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Exactly Solvable Model for Two Dimensional Topological Superconductor

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In this paper, we present an exactly solvable model for two dimensional topological superconductor with helical Majorana edge modes protected by time-reversal symmetry. Our construction is based on the idea of decorated domain walls and makes use of the Kasteleyn orientation on a two dimensional lattice, which were used for the construction of the symmetry protected fermion phase with Z_2 symmetry by Tarantino et al. and Ware et al. By decorating the time-reversal domain walls with spinful Majorana chains, we are able to construct a commuting projector Hamiltonian with zero correlation length ground state wave function that realizes a strongly interacting version of the two dimensional topological superconductor. From our construction, it can be seen that the $T^2 = -1$ transformation rule for the fermions is crucial for the existence of such a nontrivial phase; with $T^2 = 1$, our construction does not work.

Introduction – The discovery of topological insulators and superconductors[1–6] demonstrates that a fermionic system can exhibit nontrivial topological properties if the fermions occupy a band structure with nontrivial topology. The topological nature of the systems is manifested physically in the existence of gapless edge modes around a gapped bulk, which cannot be removed unless certain symmetry is explicitly or spontaneously broken. It is also manifested at symmetry defects on the boundary of the system. For example, in a 2D topological superconductor, a time-reversal domain wall on the 1D boundary hosts a Majorana zero mode and in a 3D topological superconductor, a time- reversal domain wall on the 2D boundary hosts a chiral Majorana mode. A complete classification of topological insulators and superconductors in free fermion systems was given in Ref.7 and 8. Such 'Symmetry Protected Topological (SPT)' order was found in interacting boson systems as well. A whole class of exactly solvable models with commuting projector Hamiltonian and zero correlation length ground state wave function were constructed to realize such bosonic SPT order[9,10].

Can topological insulators and superconductors discovered in the free fermion setup be realized with exactly solvable models as well? This question is interesting not only out of pure theoretical curiosity; it is also crucial for formulating a general framework for both fermionic and bosonic SPT phases which may lead to the discovery of new phases and a complete classification. Moreover, it can be useful in answering questions regarding many-body localization in such phases when strong disorder is present[11]. In this paper, we focus on the case of 2D topological superconductor.

If an exactly solvable model is possible, it necessarily involves interactions as the free fermion ground states always have a nonzero correlation length due to the nontrivial topology of the band structure[11]. Ref. 12 and 13 gave the exactly solvable model realization of a large class of fermionic SPT phases which are protected by symmetries of the form $G_b \times Z_2^f$, where G_b denotes symmetry transformation on some bosonic degrees of freedom in the system and Z_2^f is the fermion parity part of the symmetry. The symmetry protecting the topological superconductor falls out of this class. In the topological superconductor, time-reversal symmetry acts as $T^2 = P_f$, where P_f is the fermion parity operator generating the Z_2^f symmetry group. Therefore, the total symmetry group is Z_4, with the odd group elements being anti-unitary.

The decorated domain wall construction provides a different approach for constructing exactly solvable models for SPT phases.[14]. In this approach, the ground state wave function is written as a superposition of all possible symmetry breaking configurations with the symmetry breaking domain walls decorated with SPT states of one lower dimension, as shown in Fig.1 (a). The superposition guarantees that the total wave function is symmetric. Moreover, when symmetry is broken into opposite domains, the domain wall carries the lower dimensional SPT state. When the domain wall ends on the boundary of the system, the end point hence hosts the edge state of the lower dimensional SPT state, reflecting the nontrivial nature of the original SPT order, as shown in Fig.1 (b).

In a topological superconductor with helical Majorana edge mode, a mass term can gap out the edge mode while breaking time-reversal symmetry. On the symmetry domain wall, there is an isolated Majorana mode. Therefore, if the topological superconductor can be written in the decorated domain wall way, we should decorate the time-reversal domain walls with Majorana chains.

