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Phys. Rev. B **98**, 081118 — Published 31 August 2018

DOI: 10.1103/PhysRevB.98.081118

Chiral Lattice Supersolid on Edges of Quantum Spin Hall Samples

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(Dated: May 17, 2018)

We show that edges of Quantum Spin Hall topological insulators represent a natural platform for realization of exotic magnetic phase which has all properties of lattice supersolid. On one hand, fermionic edge modes are helical due to the nontrivial topology of the bulk. On the other hand, a disorder at the edge or magnetic adatoms may produce a dense array of localized spins interacting with the helical electrons. The spin subsystem is magnetically frustrated since the indirect exchange favors formation of helical spin order and the direct one favors (anti)ferromagnetic ordering of the spins. At a moderately strong direct exchange, the competition between these spin interactions results in the spontaneous breaking of parity and in the Ising type order of the z-components at zero temperature. If the total spin is conserved the spin order does not pin a collective massless helical mode which supports the ideal transport. In this case, the phase transition converts the helical spin order to the order of a chiral lattice supersolid. This represents a radically new possibility for experimental studies of the elusive supersolidity.

PACS numbers: 71.10.Pm, 73.43.-f 75.30.Hx, 67.80.K-, 67.80.kb

Supersolid is an exotic phase where, very counterintuitively, crystal order and an ideal transport coexist in one and the same physical system [1]. Dating back to the 50ties, first discussions of supersolidity resulted in arguments against its existence [2]. It was realized later that the quantum bosonic statistics could provide necessary conditions for formation of supersolids. Starting from the 60ties, studies were concentrated on interacting bosons, in particular, on ^{4}He [3–6]. It can crystallize at a high pressure and is expected to combine broken translational invariance with superfluidity. In spite of large interest and intense experimental efforts, the supersolid phase has not been convincingly realized in helium [7–9]. This failure calls for a search for alternative physical platforms for supersolidity. Recent experiments aim at realizing supersolid in cold atoms [10, 11]. Another well-known alternative is provided by a possibility to have magnetic supersolid after mapping the bosonic theory onto a magnetic (or a quantum gas) lattice model [12, 13], where both the spin rotation symmetry and the lattice symmetry can be broken simultaneously [14–16]. Longitudinal and transverse components of the antiferromagnetic order of the magnetic lattice model (or the diagonal and off-diagonal long-range order of the quantum gas) correspond respectively to the crystalline order and to superfluidity of the bosons. The transition to the supersolid phase on the lattice is similar to the Dicke and the Ising type transitions [17, 18].

In this Letter, we suggest a novel platform for lattice supersolid. It is provided by the recently discovered time reversal invariant topological insulators [19–21] which have become famous due to their virtually ideal edge transport. We will concentrate on two-dimensional topological insulators, Quantum Spin Hall samples (QSH), where transport is carried by one-dimensional (1D) helical edge modes. These modes possess lock-in relation between electron spin and momentum so that helical electrons (HEs) propagating in opposite directions have opposite spins [22–24]. This locking protects transport against disorder [25–27]: An elastic backscattering of HEs must be accompanied by spin-flip and, therefore, it can be provided only by magnetic impurities [28]. However, a single Kondo impurity is unable to change the ideal dc conductance [29] if the total spin is conserved. Under some conditions, e.g. a random magnetic anisotropy, the ballistic transport of HEs may be suppressed by coupling to a Kondo array [30–36]. The latter can be present in realistic samples due to the edge disorder which localizes a fraction of the bulk electrons close to the edge [31] such that the localized electrons become spin-1/2 local moments. Alternatively, the Kondo array can be generated by magnetic adatoms located close to the edge [37].

While transport of the 1D HEs coupled to a dense Kondo array has been intensively studied, magnetic properties of these systems have attracted less attention. It is known that helical spin ordering, similar to that caused by dynamical instabilities [38], can result from the indirect Ruderman–Kittel–Kasuya–Yosida (RKKY) spin interaction mediated by HEs [31, 33, 39]. However, the direct Heisenberg exchange interaction between the Kondo impurities has never been taken into account though one may expect it to appear at relatively high spin densities. *We will show that*, if the Heisenberg coupling, J_H, is sufficiently strong, the helical magnetic order on the QSH edge is converted to another exotic magnetic state which has all properties of lattice supersolid, see Fig.1. We will call this phase *Chiral Lattice Supersolid.*

