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Time-reversal and spatial reflection symmetry localization anomalies in (2+1)D topological phases of matter

Maissam Barkeshli[1,2] and Meng Cheng[3]

[1]*Department of Physics, Condensed Matter Theory Center,*
University of Maryland, College Park, Maryland 20742, USA
[2]*Joint Quantum Institute, University of Maryland, College Park, Maryland 20742, USA*
[3]*Department of Physics, Yale University, New Haven, CT 06511-8499, USA*

We study a class of anomalies associated with time-reversal and spatial reflection symmetry in (2+1)D bosonic topological phases of matter. In these systems, the topological quantum numbers of the quasiparticles, such as the fusion rules and braiding statistics, possess a $\mathbb{Z}_2$ symmetry which can be associated with either time-reversal (denoted $\mathbb{Z}_2^{\mathbf{T}}$) or spatial reflections. Under this symmetry, correlation functions of all Wilson loop operators in the low energy topological quantum field theory (TQFT) are invariant. However, the theories that we study possess a severe anomaly associated with the failure to consistently localize the symmetry action to the quasiparticles, precluding even defining a consistent notion of symmetry fractionalization in such systems. We present simple sufficient conditions which determine when $\mathbb{Z}_2^{\mathbf{T}}$ symmetry localization anomalies exist in general. We present an infinite series of TQFTs with such anomalies, some examples of which include $\mathrm{USp}(4)_2$ Chern-Simons (CS) theory and $\mathrm{SO}(4)_4$ CS theory. The theories that we find with these $\mathbb{Z}_2^{\mathbf{T}}$ anomalies can all be obtained by gauging the unitary $\mathbb{Z}_2$ subgroup of a different TQFT with a $\mathbb{Z}_4^{\mathbf{T}}$ symmetry. We further show that the anomaly can be resolved in several distinct ways: (1) the true symmetry of the theory is $\mathbb{Z}_4^{\mathbf{T}}$, or (2) the theory can be considered to be a theory of fermions, with $\mathbf{T}^2 = (-1)^{N_f}$ corresponding to fermion parity. Finally, we demonstrate that theories with the $\mathbb{Z}_2^{\mathbf{T}}$ localization anomaly can be compatible with $\mathbb{Z}_2^{\mathbf{T}}$ if they are "pseudo-realized" at the surface of a (3+1)D symmetry-enriched topological phase. The "pseudo-realization" refers to the fact that the bulk (3+1)D system is described by a dynamical $\mathbb{Z}_2$ gauge theory and thus only a subset of the quasi-particles are truly confined to the surface.

I. INTRODUCTION

There has recently been immense progress in understanding the interplay of global symmetries and topological degrees of freedom in physics. From the perspective of condensed matter physics, this has led to advances in our understanding of the distinct possible gapped quantum phases of matter by providing the theoretical framework for describing their universal long-wavelength properties and leading to a host of topological invariants that can distinguish such phases. On the other hand, many of these developments can be viewed entirely within the framework of quantum field theory, and have led to advances in our understanding of global symmetries in topological quantum field theory.

In two and higher spatial dimensions, the study of topological phases of matter with global symmetries is still in progress. Even without any global symmetry, gapped quantum systems can still form distinct phases of matter, characterized by their topological order. These states are distinguished by various exotic properties, including topologically non-trivial quasiparticle excitations with fractional or non-Abelian braiding statistics, robust topological ground state degeneracies, and protected gapless edge modes.[1–3]

The intrinsic topological order in (2+1)D states is believed to be fully characterized by two objects: (1) the chiral central charge c_- of the phase, which describes the chiral energy transport along the (1+1)D boundary of the system, and (2) an algebraic theory $\mathcal{C}$, known as a unitary modular tensor category (UMTC),[4, 5] which encapsulates the topological properties of the quasiparticles, such as their topological spins, fusion rules, and braiding transformations.

In the presence of a global symmetry group G, it is impor-

tant to distinguish two types of phases: (1) invertible [6, 7], or short-range entangled states,[8] and (2) long-range entangled, topologically ordered states. Invertible states have the property that given the state, there is an "inverse" state which, when the two are combined together, can be transformed into a trivial product state by a finite-depth (in the limit of infinite system size) local unitary quantum circuit (or, equivalently, by adiabatically tuning the parameters of the Hamiltonian without closing the bulk energy gap). In (2+1)D, these correspond to cases where the UMTC $\mathcal{C}$ is trivial. A special class of invertible states are symmetry-protected topological (SPT) states [9–13]. SPT states have the property that the state can be transformed into a product state by a finite-depth local unitary quantum circuit that breaks the G symmetry [8]; a non-trivial SPT state cannot be transformed into a product state by a G-symmetric finite-depth local unitary quantum circuit.

Long-range entangled, or topologically ordered, states cannot be transformed into a product state by any finite-depth local unitary quantum circuit, even in the absence of any global symmetry. In the presence of a global symmetry group G, the class of topologically ordered states is refined into symmetry-enriched topological (SET) states [1, 14–24]. Different SETs with the same intrinsic topological order differ in the way the global symmetry interplays with the topological order. This leads to different ways that the topologically non-trivial quasiparticles can carry fractional quantum numbers of the symmetry group [14, 16, 19, 21], and different topological properties of symmetry defects [19–22].

In Ref. 19, a systematic theoretical framework for characterizing symmetry-enriched topological phases was presented. Each (2+1)D topological phase has a group of symmetries

(possibly emergent), denoted $\mathrm{Aut}(\mathcal{C})$, and which we refer to as the group of *topological symmetries*. This is the group of symmetries of the long wavelength effective topological quantum field theory (TQFT). It consists of permutations of the anyon types which keeps their topological spins, fusion rules, and braiding statistics invariant (up to certain complex conjugations that are required for space-time parity reversing symmetries). Even in the absence of a global symmetry G, a topological phase of matter can have a non-trivial $\mathrm{Aut}(\mathcal{C})$, which describes the group of *emergent* symmetries of the topological quantum numbers of long wavelength degrees of freedom in the system. For example, for the $1/m$ Laughlin fractional quantum Hall (FQH) states, this includes the transformation which interchanges quasiparticles with quasiholes. For a bilayer FQH system consisting of two independent $1/m$ Laughlin FQH states, this includes the transformation which interchanges quasiparticles from different layers. For $\mathbb{Z}_2$ quantum spin liquids, this includes electric-magnetic duality, which interchanges the $\mathbb{Z}_2$ gauge charges (spinons) with the $\mathbb{Z}_2$ fluxes.[25]

The action of a global symmetry group G on the long wavelength effective TQFT is characterized first by a group homomorphism

$$[\rho] : G \to \mathrm{Aut}(\mathcal{C}). \tag{1}$$

$[\rho_{\mathbf{g}}] \in \mathrm{Aut}(\mathcal{C})$ describes how a given symmetry group element $\mathbf{g} \in G$ permutes the anyons of the system. $[\rho_{\mathbf{g}}]$ determines an action of G such that all closed anyon diagrams in the UMTC are invariant; that is, all correlation functions of Wilson loop operators in the effective TQFT description are invariant under the symmetry action.

In Ref. 19, it was shown that despite the fact that $[\rho]$ appears to define an allowed symmetry action for G in the TQFT because all correlation functions will be invariant under G, the symmetry can have a certain severe anomaly. This anomaly is associated with the inability to localize the action of the symmetry to the location of the quasiparticles in a way that is consistent with associativity of the group action. Consequently, we refer to this as a "symmetry-localization" anomaly. As we review in the subsequent section, the map $[\rho]$ defines an element $[\mho] \in \mathcal{H}^3_{[\rho]}(G, \mathcal{A})$. Here $\mathcal{A}$ is a finite Abelian group associated with the Abelian quasiparticles of $\mathcal{C}$, which form a group under fusion. $\mathcal{H}^3_{[\rho]}(G, \mathcal{A})$ is the third group cohomology of G with coefficients in $\mathcal{A}$. The subscript $[\rho]$ indicates that the cohomology depends on the action of G on $\mathcal{A}$ through $[\rho]$.

As discussed in detail in Ref. 19, when $[\mho]$ vanishes, then it is possible to consistently define a notion of symmetry fractionalization. This specifies how quasiparticles carry fractional quantum numbers of the symmetry group G; different possible symmetry fractionalization classes are related to each other by elements in $\mathcal{H}^2_{[\rho]}(G, \mathcal{A})$. However certain symmetry fractionalization classes may themselves be anomalous, in the sense that they cannot exist in purely (2+1)D, but can exist at the surface of a (3+1)D SPT state.[21, 26–40] These may be referred to as anomalous symmetry fractionalization classes, or as "SPT anomalies" because of the connection to the surface of (3+1)D SPTs. Using the language of the high energy

field theory literature, these are examples of 't Hooft anomalies in TQFTs. For space-time reflection symmetries which square to the identity, a general understanding of how to detect such anomalies was presented in Ref. 41 by studying the theory on non-orientable space-time manifolds.[42] For unitary internal or lattice translation symmetries, a general understanding was developed in Ref. 19 and 43 by solving consistency equations for the algebraic theory of symmetry defects.[44].

For unitary internal (on-site) symmetries, known examples of the $\mathcal{H}^3$ symmetry localization anomaly occur for $G = \mathbb{Z}_2$ and are associated with discrete gauge theories with gauge group $\mathbb{D}_{20}$ or $\mathbb{D}_{16}$ (the dihedral groups with 20 and 16 elements, respectively).[19, 45] Mathematically, the $\mathcal{H}^3$ obstructions in these examples have their roots in the theory of group extensions;[46] more recently, these obstructions appeared in the category theory literature, in the context of extending a fusion category by a group G.[47]

In this paper, we present and study in detail examples of the $\mathcal{H}^3$ anomaly for space-time parity odd symmetries, which include anti-unitary symmetries such as time-reversal, or unitary symmetries such as spatial reflections. Specifically, we consider cases where time-reversal symmetry $\mathbf{T}$ satisfies $\mathbf{T}^2 = \mathbb{1}$, or spatial reflection $\mathbf{R}$ satisfies $\mathbf{R}^2 = \mathbb{1}$. We refer to these symmetry groups as $\mathbb{Z}_2^{\mathbf{T}}$ and $\mathbb{Z}_2^{\mathbf{R}}$, where the superscript denotes the fact that the symmetry generator is anti-unitary or reverses the parity of space. These types of symmetries appear to be beyond what was considered in the relevant mathematical literature, and the obstructions we find are not directly related to group extension obstructions of a gauge group, as in the previously known examples.

The primary purpose of this work is to develop a deeper understanding of when and why the $\mathcal{H}^3$ symmetry localization anomaly occurs. As we review below, the $\mathcal{H}^3$ anomaly $[\mho]$ can be explicitly computed from $[\rho]$ and the F and R symbols of $\mathcal{C}$, where the F and R symbols are certain consistent data that specify the fusion and braiding properties of the anyons. However obtaining the F and R symbols of $\mathcal{C}$ is often tedious and computationally prohibitive. It is thus desirable to have a diagnostic for the presence of the $\mathcal{H}^3$ anomaly from only the modular data of the theory (the modular S matrix and topological spins). Here we provide such a set of constraints that, if violated, are sufficient to detect the existence of an $\mathcal{H}^3$ symmetry-localization anomaly.

Recently, several infinite series of time-reversal invariant TQFTs were found by studying level-rank duality in Chern-Simons theories.[48] The series of relevance to this paper are $\mathrm{USp}(2N)_N$ for N even, and $\mathrm{SO}(N)_N$ for N a multiple of 4. We show that $\mathrm{USp}(4)_2$ and $\mathrm{SO}(4)_4$ are both part of an infinite family of theories with $\mathbb{Z}_2^{\mathbf{T}}$ symmetry-localization anomalies. We further show that in these theories, the anomaly can be resolved in several distinct ways: (1) The presence of the $\mathcal{H}^3$ anomaly can be interpreted to mean that the symmetry of the theory was misidentified; the true symmetry of the theory is the larger group, $\mathbb{Z}_4^{\mathbf{T}}$. We show in our examples that $\mathbb{Z}_4^{\mathbf{T}}$ is free of the $\mathcal{H}^3$ symmetry localization anomaly. Therefore, these theories are time-reversal invariant, however $\mathbf{T}^2$ is a non-trivial unitary symmetry, while $\mathbf{T}^4 = \mathbb{1}$. (2) We show

that in the cases that we study, the anomaly can also be resolved by considering the TQFT to be a theory of fermions (i.e. a spin TQFT), such that $\mathbf{T}^2 = (-1)^{N_f}$, where $(-1)^{N_f}$ is the fermion parity of the system.

While much of our discussion is phrased in terms of time-reversal symmetry, analogous results hold also for reflection symmetry. To establish our results we will use whichever is convenient for the issues at hand, noting that in the setup of Euclidean quantum field theory, which we make use of, time-reversal and spatial reflections appear on an equal footing.

For the case where $G = \mathbb{Z}_2$ is a unitary internal (on-site) symmetry, Ref. 45 provided an example of an $\mathcal{H}^3$ anomaly that occurs for discrete gauge theory with gauge group $\mathbb{D}_{16}$. It was shown that it is possible to, in some sense, realize the theory at the surface of a (3+1)D SET with a global $G = \mathbb{Z}_2$ symmetry. In this case, the surface theory is no longer a (2+1)D theory, since some of the anyons that can exist on the surface correspond either to bulk quasiparticles or to endpoints of strings in the (3+1)D bulk. For this reason, here we refer to this as a "pseudo-realization" of the original theory at the surface of the (3+1)D SET. In this construction, the $\mathcal{H}^3_{[\rho]}(G, \mathcal{A})$ could be related to the symmetry fractionalization of string excitations in the (3+1)D system [45, 49–51].

For the case where $G = \mathbb{Z}_2^{\mathbf{T}}$, symmetry fractionalization of string excitations in (3+1)D is not well-understood, which makes the corresponding problem in this case especially intriguing. For the case of the $\mathbb{Z}_2^{\mathbf{T}}$ symmetry localization anomalies in $\mathrm{USp}(4)_2$ and $\mathrm{SO}(4)_4$ CS theories, we demonstrate that the corresponding theories can be pseudo-realized at the surface of a (3+1)D SET, whose low energy theory is a dynamical $\mathbb{Z}_2$ gauge theory and possesses a global $\mathbb{Z}_2^{\mathbf{T}}$ time-reversal symmetry.

We further provide an infinite family of TQFTs with $\mathcal{H}^3_{[\rho]}(\mathbb{Z}_2^{\mathbf{T}}, \mathcal{A})$ symmetry localization anomalies. These can all be obtained by starting with a theory with $\mathbb{Z}_4^{\mathbf{T}}$ symmetry, and gauging the unitary $\mathbb{Z}_2$ subgroup associated with $\mathbf{T}^2$. Importantly, when performing the gauging of the unitary $\mathbb{Z}_2$ subgroup, we must add a Dijkgraaf-Witten term for the $\mathbb{Z}_2$ gauge field, which is associated with the non-trivial element in $\mathcal{H}^3(\mathbb{Z}_2, \mathrm{U}(1)) = \mathbb{Z}_2$. Physically, this corresponds to stacking a $\mathbb{Z}_2$ SPT with the theory with the $\mathbb{Z}_4^{\mathbf{T}}$ symmetry. This process can be repeated indefinitely, so that we can generate an infinite series of TQFTs with $\mathcal{H}^3_{[\rho]}(\mathbb{Z}_2^{\mathbf{T}}, \mathcal{A})$ anomalies by starting with a single "root" theory with $\mathbb{Z}_4^{\mathbf{T}}$ symmetry. We provide an infinite set of such "root" theories. For the $\mathrm{USp}(4)_2$ and $\mathrm{SO}(4)_4$ examples, the root phases are $\mathrm{SU}(5)_1$ and $\mathrm{SU}(3)_1 \times \mathrm{SU}(3)_1$ CS theory, respectively. We show that $\mathrm{U}(1) \times \mathrm{U}(1)$ CS theory with K-matrix $K = \begin{pmatrix} m & n \\ n & -m \end{pmatrix}$ provides another infinite family of root theories as well.