Decorating symmetry domain walls with Majorana chains has proven to be more difficult than with bosonic chains. A breakthrough was made recently in Ref.15 and 16 where a fermionic SPT phase with $Z_2 \times Z_2^f$ symmetry was realized by decorating the Z_2 domain walls with 1D Majorana chains. Although the protecting symmetry is still of the form $G_b \times Z_2^f$, this particular phase cannot be realized using the method of Ref.12. It was realized that the incorporation of a Kasteleyn orientation on the two dimensional lattice, which corresponds to a discrete version of spin structure in 2D, is crucial for a consistent decoration.

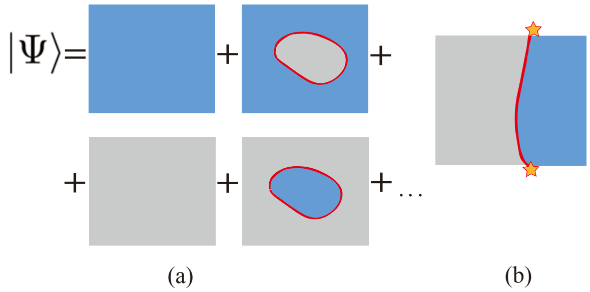

FIG. 1. The decorated domain wall approach. (a) Ground state is a superposition of all symmetry breaking domain configurations (blue and grey patches) with domain walls decorated with SPT states of one lower dimension (red curves). (b) The end point of the domain wall on the boundary (star) hosts nontrivial edge states of the lower dimensional SPT.

Using the Kasteleyn orientation, we present a decorated domain wall construction of the 2D topological superconductor in this paper. Our construction is different from that of the $Z_2 \times Z_2^f$ SPT phase in an important way. In the case of $Z_2 \times Z_2^f$, the Majorana chain used for decoration does not transform under the Z_2 part of the symmetry, which acts only on the symmetry domains. In the case of topological superconductor, time reversal acts both on the symmetry domains and on the Majorana chains decorated onto the symmetry domain walls. In fact, the way the Majorana chains transform under time reversal is crucial for the construction as we know that topological superconductivity only exists for $T^2 = -1$ fermions but not the $T^2 = +1$ ones. Indeed, after we present carefully how a zero correlation length wave function and a commuting projector Hamiltonian can be constructed for $T^2 = -1$ fermions, we will be able to see why a similar construction fails for the $T^2 = +1$ ones. Our discussion below focuses on the Honeycomb lattice, but the construction works for any trivalent lattice using the same convention as defined below.

Wave-function – Consider the planar trivalent lattice in Fig.2 together with a Kasteleyn orientation, i.e., orientation of the bonds of the lattice for which any plaquette has an odd number of clockwise-oriented bonds. There are two types of faces in the lattice: the 12-sided faces, which we will refer to as plaquettes, and the triangular faces, which we will refer to as triangles. Let $t(v)$ and $t(w)$ be the triangles that contain the vertices v and w, respectively. The bonds of the lattice also come in two types: the 'short' bonds which connect different triangles ($t(v) \neq t(w)$), and the 'long' bonds that are in the same triangle ($t(v) = t(w)$).

The Hilbert space of our model consists of a bosonic spin-1/2 located on each plaquette p, acted on by the Pauli operators τ_p^x, τ_p^y, τ_p^z, and a pair of complex fermions located on each short bond l, created and annihilated by operators $c_l^{\sigma\dagger}$ and c_l^{σ} ($\sigma = \uparrow, \downarrow$), respectively. Let $l = \langle \overrightarrow{vv'} \rangle$ be oriented from vertex v to vertex v'. Each complex fermion on l can be represented by a pair of Majorana modes

$$\begin{aligned} \gamma_v^\sigma &= c_l^{\sigma\dagger} + c_l^\sigma, \\ \gamma_{v'}^\sigma &= i(c_l^{\sigma\dagger} - c_l^\sigma), \end{aligned} \tag{1}$$

located at v and v', respectively. We can also define a fictitious spin-1/2 degree of freedom τ_t on each triangle following the majority rule: The value of τ_t is set to 1 if the majority of the three plaquettes bordering t have $\tau_p^z = 1$, and is set to -1 otherwise.

