Superconductivity in the doped t–J model: Results for four-leg cylinders
Hong-Chen Jiang, Zheng-Yu Weng, and Steven A. Kivelson
Phys. Rev. B 98, 140505 — Published 19 October 2018
DOI: 10.1103/PhysRevB.98.140505
Superconductivity in the doped $t$-$J$ model: results for four-leg cylinders

Hong-Chen Jiang,$^1$* Zheng-Yu Weng,$^2$ and Steven A. Kivelson$^3$

$^1$Stanford Institute for Materials and Energy Sciences, SLAC and Stanford University, Menlo Park, California 94025, USA
$^2$Institute for Advanced Study, Tsinghua University, Beijing, China
$^3$Physics Department, Stanford University, Stanford, CA, 94305, USA

(Dated: October 1, 2018)

We report a density-matrix renormalization group study of the lightly doped $t$-$J$ model on a 4-leg cylinder with doped hole concentrations per site $\delta = 5\% \sim 12.5\%$. By keeping an unusually large number of states and long system sizes, we are able to accurately document the interplay between superconductivity, spin and charge-density-wave orders. The long-distance behavior is consistent with that of a Luther-Emery liquid with a spin-gap and power-law charge-density-wave and superconducting correlations. This is the widest $t$-$J$ or Hubbard system in which power-law superconducting correlations have been established.

Moreover, within numerical uncertainty, as theoretically expected of a LE liquid, $K_cK_{sc} = 1$ (Fig.3) and the central charge, $c$, extracted from the scaling of the entanglement entropy, is $c = 1$ (inset of Fig.6). The SC and CDW correlations are invariant with respect to the $C_4$ symmetry of rotations about the axis of the cylinder. The SC correlations have a “d-wave-like” form factor in that the sign of the pair-field is opposite on bonds perpendicular to and along the cylinder ($Y$-directed and $X$-directed bonds). However, this is not a statement of symmetry, and indeed there is an almost equal in strength admixture of an “extended s-wave” component with the consequence that the pair-field amplitude on $Y$-bonds is two orders of magnitude larger than on $X$-bonds. (See Fig.2)

For all the dopings studied, $K_{sc} < 2$ and $K_c < 2$, which (assuming the usual emergent Lorentz invariance) implies that both the corresponding susceptibilities diverge as $T \to 0$, as $T^{-(2-K_{sc})}$ and $T^{-(2-K_c)}$ respectively. As far as we know, this is the first demonstration of power-law SC correlations on such a wide $t$-$J$ or Hubbard cylinder. As shown in Fig.3, $K_{sc}$ is an increasing function of $\delta$ and $K_c$ a decreasing function, so that the SC susceptibility is more divergent for $\delta < 0.1$ and the CDW is more divergent for $\delta > 0.1$.

In a previous study$^{16}$, we explored the same model over a wider range of parameters in which the primary focus was to explore the extent to which the nature of the ground-state depends on “microscopic details.” For the special case on which we focus here, these earlier results are generally consistent with our present results. However, the longer system sizes and the larger number of states used in the present study increase our ability to distinguish exponential-decay correlations, power-law (quasi-long-range) and true long-range order. In particular, what was previously tentatively identified as SC with a long but finite correlation length, we now identify as quasi-long-ranged SC order, albeit with exactly the previously determined form factor.

**Model and Method:** We study the hole-doped $t$-$J$...
The open squares and circles denote numerical data, while the case at doping levels \( i \)erator on site \( L \) and length \( L \). Here, we focus on cylinders with circumference \( N \) and a lattice spacing of unity. Thus, unless stated otherwise, we take periodic boundary conditions unless \( \delta \leq 50 \% \). The parameters \( t \) and \( J \) are the electron hopping integral and the spin superexchange interactions between NN sites. We take the lattice geometry to be cylindrical and a lattice spacing of unity. Thus, unless stated otherwise, we take periodic boundary conditions in the \( \hat{y} = (0, 1) \) direction and open in the \( \hat{x} = (1, 0) \) direction, although for comparison we also consider the case of anti-periodic boundary conditions corresponding to a half-quantum of flux threaded along the cylinder. Here, we focus on cylinders with circumference \( L_y = 4 \) and length \( L_x \). There are \( N = L_x \times L_y \) lattice sites and \( N_e \leq N \) electrons. The concentration of “doped holes” is defined as \( \delta = N_h/N \), where \( N_h = N - N_e \).