This paper is organized as follows. In Sec. II we provide a brief review of UMTCs and the action of global symmetry, the general definition of the $\mathcal{H}^3_{[\rho]}(G, \mathcal{A})$ anomaly, and symmetry fractionalization. In Sec. III we introduce the example of $\mathrm{USp}(4)_2$ CS theory as a theory with a $\mathcal{H}^3_{[\rho]}(\mathbb{Z}_2^{\mathbf{T}}, \mathcal{A})$ anomaly. In Sec. IV we provide a set of sufficient conditions, defined in terms of the modular data of the theory and

the way $\mathbf{T}$ permutes the anyons, for diagnosing $\mathcal{H}^3_{[\rho]}(\mathbb{Z}_2^{\mathbf{T}}, \mathcal{A})$ anomalies. In Sec. V we discuss several resolutions of the $\mathcal{H}^3$ anomaly, which include enlarging the symmetry from $\mathbb{Z}_2^{\mathbf{T}}$ to $\mathbb{Z}_4^{\mathbf{T}}$, viewing the theory as a theory of fermions, and finally how to pseudo-realize theories with such anomalies at the surface of (3+1)D SET phases. In Sec. VI we provide an infinite family of examples that generalize the $\mathrm{USp}(4)_2$ example. We conclude with a discussion of some open issues in Sec. VII.

II. REVIEW OF THE $\mathcal{H}^3_{[\rho]}(G, \mathcal{A})$ ANOMALY

In this section we briefly review the discussion of Ref. 19 regarding the $\mathcal{H}^3_{[\rho]}(G, \mathcal{A})$ symmetry localization anomaly.

A. Review of UMTC notation

Here we briefly review the notation that we use to describe UMTCs. For a more comprehensive review of the notation that we use, see e.g. Ref. 19. The topologically nontrivial quasiparticles of a (2+1)D topologically ordered state are equivalently referred to as anyons, topological charges, and quasiparticles. In the category theory terminology, they correspond to isomorphism classes of simple objects of the UMTC.

A UMTC $\mathcal{C}$ contains splitting spaces V_c^{ab}, and their dual fusion spaces, V_{ab}^c, where $a, b, c \in \mathcal{C}$ are the anyons. These spaces have dimension $\dim V_c^{ab} = \dim V_{ab}^c = N_{ab}^c$, where N_{ab}^c are referred to as the fusion rules. They are depicted graphically as:

$$(d_c/d_a d_b)^{1/4} \quad \overset{c}{\underset{a \quad b}{\wedge^\mu}} = \langle a, b; c, \mu | \in V_{ab}^c, \qquad (2)$$

$$(d_c/d_a d_b)^{1/4} \quad \overset{a \quad b}{\underset{c}{\vee_\mu}} = |a, b; c, \mu\rangle \in V_c^{ab}, \qquad (3)$$

where $\mu = 1, \ldots, N_{ab}^c$, d_a is the quantum dimension of a, and the factors $\left(\frac{d_c}{d_a d_b}\right)^{1/4}$ are a normalization convention for the diagrams.

We denote $\bar{a}$ as the topological charge conjugate of a, for which $N_{a\bar{a}}^1 = 1$, i.e.

$$a \times \bar{a} = 1 + \cdots \qquad (4)$$

Here 1 refers to the identity particle, i.e. the vacuum topological sector, which physically describes all local, topologically trivial excitations.

The F-symbols are defined as the following basis transformation between the splitting spaces of 4 anyons:

$$\overset{a \quad b \quad c}{\underset{d}{\underset{\beta}{\overset{\alpha}{\bigvee}}_e}} = \sum_{f, \mu, \nu} \left[F_d^{abc} \right]_{(e, \alpha, \beta)(f, \mu, \nu)} \overset{a \quad b \quad c}{\underset{d}{\underset{\nu}{\bigvee}_f^\mu}}. \qquad (5)$$

To describe topological phases, these are required to be unitary transformations, i.e.

$$\left[\left(F_d^{abc}\right)^{-1}\right]_{(f,\mu,\nu)(e,\alpha,\beta)} = \left[\left(F_d^{abc}\right)^{\dagger}\right]_{(f,\mu,\nu)(e,\alpha,\beta)}$$
$$= \left[F_d^{abc}\right]^{*}_{(e,\alpha,\beta)(f,\mu,\nu)}. \qquad (6)$$

The R-symbols define the braiding properties of the anyons, and are defined via the the following diagram:

$$\qquad (7)$$

The topological twist $\theta_a = e^{2\pi i h_a}$, with h_a the topological spin, is defined via the diagram:

$$\theta_a = \theta_{\bar{a}} = \sum_{c,\mu} \frac{d_c}{d_a}\left[R_c^{aa}\right]_{\mu\mu} = \frac{1}{d_a}\ \raisebox{-0.5em}{}\ , \qquad (8)$$

Finally, the modular, or topological, S-matrix, is defined as

$$S_{ab} = \mathcal{D}^{-1}\sum_c N_{\bar{a}b}^c \frac{\theta_c}{\theta_a\theta_b}d_c = \frac{1}{\mathcal{D}}\ \raisebox{-0.5em}{}\ , \qquad (9)$$

where $\mathcal{D} = \sqrt{\sum_a d_a^2}$.

B. Topological symmetry and braided auto-equivalence

An important property of a UMTC $\mathcal{C}$ is the group of "topological symmetries," which are related to "braided auto-equivalences" in the mathematical literature. They are associated with the symmetries of the emergent TQFT described by $\mathcal{C}$, irrespective of any microscopic global symmetries of a quantum system in which the TQFT emerges as the long wavelength description.

The topological symmetries consist of the invertible maps

$$\varphi : \mathcal{C} \to \mathcal{C}. \qquad (10)$$

The different φ, modulo equivalences known as natural isomorphisms, form a group, which we denote as $\text{Aut}(\mathcal{C})$.[19]

The symmetry maps can be classified according to a $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading, defined by

$$q(\varphi) = \begin{cases} 0 & \text{if } \varphi \text{ is not time-reversing} \\ 1 & \text{if } \varphi \text{ is time-reversing} \end{cases} \qquad (11)$$

$$p(\varphi) = \begin{cases} 0 & \text{if } \varphi \text{ is spatial parity even} \\ 1 & \text{if } \varphi \text{ is spatial parity odd} \end{cases} \qquad (12)$$

Here time-reversing transformations are anti-unitary, while spatial parity odd transformations involve an odd number of reflections in space, thus changing the orientation of space. Thus the topological symmetry group can be decomposed as

$$\text{Aut}(\mathcal{C}) = \bigsqcup_{q,p=0,1} \text{Aut}_{q,p}(\mathcal{C}). \qquad (13)$$

$\text{Aut}_{0,0}(\mathcal{C})$ is therefore the subgroup corresponding to topological symmetries that are unitary and space-time parity even (this is referred to in the mathematical literature as the group of "braided auto-equivalences"). The generalization involving reflection and time-reversal symmetries appears to be beyond what has been considered in the mathematics literature to date.

It is also convenient to define

$$\sigma(\varphi) = \begin{cases} 1 & \text{if } \varphi \text{ is space-time parity even} \\ * & \text{if } \varphi \text{ is space-time parity odd} \end{cases} \qquad (14)$$

A map φ is space-time parity odd if $(q(\varphi)+p(\varphi)) \mod 2 = 1$, and otherwise it is space-time parity even.

The maps φ may permute the topological charges:

$$\varphi(a) = a' \in \mathcal{C}, \qquad (15)$$

subject to the constraint that

$$N_{a'b'}^{c'} = N_{ab}^c$$
$$S_{a'b'} = S_{ab}^{\sigma(\varphi)},$$
$$\theta_{a'} = \theta_a^{\sigma(\varphi)}, \qquad (16)$$

The maps φ have a corresponding action on the F- and $R-$symbols of the theory, as well as on the fusion and splitting spaces, which we will discuss in the subsequent section.

C. Global symmetry

Let us now suppose that we are interested in a system with a global symmetry group G. For example, we may be interested in a given microscopic Hamiltonian that has a global symmetry group G, whose ground state preserves G, and whose anyonic excitations are algebraically described by $\mathcal{C}$. The global symmetry acts on the topological quasiparticles and the topological state space through the action of a group homomorphism

$$[\rho] : G \to \text{Aut}(\mathcal{C}). \qquad (17)$$

We use the notation $[\rho_{\mathbf{g}}] \in \text{Aut}(\mathcal{C})$ for a specific element $\mathbf{g} \in G$. The square brackets indicate the equivalence class of symmetry maps related by natural isomorphisms, which we define below. $\rho_{\mathbf{g}}$ is thus a representative symmetry map of the equivalence class $[\rho_{\mathbf{g}}]$. We use the notation

$$^{\mathbf{g}}a \equiv \rho_{\mathbf{g}}(a). \qquad (18)$$

We associate gradings $q(\mathbf{g})$ and $p(\mathbf{g})$ by defining

$$q(\mathbf{g}) \equiv q(\rho_{\mathbf{g}})$$
$$p(\mathbf{g}) \equiv p(\rho_{\mathbf{g}})$$
$$\sigma(\mathbf{g}) \equiv \sigma(\rho_{\mathbf{g}}) \qquad (19)$$

$\rho_{\mathbf{g}}$ has an action on the fusion/splitting spaces:

$$\rho_{\mathbf{g}} : V_{ab}^c \to V_{^{\mathbf{g}}a\,^{\mathbf{g}}b}^{^{\mathbf{g}}c}. \qquad (20)$$

This map is unitary if $q(\mathbf{g}) = 0$ and anti-unitary if $q(\mathbf{g}) = 1$.

We write this as

$$\rho_{\mathbf{g}}|a,b;c,\mu\rangle = \sum_{\nu}[U_{\mathbf{g}}(\,^{\mathbf{g}}a,\,^{\mathbf{g}}b;\,^{\mathbf{g}}c)]_{\mu\nu}K^{q(\mathbf{g})}|a,b;c,\nu\rangle,$$

(21)

where $U_{\mathbf{g}}(\,^{\mathbf{g}}a,\,^{\mathbf{g}}b;\,^{\mathbf{g}}c)$ is a $N^c_{ab} \times N^c_{ab}$ matrix, and K denotes complex conjugation.[52]

Under the map $\rho_{\mathbf{g}}$, the F and R symbols transform as well:

$$\rho_{\mathbf{g}}[F^{abc}_{def}] = U_{\mathbf{g}}(\,^{\mathbf{g}}a,\,^{\mathbf{g}}b;\,^{\mathbf{g}}e)U_{\mathbf{g}}(\,^{\mathbf{g}}e,\,^{\mathbf{g}}c;\,^{\mathbf{g}}d)F^{\,^{\mathbf{g}}a\,^{\mathbf{g}}b\,^{\mathbf{g}}c}_{\,^{\mathbf{g}}d\,^{\mathbf{g}}e\,^{\mathbf{g}}f}U^{-1}_{\mathbf{g}}(\,^{\mathbf{g}}b,\,^{\mathbf{g}}c;\,^{\mathbf{g}}f)U^{-1}_{\mathbf{g}}(\,^{\mathbf{g}}a,\,^{\mathbf{g}}f;\,^{\mathbf{g}}d) = K^{\sigma(\mathbf{g})}F^{abc}_{def}K^{\sigma(\mathbf{g})}$$

$$\rho_{\mathbf{g}}[R^{ab}_c] = U_{\mathbf{g}}(\,^{\mathbf{g}}a,\,^{\mathbf{g}}b;\,^{\mathbf{g}}c)R^{\,^{\mathbf{g}}a\,^{\mathbf{g}}b}_{\,^{\mathbf{g}}c}U_{\mathbf{g}}(\,^{\mathbf{g}}a,\,^{\mathbf{g}}b;\,^{\mathbf{g}}c)^{-1} = K^{\sigma(\mathbf{g})}R^{ab}_c K^{\sigma(\mathbf{g})},$$

(22)

where we have suppressed the additional indices that appear when $N^c_{ab} > 1$.

Importantly, we have

$$\kappa_{\mathbf{g},\mathbf{h}} \circ \rho_{\mathbf{g}} \circ \rho_{\mathbf{h}} = \rho_{\mathbf{gh}},$$

(23)

where the action of $\kappa_{\mathbf{g},\mathbf{h}}$ on the fusion / splitting spaces is defined as

$$\kappa_{\mathbf{g},\mathbf{h}}(|a,b;c,\mu\rangle) = \sum_{\nu}[\kappa_{\mathbf{g},\mathbf{h}}(a,b;c)]_{\mu\nu}|a,b;c,\nu\rangle. \quad (24)$$

The above definitions imply that

$$\kappa_{\mathbf{g},\mathbf{h}}(a,b;c) = U_{\mathbf{g}}(a,b;c)^{-1}K^{q(\mathbf{g})}U_{\mathbf{h}}(\,^{\bar{\mathbf{g}}}a,\,^{\bar{\mathbf{g}}}b;\,^{\bar{\mathbf{g}}}c)^{-1}K^{q(\mathbf{q})}U_{\mathbf{gh}}(a,b;c), \quad (25)$$

where $\bar{\mathbf{g}} \equiv \mathbf{g}^{-1}$. $\kappa_{\mathbf{g},\mathbf{h}}$ is a natural isomorphism, which means that by definition,

$$[\kappa_{\mathbf{g},\mathbf{h}}(a,b;c)]_{\mu\nu} = \delta_{\mu\nu}\frac{\beta_a(\mathbf{g},\mathbf{h})\beta_b(\mathbf{g},\mathbf{h})}{\beta_c(\mathbf{g},\mathbf{h})}, \quad (26)$$

where $\beta_a(\mathbf{g},\mathbf{h})$ are $U(1)$ phases.

D. $\mathcal{H}^3_{[\rho]}(G,\mathcal{A})$ obstruction

As discussed in detail in Ref. 19, the choice of $[\rho_{\mathbf{g}}]$ defines an element $[\mathfrak{O}] \in \mathcal{H}^3_{[\rho]}(G,\mathcal{A})$. To see this, we first define

$$\Omega_a(\mathbf{g},\mathbf{h},\mathbf{k}) = \frac{K^{\sigma(\mathbf{g})}\beta_{\rho_{\mathbf{g}}^{-1}(a)}(\mathbf{h},\mathbf{k})K^{\sigma(\mathbf{g})}\beta_a(\mathbf{g},\mathbf{hk})}{\beta_a(\mathbf{gh},\mathbf{k})\beta_a(\mathbf{g},\mathbf{h})} \quad (27)$$

It can be shown that

$$\Omega_a(\mathbf{g},\mathbf{h},\mathbf{k})\Omega_b(\mathbf{g},\mathbf{h},\mathbf{k}) = \Omega_c(\mathbf{g},\mathbf{h},\mathbf{k}), \quad (28)$$

if $N^c_{ab} \neq 0$. This then implies that[19]

$$\Omega_a(\mathbf{g},\mathbf{h},\mathbf{k}) = M^*_{a\mathfrak{O}(\mathbf{g},\mathbf{h},\mathbf{k})}, \quad (29)$$

for some $\mathfrak{O}(\mathbf{g},\mathbf{h},\mathbf{k}) \in \mathcal{A}$. Here $\mathcal{A} \subset \mathcal{C}$ is the subset of topological charges in $\mathcal{C}$ that are Abelian. These form a finite group, which we also denote $\mathcal{A}$, under fusion. Given an Abelian anyon $b \in \mathcal{A}$, M_{ab} is the braiding phase obtained by encircling b around a, as defined by the following anyon diagram:

$$(30)$$

In terms of the modular S-matrix, $M_{ab} = \frac{S^*_{ab}S_{00}}{S_{0a}S_{0b}}$.

One can show that $\mathfrak{O}$ is a 3-cocycle, and that there is a freedom in the choice of β_a which relate two different $\mathfrak{O}$'s by a 3-coboundary. Therefore $[\rho_{\mathbf{g}}]$ defines an element in $[\mathfrak{O}] \in \mathcal{H}^3_{[\rho]}(G,\mathcal{A})$.

