) = \frac{i\ell_B^2 k_F}{2} \text{sgn}(\varphi_S - \varphi_T) \psi_0(\varphi_S) \times \quad (\text{C10})$$

$$\times \left\{ \psi_0^*(\varphi_T) - \frac{1}{2\pi} \psi_0(\varphi_T) \int_{\varphi_T}^{\varphi_S} [\psi_0^*(\varphi')]^2 d\varphi' \right\} \quad (\text{C11})$$

with $\psi_0(0) = 1$. By solving Eq. (C4), we see that the zero mode with the require boundary conditions in given by

$$\psi_0(\varphi_S, \mathbf{q}) = e^{-i\ell_B^2 k_F (\mathbf{q} \times \hat{\mathbf{n}}_S)} . \quad (\text{C12})$$

The orthogonality relation with respect to the zero mode has important consequences in the form of the Green function. In fact, we are interest in the response to reasonable smooth and isotropic electromagnetic fields. In this cases, $f(\varphi_S) \sim f = \text{constant}$. As a consequence, the inhomogeneous equation of motion will have solutions provided

$$\int_0^{2\pi} \psi_0(\varphi) d\varphi = 2\pi J_0(q\ell_B^2 k_F) = 0 \quad (\text{C13})$$

Thus, the momentum should be quantized in units of $1/\ell_B^2 k_F$ and are given by the zeros of the Bessel function. At low magnetic fields, the argument of the Bessel function is large and, from the asymptotic expression we can infer that the zeros are equally spaced. We can implemented this condition introducing the quantization in the following way

$$\begin{aligned} G(\mathbf{q}, \varphi_S, \varphi_T) &= \sum_n \delta_{n, q\ell_B^2 k_F} G_m(\mathbf{q}, \varphi_S, \varphi_T) \\ &= \sum_\ell e^{i2\pi\ell(\frac{\mathbf{q} \cdot \mathbf{v}_F}{\omega_c})} G_m(\mathbf{q}, \varphi_S, \varphi_T) \end{aligned} \quad (\text{C14})$$

Finally, in order to compute D_{SS}^{-1} we need to coarse grain the Green function in the patch. For concreteness

$$D_{SS}^{-1} = \int_{\varphi_S - \Lambda/2k_F}^{\varphi_S + \Lambda/2k_F} G(\varphi_S, \varphi_T) d\varphi_T . \quad (\text{C15})$$

Disregarding terms of higher order in a $1/\ell_B k_F$ expansion, we obtain,

$$D_{SS}^{-1}(q) = \sum_\ell \frac{e^{i2\pi\ell(\mathbf{v}_S \cdot \mathbf{q}/\omega_c)}}{\hat{\mathbf{n}}_S \cdot \mathbf{q}} . \quad (\text{C16})$$

We use this expression in , Eq. (5.16), to compute the fermion propagator. Eq. (C16) also coincides with the zero frequency limit of the diagonal Green function

$$D_{SS}^{-1}(\mathbf{q}) = - \lim_{\omega \rightarrow 0} G_{SS}(\omega, \mathbf{q}) \quad (\text{C17})$$

computed in section VI.

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