For the present study, we focus on the lightly doped case at doping levels \( \delta = 5\% \sim 12.5\% \) on cylinders with length up to \( L_x = 128 \). We set \( J = 1 \) as the energy unit and report results for \( t = 3 \). We keep the total magnetization fixed at zero and perform around 60 sweeps and keep up to \( m = 150000 \) states in each DMRG block with a typical truncation error \( \epsilon \lesssim 1 \times 10^{-7} \). This leads to excellent convergence for our results when extrapolated to \( m = \infty \) limit. In all cases, but especially when computing SC correlations, it proves essential to keep very large \( m \) and to analyze the \( m \rightarrow \infty \) seriously, and in some cases, it is necessary to go to system sizes much longer than \( L_x = 48 \) in order to observe the correlations that arise in the \( L_x \rightarrow \infty \) limit. Further numerical details are presented in the Supplemental Material.}

**Theoretical expectations:** In a LE liquid phase, there is a single gapless spinless bosonic mode with linear dispersion (emergent Lorentz symmetry) - *i.e.* it is asymptotically equivalent to a 1+1 dimensional confor-
normal field theory (CFT) with $c = 1$. At long-distances the density-density correlation oscillates with a well-defined wave-vector $Q$ and decays with a power-law given by the Luttinger exponent $K_{c}$, while the dual SC correlation exhibits non-oscillatory power-law decay with exponent $K_{sc} = 1/K_{c}$. Because there is a spin-gap, spin correlations fall exponentially with a finite correlation length $\xi_{s}$, but one can still identify a wave-vector $Q_{sdw}$ which characterizes the oscillations of the SDW correlations.

These properties can be extracted in various ways from numerical data. Because the CDW is pinned by the cylinder ends, an effective method to study the CDW correlations is to compute the charge density modulations in the middle region of a finite cylinder, $\langle \hat{n}_{i} \rangle \approx (1 - \delta) + A_{sdw}(L) \cos(Qx_{i} + \theta)$ for $x_{i}$ near $Lx/2$. The SC correlation is determined from the long-distance behavior of the SC correlator $\Phi_{\alpha,\beta}(x)$ defined in Eq. (3).

The expectation is that the decay of these quantities is governed by the appropriate exponents,

$$A_{sdw}(L) \propto L^{K_{sc}/2} \text{ and } \Phi_{\alpha,\beta}(x) \propto |x|^{-K_{sc}},$$

where the second relation applies for displacements along the cylinder $1 \ll |x| \ll L_{x}$. Similarly, $Q_{sdw}$ and $\xi_{sdw}$ can be extracted directly from the decay of the density-density correlation near the cylinder ends (see Supplemental Material).22

**CDW correlations:** To describe the charge density properties of the system, we define the local rung density operator as $\hat{n}(x) = \frac{1}{N_{y}} \sum_{y=1}^{L_{y}} \hat{n}(x,y)$ and its expectation value as $n(x) = \langle \hat{n}(x) \rangle$. Fig.1 shows $n(x)$ in a central portion of cylinders with $L_{x} = 96$ for $\delta = 8.33\%$ and $\delta = 12.5\%$. Here, a stripe pattern with wavelength $\lambda = 1/2\delta$ is found, i.e., $\lambda = 4$ for $\delta = 12.5\%$, consistent with previous studies.16,24 Similar behavior (not shown) is found at other doping levels. Fig.1(b) shows examples of finite-size scaling of $A_{sdw}$ as a function of $L_{x}$. In the double-logarithmic plot, our results for all doping levels are approximately linear, which suggests that $A_{sdw}(L_{x})$ decays with a power-law and vanishes as $L_{x} \to \infty$. The exponent $K_{c}$, which is shown in Fig.3, was obtained by fitting the data points using Eq.(2). $K_{c}$ can also be obtained directly from the decay of the density-density correlation near the cylinder ends (see Supplemental Material).22

**Superconducting correlation:** Since the ground state of the system with even an number of doped holes is always found to have spin 0, we will focus on spin-singlet pairing. A diagnostic of SC order is the pair-field correlator defined as

$$\Phi_{\alpha,\beta}(x) = \frac{1}{L_{y}} \sum_{y=1}^{L_{y}} \langle \Delta_{\alpha}(x,y) \Delta_{\beta}(x+y) \rangle.$$  (3)

Here the spin-singlet pair-field creation operator is $\Delta_{\alpha}(x,y) = \frac{1}{\sqrt{2}} [c_{\alpha}^{\dagger}(x, y) c_{\alpha}(x, y) + c_{\alpha}(x, y) c_{\alpha}^{\dagger}(x, y)]$, where bond orientations are designated $\alpha = \hat{x}, \hat{y}$, $(x, y)$ is the reference bond indicated by the red oval shown in Fig.1, and $x$ is the displacement in the $\hat{x} = (1, 0)$ direction.