In Ref. 19, it was shown that $[\mathfrak{O}] \in \mathcal{H}^3_{[\rho]}(G,\mathcal{A})$ is an obstruction to symmetry localization. That is, it is an obstruction to consistently defining a notion of symmetry action on individual topological charges. More specifically, let us consider a state $|\Psi_{a_1,\cdots,a_n}\rangle$ in the full Hilbert space of the system, which consists of n anyons, $a_1,\cdots a_n$, at well-separated locations, which collectively fuse to the identity topological sector. Since the ground state is G-symmetric, we expect that the symmetry action $R_{\mathbf{g}}$ on this state decomposes as follows:

$$R_{\mathbf{g}}|\Psi_{a_1,\cdots,a_n}\rangle \approx \prod_{j=1}^{n}U^{(j)}_{\mathbf{g}}U_{\mathbf{g}}(\,^{\mathbf{g}}a_1,\cdots,\,^{\mathbf{g}}a_n;0)|\Psi_{\,^{\mathbf{g}}a_1,\cdots,\,^{\mathbf{g}}a_n}\rangle.$$

(31)

Here, $U_{\mathbf{g}}^{(j)}$ are unitary matrices that have support in a region (of length scale set by the correlation length) localized to the anyon a_j. The map $U_{\mathbf{g}}(^{\mathbf{g}}a_1, \cdots, ^{\mathbf{g}}a_n; 0)$ is the generalization of $U_{\mathbf{g}}(^{\mathbf{g}}a, ^{\mathbf{g}}b; ^{\mathbf{g}}c)$, defined above, to the case with n anyons fusing to vacuum. In contrast to the local unitaries $U_{\mathbf{g}}^{(j)}$, $U_{\mathbf{g}}(^{\mathbf{g}}a_1, \cdots, ^{\mathbf{g}}a_n; 0)$ only depends on the global topological sector of the system (i.e. on the precise fusion tree that defines the topological state). $R_{\mathbf{g}}$ is the representation of $\mathbf{g}$ acting on the full Hilbert space of the theory. The $\approx$ means that the equation is true up to corrections that are exponentially small in the distance between the anyons and the correlation length of the system. $[\mathfrak{O}] \in \mathcal{H}^3_{[\rho]}(G, \mathcal{A})$ is an obstruction to Eq. (31) being consistent when considering the associativity of three group elements.[19]

When $\rho_{\mathbf{g}}$ does not permute any anyons, i.e. $\rho_{\mathbf{g}}(a) = a$ for all a, we expect that $\rho_{\mathbf{g}}$ must be a natural isomorphism. This has so far been proven rigorously for the case where $\mathcal{C}$ is an Abelian theory. One can show that this implies that the associated $\mathcal{H}^3$ obstruction is always vanishing. With this assumption, non-vanishing $\mathcal{H}^3$ obstructions therefore require $\rho_{\mathbf{g}}$ to have a non-trivial permutation action on the anyons.

E. Symmetry fractionalization

When the $\mathcal{H}^3_{[\rho]}(G, \mathcal{A})$ symmetry-localization obstruction vanishes, then one can define a consistent notion of symmetry fractionalization. Symmetry fractionalization determines how the anyons in the system carry fractional symmetry quantum numbers. In general, the distinct allowed patterns of symmetry fractionalization are in one-to-one correspondence with elements in $\mathcal{H}^2_{[\rho]}(G, \mathcal{A})$.[19]

For the case of time-reversal symmetry, an important set of data that characterizes time-reversal symmetry fractionalization is as follows. When $a = {}^{\mathbf{T}}a$, one can define a quantity $\eta_a^{\mathbf{T}} = \pm 1$. This determines whether locally the action of $\mathbf{T}^2$ on a is equal to ± 1.[15, 19] This, in turn, determines whether a carries a "local Kramers degeneracy." If $\eta_a^{\mathbf{T}} = -1$, then a carries with it an internal local multi-dimensional Hilbert space whose degeneracy (the Kramers degeneracy) is protected by time-reversal symmetry. The quantities $\eta_a^{\mathbf{T}}$ must satisfy a number of highly non-trivial consistency relations. For example, one can show:[19]

$$[\kappa_{\mathbf{T}}(a, b; c)]_{\mu\nu} = \delta_{\mu\nu} \frac{\eta_a^{\mathbf{T}} \eta_b^{\mathbf{T}}}{\eta_c^{\mathbf{T}}}. \tag{32}$$

Moreover, if N_{ab}^c is odd and $^{\mathbf{T}}a = a$, $^{\mathbf{T}}b = b$, and $^{\mathbf{T}}c = c$, then

$$\eta_a^{\mathbf{T}} \eta_b^{\mathbf{T}} = \eta_c^{\mathbf{T}}. \tag{33}$$

Similarly, if $N_{a\,^{\mathbf{T}}a}^b$ is odd and $^{\mathbf{T}}b = b$ one can prove

$$\eta_b^{\mathbf{T}} = \theta_b. \tag{34}$$

The case of reflection symmetry is analogous to that of time-reversal symmetry. Here, when $a = {}^{\mathbf{R}}\bar{a}$, then we can define a symmetry fractionalization quantum number $\eta_a^{\mathbf{R}} = \pm 1$.

Anyon label, a	1	ε	ϕ_1	ϕ_2	ψ_+	ψ_-
Quantum dimension, d_a	1	1	2	2	$\sqrt{5}$	$\sqrt{5}$
Topological twist, θ_a	1	1	$e^{\frac{4\pi i}{5}}$	$e^{-\frac{4\pi i}{5}}$	i	$-i$
Time-reversal action, $^{\mathbf{T}}a$	1	ε	ϕ_2	ϕ_1	ψ_-	ψ_+

TABLE I. Anyon content of $\mathrm{USp}(4)_2$.

$\eta_a^{\mathbf{R}}$ can be understood as follows. We place a and $\bar{a}$ away from each other, such that the action of reflection $\mathbf{R}$ interchanges their positions. If $a = {}^{\mathbf{R}}\bar{a}$, then the system is reflection invariant, and $\eta_a^{\mathbf{R}} = \pm 1$ corresponds to the eigenvalue of the action of reflection on this state. Alternatively, we can consider taking the spatial manifold to be a cylinder, with topological charge a and $\bar{a}$ on the two ends of the cylinder, such that the action of reflection interchanges their position. If $a = {}^{\mathbf{R}}\bar{a}$, then we can view this system as a (1+1)D reflection symmetry SPT system, which has a $\mathbb{Z}_2$ classification. $\eta_a^{\mathbf{R}}$ can be related to whether this (1+1)D reflection SPT is trivial or non-trivial (when compared to the case where a is the identity particle). See Ref. 41 for more details.

III. EXAMPLE: $\mathrm{USp}(4)_2$ CHERN-SIMONS THEORY

An explicit example of a theory with a $\mathcal{H}^3(\mathbb{Z}_2^{\mathbf{T}}, \mathcal{A})$ anomaly is Chern-Simons theory with gauge group $\mathrm{USp}(4)_2$. Here $\mathrm{USp}(2n)$ is the symplectic group, where $\mathrm{USp}(2) = \mathrm{SU}(2)$.[53] The anyon content of $\mathrm{USp}(4)_2$ CS theory coincides with the integrable highest weight representations of the affine lie aglebra $\mathfrak{so}(5)_2$.[5, 54] It consists of 6 particles, which we can label $1, \epsilon, \phi_1, \phi_2, \psi_+, \psi_-$. The fusion rules are given by

$$\epsilon \times \epsilon = 1, \quad \epsilon \times \phi_i = \phi_i, \quad \epsilon \times \psi_+ = \psi_-$$
$$\phi_i \times \phi_i = 1 + \epsilon + \phi_{\min(2i, 5-2i)},$$
$$\phi_1 \times \phi_2 = \phi_1 + \phi_2 \tag{35}$$
$$\psi_+ \times \psi_+ = 1 + \phi_1 + \phi_2.$$

Here $i = 1, 2$. We also list the quantum dimensions and topological twists in Table I, from which one can construct the modular S matrix:

$$S = \frac{1}{\sqrt{20}} \begin{pmatrix} 1 & 1 & 2 & \sqrt{5} & \sqrt{5} & 2 \\ 1 & 1 & 2 & -\sqrt{5} & -\sqrt{5} & 2 \\ 2 & 2 & \sqrt{5}-1 & 0 & 0 & -\sqrt{5}-1 \\ \sqrt{5} & -\sqrt{5} & 0 & \sqrt{5} & -\sqrt{5} & 0 \\ \sqrt{5} & -\sqrt{5} & 0 & -\sqrt{5} & \sqrt{5} & 0 \\ 2 & 2 & -\sqrt{5}-1 & 0 & 0 & \sqrt{5}-1 \end{pmatrix}, \tag{36}$$

where we have presented S in the basis $(1, \epsilon, \phi_1, \psi_+, \psi_-, \phi_2)$.

The F and R symbols of this theory are tabulated in Ref. 55. We see that there is only one possible action under time-reversal, summarized in Table I.

In this case, $\mathcal{A} = \mathbb{Z}_2$, and $\mathcal{H}^3(\mathbb{Z}_2^{\mathbf{T}}, \mathbb{Z}_2) = \mathbb{Z}_2$. By direct computation, following the procedure outlined in the previous section, we find that the choice of $[\rho_{\mathbf{T}}]$ described above

leads to a non-trivial obstruction $[\mathfrak{O}] \in \mathcal{H}^3(\mathbb{Z}_2^{\mathbf{T}}, \mathbb{Z}_2)$. In fact, one finds that the representative 3-cocycle is given by $\mathfrak{O}(\mathbf{T}, \mathbf{T}, \mathbf{T}) = \epsilon$ (all others are 1). Therefore USp$(4)_2$ possesses an $\mathcal{H}^3(\mathbb{Z}_2^{\mathbf{T}}, \mathbb{Z}_2)$ symmetry-localization anomaly.

IV. SUFFICIENT CONDITIONS FOR THE PRESENCE OF $\mathcal{H}^3_{[\rho]}(\mathbb{Z}_2^{\mathbf{T}}, \mathcal{A})$ ANOMALY

A. Overview

The above discussion of the $\mathcal{H}^3$ symmetry-localization anomaly requires detailed knowledge of the F and R symbols of the theory in order to determine whether any symmetry G and map $[\rho]$ possesses the anomaly. However obtaining the F and R symbols given the modular data (the S matrix and the topological twists) of a TQFT is often a computationally prohibitive problem. Below we will discuss some simple conditions that must be satisfied for a theory to be free of the $\mathcal{H}^3_{[\rho]}(\mathbb{Z}_2^{\mathbf{T}}, \mathcal{A})$ anomaly.

First we define the quantities:

$$\mathcal{Z}(\mathbb{RP}^4) = \sum_{\{a|a=\mathbf{T}a\}} S_{0a}\theta_a\eta_a^{\mathbf{T}},$$

$$M_a = \sum_{\{x|x=\mathbf{T}x\}} S_{ax}\eta_x^{\mathbf{T}}, \qquad (37)$$

whose significance will be described in the subsequent sections.

The conditions that must be satisfied are as follows:

1. $\mathcal{Z}(\mathbb{RP}^4) = \pm 1$.

2. M_a is a non-negative integer for all a.

3. $\theta_a = \pm 1$ and θ_a is independent of a, for all a such that $M_a > 0$.

Given the modular S-matrix, the topological twists, and an action of $\mathbf{T}$ that permutes the anyons, there must be a choice of $\{\eta_a^{\mathbf{T}}\}$ such that conditions (1)-(3) are satisfied. If not, the theory possesses an $\mathcal{H}^3_{[\rho]}(\mathbb{Z}_2^{\mathbf{T}}, \mathcal{A})$ anomaly.

For the case of reflection symmetry $\mathbf{R}$, we replace $^{\mathbf{T}}a$ with $^{\mathbf{R}}\bar{a}$, and $\eta_a^{\mathbf{T}}$ with $\eta_a^{\mathbf{R}}$ in the above formulae:

$$\mathcal{Z}(\mathbb{RP}^4) = \sum_{\{a|a=\mathbf{R}\bar{a}\}} S_{0a}\theta_a\eta_a^{\mathbf{R}},$$

$$M_a = \sum_{\{x|x=\mathbf{R}\bar{x}\}} S_{ax}\eta_x^{\mathbf{R}}. \qquad (38)$$

In the subsequent sections we will discuss these conditions and their origin in detail. It will be convenient to phrase the discussion in terms of spatial reflection symmetry $\mathbf{R}$, and then to obtain the results for time reversal $\mathbf{T}$ by replacing $\mathbf{R}$ with $\mathbf{CT}$ (where $\mathbf{C} : a \rightarrow \bar{a}$ is topological charge conjugation) and $\{\eta_a^{\mathbf{R}}\}$ with $\{\eta_a^{\mathbf{T}}\}$.

In Sec. IV E, we also provide an additional diagnostic by considering the theory obtained by condensing certain bosons in the TQFT of interest, where the inconsistency arises by finding conflicting constraints on $\{\eta_a^{\mathbf{T}}\}$.

In order to connect the constraints above to $\mathcal{H}^3_{[\rho]}(G, \mathcal{A})$ symmetry-localization anomalies, we need to make an important assumption that we discuss below. Let us consider a (2+1)D topological phase with symmetry G whose symmetry action $[\rho]$ is free of the $\mathcal{H}^3_{[\rho]}(G, \mathcal{A})$ symmetry-localization anomaly. Depending on the symmetry fractionalization class, which recall is classified by $\mathcal{H}^2_{[\rho]}(G, \mathcal{A})$, the system may possess an SPT (t 'Hooft) anomaly, in the sense that the fractionalization class can only occur if the (2+1)D system is realized at the surface of a (3+1)D SPT state. The (3+1)D SPT cancels the anomaly from the symmetry fractionalization of the (2+1)D surface. In this way, a given (2+1)D SET determines a (3+1)D SPT state. If the (2+1)D theory has no anomaly at all, then the bulk (3+1)D system is a trivial SPT.

In the following, we will show that the choice of symmetry action $[\rho]$ alone can be enough to preclude the (2+1)D system from existing at the surface of a (3+1)D SPT state. In the example of USp$(4)_2$ CS theory, we find that the existence of the $\mathcal{H}^3_{[\rho]}(G, \mathcal{A})$ anomaly is accompanied by the impossibility of the theory to be consistent at the surface of any (3+1)D $\mathbb{Z}_2^{\mathbf{T}}$ SPT state. We expect that this is a general phenomenon: the impossibility of the theory, with a specified action of $[\rho]$, to exist at the surface of a (3+1)D SPT state with symmetry group G signals the existence of the $\mathcal{H}^3_{[\rho]}(G, \mathcal{A})$ anomaly. In principle, for space-time reflection symmetries we have not ruled out the possibility that the failure of the (2+1)D system to exist at the surface of a (3+1)D SPT state could signal an additional anomaly associated with $[\rho]$, which is independent of the $\mathcal{H}^3_{[\rho]}(G, \mathcal{A})$ obstruction. However this appears to be unlikely, given that this is known not to be the case for unitary internal symmetries.[19, 47]

B. Condition (1): (3+1)D Path integral on $\mathbb{RP}^4$

In the case of $\mathbb{Z}_2^{\mathbf{R}}$ (or $\mathbb{Z}_2^{\mathbf{T}}$) symmetry, bosonic SPTs in (3+1)D have a $\mathbb{Z}_2 \times \mathbb{Z}_2$ classification. The 4 distinct SPT states can be distinguished by the value of their topological path integrals on $\mathbb{RP}^4$ and $\mathbb{CP}^2$:[12]

$$\mathcal{Z}(\mathbb{RP}^4) = \pm 1,$$

$$\mathcal{Z}(\mathbb{CP}^2) = \pm 1. \qquad (39)$$

As mentioned above, a given (2+1)D SET determines a (3+1)D SPT state. In Ref. 41, it was shown that if the symmetry of the system is $\mathbb{Z}_2^{\mathbf{R}}$, then one can compute the path integral on $\mathbb{RP}^4$ entirely in terms of the properties of the (2+1)D theory through the following formula:

$$\mathcal{Z}(\mathbb{RP}^4) = \sum_{\{a|a=\mathbf{R}\bar{a}\}} S_{0a}\theta_a\eta_a^{\mathbf{R}}. \qquad (40)$$

Here, the sum is over all anyons that are invariant under the action of reflection, $\mathbf{R}$, and topological charge conjugation.