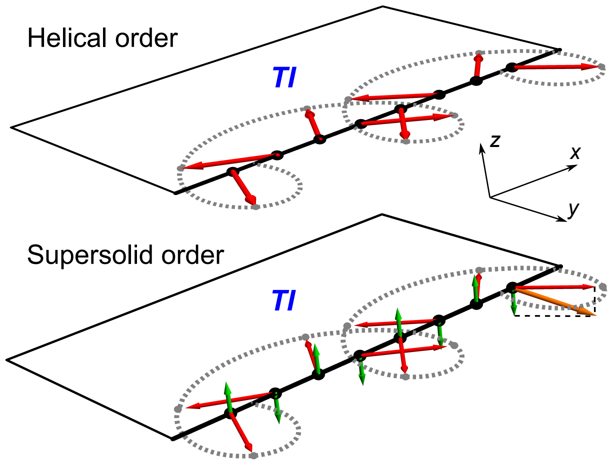

FIG. 1. Illustration of helical (upper panel) and supersolid (lower panel) spin orders. Red and green arrows show in-plane- and z-components of the spins, respectively. The orange arrow exemplifies the total spin orientation in the supersolid phase. Dotted line stands for the helix; black circles are the lattice sites. We have assumed that the isotropy of the xy-plane is inherited from the bulk of QSH.

Our prediction is prompted by the recent theory for Kondo-Heisenberg models which states that a competition of RKKY with the Heisenberg exchange may lead to the Ising-type phase transition [40]. Time-reversal and parity symmetries are spontaneously broken in the ordered phase and, if the system is SU(2) symmetric, spins form the isotropic scalar chiral spin order. It is characterized by an exotic order parameter which involves three neighboring spins [41, 42], Eq.(13).

The lattice which we consider is very unusual: SU(2) symmetry is broken at the QSH edges by helicity of the electrons. Therefore, the RKKY/Heisenberg competition leads to the formation of a different exotic state. It combines: (i) the helical in-plane and (anti)ferromagnetic spin orders which are counterparts of off-diagonal and diagonal supersolid orders, respectively; (ii) the helical transport which is supported by collective modes of HEs coupled to the transverse spin fluctuations. These modes are slow due to the strong electron-spin coupling. Even more importantly, they are gapless, i.e. transport is ideal, provided the total spin is conserved. Thus, the electrons play an auxiliary róle for formation of the spin orders but their helicity is crucially important for the ideal transport. We come across all properties of supersolidity described above; the name "Chiral Lattice Supersolid" reflects their unique combination peculiar to the QSH edges. These are our *main results*.

Both spin orders appear only at $T = 0$. At finite temperature, the corresponding correlation lengths are finite but diverge at $T \to 0$. Hence there is a region of temperatures $T < \Delta$ where the correlation lengths are large in comparison with the scale $O(1/\Delta)$; Δ is the energy scale below which the coupling between the electrons and the localized moments becomes strong, see Eq.(7). In this temperature range, the proximity to the ordered state is strongly felt and the spin order is present [43].

The model and key steps of our approach [44]: The Hamiltonian of HEs coupled to an array of interacting localized spins is: $\hat{H} = \hat{H}_0 + \hat{H}_{\rm int} + \hat{H}_H + \hat{H}_K$, where $\hat{H}_{0,\rm int}$ describe the free fermions and the interaction between them, respectively:

$$\hat{H}_0 = -iv_F \int \mathrm{d}x \sum_{\eta=\pm} \eta\, \psi^\dagger_\eta(x)\partial_x\psi_\eta(x), \tag{1}$$

$$\hat{H}_{\rm int} = \frac{g}{2\nu}\int \mathrm{d}x\ (\rho_+ + \rho_-)^2, \quad \rho_\pm \equiv \psi^\dagger_\pm\psi_\pm. \tag{2}$$

Here ψ_+ (ψ_-) describes spin-up right moving (spin-down left moving) in the x-direction HEs $\psi_{R,\uparrow}$ $(\psi_{L,\downarrow})$; v_F is the Fermi velocity, ν is the density of states of HEs and g is the dimensionless interaction strength which governs the Luttinger parameter $K = 1/\sqrt{1+g}$ [45].

Without loss of generality, we consider the isotropic short range antiferromagnetic exchange interaction between neighboring spins described by the Hamiltonian:

$$\hat{H}_H = J_H \sum_m \boldsymbol{S}(x_{m+1})\boldsymbol{S}(x_m),\ x_m = \xi m,\ J_H > 0; \tag{3}$$

$\boldsymbol{S}$ are s-spin operators on the lattice sites x_m. The sum runs over sites of the spin array; for the sake of simplicity, we will not distinguish constants of crystalline and spin lattices, ξ.