Our system has a time-reversal symmetry T, which acts on both the plaquette spins and the complex fermions. In the eigenbasis of τ_p^z, T maps between the two eigenstates of τ_p^z:

$$T: |1\rangle \to |-1\rangle, \quad |-1\rangle \to |1\rangle, \tag{2}$$

together with the complex conjugation operation in this basis. The fictitious spins on the triangles will also be flipped due to the majority rule. Since any fixed plaquette spin configuration in the τ^z basis breaks time-reversal symmetry, we will refer to a domain of plaquette spins in the same τ^z basis state as a time-reversal domain. Furthermore, c_l^σ transforms as a Kramers doublet under T: $c_l^\uparrow \to c_l^\downarrow$, $\quad c_l^\downarrow \to -c_l^\uparrow$. Written in terms of the Majorana modes, we have:

$$T: \begin{cases} \gamma_v^\uparrow \to \gamma_v^\downarrow \\ \gamma_v^\downarrow \to -\gamma_v^\uparrow, \end{cases} \quad \begin{cases} \gamma_{v'}^\uparrow \to -\gamma_{v'}^\downarrow \\ \gamma_{v'}^\downarrow \to \gamma_{v'}^\uparrow. \end{cases} \tag{3}$$

where the Kasteleyn orientation points from v to v'.

Now we describe in detail how we decorate the time-reversal domain walls with Majorana chains. Away from the domain wall, we pair up Majorana modes that share a short bond $\langle \overrightarrow{vv'} \rangle$ as $i\gamma_v^\uparrow \gamma_{v'}^\uparrow + i\gamma_v^\downarrow \gamma_{v'}^\downarrow$. On a domain wall, we pick out one Majorana mode $\gamma_v^{\sigma_v}$ from each vertex v and pair them along the long bonds $\langle \overrightarrow{vw} \rangle$ as $i\gamma_v^{\sigma_v}\gamma_w^{\sigma_w}$ so that they form a Majorana chain. The spin label σ_v is determined as follows: If the left hand side of the short bond is a $|1\rangle$ domain, $\sigma_v = \uparrow$; otherwise, $\sigma_v = \downarrow$. After the Majorana modes of the σ_v species pair into Majorana chains, we are left with exactly one unpaired Majorana mode on each vertex on the domain wall. The two unpaired Majorana modes that share a short bond $\langle \overrightarrow{vv'} \rangle$ will have the same spin $\bar{\sigma}_v$ which can be paired as $i\gamma_v^{\bar{\sigma}_v}\gamma_{v'}^{\bar{\sigma}_{v'}}$. This is the same kind of coupling as that away from the domain wall, but with only one species of Majorana modes. Fig.2 (b) and (c) give a pictorial illustration of these coupling rules.

The ground state wave function of a topological superconductor is then given by the superposition of all possible time-reversal domain configurations with domain walls decorated with Majorana chains. It satisfies the following properties: It's time-reversal invariant, and every configuration in the superposition has the same fermion parity. The latter fact is ensured by the Kasteleyn orientation. The reason for this is very similar to that presented in Ref.15 and 16 although here we have two species of fermion modes.