Similarly, one can compute the path integral on $\mathbb{CP}^2$:

$$\mathcal{Z}(\mathbb{CP}^2) = \frac{1}{\mathcal{D}} \sum_a d_a^2 \theta_a = e^{2\pi i c/8}, \qquad (41)$$

where c is the chiral central charge of the UMTC.

Therefore, for any (2+1)D topological phase that admits a $\mathbb{Z}_2^{\mathbf{R}}$ symmetry with a given action $[\rho]$, there must be a choice of $\{\eta_a^{\mathbf{R}}\}$ such that (39) is satisfied. Failure to sastify (39) for any choice of $\{\eta_a^{\mathbf{R}}\}$ implies that the symmetry action $[\rho]$ cannot be realized in a way that allows the system to exist at the surface of a (3+1)D SPT state.

For the case of the $\mathrm{USp}(4)_2$ CS theory, performing the computation for $\mathbb{RP}^4$ (using time-reversal $\mathbb{Z}_2^{\mathbf{T}}$ instead of reflection $\mathbf{Z}_2^{\mathbf{R}}$ as the example), we find:

$$\mathcal{Z}(\mathbb{RP}^4) = \frac{1}{\mathcal{D}}(1 + \eta_\epsilon^{\mathbf{T}}). \qquad (42)$$

We can see that if $\eta_\epsilon^{\mathbf{T}} = -1$, then $\mathcal{Z}(\mathbb{RP}^4) = 0$. If $\eta_\epsilon^{\mathbf{T}} = 1$, then $\mathcal{Z}(\mathbb{RP}^4) = \frac{2}{\mathcal{D}} = \frac{1}{\sqrt{5}}$. We see that in both cases, the bulk (3+1)D system cannot be a $\mathbb{Z}_2^{\mathbf{T}}$ SPT state.

C. Condition (2): constraints from $\mathbb{RP}^2 \times D^2$

Let us suppose that we are given a $\mathbb{Z}_2^{\mathbf{R}}$ SET state that can exist at the surface of a (3+1)D SPT state. Consider the path integral for the (3+1)D theory on $\mathbb{RP}^2 \times D^2$, where D^p denotes the p-dimensional disk. We choose boundary conditions on the boundary, $\mathbb{RP}^2 \times S^1$, such that there is a Wilson loop of a encircling the S^1 at a particular point on the $\mathbb{RP}^2$. In Ref. 41, this path integral, denoted as $\mathcal{Z}(\mathbb{RP}^2 \times D^2)[l_a]$, was computed to be

$$\mathcal{Z}(\mathbb{RP}^2 \times D^2)[l_a] \equiv M_a = \sum_{\{x | x = {}^{\mathbf{R}}\bar{x}\}} S_{ax} \eta_x^{\mathbf{R}}. \qquad (43)$$

Below we derive the following non-trivial constraint, that M_a must be a non-negative integer:

$$M_a \in \mathbb{Z}_{\geq 0}. \qquad (44)$$

To derive (44), let us first consider the case where the bulk (3+1)D theory is a trivial SPT, i.e. when $\mathcal{Z}(\mathbb{RP}^4) = \mathcal{Z}(\mathbb{CP}^2) = 1$. In this case, all path integrals of the (3+1)D theory depend solely on the (2+1)D boundary. Thus M_a can be intepreted to be solely a property of the boundary of $\mathbb{RP}^2 \times D^2$:

$$M_a = \mathcal{Z}_{2+1;a}(\mathbb{RP}^2 \times S^1), \qquad (45)$$

where the subscript (2+1) emphasizes that this is purely a property of the (2+1)D theory. This, in turn, corresponds to the dimension of the Hilbert space of the (2+1)D theory on $\mathbb{RP}^2$, with a puncture labelled by a. By definition the dimension of a Hilbert space must be a non-negative integer; thus M_a must be a non-negative integer.

Now let us consider the case when the bulk (3+1)D theory is a non-trivial SPT, i.e. when either $\mathcal{Z}(\mathbb{RP}^4) = -1$ or

$\mathcal{Z}(\mathbb{CP}^2) = -1$. In this case we argue for (44) in two steps. We first prove that $2M_a \in \mathbb{Z}_{\geq 0}$, after which we prove that $M_a M_b \in \mathbb{Z}$. Together these imply (44).

To prove that $2M_a \in \mathbb{Z}_{\geq 0}$, we argue as follows. Let us refer to the bulk (3+1)D theory of interest as A, which can be obtained from the UMTC $\mathcal{C}$, which below we denote as $\mathcal{C}_A$. Let us now consider a second bulk (3+1)D theory, denoted B, which can be obtained from a second UMTC $\mathcal{C}_B$, and which satisfies $\mathcal{Z}^B(\mathbb{RP}^4) = \mathcal{Z}^A(\mathbb{RP}^4)$, and $\mathcal{Z}^B(\mathbb{CP}^2) = \mathcal{Z}^A(\mathbb{CP}^2)$.

Next, we consider the combined theory, denoted AB, such that

$$\mathcal{Z}^{AB}(M^4) = Z^A(M^4) Z^B(M^4), \qquad (46)$$

where M^4 is any closed 4-manifold. Now consider

$$M_{(a,b)}^{AB} = M_a^A M_b^B, \qquad (47)$$

where $M_{(a,b)}^{AB} = \mathcal{Z}^{AB}(\mathbb{RP}^2 \times D^2)[l_{(a,b)}]$, $a \in \mathcal{C}_A$, and $b \in \mathcal{C}_B$. By our previous argument, we have $M_{(a,b)}^{AB} \in \mathbb{Z}_{\geq 0}$.

When $\mathcal{Z}^B(\mathbb{RP}^4) = -1$ and $\mathcal{Z}^B(\mathbb{CP}^2) = 1$, we can take $\mathcal{C}_B$ to be the $\mathbb{Z}_2$ toric code model, which has four particles, $\{1, e, m, \psi\}$, with $\psi = e \times m$ the fermion. Furthermore, we consider the case where $\eta_e^{\mathbf{R}} = \eta_m^{\mathbf{R}} = -1$. For this theory, $M_b^B = 2\delta_{b\psi}$. Since we also have $M_{(a,b)}^{AB} \in \mathbb{Z}_{\geq 0}$, it follows that $2M_a^A \in \mathbb{Z}_{\geq 0}$.

When $\mathcal{Z}^B(\mathbb{RP}^4) = -1$ and $\mathcal{Z}^B(\mathbb{CP}^2) = -1$, we can take $\mathcal{C}_B$ to be the three-fermion theory, i.e. $\mathrm{SO}(8)_1$ CS theory. This is an Abelian theory with four particle types, $\{1, f_1, f_2, f_3\}$, where f_i are fermions ($\theta_{f_i} = -1$), with $S_{f_i, f_j} = -1/2$ for $i \neq j$. For this theory, $M_b^B = 2\delta_{b1}$, and therefore again $2M_a^A \in \mathbb{Z}_{\geq 0}$ in this case as well.

Finally, when $\mathcal{Z}^B(\mathbb{RP}^4) = 1$ and $\mathcal{Z}^B(\mathbb{CP}^2) = -1$, we can take $\mathcal{C}_B$ to consist of the three-fermion model, with $\eta_{f_1}^{\mathbf{R}} = -1$ and $\eta_{f_2}^{\mathbf{R}} = 1$. Then we get $M_b^B = 2\delta_{bf_2}$, which again proves that $2M_a^A \in \mathbb{Z}_{\geq 0}$.

Next, we must prove that $M_a M_b \in \mathbb{Z}$. This follows from the fact that

$$\mathcal{Z}(\mathbb{RP}^2 \times S^1 \times I)[l_a \cup l_b] = M_a M_b, \qquad (48)$$

where the boundary conditions (denoted in the square brackets) are such that there is a loop of a encircling the S^1 at $\mathbb{RP}^2 \times \{0\}$ and a loop of b encircling the S^1 at $\mathbb{RP}^2 \times \{1\}$. Here we take the interval $I = [0, 1]$, so that $\{0\}$ and $\{1\}$ are the boundaries of I. $\mathcal{Z}(\mathbb{RP}^2 \times S^1 \times I)[l_a \cup l_b]$ can be interpreted as the dimension of the Hilbert space of the (3+1)D system on $\mathbb{RP}^2 \times I$ with two anyons a and b on the two ends of I. Since this is the dimension of a Hilbert space, $M_a M_b$ must be a non-negative integer.

It remains to prove (48). We can demonstrate (48) as follows, using ideas from Ref. 56.[57] We refer the reader to Ref. 41 for a detailed discussion of these computations written for physicists. First we note that the (3+1)D path integrals on closed manifolds are all bordism invariant when constructed from UMTCs. Therefore,

$$\mathcal{Z}(\mathbb{RP}^2 \times S^2) = 1, \qquad (49)$$

because $\mathbb{RP}^2 \times S^2$ is bordant to the empty manifold, as it is the boundary of $\mathbb{RP}^2 \times D^3$. Next, we use the fact that the Hilbert space of the (3+1)D theory on any closed 3-manifold is one-dimensional, which follows from the fact that

this is a bulk (3+1)D SPT, with no intrinsic topological order. Therefore, using the gluing formula for topological path integrals,[41, 56]

$$\mathcal{Z}(\mathbb{RP}^2 \times S^2) = \frac{\mathcal{Z}(\mathbb{RP}^2 \times D^2)[l_a]\mathcal{Z}(\mathbb{RP}^2 \times S^1 \times I)[l_a \cup l_b]\mathcal{Z}(\mathbb{RP}^2 \times D^2)[l_b]}{\langle l_a | l_a \rangle_{\mathcal{V}(\mathbb{RP}^2 \times S^1)} \langle l_b | l_b \rangle_{\mathcal{V}(\mathbb{RP}^2 \times S^1)}}. \tag{50}$$

Here, we consider obtaining $\mathbb{RP}^2 \times S^2$ by gluing $\mathbb{RP}^2 \times S^1 \times I$ to two copies of $\mathbb{RP}^2 \times D^2$, each one along $\mathbb{RP}^2 \times S^1$. Thus the inner product is in the Hilbert space $\mathcal{V}(\mathbb{RP}^2 \times S^1)$, as indicated by the subscript. Using (49), moving the denominator to the LHS, and using the definition of the inner product in terms of the path integral[41, 56] we obtain

$$\mathcal{Z}(\mathbb{RP}^2 \times S^1 \times I)[l_a \cup l_{\bar{a}}]\mathcal{Z}(\mathbb{RP}^2 \times S^1 \times I)[l_b \cup l_{\bar{b}}]$$
$$= M_a M_b \mathcal{Z}(\mathbb{RP}^2 \times S^1 \times I)[l_a \cup l_b]. \tag{51}$$

Together with the identity $M_a = M_{\bar{a}}$, this implies (48).

D. Condition (3): Topological twists and M_a

The third non-trivial constraint summarized above is

$$\theta_a = \pm 1, \quad \text{and independent of } a, \text{ for } M_a > 0. \tag{52}$$

In the special case that $M_1 > 0$, this implies that $\theta_a = 1$ for all a such that $M_a > 0$.

One way to derive (52) is to use the following identity, shown in Ref. 41:

$$\frac{\sum_a \theta_a M_a^2}{\sum_a M_a^2} = \mathcal{Z}(\mathbb{RP}^4)\mathcal{Z}(\mathbb{CP}^2). \tag{53}$$

Since M_a^2 is a non-negative integer, and $\mathcal{Z}(\mathbb{RP}^4)\mathcal{Z}(\mathbb{CP}^2) = \pm 1$, the condition (52) follows trivially.

To better understand the origin of (52), below we will prove it through a different approach, which does not make use of the identity in Eq. (53). Let us again consider the path integral of the (3+1)D system, $\mathcal{Z}(\mathbb{RP}^2 \times S^1 \times I)[l_a \cup l_b] = M_a M_b$. As in the previous section, the boundary conditions denoted in square brackets consist of a Wilson loop of a along the S^1 direction on the first $\mathbb{RP}^2 \times S^1$, and a Wilson loop of b along the S^1 direction on the other $\mathbb{RP}^2 \times S^1$.

Now, we can consider the following procedure. If $M_a M_b > 0$, we can consider a state $|\Psi_{a,b}\rangle$ from the Hilbert space on $\mathbb{RP}^2 \times I$, which contains a puncture of a and b on the top and bottom surfaces. Next, we can cut out a tube that encircles a on the top $\mathbb{RP}^2$ and b on the bottom surface, rotate the tube by 2π, and glue it back in. This leads to an operation $\mathcal{T}$ on the state $|\Psi_{a,b}\rangle$ which corresponds to a Dehn twist on the top surface, and the oppositely oriented Dehn twist on the bottom surface. Thus we obtain:

$$\mathcal{T}|\Psi_{ab}\rangle = \theta_a \theta_b^* |\Psi_{ab}\rangle. \tag{54}$$

Since the Dehn twist around the cross-cap on $\mathbb{RP}^2$ is isotopic to the identity (see e.g. Ref. 41 for a discussion of this), it is a trivial operation and therefore we must have that $\mathcal{T}|\Psi_{ab}\rangle = |\Psi_{ab}\rangle$. In particular, we must have

$$\theta_a \theta_b^* = 1. \tag{55}$$

This implies that $\theta_a = \theta_b$ whenever $M_a M_b > 0$; that is, that θ_a is independent of a for all a such that $M_a > 0$.

Next, we can consider taking b through the cross-cap on the bottom $\mathbb{RP}^2$, obtaining $^{\mathrm{R}}b$ and yielding the state $|\Psi_{a, \, ^{\mathrm{R}}b}\rangle$. The Dehn twist procedure now gives

$$\mathcal{T}|\Psi_{a \, ^{\mathrm{R}}b}\rangle = \theta_a \theta_{^{\mathrm{R}}b}^* |\Psi_{a \, ^{\mathrm{R}}b}\rangle \tag{56}$$

Again, since the Dehn twist operation is trivial, this gives

$$\theta_a = \theta_{^{\mathrm{R}}b}. \tag{57}$$

In particular, if we take $b = a$, we find:

$$\theta_a = \theta_{^{\mathrm{R}}a}. \tag{58}$$

Since we also know that the reflection action must satisfy

$$\theta_a = \theta_{^{\mathrm{R}}a}^*, \tag{59}$$

we find that θ_a must be real. This proves (52).

Examining the example of $\mathrm{USp}(4)_2$ (again using time-reversal $\mathbb{Z}_2^{\mathbf{T}}$ as the example), we find:

$$M_a = S_{a1} + S_{a\epsilon}\eta_\epsilon^{\mathbf{T}}. \tag{60}$$

We see that M_a will all be integer only if we set $\eta_\epsilon^{\mathbf{T}} = -1$. However, this then implies that $M_{\psi_+} = M_{\psi_-} > 0$. But $\theta_{\psi_+} = \theta_{\psi_-}^* = i \neq \pm 1$. We thus conclude again that $\mathrm{USp}(4)_2$ cannot be a $\mathbb{Z}_2^{\mathbf{T}}$ time-reversal invariant topologial phase that exists on the surface of a (3+1)D $\mathbb{Z}_2^{\mathbf{T}}$ SPT.

E. Conflicting constraints on symmetry fractionalization quantum numbers

Here we point out that another symptom of the $\mathcal{H}_{[\rho]}^3(\mathbb{Z}_2^{\mathbf{T}}, \mathcal{A})$ symmetry localization anomaly arises in terms of conflicting constraints on the time-reversal symmetry fractionalization patterns.

On the one hand, Ref. 19 derived a set of constraints for $\eta_a^{\mathbf{T}}$, which showed that

$$\eta_a^{\mathbf{T}} = \theta_a, \quad \text{if } N_{c\,{}^{\mathbf{T}}c}^a \text{ is odd, for any } c. \tag{61}$$

In the case of $\mathrm{USp}(4)_2$, we have $\psi_+ \times \psi_- = \epsilon + \cdots$, and ${}^{\mathbf{T}}\psi_+ = \psi_-$. Therefore, we must have

$$\eta_\epsilon^{\mathbf{T}} = 1. \tag{62}$$

On the other hand, we show below that by a different argument, we must have

$$\eta_\epsilon^{\mathbf{T}} = -1. \tag{63}$$

It is the incompatibility of the above two results that signals the presence of an $\mathcal{H}^3$ anomaly for the $\mathbb{Z}_2^{\mathbf{T}}$ symmetry.

