This is the accepted manuscript made available via CHORUS. The article has been published as:

Interacting Floquet topological phases in three dimensions

Dominic Reiss, Fenner Harper, and Rahul Roy

Phys. Rev. B **98**, 045127 — Published 19 July 2018

DOI: [10.1103/PhysRevB.98.045127](https://doi.org/10.1103/PhysRevB.98.045127)

Interacting Floquet topological phases in three dimensions

Dominic Reiss, Fenner Harper, and Rahul Roy

*Mani L. Bhaumik Institute for Theoretical Physics, Department of Physics and Astronomy,
University of California at Los Angeles, Los Angeles, CA 90095, USA*

(Dated: June 26, 2018)

In two dimensions, interacting Floquet topological phases may arise even in the absence of any protecting symmetry, exhibiting chiral edge transport that is robust to local perturbations. We explore a similar class of Floquet topological phases in three dimensions, with translational invariance but no other symmetry, which also exhibit anomalous transport at a boundary surface. By studying the space of local 2D unitary operators, we show that the boundary behavior of such phases falls into equivalence classes, each characterized by an infinite set of reciprocal lattice vectors. In turn, this provides a classification of the 3D bulk, which we conjecture is complete. We demonstrate that such phases may be generated by exactly-solvable ‘exchange drives’ in the bulk. In the process, we show that the edge behavior of a general exchange drive in two or three dimensions can be deduced from the geometric properties of its action in the bulk, a form of bulk-boundary correspondence.

I. INTRODUCTION

Driving a system periodically in time can generate remarkable behavior with an intrinsically dynamical character. In this rapidly evolving field of Floquet systems, recent advances include the prediction of phases which exhibit an analog of symmetry breaking in the time domain, known as discrete time crystals or π -spin glasses [1–6], as well as a host of novel topological phases that lie beyond any static characterization [7–24]. These theoretical works have been complemented by significant experimental advances, with analogs of Floquet topological phases being realized in a variety of different settings [25–31].

A particularly surprising set of Floquet topological phases are those which are robust even in the absence of symmetry [9, 32–35]. In the presence of interactions, 2D systems in this class have been shown to exhibit robust Hilbert space translation at the boundary of an open system [32, 33], and may be combined with bulk topological order to generate Floquet enriched topological phases [34, 35]. Despite their range of novel features, systems in this class have so far only been studied in 2D; in this paper, we set out to find and classify the Floquet topological phases that exist in 3D, under the assumption of translational invariance.

Similar to the classification of the related 2D phases, our approach is to first identify the distinct types of boundary behavior that these 3D Floquet systems may exhibit. By invoking ideas from Ref. 36, we find that local, translationally invariant unitary operators in two dimensions form distinct equivalence classes with representative ‘shift’ (or translation) actions. In turn, this boundary classification partitions the space of 3D unitary evolutions in the bulk into distinct classes. Each class may be labeled by a topological invariant (in this case, an infinite set of reciprocal lattice vectors), with evolutions that are members of the same class being topologically equivalent at a boundary. We construct exactly solvable bulk drives which populate these equivalence classes, and in the process, identify a geometric property of such a drive

that determines its anomalous behavior at an arbitrary boundary, a result that also applies to 2D. We conjecture that there are no intrinsically 3D Floquet topological phases (without symmetry), which would make this classification complete.

The structure of this paper is as follows: We begin with a brief review of Floquet systems and phases in Sec. II and provide some additional motivation for the work. In Sec. III, we describe and classify local, translationally invariant unitary operators with no symmetry in two dimensions, and show how this classification may be applied to the boundaries of 3D Floquet systems. In Sec. IV, we describe a modification of the exactly solvable ‘exchange drives’ introduced in Refs. 9, 32, and 33, and show that these have bulk geometric properties directly related to their action at a boundary. In Sec. V, we extend these models to 3D, and demonstrate that they may be used to generate boundary behavior from all equivalence classes. We summarize and discuss our results in Sec. VI.