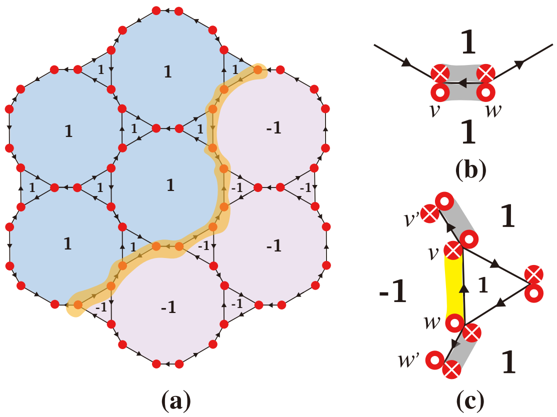

FIG. 2. (a) illustrates the lattice structure and degrees of freedom in our model. Here 1 and −1 denote the eigenstates of τ_p^z with eigenvalues 1 and −1, respectively. The blue bonds indicate the time-reversal domain wall. The solid red circles denote the Majorana modes γ_v^σ ($\sigma = \uparrow, \downarrow$). The arrow at each bond denotes the Kasteleyn orientation of the bond. (b) (resp. (c)) is a detailed illustration of the coupling of Majorana modes away from (resp. on) the domain wall. The dots and crosses on the solid red circles indicate the up (↑) and down (↓) spins of the Majorana modes, respectively. The yellow (resp. grey) bond denotes the coupling of Majorana modes that share a long (resp. short) bond.

To see the time-reversal invariance, we note that time reversal acts by flipping the plaquette spins, and transforms the Majorana modes in a way that conforms to the decoration rules introduced above. In particular, for Majorana modes not on a domain wall, they pair as $i\gamma_v^\uparrow\gamma_{v'}^\uparrow + i\gamma_v^\downarrow\gamma_{v'}^\downarrow$ on a short bond which is invariant under time reversal. For Majorana modes on a domain wall, the decoration rule says that the modes that form (do not form) Majorana chains flip their spin when the plaquette spins are flipped, which is consistent with the time-reversal transformation action. Moreover, the pairing terms along the domain wall, whose signs are fixed by the Kasteleyn orientation, exactly map into each other under time reversal without any sign ambiguity. To see this, first notice that for the modes which do not form Majorana chains, the pairing maps from $i\gamma_v^{\sigma_v}\gamma_{v'}^{\sigma_{v'}}$ to $i\gamma_v^{\bar{\sigma}_v}\gamma_{v'}^{\bar{\sigma}_{v'}}$, which are both consistent with the Kasteleyn orientation. Secondly, for the modes that are involved in forming Majorana chains, one can check that the pairing term $i\gamma_v^{\sigma_v}\gamma_w^{\sigma_w}$ is mapped into $i\gamma_v^{\bar{\sigma}_v}\gamma_w^{\bar{\sigma}_w}$ which are both consistent with the Kasteleyn orientation.[17] Therefore, we can conclude that time reversal maps from one to another the decorated domain wall configurations in the superposition. The whole superposition is then time-reversal invariant if the weight of the time-reversal partner configurations are complex conjugate of each other. This will be demonstrated in detail in the Supplemental Material[18].

Hamiltonian – The Hamiltonian of our model can be written as

$$H = H_{\text{decorate}} + H_{\text{tunnel}}, \tag{4}$$

where H_{decorate} will be defined to realize the domain wall decoration described in the above section for each plaquette spin configuration, and H_{tunnel} will be defined to tunnel between the different plaquette spin configurations.