The coupling between the spins and HEs is described by the backscattering Hamiltonian:

$$\hat{H}_K = \int \mathrm{d}x\, \rho_s J_K \left[S^+ e^{2ik_F x}\psi^\dagger_-\psi_+ + h.c.\right]; \tag{4}$$

where k_F is the Fermi momentum; J_K is xy-isotropic coupling constant; $S^\pm \equiv S_x \pm iS_y$. The dimensionless impurity density ρ_s is used to convert the sum over the lattice sites to the integral. We omit the forward-scattering term $\sim J_z S_z$ since a unitary transformation of the Hamiltonian allows one to map the theory with the parameters $\{K, J_z \neq 0\}$ to the equivalent theory with the effective parameters $\tilde{K} = K(1 - \xi J_z\nu/2K)^2$ and $\tilde{J}_z = 0$ [46, 47]. Thus, $H_{\rm int}$ can take into account both the direct electron-electron interaction and the interaction mediated by the z-coupling to the Kondo impurities. The coupling constants are assumed to be small, $sJ_{H,K} \ll u/\xi, D$. Here D is the UV energy cutoff which is of the order of the bulk gap in the QSH sample and u denotes the excitation velocity renormalized by the electron interaction.

The model Eqs.(1,2,4) with $J_H = 0$ was studied in Ref.[33]. Let us briefly recapitulate key points of that paper and generalize it for finite J_H. Our goal is to derive the effective low energy theory. This can be conveniently done after parameterizing the spins by unit vectors:

$$S^\pm(x_m) = s\sqrt{1 - n_z^2(x_m)}\, \mathrm{e}^{\mp 2ik_F x_m \pm i\alpha(x_m)},$$

$$S^z(x_m) = (-1)^m s\, n_z(x_m)\,. \tag{5}$$

Here, we have singled out slow spin variables α, n_z. Next, we change from the Hamiltonian to the action. The parametrization Eq.(5) requires the Wess-Zumino term in the Lagrangian [48], $\mathcal{L}_{\rm WZ} = -is(\rho_s/\xi)n_z\partial_\tau\alpha$; where τ is the imaginary time. Performing the gauge transformation of the fermionic fields: $\psi_\eta e^{-i\eta\alpha/2} \to \psi_\eta$, we reduce the noninteracting fermionic part of the Hamiltonian, Eqs.(1,4), to the following Lagrangian density:

$$\mathcal{L}_0 = \sum_{\eta=\pm}\Big[\bar\psi_\eta\partial_\eta\psi_\eta + s\rho_s J_K\sqrt{1-n_z^2}\,\bar\psi_{-\eta}\psi_\eta\Big] + \frac{\mathcal{L}_{\rm LL}[\alpha, v_F]}{4};$$
$$\mathcal{L}_{\rm LL}[\alpha, v_F] \equiv [(\partial_\tau\alpha)^2 + (v_F\partial_x\alpha)^2]/(2\pi v_F)\,. \tag{6}$$

$\partial_\eta \equiv \partial_\tau - i\eta v_F\partial_x$ denotes the chiral derivative and $\mathcal{L}_{\rm LL}$ is the hydrodynamic Lagrangian of the Luttinger liquid model. $\mathcal{L}_{\rm LL}$ has been generated by the anomaly of the fermionic gauge transformation [49].

A mean value $\mathcal{M} \equiv \langle\sqrt{1-n_z^2}\rangle = \text{const}$ yields a constant gap in the spectrum of the electrons, $\Delta_0 = \bar\Delta_0\mathcal{M}$ with $\bar\Delta_0 \equiv s\rho_s J_K$, which is opened by backscattering, Eq.(4). By combining the functional bosonization approach [50] with scaling arguments, one can show that the main effect of the weak electron interaction, $|\delta K| \ll 1$ with $\delta K \equiv 1 - \tilde K$, is renomalization of the Luttinger liquid parameters, $\mathcal{L}_{\rm LL}[\alpha, v_F] \to \mathcal{L}_{\rm LL}[\alpha, u]/\tilde K$, and of the gap $\Delta_0 \to \Delta \ll D$:

$$\frac{\Delta}{D} \simeq \left(\frac{\Delta_0}{D}\right)^{\frac{1}{2-\tilde K}} \simeq \mathcal{M}\Big[1 - \delta K\log(\mathcal{M})\Big]\left(\frac{\bar\Delta_0}{D}\right)^{\frac{1}{2-\tilde K}}. \tag{7}$$

We will not consider the case $\mathcal{M} \to 0$ and, therefore, correction $O(\delta K)$ can be neglected in Eq.(7).