II. PRELIMINARY DISCUSSION

We begin by recalling some concepts from the study of time-dependent systems that we will use throughout the paper. We are primarily interested in Floquet systems, whose Hamiltonians are periodic in time (with $H(t+T) = H(t)$). The behavior of such a system is captured by the unitary time-evolution operator, defined by

$$U(t) = \mathcal{T} \exp \left[-i \int_0^t H(t') dt' \right], \quad (1)$$

where $\mathcal{T}$ indicates time ordering. Without loss of generality, we consider our system prepared at $t = 0$ and refer to the unitary path $U(t)$, between $t = 0$ and $t = T$ as a unitary ‘drive’. Although the system Hamiltonian can in general vary continuously with time, the models we consider in this paper will have Hamiltonians that are piecewise constant. For these systems, the complete

unitary evolution operator is a product of unitary evolutions corresponding to each step, applied in chronological order.

We will classify these dynamical systems using the homotopy approach introduced in Ref. 24, which is concerned with identifying topologically distinct paths $U(t)$ within the global space of local unitary evolutions. This framework has the advantage that it disentangles questions about the topology of the path $U(t)$ from questions about the stability of the resulting phase. For example, interacting Floquet systems are believed to be inherently unstable to heating, since energy is pumped into the system with every driving cycle [37–39]. To prevent heating to infinite temperature, strong disorder may be added so that the system becomes many-body localized [40–44] (see Ref. 45 for a review of many-body localization (MBL)). Alternatively, even in the absence of MBL, Floquet systems can retain their topological characterization over a (quasi-)exponential prethermal time scale during which heating is negligible [46–48]. In the homotopy approach, the topology of an evolution is well defined independently of these heating effects, which would nevertheless need to be considered in a physical realization of the model, or to formally define a ‘phase’ [24].

The homotopy approach also allows a distinction to be made between static topological order, which depends only on the end point of the unitary evolution $U(T)$, and inherently dynamical topological order, which depends on the complete path of the evolution $U(t)$. As argued in Ref. 24, this latter dynamical order can be completely classified by studying a subset of unitary evolutions known as unitary loops which, for a closed system, satisfy $U(0) = U(T) = \mathbb{I}$. Although unitary loops are somewhat peculiar, and do not usually describe physically realistic drives, they are a useful theoretical tool that can be used to isolate the dynamical part of a drive.

In brief, if a generic unitary evolution $U(t)$ has an endpoint of the form $U(T) = e^{-iH_F}$, where H_F is a well-defined (Floquet) Hamiltonian, then the complete evolution can be expressed as a unitary loop, followed by an evolution with the static Hamiltonian H_F . The unitary loop component is responsible for the ‘dynamical’ order, while the constant evolution component is responsible for the ‘static’ order.¹ An explicit method for constructing a unitary loop from an arbitrary MBL drive was given in Ref. 32, although we emphasize that unitary loops are applicable more generally than this [24].

For our purposes, we therefore restrict our view to the classification of unitary loops, which, given the above decomposition, can capture all inherently dynamical topological evolutions. Specifically, we will classify unitary loops by studying their edge behavior, which can be observed when a loop drive is applied to a system with a

FIG. 1. Effective edge unitary for a loop evolution applied to an open system. (a) In a closed system, the loop evolution acts like the identity after a complete cycle. The open system Hamiltonian can be formed from the closed system Hamiltonian by removing terms which connect sites across the boundary cut (purple line). (b) The open-system evolution differs only from the identity in a narrow region in the vicinity of the cut, indicated by red and blue shading. The action of the open-system unitary is captured by an effective edge unitary U_{eff} for each boundary. See main text for further details.

boundary: In an open system a loop evolution will not necessarily return to the identity at $t = T$, and may instead generate topologically protected edge modes.