To see how Eq. (63) arises, we observe that since ϵ is an Abelian boson, we can condense it, leading us to a new topological phase, which in this case corresponds to $\mathrm{SU}(5)_1$ CS theory. To be more explicit, we label the anyons in $\mathrm{SU}(5)_1$ by $[j]$, with $j = 0, \ldots, 4$ (mod 5). They form a $\mathbb{Z}_5$ group under fusion. The topological twists are $\theta_j = e^{\frac{4\pi i}{5}j^2}$. This theory has a charge conjugation symmetry $\mathbf{C} : [j] \leftrightarrow [-j]$.

In addition, we observe that $\mathrm{SU}(5)_1$ CS theory has a $\mathbb{Z}_4^{\mathbf{T}}$ symmetry, where

$$\mathbf{T} : [j] \to [2j], \tag{64}$$

so that $\mathbf{T}^2 = \mathbf{C}$, and $\mathbf{C}^2 = 1$. Notice that similar to $\mathrm{USp}(4)_2$, the chiral central charge $c = 4$ implying that the theory can only be time-reversal invariant at the surface of a (3+1)D theory.

The inverse process of condensing an Abelian $\mathbb{Z}_2$ boson in $\mathrm{USp}(4)_2$ is to gauge the unitary $\mathbb{Z}_2$ symmetry $\mathbf{C}$ in $\mathrm{SU}(5)_1$. The general procedure for gauging is discussed in Ref. 19. To see how gauging $\mathbf{C}$ in $\mathrm{SU}(5)_1$ yields $\mathrm{USp}(4)_2$, let us enumerate the anyons in the gauged theory. First we enlarge the theory to include extrinsic $\mathbb{Z}_2$ symmetry defects, denoted by ψ. ψ satisfies the following fusion rules:

$$\begin{aligned} \psi \times \psi &= [0] + [1] + [2] + [3] + [4], \\ \psi \times [a] &= \psi. \end{aligned} \tag{65}$$

We then need to project the whole theory to the $\mathbb{Z}_2$-invariant subspace, which in the mathematics literature is referred to as "equivariantization." This amounts to (a) reorganizing objects into orbits under the symmetry, and (b) including symmetry charges. In the present case, the nontrivial anyons in $\mathrm{SU}(5)_1$ form two orbits: $\{[1], [4]\}$ and $\{[2], [3]\}$, which correspond to ϕ_1 and ϕ_2. Both the identity and the defect split into two, carrying opposite $\mathbb{Z}_2$ charges, which are $1, \epsilon$ for the identity and ψ_+, ψ_- for ψ.

Since $\mathbf{T}^2$ is gauged, and since ϵ is the $\mathbb{Z}_2$ gauge charge, we must have that

$$\eta_\epsilon^{\mathbf{T}} = -1, \tag{66}$$

because $\eta_\epsilon^{\mathbf{T}}$ is precisely the local $\mathbf{T}^2$ value of ϵ. The anyon permutation given in Eq. (64) becomes $\phi_1 \leftrightarrow \phi_2$ in the gauged theory.

a	1	ϵ	ϕ_1	ϕ_2	ψ_+	ψ_-
θ_a	1	1	$e^{\frac{4\pi i}{5}}$	$e^{-\frac{4\pi i}{5}}$	1	-1
d_a	1	1	2	2	$\sqrt{5}$	$\sqrt{5}$
${}^{\mathbf{T}}a$	1	ϵ	ϕ_2	ϕ_1	ψ_+	ψ_-

TABLE II. Anyon types, topological spins, and quantum dimensions for $\mathrm{USp}(4)_2^\vee$.

Another possible way to see that $\eta_\epsilon^{\mathbf{T}} = -1$ follows from the general constraints of Eq. (32). While we do not pursue this analysis in detail here, we note that a similar analysis was performed in the case of $\mathrm{D}(\mathbb{S}_3)$ in the Appendix of Ref. 41. Here $\mathrm{D}(\mathbb{S}_3)$ refers to the quantum double of $\mathbb{S}_3$, the permutation group on three elements.

F. Non-anomalous cousins

We have glossed over an important detail of the gauging procedure, upon which we now elaborate. As discussed in Ref. 19, once the $\mathbb{Z}_2$ symmetry fractionalization class is chosen, the remaining $\mathbb{Z}_2$ symmetry enriched phases are related to each other by elements of $\mathcal{H}^3(\mathbb{Z}_2, \mathrm{U}(1)) = \mathbb{Z}_2$. The difference between these two can be thought of as stacking a $\mathbb{Z}_2$ (2+1)D SPT before gauging (in high energy field theory language, this corresponds to adding a Dijkgraaf-Witten[58] term for the $\mathbb{Z}_2$ gauge field). The SPT phase does not change either the bulk anyons or the chiral central charge. The only effect is that the topological twist factors of the $\mathbb{Z}_2$ gauge fluxes $\psi_\pm$ are modified by a factor of i. Namely, we will have $\theta_{\psi_+} = 1, \theta_{\psi_-} = -1$ instead.

This implies that $\mathrm{USp}(4)_2$ has a partner theory with the same fusion rules, where $\psi_\pm$ are invariant under the action of time-reversal (see Table II). Let us refer to this theory as $\mathrm{USp}(4)_2^\vee$. We can readily compute for this theory that

$$\mathcal{Z}(\mathbb{RP}^4) = \frac{1}{\sqrt{20}}(1 + \eta_\epsilon^{\mathbf{T}} + \sqrt{5}\eta_{\psi_+}^{\mathbf{T}} - \sqrt{5}\eta_{\psi_-}^{\mathbf{T}}). \tag{67}$$

Using $\eta_\epsilon^{\mathbf{T}} = -1$ and $\eta_{\psi_-}^{\mathbf{T}} = \eta_\epsilon^{\mathbf{T}}\eta_{\psi_+}^{\mathbf{T}}$, we obtain:

$$\mathcal{Z}(\mathbb{RP}^4) = \eta_{\psi_+}^{\mathbf{T}} = \pm 1. \tag{68}$$

Since ψ_+ and ψ_- are both time-reversal invariant, we no longer have the constraint $\eta_\epsilon^{\mathbf{T}} = \theta_\epsilon$. Therefore, $\mathrm{USp}(4)_2^\vee$ does not lead to conflicting constraints on $\eta_\epsilon^{\mathbf{T}}$. We conclude that $\mathrm{USp}(4)_2^\vee$ does not possess the $\mathcal{H}^3(\mathbb{Z}_2^{\mathbf{T}}, \mathcal{A})$ anomaly. Depending on the value of $\eta_{\psi_+}^{\mathbf{T}}$, it does have the SPT ('t Hooft) anomalies associated with $\mathcal{Z}(\mathbb{RP}^4)$ and $\mathcal{Z}(\mathbb{CP}^2)$, which indicate which (3+1)D $\mathbb{Z}_2^{\mathbf{T}}$ SPT state hosts it at the surface. In particular, if we see $\eta_{\psi_+}^{\mathbf{T}} = 1$, the $\mathrm{USp}(4)_2^\vee$ theory only has the $\mathcal{Z}(\mathbb{CP}^2)$ anomaly, which matches the SPT (t' Hooft) anomaly in $\mathrm{SU}(5)_1$.

Another way of stating the relation between $\mathrm{USp}(4)_2$ and $\mathrm{USp}(4)_2^\vee$ is as follows. We can stack a double semion state[59] on top of $\mathrm{USp}(4)_2^\vee$. Note that the double semion state is described by $\mathrm{U}(1)_2 \times \mathrm{U}(1)_{-2}$ CS theory. We denote the four

anyons in the double semion state by $1, s, s', b = s \times s'$, where s (s') is a semion(anti-semion). The resulting theory has 24 particles. In particular, it has a boson (b, ϵ), consisting of the boson from the double semion state and the ϵ from the $\mathrm{USp}(4)_2^\vee$ state. If we condense the composite (b, ϵ), the following particles remain deconfined after the condensation:

$$1 \sim (b, \epsilon), (1, \epsilon) \sim (b, 1), (1, \phi_i) \sim (b, \phi_i)$$
$$(s, \psi_+) \sim (s', \psi_-), (s, \psi_-) \sim (s', \psi_+), \tag{69}$$

where $\sim$ here means differing by the condensed particle. We can readily see that the resulting theory is $\mathrm{USp}(4)_2$.

However, this condensation process must break the $\mathbb{Z}_2^\mathbf{T}$ symmetry. This is because the b boson must have $\eta_b^\mathbf{T} = \theta_b = 1$, because $b = s \times s'$ and $s' = {}^\mathbf{T}s$ due to the opposite topological spins. Therefore, the bound state (b, ϵ) also has $\eta_{(b,\epsilon)}^\mathbf{T} = -1$, i.e. it carries a local Kramers degeneracy. Condensing (b, ϵ) must therefore break time-reversal symmetry.

To summarize, we have found a cousin theory $\mathrm{USp}(4)_2^\vee$ of $\mathrm{USp}(4)_2$ which has at most t'Hooft anomaly.

V. THREE RESOLUTIONS

Above we saw that the $\mathbb{Z}_2^\mathbf{T}$ symmetry in $\mathrm{USp}(4)_2$ possesses an $\mathcal{H}^3$ anomaly, implying that the symmetry action cannot be consistently localized to the quasiparticles. In the following we discuss three possible resolutions of this anomaly.

A. Enlarging the symmetry from $\mathbb{Z}_2^\mathbf{T}$ to $\mathbb{Z}_4^\mathbf{T}$

One resolution of the $\mathbb{Z}_2^\mathbf{T}$ symmetry localization anomaly is that the theory actually does not have a $\mathbb{Z}_2^\mathbf{T}$ symmetry, but rather the true symmetry is $\mathbb{Z}_4^\mathbf{T}$.

Mathematically, one can show straightforwardly that if the symmetry group is enlarged to $\mathbb{Z}_4^\mathbf{T}$, the obstruction class we found earlier (naturally embedded into the larger symmetry group) becomes trivial. Here instead we will present a physical argument, explicitly constructing the $\mathrm{USp}(4)_2$ theory in a system with $\mathbb{Z}_4^\mathbf{T}$ time-reversal symmetry.

Let us start from a system of bosons where each boson, ϕ, has $\mathbf{T}^2 = -1$ (e.g. spin-1/2 bosons). Thus globally $\mathbf{T}^2 = (-1)^{N_\phi}$ on the whole system, where N_ϕ is the number of bosons. Microscopically the system therefore possesses a $\mathbb{Z}_4^\mathbf{T}$ symmetry. Pairs of bosons in this system thus locally have $\mathbf{T}^2 = 1$. Let us suppose that the paired bosons realize a topological phase described at long wavelengths by two decoupled theories, consisting of the double semion state and the $\mathrm{USp}(4)_2^\vee$ state. Now consider the composite (b, ϵ) (recall b is the topologically non-trivial boson of the double semion state). Depending on whether we attach the fundamental $\mathbf{T}^2 = -1$ boson ϕ, the composite (b, ϵ) can be a Kramers singlet, so that its condensation no longer breaks any symmetry. This way we have a realization of $\mathrm{USp}(4)_2$ with $\mathbb{Z}_4^\mathbf{T}$ symmetry (up to the chiral central charge anomaly, which can be cancelled by a bulk (3+1)D SPT).

We notice that similar resolutions apply to unitary symmetry groups as well. For example, the $\mathcal{H}^3$ obstruction for a $\mathbb{Z}_2$ symmetry in the $\mathbb{D}_{16}$ gauge theory can be avoided if the symmetry group is actually $\mathbb{Z}_4$.[45]

B. $\mathrm{USp}(2N)_N$ CS theory as a fermionic (spin) theory with SPT (t 'Hooft) anomaly

In Ref. 48, it was proposed that $\mathrm{USp}(2N)_N$ CS theory is time-reversal invariant as a fermionic theory. Here we consider the case where $\mathbf{T}^2 = -1$ on the electron creation/annihilation operators, so that globally $\mathbf{T}^2 = (-1)^{N_f}$ on the state of the system, with $(-1)^{N_f}$ being the fermion parity of the system. We show that $\mathrm{USp}(4)_2$ possesses a time-reversal anomaly that can be cancelled by a bulk (3+1)D electronic time-reversal-invariant topological superconductor in class DIII. In other words, the theory no longer has any $\mathcal{H}^3$ anomaly, but it does have an SPT anomaly. This may look similar to the $\mathbb{Z}_4^\mathbf{T}$ symmetry we discussed in the previous section, however we cannot use the same argument to resolve the anomaly due to the fermionic statistics.

Recall that $\mathrm{USp}(4)_2$ has a cousin $\mathrm{USp}(4)_2^\vee$ that is free of the $\mathcal{H}^3(\mathbb{Z}_2^\mathbf{T}, \mathcal{A})$ anomaly. As discussed earlier in Sec. IV F, the two theories are related by stacking with a double semion state and condensing certain bosonic quasiparticles. To preserve the time-reversal symmetry, the boson b in the double semion has to have $\eta_b^\mathbf{T} = -1$, which is impossible in bosonic systems in two dimensions. However, if the double semion theory is viewed as a state arising from microscopic degrees of freedom that contain fermions (i.e. as a spin theory), then time-reversal symmetry can be implemented differently:

$$\mathbf{T} : s \leftrightarrow s \times f,$$
$$s' \leftrightarrow s' \times f. \tag{70}$$

Here f denotes the local fermion, which is transparent under braiding with the anyons of the system. With this action of $\mathbf{T}$, time-reversal symmetry changes the local fermion parity on the semion and anti-semion. As shown in Ref. 35, if an anyon a transforms as $a \to a \times f$ under $\mathbf{T}$, then (i) f must have $\mathbf{T}^2 = -1$, i.e. the fermions are Kramers doublets; and (ii) a has a well-defined "$\mathbf{T}^2$" value which can now be $\pm i$ and which will also be denoted by $\eta_a^\mathbf{T}$. Such anyons are said to carry a "Majorana Kramers doublet". However, unlike in bosonic systems, for two anyons a and b both having Majorana Kramers doublets, and c with $N_{ab}^c = 1$, one can show that

$$\eta_c^\mathbf{T} = -\eta_a^\mathbf{T} \eta_b^\mathbf{T}. \tag{71}$$

In the double semion state above, since both s and s' carry Majorana Kramers doublets we have $\eta_b^\mathbf{T} = -\eta_s^\mathbf{T} \eta_{s'}^\mathbf{T}$. Therefore, if $\eta_s^\mathbf{T} = i, \eta_{s'}^\mathbf{T} = -i$, we have $\eta_b^\mathbf{T} = -1$. This is precisely the surface topological order of a $\nu = 4$ class DIII topological superconductor ($\nu = 1$ is the root phase, having a single Majorana cone on the surface without any interactions).

With this anomalous fermionic double semion theory, we can condense (b, ϵ) in $\mathrm{USp}(4)_2^\vee$ stacked with the double

semion state, obtaining a **T**-invariant $\mathrm{USp}(4)_2$ theory with $\mathbf{T}^2 = (-1)^{N_f}$. This provides a surface topological order for the $\nu = 4$ class DIII topological superconductor. Under **T**, the anyons are transformed according to

$$\mathbf{T} : \phi_1 \leftrightarrow \phi_2, \psi_\pm \leftrightarrow \psi_\pm f. \tag{72}$$

We note that this conclusion is consistent with the use of the fermionic anomaly indicator formula discussed in Ref. 60 and 61.

C. Pseudo-2D realization at the surface of (3+1)D SETs

While strictly speaking $\mathbb{Z}_2^{\mathbf{T}}$ cannot be a symmetry of theories with an $\mathcal{H}^3$ anomaly, below we will demonstrate a precise sense in which these theories can admit a $\mathbb{Z}_2^{\mathbf{T}}$ symmetry, provided they are "pseudo-realized" at the surface of a (3+1)D topologically ordered bulk state. From the discussion of Sec. IV F, we see that it is sufficient to show that the anomalous double semion state (where $\eta_b^{\mathbf{T}} = -1$) can be "pseudo-realized."