It is known that α is gapless at $J_H = 0$ if the total spin is conserved [31, 33]. We will show that this holds true even at finite J_H. Thus, Eq.(6) describes the connection between gapped- and gapless sectors which is mediated by fluctuations of n_z. The energy scale Δ establishes a crossover from the weak to strong coupling between HEs and the spins. In the strong coupling regime, they form a single Luttinger liquid where the low energy charge excitations and the in-plane spin excitations are described by the same field α.

Transition between the helical phase and supersolid can be identified after treating n_z and α as the slow variables and integrating out the gapped fermions. This yields the density of the effective potential $\mathcal{E}(\mathcal{M})$ per one unit cell. Restoring now finite J_H, we find in the leading order in sJ_K/D:

$$\mathcal{E}(\mathcal{M}) \simeq -\big(\xi\Delta^2/2\pi u\big)\log(D/\Delta) + \tag{8}$$
$$+\, s^2 J_H\mathcal{M}^2\Big(1 + \cos[2k_F\xi]\Big) + \text{const};$$

gradient terms are discussed below. Minima of $\mathcal{E}(\mathcal{M})$ determine the ground state configuration of the magnetization field, n_z .

If J_H is smaller than the critical value J_H^*, the minimum is at $\mathcal{M} = \mathcal{M}_h = 1$ (i.e. $\langle n^z\rangle = 0$). The spins are in the xy-plane [the upper panel of Fig.1]. When the Heisenberg exchange exceeds J_H^* a nontrivial minimum appears at $\mathcal{M} = \mathcal{M}_s < 1$:

$$\mathcal{M}_s = \frac{D}{\bar\Delta}\exp\left\{-4\pi s^2\frac{J_H u}{\xi\bar\Delta^2}\cos^2(k_F\xi_0) - \frac{1}{2}\right\}; \tag{9}$$

where $\bar\Delta = \Delta/\mathcal{M}$ is the $\mathcal{M}$-independent part of Δ. The critical value, J_H^*, is defined by the equation

$$\mathcal{M}_s(J_H^*) = 1 \;\Rightarrow\; J_H^* \simeq \frac{\xi\bar\Delta^2\log\big(D/\bar\Delta\big)}{4\pi s^2 u\cos^2(k_F\xi_0)}. \tag{10}$$

We consider small Heisenberg couplings. Therefore, the nontrivial minimum can be realized only if $sJ_H^* \ll u/\xi, D$. This implies, in particular, the case $\cos(k_F\xi_0) \to 0$ must be excluded from the consideration.

The solution Eq.(9) corresponds to the staggered magnetization [the lower panel of Fig.1]. Since $\mathcal{E}(\mathcal{M})$ is invariant with respect to inverting the spin components S_z, $S_z(m) \to -S_z(m)$ for all lattice sites, the ground state is double-degenerate. This degeneracy is lifted at $T = 0$ by a spontaneous breaking of the corresponding Z_2 as in 1D Ising model.

With a further increase of J_H, the system approaches the regime of isotropic Heisenberg magnet which is beyond the scope of the present paper.

Fluctuations of S^z are gapped for all values of J_H excluding its critical value J_H^*. Therefore, the corresponding correlation functions are short ranged. The effective action for α, $\mathcal{L}_\alpha$, can be derived by integrating out massive modes: the fermions and the n_z fluctuations [31, 33]. If the total spin is conserved, this yields for the energies below Δ: $\mathcal{L}_\alpha = \mathcal{L}_{\rm LL}[\alpha, u_\alpha]/4K_\alpha$ with $u_\alpha/K_\alpha \simeq u/\tilde K$. Parameters of $\mathcal{L}_\alpha$ are substantially influenced by the electron-spin interactions such that $K_\alpha \ll \tilde K$ and $u_\alpha \ll u$. $\mathcal{L}_\alpha$ contains only gradients $\partial_{x,\tau}\alpha$, hence, fluctuations of α *are massless.* One can say that the massless excitations of our model are slow spinons dressed by localized electrons. They govern spin correlations at $T \ll \Delta$:

$$\langle\langle S^+(\tau,x)S^-(0,0)\rangle\rangle \sim \mathcal{M}^2 e^{-2ik_F x}\langle e^{i[\alpha(\tau,x)-\alpha(0,0)]}\rangle =$$
$$= \mathcal{M}^2 e^{-2ik_F x}\left[\frac{(\pi T\xi/u_\alpha)^2}{\sin^2(\pi T\tau) + \sinh^2(\pi T x/u_\alpha)}\right]^{K_\alpha}. \tag{11}$$