Explicitly, we can write a generic closed system Hamiltonian which generates a unitary loop as

$$H_{\text{closed}}(t) = H_{\text{open}}(t) + H_{\text{edge}}(t), \quad (2)$$

where H_{edge} connects sites across a predetermined boundary and H_{open} connects sites away from this boundary. We can then evolve with $H_{\text{open}}(t)$ for a complete cycle to obtain the action of the drive for the open system. However, if the Hamiltonian satisfies some notion of locality, then it will have a corresponding Lieb-Robinson velocity [49]. In turn, this implies that the open system evolution differs from the closed system evolution only in a finite region in the vicinity of the boundary, and so both evolutions must act as the identity deep within the bulk (see Fig. 1). We can formally restrict the open system evolution to this boundary region, and we refer to this restricted unitary operator as the ‘effective edge unitary’ U_{eff} . A proof that U_{eff} can always be defined is given in Ref. 33.

In this paper, our first aim is to obtain a complete characterization of effective edge unitaries described by local, two-dimensional unitary operators with translational invariance. We will then show that these distinct effective edge unitaries may be used to classify unitary loops in the 3D bulk, and will obtain an explicit set of loop drives which generate the different boundary behaviors. Although the unitary loops we introduce may seem somewhat fine-tuned, we will argue that any chiral Floquet phase is topologically equivalent to one of these representative drives, in the sense that their edge behaviors are equivalent.

¹ The uniqueness of this decomposition depends on the nature of the Floquet Hamiltonian H_F , however. We leave a discussion of this subtlety to Ref. 24.

III. BOUNDARY CLASSIFICATIONS IN 2D AND 3D

Dynamical phases of 2D Floquet systems with no symmetry were classified based on their boundary behavior in Refs. 32 and 33, building on a rigorous classification of 1D unitary operators from Ref. 36. The aim of this paper is to obtain a similar classification of 3D Floquet systems by considering the distinct behaviors that may arise at a 2D boundary. To this end, we now briefly review the classification of unitary operators at a 1D boundary, before going on to discuss the 2D case.

A. Effective unitary operators at a 1D boundary: single-particle case

As motivated in Sec. II, dynamical Floquet phases are described by unitary loops, which in turn may be classified by their edge behavior in an open system. For 2D phases, this edge behavior is encapsulated in a 1D unitary operator U_{eff} . Since the underlying Hamiltonian which generates the evolution should be physically motivated, the only restriction on the form of U_{eff} is that it should be *local*—i.e., it should map local operators onto other local operators.² There is no requirement, however, that it be possible to generate U_{eff} with a local *one-dimensional* Hamiltonian. This potential anomaly partitions the space of 1D edge behaviors into different equivalence classes.

Our ultimate aim is to classify the equivalence classes of many-body unitaries. However, much of the intuition required for this can be developed in the noninteracting case, which we review here. We consider a 1D lattice $\Gamma = \mathbb{Z}$, whose Hilbert space is spanned by a basis of single-particle on-site occupation states, $\{|i\rangle\}$ with $i \in \mathbb{Z}$. If this is the boundary of a 2D Floquet loop drive, then the effective edge unitary U_{eff} is a local 1D unitary operator. In the language of Ref. 36, U_{eff} is a *quantum walk*.

As a motivating example, we consider a unitary operator τ which acts as a unit translation to the right, $\tau|i\rangle = |i+1\rangle$. The operator τ is clearly local, since for a generic local operator $\hat{O}$, conjugation through $\tau\hat{O}\tau^{-1}$ simply translates $\hat{O}$ to the right, yielding another local operator. However, τ cannot be generated in finite time by any local 1D Hamiltonian, and is therefore anomalous [36]. More general unitary operators of this form, known as *shifts*, can be obtained by taking τ^p , where p is an integer.

The authors of Ref. 36, building on earlier work by Kitaev [50] (and others), form a rigorous classification of

quantum walks in 1D, finding that a generic local quantum walk is a shift operator τ^p combined with a locally generated 1D unitary. These form equivalence classes labeled by the integer p describing the shift, with different unitary operators within each equivalence class related by locally generated 1D unitary operators. In this way, the ‘pure’ shifts τ^p form representative effective edge unitaries which populate all distinct equivalence classes.