1. Pseudo-realizing double semion with $\eta_b^{\mathbf{T}} = -1$ on the surface of a (3+1)D SET

As discussed in the preceding sections, in (2+1)D a double semion state with $\mathbb{Z}_2^{\mathbf{T}}$ time-reversal symmetry must satisfy $\eta_b^{\mathbf{T}} = \theta_b = 1$ because $N_{s\,\mathbf{T}\,s}^b = 1$.

Nevertheless, below we will show how one can realize a state that is closely related (but strictly speaking not identical) to the double semion state with $\eta_b^{\mathbf{T}} = -1$ on the surface of a bulk (3+1)D system with long-range entanglement. To understand this, it is helpful to consider spatial reflection symmetry, **R**, instead of time-reversal symmetry. In the TQFT, we replace **T** by **CR**. Since charge conjugation **C** acts trivially here, the action of **T** can just be replaced with the action of **R**.

2. $\mathbb{Z}_2$ gauge theory on the bulk mirror plane

In Ref. 33, it was shown how (3+1)D reflection invariant SPTs (with $\mathbf{R}^2 = \mathbb{1}$), can be understood by restricting attention to the mirror plane. On the mirror plane, the reflection symmetry acts like an on-site $\mathbb{Z}_2$ symmetry. Thus, to understand (3+1)D SPTs with $\mathbb{Z}_2^{\mathbf{R}}$ reflection symmetry, one is led to considering the possible existence of non-trivial (2+1)D SPT states with on-site (internal) $\mathbb{Z}_2$ symmetry existing on the mirror plane.

Here, we instead consider a (2+1)D SET on the mirror plane. Specifically, on the mirror plane we consider a $\mathbb{Z}_2$ toric code state, which is described by a dynamical $\mathbb{Z}_2$ (untwisted) gauge theory. Furthermore, we consider a global $\mathbb{Z}_2^{\mathbf{R}}$ reflection symmetry, which acts as an on-site $\mathbb{Z}_2$ symmetry on the mirror plane. The anyons in $\mathbb{Z}_2$ gauge theory are denoted by $\{1, e, m, \psi\}$ where e (m) is the $\mathbb{Z}_2$ gauge charge (flux). We
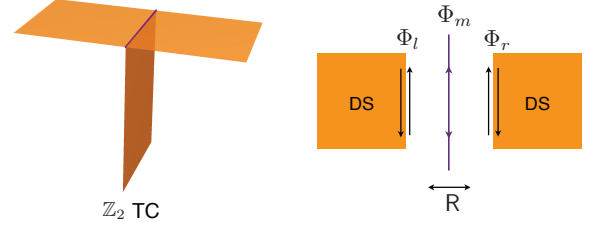

FIG. 1. Illustration of the construction of the anomalous double semion state on the surface of a $\mathbb{Z}_2$ toric code on the mirror plane. Left panel: 3D setup, where a $\mathbb{Z}_2$ gauge theory harbors the mirror plane. Right panel: top view of the surface.

further consider the case where the e and m particles of the $\mathbb{Z}_2$ gauge theory both carry fractional $\mathbb{Z}_2$ charge under the global $\mathbb{Z}_2$ symmetry.

Next, we consider a (2+1)D surface of the (3+1)D bulk system. At a given time-slice, the (2+1)D surface forms a plane which is perpendicular to the mirror plane of the (3+1)D bulk on which the $\mathbb{Z}_2$ gauge theory lives (see Fig. 1). We consider the case where the surface theory is a double semion state. Below we will demonstrate that this surface theory can have $\eta_b^{\mathbf{R}} = -1$.

To demonstrate this explicitly, we imagine cutting the system along the mirror axis on the surface, so that we have three subsystems: the double semion edge states on the left and right of the mirror plane, and the $\mathbb{Z}_2$ gauge theory on the mirror plane. We need to show that at the $(1 + 1)$ dimensional intersection between the surface and the bulk mirror plane, the three pairs of edge modes can be fused together in a way which preserves the global symmetry and which is gapped everywhere. Note that with the on-site $\mathbb{Z}_2$ symmetry, the $\mathbb{Z}_2$ gauge theory has gapless edge modes, since a gapped edge has to correspond to either e or m condensation, which necessarily breaks the symmetry because both of them carry half charge.

To this end, we see that the $(1 + 1)$D theory at the junction consists of the edge theories of the three subsystems. Each of them admits a description as a non-chiral Luttinger liquid, and altogether can be compactly written as

$$\mathcal{L} = \frac{1}{4\pi} \partial_t \Phi^{\mathsf{T}} K \partial_x \Phi - \dots \tag{73}$$

here $\Phi = (\phi_{1l}, \phi_{2l}, \phi_{1r}, \phi_{2r}, \phi_{1m}, \phi_{2m})^{\mathsf{T}}$, where $\phi_{l/r}$ refer to the edge bosonic fields of the double semion states on the left/right, and ϕ_m refer to the edge fields of the $\mathbb{Z}_2$ gauge theory on the mirror plane. The K-matrix reads

$$K = \begin{pmatrix} 2\sigma^z & 0 & 0 \\ 0 & 2\sigma^z & 0 \\ 0 & 0 & 2\sigma^x \end{pmatrix} \tag{74}$$

The $\pm 2\sigma^z$ parts describe the contribution from the double semion state on either side of the interface. The $2\sigma^x$ part describes the contribution from the $\mathbb{Z}_2$ gauge theory on the mirror plane. We adopt the convention that the e particle at

the mirror plane corresponds to the operator $e^{i\phi_{1m}}$, and the m particle corresponds to the operator $e^{i\phi_{2m}}$. The $\mathbb{Z}_2^{\mathbf{R}}$ symmetry acts in the following way on these fields:

$$\phi_{1m} \rightarrow \phi_{1m} + \frac{\pi}{2}$$
$$\phi_{2m} \rightarrow \phi_{2m} + \frac{\pi}{2}$$
$$\phi_{1l} \leftrightarrow \phi_{1r}$$
$$\phi_{2l} \leftrightarrow \phi_{2r}$$

The first two transformations on ϕ_{1m} and ϕ_{2m} encode the fact that the e and m particles carry half $\mathbb{Z}_2$ charge. The boson operator on one side of the interface is $b = e^{i(\phi_{1l}+\phi_{2l})}$, and on the other side of the interface is $b = e^{i(\phi_{1r}+\phi_{2r})}$.

The idea in this construction will be that the boson on the surface will be the $\mathbb{Z}_2$ gauge charge on the mirror plane. Therefore we consider the following gapping terms:

$$\begin{aligned}\delta\mathcal{L} = &- u\cos 2(\phi_{1l} + \phi_{2l} - \phi_{1m}) \\ &+ u\cos 2(\phi_{1r} + \phi_{2r} - \phi_{1m}) \\ &- v\cos 2(\phi_{1l} - \phi_{2l} + \phi_{1r} - \phi_{2r} - 2\phi_{2m})\end{aligned} \quad (75)$$

The gapping terms preserve the reflection symmetry defined in Eq. (75). The first two terms show that the combination $b \times e$ is condensed at the junction, which implies that at the junction, b can continue into the mirror plane as the e particle, which is the $\mathbb{Z}_2$ gauge charge.

The above gapping terms can be written as

$$\delta\mathcal{L} = \sum_a t_a \cos(\mathbf{\Lambda}_a^{\mathsf{T}} K \Phi), \quad (76)$$

where the integer vectors $\mathbf{\Lambda}_a$ determine the gapping terms. In the present case, we have

$$\begin{aligned}\mathbf{\Lambda}_1 &= (1, 1, 0, 0, -1, 0) \\ \mathbf{\Lambda}_2 &= (0, 0, 1, 1, -1, 0) \\ \mathbf{\Lambda}_3 &= (1, -1, 1, -1, 0, -1)\end{aligned} \quad (77)$$

and $t_1 = -t_2 = u, t_3 = v$. One can see that these vectors are null vectors for the K-matrix, as they satisfy

$$\mathbf{\Lambda}_a^{\mathsf{T}} K \mathbf{\Lambda}_b = 0 \quad (78)$$

for all $a, b = 1, 2, 3$. We should also check that the global $\mathbb{Z}_2$ symmetry is not spontaneously broken in the ground state. Using the criteria derived in Ref. 15, we can in fact show that the gapping terms lead to a unique gapped ground state, thus excluding the possibility of spontaneous symmetry breaking.

Importantly, observe that the operator $e^{i(\phi_{1l}-\phi_{2l})+i(\phi_{1r}-\phi_{2r})}$ creates a pair of b bosons on either side of the mirror plane, in a mirror-symmetric way. Due to the third gapping term above, we see that since $e^{i(\phi_{1l}-\phi_{2l}+\phi_{1r}-\phi_{2r}+2\phi_{2m})}$ is condensed, then $e^{i(\phi_{1l}-\phi_{2l})+i(\phi_{1r}-\phi_{2r})}$ can be replaced by $e^{2i\phi_{2m}}$. Since under the global $\mathbb{Z}_2^{\mathbf{R}}$ symmetry $\phi_{2m} \rightarrow \phi_{2m} + \pi/2$, we see that this operator goes to minus itself, which is precisely the definition of the reflection eigenvalue $\eta_b^{\mathbf{R}} = -1$.
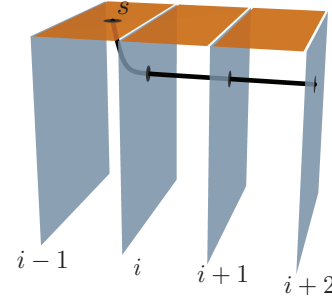

FIG. 2. Illustration of the layer construction. Each vertical layer denotes a $\mathbb{Z}_2$ toric code, and neighboring layers are coupled such that pairs of e particles are condensed. The surface consists of a double semion state, joined to the vertical layers at each (1+1)D intersection by condensing $b \times e$.

3. $\mathbb{Z}_2$ gauge theory in the bulk (3+1)D system

In the previous section we showed how the double semion state with $\eta_b^{\mathbf{R}} = -1$ can be realized at the surface of a bulk (3+1)D system with a certain $\mathbb{Z}_2$ gauge theory on the bulk mirror plane. Here we will briefly point out that with this starting point, one can then consider a layer construction, where we stack (2+1)D $\mathbb{Z}_2$ toric code states on planes parallel to the mirror plane (see Fig. 2). We condense pairs of e particles from neighboring planes, so that the e particle can propagate in three dimensions. The other deconfined excitations are strings of m particles from each layer. This way we get a bulk (3+1)D $\mathbb{Z}_2$ gauge theory, with a global $\mathbb{Z}_2^{\mathbf{R}}$ reflection symmetry.

At the interface of the bulk planes with the (2+1)D surface we condense pairs of e and b particles together, so that the b particle can just propagate from the surface into the bulk. Because of the condensation of $b \times e$ at each intersection, if a semion/anti-semion were to propagate on the surface, each time it passes through an intersection it has to bind with a m particle from the layer. In other words, due to the mutual statistics of the particles, the condensation of $b \times e$ confines the semions/anti-semions s and s' to the end points of the m-strings.

We refer to the above as a "pseudo-realization" of the double semion state with $\eta_b^{\mathbf{R}} = -1$ at the surface of a (3+1)D $\mathbb{Z}_2$ gauge theory. The reason we call it a "pseudo-realization" is because strictly speaking, the double semion is not confined to live on the surface of the (3+1)D system anymore. The boson b can propagate into the bulk, while the semions at the surface are bound to the end-points of $\mathbb{Z}_2$ flux strings in the bulk. In this case, since the endpoints of the flux strings do not have a well-defined topological spin, it is no longer meaningful to associate them with semions.

We thus find that while the double semion state with $\eta_b^{\mathbf{R}} = -1$ is strictly speaking not allowed, it can be "pseudo-realized" at the surface of a bulk (3+1)D SET with global $\mathbb{Z}_2^{\mathbf{R}}$ reflection symmetry.

D. Pseudo-realizing $\mathrm{USp}(4)_2$

Combining the above results, we can now conclude the following. $\mathrm{USp}(4)_2$ possesses an $\mathcal{H}^3(\mathbb{Z}_2^{\mathbf{T}}, \mathcal{A})$ time-reversal anomaly. It has a cousin $\mathrm{USp}(4)_2^{\vee}$ which does not possess this $\mathcal{H}^3$ time-reversal anomaly. One can obtain $\mathrm{USp}(4)_2$ from $\mathrm{USp}(4)_2^{\vee}$ by stacking a double semion on the latter, and condensing the combination of $\epsilon \times b$. If we demand that $\eta_b^{\mathbf{T}} = -1$, then the bound state will have $\eta_{b\epsilon}^{\mathbf{T}} = 1$, and therefore condensing it will not break $\mathbf{T}$. However such a double semion state with $\mathbb{Z}_2^{\mathbf{T}}$ symmetry can only be pseudo-realized at the surface of a bulk $(3+1)\mathrm{D}$ that contains a dynamical $\mathbb{Z}_2$ gauge field. Therefore $\mathrm{USp}(4)_2$ can be pseudo-realized with $\mathbb{Z}_2^{\mathbf{T}}$ symmetry at the surface of a bulk $(3+1)D$ $\mathbb{Z}_2$ gauge theory.

VI. MORE EXAMPLES OF THEORIES WITH $\mathcal{H}^3$ ANOMALIES

We have seen that the $\mathcal{H}^3_{[\rho]}(\mathbb{Z}_2^{\mathbf{T}}, \mathcal{A})$ anomaly in $\mathrm{USp}(4)_2$ CS theory is intimately associated with the $\mathbb{Z}_4^{\mathbf{T}}$ symmetry in $\mathrm{SU}(5)_1$ CS theory. Below we will see that this is a general phenomenon. We provide an infinite series of TQFTs with $\mathcal{H}^3_{[\rho]}(\mathbb{Z}_2^{\mathbf{T}}, \mathcal{A})$ anomalies, obtained by gauging $\mathbf{T}^2$ in a theory with $\mathbb{Z}_4^{\mathbf{T}}$ symmetry.

The essential point is that when we gauge $\mathbf{T}^2$, we can consider adding a Dijkgraaf-Witten (DW) term for the gauge field associated with the $\mathbf{T}^2$ symmetry. Physically this corresponds to stacking a $\mathbb{Z}_2$ SPT (associated with $\mathbf{T}^2$) together with the theory with the $\mathbb{Z}_4^{\mathbf{T}}$ symmetry. This combined system no longer possesses $\mathbb{Z}_4^{\mathbf{T}}$ symmetry, because a $\mathbb{Z}_2$ SPT is incompatible with $\mathbb{Z}_4^{\mathbf{T}}$ symmetry if the $\mathbb{Z}_2$ SPT is to be protected by $\mathbf{T}^2$. We will see that the theory obtained by gauging $\mathbf{T}^2$ under these circumstances thus leads us to another theory that contains the $\mathcal{H}^3$ anomaly.

A. Gauging $\mathbf{T}^2$ in $\mathrm{USp}(4)_2$ CS theory

Given that $\mathrm{USp}(4)_2$ supports a $\mathbb{Z}_4^{\mathbf{T}}$ symmetry, we can study the time-reversal symmetry properties of the theories obtained by gauging the global unitary $\mathbb{Z}_2$ symmetry $\mathbf{T}^2$.[19]

The symmetry fractionalization associated with the $\mathbb{Z}_2$ global symmetry is classified by $\mathcal{H}^2(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$. In one case, none of the particles of $\mathrm{USp}(4)_2$ carry fractional quantum numbers; naively gauging the $\mathbb{Z}_2$ gives rise to two decoupled theories: $\mathrm{USp}(4)_2 \times D(\mathbb{Z}_2)$, where $D(\mathbb{Z}_2)$ refers to a $\mathbb{Z}_2$ discrete gauge theory. Depending on whether there is a Dijkgraaf-Witten (DW) term in the effective action for the $\mathbb{Z}_2$ gauge field, the $D(\mathbb{Z}_2)$ factor will correspond to a toric code or double semion theory. One then runs into the contradiction that since the $\mathbb{Z}_2$ gauge theory is decoupled from $\mathrm{USp}(4)_2$, the theory continues to have the same $\mathcal{H}^3_{[\rho]}(\mathbb{Z}_2^{\mathbf{T}}, \mathcal{A})$ anomaly, which is impossible if we start from an anomaly-free theory.