At $T = 0$, the correlations in Eq.(11) decay as power law which is a signature of a quasi long range order of these components. The correlations are cut by the thermal length, $L_T = u_\alpha/T$, at $T \neq 0$.

Helical phase, $J_H < J_H^*$ and $\mathcal{M} = \mathcal{M}_h = 1$: The correlation function of $S^\pm$ spin components is given by Eq.(11) with fluctuations being centered at the wave vector $-2k_F$ (not at $+2k_F$). This asymmetry is bound to the certain helicity of the fermions at the edge of QSH,

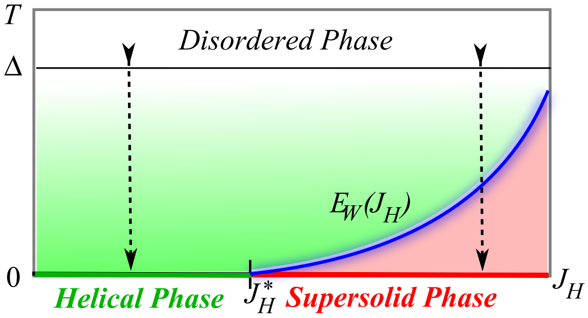

FIG. 2. Phase diagram of the dense Kondo-Heisenberg array coupled to the interacting HEs at the QSH edge. Green and red lines show phases with helical and supersolid order at $T = 0$, respectively. Light-green and light-red regions mark regimes where these orders are felt at finite T. The supersolid order disappears at $T \sim E_W$, Eq.(14). The system becomes completely disordered at $T \sim \Delta$, Eq.(7). Dashed lines exemplify measurement protocols which could reveal different phases, see Conclusions.

see Eq.(4): fermionic helicity governs orientation (right or left handed) of the spin helix. The phase has a nematic (or vector chiral) order parameter reflecting the helical spin structure:

$$\vec{\mathcal{O}}_h = [\boldsymbol{S}(x) \times \boldsymbol{S}(x+\xi)], \ [\vec{\mathcal{O}}_h]_z \sim s^2 \sin(2k_F\xi). \qquad (12)$$

The helical order is felt at $\xi \ll u/\Delta \ll L < L_T$ where L is the system size. It becomes suppressed at $u/\Delta \ll L_T < L$ and is completely destroyed by the thermal fluctuation at $T \sim \Delta$, see Fig.2.

Helimagnetic Lattice phase, $J_H > J_H^*$ and $\mathcal{M} = \mathcal{M}_s < 1$: in addition to the helical order, a new order appears in the system via the Ising type transition: $\langle S_z \rangle$ becomes staggered and forms a magnetic lattice. The new phase has some special features inherited from the helical phase. The excitations are again centered at $-2k_F$ and not $2k_F$ [see Eq.(11)] and, therefore, are helical. The origin of this asymmetry is the same: a certain helicity of the edge fermions caused by nontrivial topology of the QSH bulk. Moreover, the combination of the helical order with the staggered magnetization trivially produces non-zero scalar chiral order parameter:

$$\mathcal{O}_c = \Big(\boldsymbol{S}(x-\xi), \vec{\mathcal{O}}_h\Big). \qquad (13)$$

The finite temperature suppresses the staggered magnetization via formation of domain walls. The energy of the single wall can be estimated by the height of the potential barrier in the potential $\mathcal{E}(\mathcal{M})$: $E_W \sim \mathcal{E}(\mathcal{M}_h) - \mathcal{E}(\mathcal{M}_c)$. For J_H close to J_c, E_W simplifies to:

$$E_W \sim \big[(J_H - J_c)/\bar{\Delta}\big]^2 \times (\xi/u). \qquad (14)$$

Order of the helimagnetic lattice can be felt if $T < E_W$, see Fig.2, which ensures the exponentially large correlation length of the field n_z: $L_z \propto \exp(E_W/T)$. The Z_2 symmetry is restored beyond the scale L_z.