The index defining an equivalence class can be interpreted as the net particle flow along the lattice effected by a unitary operator in the class. For example, the shift operator τ^p translates a single particle p sites to the right, and so p is a measure of the charge transported by the drive. For a generic unitary operator, the index p can be extracted directly from the real-space unitary.³ If the system has translational symmetry, then the integer p is equivalent to the Brillouin-zone winding number that describes the wrapping of $U(k)$ around the unit circle as a function of momentum,

$$p = \frac{1}{2\pi i} \int dk \operatorname{tr} [U^{-1}(k) \partial_k U(k)]. \quad (3)$$

An example of a 2D drive that generates this chiral edge behavior was introduced in Ref. 9.

B. Effective unitary operators at a 1D boundary: many-body case

In the many-body case the underlying Hilbert space is much larger, but the different possible edge behaviors again form distinct equivalence classes characterized by quantized chiral transport. The 1D boundary is still described by a lattice $\Gamma = \mathbb{Z}$, but each lattice site $x \in \Gamma$ is now associated with an (identical) d -dimensional Hilbert space $\mathcal{H}_x$. The entire Hilbert space is the tensor product of all on-site Hilbert spaces,

$$\mathcal{H} = \bigotimes_{x \in \Gamma} \mathcal{H}_x. \quad (4)$$

For concreteness, we can assume each factor $\mathcal{H}_x$ corresponds to a single spin- $(d-1)/2$ particle. As before, a physically reasonable effective edge unitary U_{eff} acting on this Hilbert space should be a local operator, which will now generically involve interactions. In the language of Ref. 36, an operator of this form is equivalent to a (quantum) *cellular automaton*.

In Ref. 36, 1D cellular automata were classified according to the net flow of quantum information through the system. As in the single-particle case, unitary operators which enact chiral flow are known as *shifts*. However, a many-body shift translates the entire on-site Hilbert

² For simplicity, we consider local operators to be those expressible as a sum over terms which each act as the identity everywhere except on a region of finite size. We expect that our results may be easily extended to more flexible definitions of locality.

³ See Eq. (16) in Ref. 36 and the surrounding discussion for an explicit formula.

FIG. 2. The action of the shift operator $\sigma_3 \otimes \sigma_2^{-1}$. The on-site Hilbert space has total dimension six, and may be expressed as the tensor product of a dimension-3 Hilbert space (blue) with a dimension-2 Hilbert space (red). The unitary acts as a shift to the right on the dimension-3 space but as a shift to the left of the dimension-2 space. This shift is the representative 1D edge unitary in the equivalence class with index $\text{ind}(U) = 3/2$.

space $\mathcal{H}_x$, or more generally a subspace of $\mathcal{H}_x$, for example through

$$\mathcal{H}_{x,1} \otimes \mathcal{H}_{x,2} \longrightarrow \mathcal{H}_{x+1,1} \otimes \mathcal{H}_{x,2}. \quad (5)$$

In this way, a general shift operator may involve a tensor product of Hilbert spaces moving in different directions. As an example of this, Fig. 2 shows the action of the shift operator $\sigma_3 \otimes \sigma_2^{-1}$, where the factor σ_p indicates the translation of a Hilbert space of dimension p to the right (and the inverse shift σ_p^{-1} to the left). Since the two Hilbert space factors have different dimensions, there is net chiral flow to the right.

Similar to the 1D case, the authors of Ref. 36 demonstrate that a generic 1D cellular automaton is a shift operator $\sigma_p \otimes \sigma_q^{-1}$ (with p and q coprime) followed by a locally generated 1D unitary. In particular, adding additional factors to the tensor product or acting with different unitary operators in succession does not modify this general structure [33]. In this way, local many-body unitary operators in 1D form equivalence classes which are each labeled by a rational number p/q , with p and q describing the underlying shift operator. Different unitary operators within the same equivalence classes are related by a locally generated (in 1D) unitary evolution (or more formally, a locally generated unitary circuit of finite depth). The set of ‘pure’ shifts of the form $\sigma_p \otimes \sigma_q^{-1}$ populate all distinct equivalence classes. In the context of Floquet systems, these equivalence classes correspond to effective edge unitaries with distinct topological invariants.