More explicitly, let $D_{\langle\overrightarrow{vw}\rangle} = \frac{1}{2}\left(1 - \tau^z_{f_{\overrightarrow{vw}}}\tau^z_{f'_{\overrightarrow{vw}}}\right)$ be the operator which detects if the bond (either short or long) $\langle\overrightarrow{vw}\rangle$ is on a domain wall. $f_{\overrightarrow{vw}}$ denotes the left-hand-side face of the bond $\langle\overrightarrow{vw}\rangle$; $f'_{\overrightarrow{vw}}$ denotes the right-hand-side one. If $\langle\overrightarrow{vw}\rangle$ is a long bond, we denote by $\langle\overline{vv'}\rangle$ ($\langle\overline{ww'}\rangle$) the short bond that includes vertex $v(w)$.[19] We can define two operators $W^{\pm}_{vw} = \frac{1}{4}\left(1 \pm \tau^z_{f_{\overline{vv'}}}\right)\left(1 \mp \tau^z_{f'_{\overline{ww'}}}\right)$ to determine which $\gamma^s_{v,w}$($s = \uparrow, \downarrow$) to pair in the Majorana chain on the domain wall. If $W^+_{vw} = 1$, $W^-_{vw} = 0$, the pairing over the long bond $\langle\overrightarrow{vw}\rangle$ is $i\gamma_v^\uparrow\gamma_w^\downarrow$; if $W^-_{vw} = 1$, $W^+_{vw} = 0$, it is $i\gamma_v^\downarrow\gamma_w^\uparrow$. If both are zero, $\langle\overrightarrow{vw}\rangle$ is not on a domain wall.

Now we write the decoration part of the Hamiltonian as

$$\begin{aligned}
H_{\text{decorate}} &= -\sum_{\substack{\langle\overrightarrow{vw}\rangle \\ t(v)=t(w)}} [iD_{\langle\overrightarrow{vw}\rangle}W^+_{vw}\gamma_v^\uparrow\gamma_w^\downarrow + iD_{\langle\overrightarrow{vw}\rangle}W^-_{vw}\gamma_v^\downarrow\gamma_w^\uparrow)] \\
&\quad - \sum_{\substack{\langle\overrightarrow{vw}\rangle \\ t(v)\neq t(w)}} [iD_{\langle\overrightarrow{vw}\rangle}\left(\frac{1+\tau^z_f}{2}\right)\gamma_v^\downarrow\gamma_w^\downarrow + iD_{\langle\overrightarrow{vw}\rangle}\left(\frac{1-\tau^z_f}{2}\right)\gamma_v^\uparrow\gamma_w^\uparrow \\
&\quad + i\left(\frac{1-D_{\langle\overrightarrow{vw}\rangle}}{2}\right)(\gamma_v^\uparrow\gamma_w^\uparrow + \gamma_v^\downarrow\gamma_w^\downarrow)],
\end{aligned} \tag{5}$$

where $t(v)$ (resp. $t(w)$) denotes the triangular face that includes the vertex v (resp. w). H_{tunnel} can be defined by

$$H_{\text{tunnel}} = \sum_p \tau^x_p X_p, \tag{6}$$

where the sum over p only involves the plaquettes, not the triangles. The plaquette term X_p rearranges the Majorana chains to comply with the domain wall decoration rules defined above after τ^x_p is applied. Specifically,

$$X_p = \sum_{\substack{\mu_p=\pm1 \\ \{\mu_q=\pm1\}}} V_p^{\{\mu_{p,q}\}}\Pi_p P_p^{\{\mu_{p,q}\}}, \tag{7}$$

where the sum over $\{\mu_q = \pm1\}$ denotes the summation over all the adjacent plaquette spin configurations around p .[20] The operators $P_p^{\{\mu_{p,q}\}}$ and Π_p are projectors: $P_p^{\{\mu_{p,q}\}}$ projects onto bosonic spin states with precisely $\tau^z_p = \mu_p$ and $\tau^z_q = \mu_q$, and Π_p projects onto states in the fermionic Hilbert space that conform to those spin configurations:

$$P_p^{\{\mu_{p,q}\}} = \left(\frac{1+\tau^z_p\mu_p}{2}\right)\prod_{\{q\}}\left(\frac{1+\tau^z_q\mu_q}{2}\right) \tag{8}$$