The resolution is that we have not been careful enough in gauging. The correct theory obtained from gauging can be most easily obtained from the physical construction in Sec.

V A. There we observed that $\mathrm{USp}(4)_2$ with $\mathbb{Z}_4^{\mathbf{T}}$ symmetry can be obtained as follows. We start with a theory of physical bosons ϕ, which are the local degrees of freedom of the theory, and which have $\mathbf{T}^2 = -1$. Then we imagine that pairs of bosons form a topological state described by $\mathrm{USp}(4)_2^{\vee} \times \mathrm{DS}$. Next we condense the particle $b\epsilon$ combined with the local boson ϕ. Since $b\epsilon$ has $\mathbf{T}^2 = -1$, combining it with ϕ yields a particle which is a Kramers singlet, which can then be condensed without breaking the $\mathbb{Z}_4^{\mathbf{T}}$ symmetry.

Thus to gauge $\mathbf{T}^2$ in $\mathrm{USp}(4)_2$, we first start from $\mathrm{USp}(4)_2^{\vee} \times \mathrm{DS}$, then we gauge the $\mathbf{T}^2$ symmetry and subsequently condense the particle that corresponds to the fusion of $b\epsilon$ and the $\mathbb{Z}_2$ charge of the $\mathbf{T}^2$ gauge field.

Since we are considering the case with no DW term and trivial symmetry fractionalization classes, the result for gauging $\mathbf{T}^2$ in $\mathrm{USp}(4)_2^{\vee} \times \mathrm{DS}$ is $\mathrm{USp}(4)_2^{\vee} \times \mathrm{DS} \times D(\mathbb{Z}_2)$. Here $D(\mathbb{Z}_2)$ refers to an untwisted $\mathbb{Z}_2$ gauge theory. Denote the four particles in $D(\mathbb{Z}_2)$ as $\{1, e, m, \psi = e \times m\}$ where e is interpreted as the gauge charge and therefore has $\mathbf{T}^2 = -1$. Thus we now condense ϵbe. After condensation, we can identify an Abelian subsector

$$1, b \sim \epsilon e, sm \sim s'\psi\epsilon, s'm \sim s\psi\epsilon$$
$$\epsilon, \epsilon b \sim e, \epsilon sm \sim s'\psi, \epsilon s'm \sim s\psi \tag{79}$$

The first line is nothing but a DS theory. We can further identify another sector

$$1, \epsilon, \psi_+ m, \psi_- m, \phi_1, \phi_2, \tag{80}$$

which is closed under fusion and can again be identified as $\mathrm{USp}(4)_2^{\vee}$. Thus the resulting theory is $\mathrm{USp}(4)_2^{\vee} \times \mathrm{DS}$, and $\mathbf{T}$ acts diagonally on the two sectors. This theory is free of any $\mathcal{H}^3$ anomaly, as it should be. If we further add a DW term in gauging the $\mathbf{T}^2$ symmetry, we would obtain a theory with $\mathcal{H}^3$ anomaly again, which turns out to be $\mathrm{USp}(4)_2 \times D(\mathbb{Z}_2)$.

In the case where the $\mathrm{USp}(4)_2$ particles do carry fractional quantum numbers (corresponding to the non-trivial $\mathbb{Z}_2$ fractionalization class), we can obtain the gauged theory as follows.

In the physical construction of Sec. V A, we let the s and s' particles in the DS sector both have fractional quantum numbers, i.e.

$$\eta_s(\mathbf{T}^2, \mathbf{T}^2) = \eta_{s'}(\mathbf{T}^2, \mathbf{T}^2) = -1. \tag{81}$$

One can easily see that after condensing $b\epsilon$ together with a physical boson ϕ, we do get a $\mathrm{USp}(4)_2$ where $\psi_{\pm}$ have fractional quantum numbers under $\mathbf{T}^2$.

The gauged theory can be obtained as follows: first we add a gauge flux σ. Due to the fractionalization of $\mathbf{T}^2$, the flux satisfies the fusion rule $\sigma^2 = b$. The topological twist of this flux can be chosen to be either $\theta_{\sigma} = 1$ or $\theta_{\sigma} = i$, depending on whether a DW term is included or not. Without loss of generality we also set

$$M_{s,\sigma} = i, M_{s',\sigma} = -i, \tag{82}$$

where recall M_{ab} is the braiding phase between a and b, defined in Eq. 30. Further we also need to add a bosonic $\mathbf{T}^2$

gauge charge e, such that $s^2 = s'^2 = e$. Notice that we still define $b = ss'$.

When $\theta_\sigma = 1$, we can identify the gauged theory as $\mathrm{USp}(4)_2^\vee \times \mathrm{D}(\mathbb{Z}_4)$. The $\mathbb{Z}_4$ gauge charge, which we label as $(1,0)$, is identified with σ. The $\mathbb{Z}_4$ gauge flux $(0,1)$ can be identified with $s'\sigma e = s\sigma b$. The $\mathbf{T}^2$ gauge charge e becomes $(2,2)$. For reference, the topological twist of an anyon (a,b) in $\mathrm{D}(\mathbb{Z}_4)$ is

$$\theta_{(a,b)} = i^{ab}. \tag{83}$$

Under time reversal symmetry $\mathbf{T}$ (which now satisfies $\mathbf{T}^2 = \mathbf{1}$), we find that e and σ are invariant, while $s\sigma b \to s\sigma e = \overline{s\sigma b}$. We have $\eta_\sigma^{\mathbf{T}} = 1$, $\eta_b^{\mathbf{T}} = 1$, $\eta_e^{\mathbf{T}} = -1$, and $\eta_{be}^{\mathbf{T}} = -1$, which form a consistent set of time-reversal symmetry fractionalization quantum numbers. Therefore, as expected, the case where $\theta_\sigma = 1$, which corresponds to no added DW term, does not have a $\mathcal{H}^3$ anomaly.

When $\theta_\sigma = i$, we again define $(1,0) \equiv \sigma, (0,1) \equiv s\sigma b$, with $\theta_{(0,1)} = \theta_{(1,0)} = i$. A general anyon (a,b) in this theory has topological twist

$$\theta_{(a,b)} = i^{a^2 + b^2 - ab}. \tag{84}$$

Let us denote it by $\mathrm{D}'(\mathbb{Z}_4)$.

Now we determine the time-reversal transformation in this case. Using the consistency of braiding, we can uniquely fix

$$\mathbf{T}\sigma = \overline{\sigma}e.$$
$$\mathbf{T}s\sigma b = s\sigma. \tag{85}$$

This theory $\mathrm{D}'(\mathbb{Z}_4)$ has a $\mathcal{H}^3$ anomaly: on the one hand, $\eta_e^{\mathbf{T}} = -1$ due to the fact that e is a $\mathbf{T}^2$ gauge charge of the original theory. However, on the other hand we have $\eta_{be}^{\mathbf{T}} = \theta_{be} = 1$, because $be = \mathbf{T}\sigma \times \sigma$. Since $\eta_b^{\mathbf{T}} = 1$, it follows that $\eta_e^{\mathbf{T}} = \eta_{be}^{\mathbf{T}} \eta_b^{\mathbf{T}} = 1$, which is a contradiction.

We now start from $\mathrm{USp}(4)_2^\vee \times \mathrm{D}(\mathbb{Z}_4)$ or $\mathrm{USp}(4)_2^\vee \times \mathrm{D}'(\mathbb{Z}_4)$ depending on whether $\theta_\sigma = 1$ or i, and condense $\epsilon be \equiv \epsilon \times (0,2)$. Notice that while $\mathrm{D}(\mathbb{Z}_4)$ and $\mathrm{D}'(\mathbb{Z}_4)$ generally have different braiding structures, the braiding of $(0,2)$ with other anyons are the same:

$$M_{(a,b),(0,2)} = (-1)^a. \tag{86}$$

Therefore we can treat the condensation uniformly for both cases. The resulting gauged theory contains 24 particles, with the spins and quantum dimensions shown in Table III.

When $\theta_\sigma = i$, from Table III we see that all of the anyons must be permuted by $\mathbf{T}$ because of their complex twists, except for $(0,0), (2,0), (0,2)$ and $(2,2)$, which are invariant under $\mathbf{T}$. This implies that

$$\mathcal{Z}(\mathbb{RP}^4) = \frac{1}{2\sqrt{20}}(1 + \eta_{(2,0)}^{\mathbf{T}} + \eta_{(0,2)}^{\mathbf{T}} + \eta_{(2,2)}^{\mathbf{T}}) \neq \pm 1. \tag{87}$$

We thus conclude that this theory possesses a $\mathcal{H}_{[\rho]}^3(\mathbb{Z}_2^{\mathbf{T}}, \mathcal{A})$ anomaly.

To summarize, we have found so far that $\mathrm{USp}(4)_2$ CS theory possesses an $\mathcal{H}^3(\mathbb{Z}_2^{\mathbf{T}}, \mathcal{A})$ anomaly. As discussed above,

Label	d	θ	Remark
(a,b)	1	$\theta_{(a,b)}$	$a \in \{0,2\}, b \in \{0,1,2,3\}$
$\psi_+ \times (a,b)$	$\sqrt{5}$	$\theta_{(a,b)}$	$a \in \{1,3\}, b \in \{0,1,2,3\}$
$\phi_1 \times (a,b)$	2	$e^{\frac{4\pi i}{5}}\theta_{(a,b)}$	$a \in \{0,2\}, b \in \{0,1\}$
$\phi_2 \times (a,b)$	2	$e^{-\frac{4\pi i}{5}}\theta_{(a,b)}$	$a \in \{0,2\}, b \in \{0,1\}$

TABLE III. Particle types, quantum dimensions, and topological twists for the theory obtained by gauging a unitary global $\mathbb{Z}_2$ symmetry in $\mathrm{USp}(4)_2$. We take the case where the symmetry fractionalization class is non-trivial. There are thus two remaining distinct choices, $\theta_\sigma = 1, i$, encoded in the two expressions for $\theta_{(a,b)}$.

a	1	$(1/2,1/2)$	$(0,1)$	$(1,0)$	$(1/2,3/2)$	$(1,1)_+$	$(1,1)_-$	$(0,2)$
θ_a	1	i	$e^{\frac{2\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$	$-i$	$e^{-\frac{2\pi i}{3}}$	$e^{-\frac{2\pi i}{3}}$	1
d_a	1	3	2	2	3	2	2	1

TABLE IV. Anyon types, topological spins, and quantum dimensions for $\mathrm{SO}(4)_4$.

one resolution to this is that the symmetry can be taken to be $\mathbb{Z}_4^{\mathbf{T}}$. This allows us to consider the theories obtained by gauging $\mathbf{T}^2$, which leads us to new theories that also possess $\mathcal{H}^3(\mathbb{Z}_2^{\mathbf{T}}, \mathcal{A})$ anomalies, for example as summarized in Table III. Again this means that the true symmetry of the new gauged theory is $\mathbb{Z}_4^{\mathbf{T}}$. Thus we can again consider gauging $\mathbf{T}^2$. This procedure can continue indefinitely, yielding an infinite family of theories with $\mathcal{H}^3(\mathbb{Z}_2^{\mathbf{T}}, \mathcal{A})$ anomalies.

B. $\mathrm{SO}(4)_4$ CS theory

Another example of a theory with an $\mathcal{H}^3(\mathbb{Z}_2^{\mathbf{T}}, \mathcal{A})$ anomaly is $\mathrm{SO}(4)_4$ CS theory.

$\mathrm{SO}(4)_4$ CS theory can be obtained from $\mathrm{SU}(2)_4 \times \mathrm{SU}(2)_4$ CS theory by a condensation process as follows. The 5 particle types of $\mathrm{SU}(2)_4$ CS theory can be organized in terms of $\mathrm{SU}(2)$ representations: $0, 1/2, 1, 3/2, 2$, with the spin 2 particle being the Abelian anyon. Thus the particle types of $\mathrm{SU}(2)_4 \times \mathrm{SU}(2)_4$ can be written as (a,b), with $a, b = 0, 1/2, 1, 3/2, 2$. $\mathrm{SO}(4)_4$ CS theory can be obtained from $\mathrm{SU}(2)_4 \times \mathrm{SU}(2)_4$ CS theory by condensing the spin $(2,2)$ Abelian boson.[62] This leads to a theory with 8 types of particles, summarized in Table IV.

$\mathrm{SO}(4)_4$ has an Abelian particle $\epsilon = (1,1)$. Condensing ϵ takes us to a new theory, $\mathrm{SU}(3)_1 \times \mathrm{SU}(3)_1$ CS theory. $\mathrm{SU}(3)_1$ CS theory is an Abelian theory with 3 particle types. Thus we can label the anyons of $\mathrm{SU}(3)_1 \times \mathrm{SU}(3)_1$ as (a,b), for $a, b = 1, \cdots 3 \pmod 3$. This theory has a $\mathbb{Z}_4^{\mathbf{T}}$ symmetry associated with the following transformation on the anyons:

$$\mathbf{T}: (a,b) \to (2a+b, a+b) \tag{88}$$

Note that the above transformation induces the following action: $(1,0) \to (2,1) \to (2,0) \to 1,2) \to (1,0)$, and $(0,1) \to (1,1) \to (0,2) \to (2,2) \to (0,1)$. It is easy to verify that $\mathbf{T}^2 = \mathbf{C}$, and therefore $\mathbf{T}$ generates a $\mathbb{Z}_4^{\mathbf{T}}$ symmetry. We note that it is possible to obtain another $\mathbb{Z}_4^{\mathbf{T}}$ symmetry

by defining $\mathbf{T}' = \mathbf{LTL}$, where $\mathbf{L}$ is the $\mathbb{Z}_2$ layer-exchange symmetry, $\mathbf{L} : (a, b) \rightarrow (b, a)$.

This theory has all of the same essential features that appeared in our preceding analysis of $\mathrm{USp}(4)_2$ CS theory. The possible actions of $\mathbf{T}$ preclude the constraints discussed in Sec. IV from being satisfied, in a similar manner to that of $\mathrm{USp}(4)_2$ CS theory. For example,

$$\mathcal{Z}(\mathbb{RP}^4) = \frac{1}{6}(1 + \eta_{(0,2)}^{\mathbf{T}}) \neq \pm 1. \tag{89}$$

Moreover, it is straightforward to check that all of the same resolutions as discussed for $\mathrm{USp}(4)_2$ apply in this case as well.

As in the $\mathrm{USp}(4)_2$ case, we can now consider enlarging the symmetry to $\mathbb{Z}_4^{\mathbf{T}}$, which does not possess the $\mathcal{H}^3$ anomaly, and subsequently gauge the $\mathbf{T}^2$ symmetry. Through this process, one can again generate an infinite series of TQFTs with $\mathcal{H}^3$ anomalies, with now $\mathrm{SU}(3)_1 \times \mathrm{SU}(3)_1$ CS theory being the "root" phase, instead of $\mathrm{SU}(5)_1$ CS theory.

C. Infinite family of root phases

Given the understanding developed in the preceding sections of when $\mathcal{H}^3$ anomalies arise, we can now provide an infinite family of "root phases" with $\mathbb{Z}_4^{\mathbf{T}}$ symmetry which, upon gauging $\mathbf{T}^2$, each lead to a series of theories with $\mathcal{H}^3$ anomalies. Let us consider a two-component Abelian $\mathrm{U}(1) \times \mathrm{U}(1)$ CS theory:

$$\mathcal{L}_{\mathrm{CS}} = \frac{1}{4\pi} \sum_{I,J=1}^{2} \varepsilon^{\mu\nu\lambda} a_{I\mu} K_{IJ} \partial_\nu a_{J\lambda} + \cdots, \tag{90}$$

with the K-matrix

$$K = \begin{pmatrix} m & n \\ n & -m \end{pmatrix}. \tag{91}$$

where n, m are integers, and m is even to describe a bosonic (non-spin) theory. The quasiparticles of this theory are described by 2-component integer vectors $\vec{l}$.