**Spin-spin correlation:** To describe the magnetic properties of the ground state, we have also calculated the spin-spin correlation functions defined as

$$F(x) = \frac{1}{L_{y}} \sum_{y=1}^{L_{y}} \langle |\vec{S}_{x+y} \cdot \vec{S}_{x}| \rangle.$$  (4)

Here $\vec{S}_{x,y}$ denotes the spin operator on site $i = (x,y)$, $(x, y)$ is the reference site indicated by the red oval shown in Fig.1, and $x$ is the displacement in the $\hat{x} = (1, 0)$ direction. As we did for $A_{sdw}$ and $\Phi_{yy}$, we first extrapolate $F(L_{x}/2)$ to the limit $m = \infty$, and then analyze the functional dependence of the result on $L_{x}$. As shown in Fig.4, $F(L_{x}/2)$ decays exponentially with $L_{x}$, i.e., $F(L_{x}/2) \propto e^{-L_{\xi_{s}}/2}$. The corresponding correlation length is $\xi_{s} = 4 \sim 5$ lattice spacings. We conclude that the spin correlations are short-ranged and consequently that there is a spin-gap.

**Anti-periodic boundary condition:** We have also considered cylinders with anti-periodic boundary condition (ABC) in the $\hat{y}$ direction in order to test the extent to which our results are representative of the 2D limit. As shown in Fig.5 and previous studies16, the influence of changing boundary condition around the cylinder is significant. For example, the ground state of short cylinders with length $L_{x} \leq 48$, e.g., the $L_{x} = 32$ cylinder in...
Fig. 5: (Color online) The charge density profile $n(x)$ at $\delta = 12.5\%$ and ABC in the $\hat{y}$ direction for $L_x = 32$ and $L_x = 160$ cylinders.

Fig. 6: (Color online) Von Neumann entanglement entropy $S$ with $\delta = 8.33\%$, 10% and 12.5%. Inset: The extracted central charge $c$ as a function of $\delta$. Dashed line marks $c = 1$.

It is still unclear how the interplay between SC and CDW order should be expected to evolve with increasing cylinder circumference $L_y$. This uncertainty is exacerbated by the large number of nearly degenerate ground-state phases that were found previously$^{15}$ to be stabilized by relatively small changes in the microscopic parameters of the model. The subtlety of the interplay between multiple phases is illustrated by changing the boundary conditions on the electronic wave-functions from periodic to anti-periodic. As shown in Fig. 5, on shorter cylinders (e.g. $L_x = 32$), a distinct CDW state with $Q = 2\pi\delta$ is stabilized. This state is reminiscent of the “filled” stripes found in Hartree-Fock calculations$^{27-29}$ (where it is accompanied by long-range SDW order) and using various approximate methods$^4$ used in studies of the 2D Hubbard model$^{30}$. In the present case, we find that while even for much longer flux-pierced cylinders, while the filled stripe state is observable locally for a finite region near the ends of the cylinders, far from the ends the CDW correlations have the same $Q = 4\pi\delta$ ordering vector as in the fluxless cylinder.

One big question is the fate of the magnetic correlations in the 2D limit. For $\delta = 0$, on theoretical grounds$^{30,31}$ we know that $\xi_s$ should diverge with $L_y \rightarrow \infty$ since the ground-state of the spin-1/2 Heisenberg model is magnetically ordered in 2D. The shorter correlation lengths of the doped systems suggests, but does not establish, that long-range antiferromagnetic order is unlikely to persist in 2D for even relatively modest values of $\delta$.

Acknowledgement: We would like to thank T. Devoreaux, D. J. Scalapino, J. Tranquada, J. Zaanen, A. Broido, Y. F. Jiang and J. Dodaro for insightful discussions. This work was supported by the Department of Energy, Office of Science, Basic Energy Sciences, Materials Sciences and Engineering Division, under Contract DE-AC02-76SF00515.
Electronic address: hcjiang@stanford.edu

33 Such a state was also found for 4-leg cylinders with t' ≈ 0.06 – 0.08t in Ref.16.