$$\Pi_p = \prod_{\substack{\langle \overrightarrow{vw}\rangle\in\partial' p\\ t(v)=t(w)}} D_{\langle\overrightarrow{vw}\rangle}\left[W^+_{vw}\left(\frac{1+i\gamma_v^\uparrow\gamma_w^\downarrow}{2}\right)+W^-_{vw}\left(\frac{1+i\gamma_v^\downarrow\gamma_w^\uparrow}{2}\right)\right]$$
$$\prod_{\substack{\langle \overrightarrow{vw}\rangle\in\partial' p\\ t(v)\neq t(w)}} \Bigg\{\left(\frac{1-D_{\langle\overrightarrow{vw}\rangle}}{2}\right)\left(\frac{1+i\gamma_v^\uparrow\gamma_w^\uparrow}{2}\right)\left(\frac{1+i\gamma_v^\downarrow\gamma_w^\downarrow}{2}\right)+$$
$$D_{\langle\overrightarrow{vw}\rangle}\left[\left(\frac{1+\tau^z_{f_{vw}}}{2}\right)\left(\frac{1+i\gamma_v^\downarrow\gamma_w^\downarrow}{2}\right)+\left(\frac{1-\tau^z_{f_{vw}}}{2}\right)\left(\frac{1+i\gamma_v^\uparrow\gamma_w^\uparrow}{2}\right)\right]\Bigg\} \tag{9}$$

Here $\partial' p$ includes the 36 Majoranas in the triangles surrounding the plaquette p, as shown in Fig.3(a). The first line and third line of Eq.(9) enforce the pairing of Majorana modes on the domain wall, and the second line of Eq.(9) enforces the pairing of Majorana modes away from the domain wall.

The third part in the definition of X_p is

$$V_p^{\{\mu_{p,q}\}} = 2^{-\frac{n+1}{2}}(1+is_{2,3}\gamma_2^{\sigma_2}\gamma_3^{\sigma_3})(1+is_{4,5}\gamma_4^{\sigma_4}\gamma_5^{\sigma_5})\ldots (1+is_{2n,1}\gamma_{2n}^{\sigma_{2n}}\gamma_1^{\sigma_1}). \tag{10}$$

which takes the initial fermion configuration $|\Psi_i\rangle$ determined by Π_p corresponding to a fixed bosonic configuration determined by $P_p^{\{\mu_{p,q}\}}$, and maps it to $|\Psi_f\rangle$. The constant in the front is chosen so that $|\Psi_f\rangle$ has the same norm as $|\Psi_i\rangle$. The labels σ_i ($i = 1, 2...2n$) can take values $\uparrow$ and $\downarrow$, specifying the spins of the Majorana modes, and are determined by the bosonic spin configuration on and around the plaquette p following the aforementioned decoration rules. The Majorana modes γ_i are arranged so that the initial state satisfy $is_{2i-1,2i}\gamma_{2i-1}^{\sigma_{2i-1}}\gamma_{2i}^{\sigma_{2i}} = 1$. Then $V_p^{\{\mu_{p,q}\}}$ maps this state into a state $|\Psi_f\rangle$ with $is_{2i,2i+1}\gamma_{2i}^{\sigma_{2i}}\gamma_{2i+1}^{\sigma_{2i+1}} = 1$. Here $s_{i,j} = 1$ if the edge $\langle v_i v_j\rangle$ points from v_i to v_j and $s_{i,j} = -1$ otherwise. A pictorial illustration is given in Fig.3(b).

$V_p^{\{\mu_{p,q}\}}$ defined above determines the relative weight and phase factor of different configurations. With repeated application of V_p and τ_p^x, we can start from any initial configuration (including both boson and fermion degrees of freedom) satisfying H_{decorate}, and reach any other final configuration. The total ground state wave function is then a superposition of all the configurations obtained in this way. The fact that the relative weight and phase factor of different configurations can be uniquely and consistently determined is guaranteed by the commutativity of different V_p terms, which we prove in the Supplemental Material[18]. Moreover, as we discuss in the Supplemental Material, the Hamiltonian as defined is time-reversal invariant and ensures the time-reversal invariance of the ground state wave function.