Chiral Lattice Supersolid: Let us show that the helimagnetic lattice is a peculiar lattice supersolid. The spin correlation function Eq.(11) is $\propto \mathcal{M}^2$ and possess the quasi long range order. Simultaneously, nonzero value of $\mathcal{M}$ provides the ideal helical transport of electron/spinon complexes [31, 33]. This suggest that, in our model, $\mathcal{M}$ plays the role of the superfluid density with off-diagonal order being reflected by $\langle\langle S^+S^-\rangle\rangle$ correlations. The staggered magnetization breaks translational symmetry of S_z in the magnetic subsystem and, therefore, reflects diagonal order which does not suppress the ideal transport.

Since diagonal spin order coexists with the off-diagonal one and with the gapless excitations, the helimagnetic lattice is lattice supersolid. This concludes the proof of our main result. To emphasize the complex nature of the new lattice supersolid, we refer to is as "Chiral Lattice Supersolid". The QSH samples are probably the unique platform for realization of this phase.

To summarize, we have demonstrated that, being coupled to a dense array of localized quantum spins, helical edge modes of a Quantum Spin Hall topological insulator can host an exotic magnetic order at $T = 0$. The system possesses a characteristic energy scale Δ related to the backscattering of the helical electrons from the local spins. This energy scale signifies a crossover from weak to strong coupling. In the strong coupling regime the system remains critical, but the spin fluctuations are absorbed into the electronic ones.

The temperature region $T < \Delta$ can be characterized by the proximity to the helical spin order existing at $T = 0$. Its underlying mechanism is based on the RKKY interaction of the spins mediated by HEs. A competition of the RKKY indirect exchange with the direct Heisenberg one may lead at $T = 0, J_H > J_H^*$ [see Eq.(10)] to the Ising type phase transition and to the appearance of the additional order which is the staggered magnetization. If the total spin is conserved these two spin orders coexist with the gapless excitation being able to support a symmetry protected (virtually ideal) transport. This is the principal difference of our results from theories describing an interaction induced spontaneous breaking of time reversal symmetry which removes the symmetry protection of the ideal transport [22, 23, 31, 33]. We have shown that there is one-to-one correspondence between the new phase and the magnetic lattice supersolidity. Thus, the phase which we have described is also a kind of lattice supersolid which inherits peculiar features of the helical magnetic phase. The latter has the nontrivial vector chiral order parameter, Eq.(12). That is why supersolid hosted by QSH samples can be called "Chiral Lattice Supersolid".

A weak disorder in the spin lattice can suppress neither the helical spin order nor the protected ideal transport [31]. Clearly, the staggered magnetization can also appear in the weakly disordered Kondo-Heisenberg array coupled to HEs. Thus, such a disorder can lead only to some quantitative changes and is unable to destroy the Chiral Lattice Supersolid.

Our findings suggest that magnetically doped QSH edges provide a principally new possibility to study elusive supersolidity. Coupling constants $J_{K,H}$ can be controlled by varying the proximity of the magnetic adatoms to the helical edge and their density, respectively. Experimental detection of the Chiral Lattice Supersolid can be based on spin correlations, i.e. spin susceptibilities, which have no pronounced peaks in the disordered phase. In the proximity to the helical phase [left dashed line in Fig.2 at $T < \Delta$], correlation functions of xy-spin components acquire peaks at the wave vector $Q_h = \pm 2k_F$ with the sign being defined by helicity of the electrons. The correlation function of z-components is expected to be structureless in the helical phase but must show new peaks at the Neel vector, $Q_a = \pi/\xi$, in the proximity to the supersolid phase [right dashed line in Fig.2 at $T < E_W$]. Thus, measuring the spin susceptibilities at different temperatures can fully characterize the system.

We have considered purely 1D system and, therefore, the spin order is only algebraic even in the limit $T \to 0$. One promising generalization could include the study of the Kondo-Heisenberg array coupled to the 2D edge of a 3D topological insulator. The influence of fluctuations is weaker in 2D and, if lattice supersolid can be realized in this setup, its spin order could become long-range.

Acknowledgments: O.M.Ye. acknowledges support from the DFG through the grant YE 157/2-1. A.M.T. was supported by the U.S. Department of Energy (DOE), Division of Materials Science, under Contract No. DE-SC0012704. We are grateful to Boris Altshuler for useful discussions.

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the derivation of the effective low energy theory there is no need to present details of the secondary importance; one can restore them straightforwardly with the help of Refs.[33, 40].
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