This theory has a time-reversal symmetry, whose action on the gauge fields is given by

$$\mathbf{T} : \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \rightarrow \begin{pmatrix} a_2 \\ -a_1 \end{pmatrix}. \tag{92}$$

It is clear that

$$\mathbf{T}^2 = -\mathbb{1}, \tag{93}$$

which takes all quasiparticles $\vec{l} \rightarrow -\vec{l}$. Therefore, $\mathbf{T}^2$ is a $\mathbb{Z}_2$ charge-conjugation symmetry, while $\mathbf{T}$ generates a $\mathbb{Z}_4^{\mathbf{T}}$ symmetry.

It is important to know that the $\mathbb{Z}_4^{\mathbf{T}}$ symmetry of the K-matrix in Eq. (91) does not possess any anomaly. In Appendix A, we give an explicit microscopic construction of this

phase with a $\mathbb{Z}_4^{\mathbf{R}}$ reflection symmetry in (2+1)D, to explicitly demonstrate the absence of any anomaly.

We note that the case where $(m, n) = (0, 3)$ gives rise to $\mathbb{Z}_3$ gauge theory. Gauging the unitary $\mathbb{Z}_2$ particle-hole symmetry gives rise to $\mathbb{S}_3$ gauge theory (the permutation group on three elements), which contains 8 particles.[19] It is a close cousin of the $\mathrm{SO}(4)_4$ example, but with chiral central charge $c = 0$.

Similarly, the case where $(m, n) = (2, 1)$ gives rise to a $\mathbb{Z}_5$ anyon theory (i.e. the fusion rules form a $\mathbb{Z}_5$ group. This is a close cousin of $\mathrm{SU}(5)_1$ CS theory). Gauging the unitary $\mathbb{Z}_2$ particle-hole symmetry gives rise to a theory with 6 particles, which is closely related to $\mathrm{USp}(4)_2$, but with $c = 0$.

In fact we can define a $\mathbb{Z}_4^{\mathbf{T}}$ symmetry for an even more general class of anyon models denoted as $\mathbb{Z}_N^{(p)}$, for $N \equiv 1 \pmod 4$. $\mathbb{Z}_N^{(p)}$ contains Abelian anyons with $\mathbb{Z}_N$ fusion rules and with R symbols $R_{[a+b]}^{[a][b]} = e^{2\pi i p a b / N}$. The $\mathbf{T}$ symmetry acts on an anyon $[j]$ as

$$\mathbf{T} : [j] \rightarrow [tj], \tag{94}$$

such that $\theta_{[tj]} = \theta_{[j]}^*$, which implies $t^2 \equiv -1 \pmod N$. Since N is odd, t has to be even in order to satisfy this condition. As a result, we must have $N \equiv 1 \pmod 4$ as a necessary condition. $\mathbf{T}^2$ is automatically the charge-conjugation symmetry.

1. Gauging the unitary $\mathbb{Z}_2$ symmetry

Let us now consider gauging the unitary $\mathbb{Z}_2$ charge conjugation symmetry associated with $\mathbf{T}^2$ in the theories described above. We will show that this leads to a theory with an $\mathcal{H}_{[\rho]}^3(\mathbb{Z}_2^{\mathbf{T}}, \mathcal{A})$ anomaly. As discussed in Ref. 19, there are in principle two distinct ways to gauge the $\mathbb{Z}_2$ symmetry, corresponding to a choice of group element in $\mathcal{H}^3(\mathbb{Z}_2, \mathrm{U}(1)) = \mathbb{Z}_2$. This is associated with whether we include a Dijkgraaf-Witten term for the $\mathbb{Z}_2$ gauge field in the effective Lagrangian description. (Note that for this theory this is the only choice that needs to be made in the gauging process, because the symmetry fractionalization class $\mathcal{H}_{[\rho]}^2(\mathbb{Z}_2^{\mathbf{T}}, \mathcal{A})$ is trivial in this case[19]).

Here we mainly focus on the case $\gcd(m, n) = 1$. In this case, the K-matrix defined above defines an Abelian theory with $\mathbb{Z}_{m^2+n^2}$ fusion rules.[63] The $m^2 + n^2$ quasiparticles can be taken to be $\vec{l}_a = (a, 0)$, for $a = 0, \cdots, m^2 + n^2 - 1$. The mutual braiding statistics between anyons labelled by $\vec{l}_a$ and $\vec{l}_b$ is therefore $e^{\frac{2\pi i a b m}{m^2+n^2}}$.

Since we are interested in bosonic theories, we require m to be even. In order for $\gcd(m, n) = 1$, we require that n be odd. This implies that that $m^2 + n^2$ is odd. This corresponds to the anyon theory $\mathbb{Z}_N^{(p)}$ for $N = m^2 + n^2$ and $p = m/2$.

When gauging charge conjugation in this theory, the resulting theory has $(N + 7)/2$ anyons.[19, 64] The anyons $[a], [-a]$ for $a = 1, \cdots, (N - 1)/2$ are grouped into non-Abelian anyons with quantum dimension 2. The identity particle splits into two particles, $1, \epsilon$, with ϵ being the $\mathbb{Z}_2$ charge. After gauging there are two types of twist fields, $\sigma_{\pm}$. These

have fusion rules

$$\sigma_+ \times \sigma_+ = 1 + \sum_a \phi_a,$$

$$\sigma_+ \times \sigma_- = \epsilon + \sum_a \phi_a. \tag{95}$$

Let us compute θ_σ^2. As shown in Ref. 19, we have

$$\theta_\sigma^2 = \frac{\kappa_\sigma}{\sqrt{N}} \sum_{j=0}^{N-1} e^{\frac{2\pi i}{N} \frac{p(N-1)}{2} j^2} \equiv \kappa_\sigma G(N, p), \tag{96}$$

where[65]

$$G(N, p) = \left(\frac{p(N-1)/2}{N} \right) G(N, 1), \tag{97}$$

$\left(\frac{a}{b} \right)$ is the Jacobi symbol, and

$$G(N, 1) = \begin{cases} 1 & N \equiv 1 \,(\text{mod } 4) \\ i & N \equiv 3 \,(\text{mod } 4) \end{cases}. \tag{98}$$

$\kappa_\sigma = \pm 1$ is the Frobenius-Schur indicator of σ, which depends on the choice of $\mathcal{H}^3(\mathbb{Z}_2, \mathrm{U}(1)) = \mathbb{Z}_2$.[19] Therefore, for $N \equiv 1 \,(\text{mod } 4)$, we find $\theta_\sigma^2 = \pm 1$.

Depending on the choice of $\mathcal{H}^3(\mathbb{Z}_2, \mathrm{U}(1)) = \mathbb{Z}_2$ class for the gauging process, we have $\theta_{\sigma_\pm} = \pm 1$ or $\theta_{\sigma_\pm} = \pm i$. It is clear that the latter case gives rise to the inconsistency in $\eta_\epsilon^{\mathbf{T}}$, as discussed in Sec. IV E. The latter case also implies that

$$\mathcal{Z}(\mathbb{RP}^4) = \frac{1}{2N}(1 + \eta_\epsilon^{\mathbf{T}}) \neq \pm 1. \tag{99}$$

Thus gauging $\mathbf{T}^2$ gives us a theory with an $\mathcal{H}^3$ anomaly. As discussed in Sec. V A, this can be resolved by taking the true symmetry to be $\mathbb{Z}_4^{\mathbf{T}}$. As in the examples of Sec. VI A, we can now continue to gauge $\mathbf{T}^2$ again, and in this way generate an infinite series of theories with $\mathcal{H}^3$ anomalies.

In the more general case where $\gcd(m, n) = f$, the fusion rules of the theory split as $\mathbb{Z}_{N/f} \times \mathbb{Z}_f$, with $N = m^2 + n^2$.[66] We leave a detailed analysis of this case for future work.

VII. DISCUSSION

We have demonstrated a series of TQFTs which possess a $\mathbb{Z}_2^{\mathbf{T}}$ time-reversal symmetry localization anomaly, which is classified by $\mathcal{H}_{[\rho]}^3(\mathbb{Z}_2^{\mathbf{T}}, \mathcal{A})$. As described in Ref. 19, the general diagnostic of the existence of an $\mathcal{H}_{[\rho]}^3(\mathbb{Z}_2^{\mathbf{T}}, \mathcal{A})$ requires the full F and R symbols of the theory and their transformation properties under time-reversal. We have further provided a series of simpler constraints, which only depend on the modular data (the topological spins and modular S-matrix), that must be satisfied for any (2+1)D theory to be free of this $\mathbb{Z}_2^{\mathbf{T}}$ localization anomaly. All of the theories that we considered violated these constraints, signalling the existence of their $\mathcal{H}^3$ anomaly. Analogous results hold for $\mathbb{Z}_2^{\mathbf{R}}$ reflection symmetry, which we obtain by replacing the action of $\mathbf{T}$ with $\mathbf{CR}$.

There are a number of interesting questions that we leave for future work. In particular, it is unclear whether theories with $\mathbb{Z}_2^{\mathbf{T}}$ symmetry localization anomalies always fall within the framework found in this paper. For example, do all theories with $\mathcal{H}_{[\rho]}^3(\mathbb{Z}_2^{\mathbf{T}}, \mathcal{A})$ anomalies always violate the constraints that we have found and, if so, are any of the constraints always violated together? In the examples that we have studied many of the constraints that we have found are violated simultaneously. Furthermore, in our examples, in addition to the example of Ref. 45 which studied a theory with unitary $\mathbb{Z}_2$ symmetry, the anomalous theories can be "pseudo-realized" at the surface of a (3+1)D dynamical $\mathbb{Z}_2$ gauge theory with bosonic gauge charge; it would be interesting to know whether cases with fermionic gauge charge exist as well.

We note that the existence of a subgroup of Abelian anyons $\mathcal{A}$ implies the existence of a one-form symmetry with symmetry group $\mathcal{A}$.[67] The cohomology class $\mathcal{H}_{[\rho]}^3(G, \mathcal{A})$ also appears in the study of 2-groups, where G is interpreted as the 0-form symmetry, and $\mathcal{A}$ is interpreted as the 1-form symmetry.[68] This suggests another possible interpretation of the $\mathcal{H}_{[\rho]}^3(G, \mathcal{A})$ anomaly as implying that the true symmetry of the TQFT is a non-trivial 2-group symmetry, where the existence of the $\mathcal{H}_{[\rho]}^3(G, \mathcal{A})$ class implies that the 1-form symmetry cannot be broken without also breaking the 0-form symmetry. This 2-group symmetry then could potentially possess a t' Hooft anomaly, which would be cancelled by a 2-group SPT in (3+1)D.[68, 69]

However, an important point regarding the 2-group symmetry perspective is that 1-form symmetries apparently cannot be exact symmetries of a system whose ultraviolet (UV) degrees of freedom are described by a Hilbert space that decomposes into a tensor product of local Hilbert spaces. Such models always contain dynamical matter fields in their effective quantum field theory description, which spoils the 1-form symmetry at high enough energies. Therefore it appears that the 2-group symmetry perspective cannot resolve the possibility of whether the symmetric topological phases of interest can exist in a system whose UV degrees of freedom form a Hilbert space that decomposes into a tensor product of local Hilbert spaces.

Furthermore, we note that Ref. 69 studied t 'Hooft anomalies associated with 2-group symmetries with non-trivial $\mathcal{H}^3$ class. There, the possibility of such anomalies being cancelled by a bulk (3+1)D 2-group SPT via a generalization of anomaly in-flow was studied. Here we emphasize that the $\mathcal{H}^3$ anomalies described in Ref. 69 are distinct from the $\mathcal{H}_{[\rho]}^3(G, \mathcal{A})$ discussed in this paper and those of Ref. 19. The $\mathcal{H}_{[\rho]}^3(G, \mathcal{A})$ anomalies that we discuss here are defined solely by the action $[\rho]$, defined in Ref. 19 and reviewed in Sec. II of this paper. In particular, all Abelian topological phases where the symmetries do not permute the anyon types are free of such $\mathcal{H}_{[\rho]}^3(G, \mathcal{A})$ anomalies. The examples studied in Ref. 40 and 69 uncover a different $\mathcal{H}^3$ structure in a purely Abelian topological phase with symmetries that do not permute the anyon types. There the obstruction arises once certain choices for the symmetry fractionalization class are made for a subset of the anyons, and one attempts to lift that choice to a symmetry

fractionalization class for the full theory. It would be interesting to understand whether and how the 2-group anomaly in-flow picture also applies for $\mathcal{H}^3_{[\rho]}(G, \mathcal{A})$ obstructions studied in this paper and in Ref. 19.

Appendix A: Explicit Construction of Abelian Theories with $\mathbb{Z}_4^{\mathbf{R}}$ Symmetry

In Sec. VIC we presented a series of $U(1) \times U(1)$ CS theories described by a 2×2 K-matrix $K = \begin{pmatrix} m & n \\ n & -m \end{pmatrix}$, which have a $\mathbb{Z}_4^{\mathbf{R}}$ reflection symmetry. Here we provide a construction of this theory that demonstrates that it possesses no SPT (t' Hooft) anomaly. The idea is similar to ideas used in Ref. 33 and in Sec. VC.

We use the fact that the K-matrix theory can be realized as the effective theory of a system with a microscopic charge-conjugation symmetry. That is, $\mathbf{C} = -\mathbb{1}$ is not only a symmetry of the effective topological field theory, but also corresponds to a microscopic on-site (internal) $\mathbb{Z}_2$ symmetry (which will also be denoted by $\mathbf{C}$ for convenience) of some lattice realization of the phase. This follows from the fact that in this theories, $\mathbf{C}$ is free of both the $\mathcal{H}^3_{[\rho]}(G, \mathcal{A})$ and $\mathcal{H}^4(G, U(1))$ anomalies.[19]

We imagine cutting the two-dimensional space of the system into two regions, left and right, which are mapped to each other under $\mathbf{R}$. By defining an appropriate action of $\mathbf{R}$, we show that the edge modes along the interface can be gapped in a way which preserves the symmetry, without needing any additional degrees of freedom on the mirror plane of any bulk $(3+1)$D system. This will show that the action of $\mathbf{R}$ that we define is free of any SPT (t 'Hooft) anomalies.

We define the $\mathbf{R}$ symmetry by composing the usual reflection, which only acts on the spatial degrees of freedom, with $\mathbf{C}$ restricted to the right half of the system (this is well-defined because $\mathbf{C}$ is on-site). On the edge fields, we have

$$\mathbf{R} : \Phi_l \to -\Phi_r, \Phi_r \to \Phi_l. \tag{A1}$$

Such a $\mathbf{R}$ transformation generates a $\mathbb{Z}_4^{\mathbf{R}}$ symmetry group.

Using the $\mathbf{C}$ symmetry, the entire K matrix of the interface edge theory is given by

$$\mathcal{K} = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}. \tag{A2}$$

We define the following two null vectors:

$$\begin{aligned} \mathbf{\Lambda}_1 &= \frac{1}{f}(m, n, -n, m), \\ \mathbf{\Lambda}_2 &= \frac{1}{f}(n, -m, m, n). \end{aligned} \tag{A3}$$

Here $f = \gcd(m, n)$. We then add the following gapping terms to the Lagrangian:

$$\begin{aligned} \delta\mathcal{L} &= -u(\cos \mathbf{\Lambda}_1^{\mathsf{T}} K \Phi + \cos \mathbf{\Lambda}_2^{\mathsf{T}} K \Phi) \\ &= -u\left[\cos \frac{m^2+n^2}{f}(\phi_{l1} - \phi_{r2}) + \cos \frac{m^2+n^2}{f}(\phi_{l2} + \phi_{r1}) \right] \end{aligned} \tag{A4}$$

It is easy to see that $\delta\mathcal{L}$ preserves the reflection symmetry defined in Eq. (A1), and leads to a unique ground state. The cosine potentials pin $\phi_{l1} = \phi_{r2}, \phi_{l2} = -\phi_{r1}$ in the ground state, and therefore the two regions are joint together, such that the quasiparticles can tunnel between the two regions. In other words, we have realized a single topological phase described by the K matrix, with a $\mathbb{Z}_4^{\mathbf{R}}$ symmetry.

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