Why $T^2 = 1$ fermion does not work – We now discuss why our decoration procedure discussed above does not work for spinless fermions with $T^2 = 1$. Let there be one complex fermion on each short bond $l = \langle\overrightarrow{vv'}\rangle$, created and annihilated by c_l and $c_l^\dagger$, respectively. We define a pair of Majorana modes for each complex fermion by

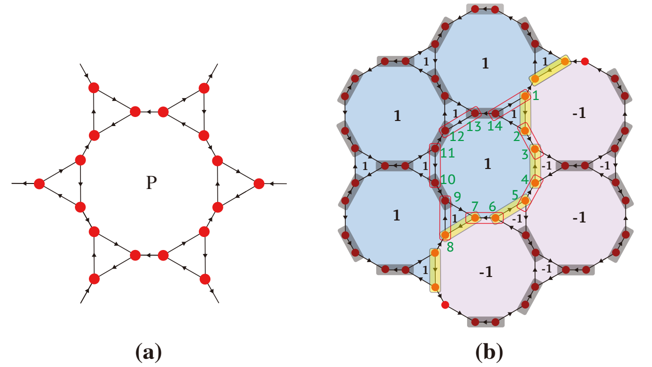

FIG. 3. (a) The 36 Majorana modes denoted by the 18 red dots in this figure are the Majorana modes surrounding the plaquette p, denoted by $\partial' p$. (b) Majorana modes (labeled $1-14$) involved in the definition of $V_p^{\{\mu_{p,q}\}}$ when flipping the middle plaquette starting from this particular initial configuration. Red rectangles correspond to the pair projector terms involved in $V_p^{\{\mu_{p,q}\}}$. Note that the spins of the involved Majorana modes are not shown in the figure.

$\gamma_v = c_l^\dagger + c_l$, $\gamma_{v'} = i(c_l^\dagger - c_l)$, located at vertices v and v', respectively. Under time reversal, $T : c_l \to c_l$. Written in terms of the Majorana modes, we have:

$$T : \gamma_v \to \gamma_v, \quad \gamma_{v'} \to -\gamma_{v'}. \tag{11}$$

We may decorate the time-reversal domain walls with Majorana chains in a way similar to our construction above and similar to Ref.15 and 16. Away from the domain wall, we pair up Majorana modes that share a short bond $l = \langle\overrightarrow{vv'}\rangle$ as $i\gamma_v\gamma_v'$. On a domain wall, we pair up Majorana modes that share a long bond $\tilde{l} = \langle\overrightarrow{vw}\rangle$ as $i\gamma_v\gamma_w$. This coupling rule ensures that different domain wall configurations have the same fermion parity. However, one can check explicitly that the domain-wall coupling terms are odd under time reversal, hence breaking time-reversal symmetry. In the Supplemental Material[18], we will further argue that modifications of the above coupling rule which preserve the time-reversal symmetry inevitably breaks the fermion parity invariance of the domain wall configurations. This strongly suggests that our construction cannot be generalized to spinless fermions with $T^2 = 1$. This is consistent with the fact that there are no nontrivial fermionic short-range entangled phases with $T^2 = 1$.

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[17] The way $\gamma_v^{\sigma_v}$ transforms into $\gamma_v^{\bar{\sigma}_v}$ depends on the orientation of the short bond $\langle vv' \rangle$ and similarly for w. One can check that with all four orientation possibilities, this conclusion is always true.

[18] See Supplemental Material at [URL will be inserted by publisher] for a proof of the commutativity of the Hamiltonian terms, a proof of the time-reversal invariance of the Hamiltonian and the wave function, and an argument of why $T^2 = 1$ fermion does not give a consistent construction in our framework.

[19] The overline on top of $\overline{vv'}$ means that if v is oriented to v', $\overline{vv'} = \overrightarrow{vv'}$, otherwise $\overline{vv'} = \overrightarrow{v'v}$.

[20] Note that by using the 'majority rule', one can extend the spin configuration from plaquettes to triangles.