Ref. 36 also demonstrates that the discrete index p/q , which we refer to as the GNVW index, is locally computable for an arbitrary unitary operator, a construction which we now review. First we define a notion of operator overlap. We consider two observable algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ which, for example, could correspond to two spatial regions of a physical system. The matrix units $|e_{ij}\rangle \cong |i\rangle\langle j|$, with $|i\rangle$ and $|j\rangle$ states from the appropriate region, form a suitable basis for these. A projector onto each algebra can be defined through

$$P_n = d_n \sum_{ij} |e_{ij}\rangle\langle e_{ij}|,$$

where d_n is the dimension of $\mathcal{A}_n$. With this setup, we define the overlap measure

$$\eta(\mathcal{A}_1, \mathcal{A}_2) = \sqrt{\text{Tr}(P_1 P_2)}, \quad (6)$$

where $\text{Tr}(M) = \frac{1}{d} \text{Tr}(M)$ is the normalised trace over the union algebra $\mathcal{A}_1 \cup \mathcal{A}_2$. In Ref. 36 it is demonstrated that the expression $\eta(\mathcal{A}_1, \mathcal{A}_2)$ gives a useful measure of the overlap between algebras $\mathcal{A}_1$ and $\mathcal{A}_2$. In particular, η is always greater than or equal to one, with equality holding only when $\mathcal{A}_1$ and $\mathcal{A}_2$ commute.

The GNVW index uses the quantity η to compare the observable algebras of the 1D boundary system before and after the action of the effective edge unitary. Explicitly, we imagine cutting the (infinite) 1D boundary lattice into left and right halves. We then choose a finite (but large) set of sites immediately to the left and to the right of the cut and denote the Hilbert space corresponding to each as L or R , respectively. We denote the corresponding observable algebras as $\mathcal{A}_L$ and $\mathcal{A}_R$. Now, a unitary operator U acts on a member of an observable algebra through conjugation: i.e., the unitary action α_U on some element M is $\alpha_U(M) = U M U^{-1}$. In this notation, the GNVW index of a unitary operator U is given by the ratio

$$\text{ind}(U) = \frac{\eta(\alpha_U(\mathcal{A}_L), \mathcal{A}_R)}{\eta(\alpha_U(\mathcal{A}_R), \mathcal{A}_L)}. \quad (7)$$

In this way, the index compares the extent to which the observable algebra in L is mapped onto the observable algebra in R , and vice versa, by the action of the 1D unitary.

In Ref. 36 it is shown that $\text{ind}(U)$ is always a positive rational number, p/q . In addition, the value of the index is independent of the choice of L and R (as long as they are sufficiently large) and independent of the location of the cut within the system. Importantly, $\text{ind}(U)$ is robust against unitary evolutions generated by local 1D Hamiltonians. In this way, $\text{ind}(U)$ is a stable topological index that defines a set of equivalence classes enumerated by positive rational numbers. In contrast to the single-particle case, the index p/q does not have a simple interpretation as the amount of charge (or particle) transport through the system. Instead, the index $\text{ind}(U)$ provides a measure of the quantum information transported, as explored in detail in Ref. 51.

It can be verified that the shift operator $\sigma_p \otimes \sigma_q^{-1}$ corresponds to the equivalence class with index $\text{ind}(U) = p/q$. A generic (local) 1D unitary operator can always be brought to a representative shift of this form through the action of a finite-depth quantum circuit (regrouping lattice sites if necessary). In the context of 2D Floquet systems, these representative shift unitaries correspond to the chiral transport of a many-body state around the 1D boundary [32, 33].

FIG. 3. Illustration of a choice of cut (red dashed line) which divides a 2D system with a compact dimension along $\mathbf{r}$ into the two halves L and R . By grouping the Hilbert space of N sites along the compact dimension, one may calculate the 1D GNVW index. This index is invariant under translations of the cut by $\mathbf{r}'$.

C. Effective unitary operators at a 2D boundary

We now turn our attention to the edge behavior of interacting 3D Floquet phases. Motivated by the lower-dimensional case, our approach will be to study the properties of local unitary operators in *two* dimensions, corresponding to the effective edge unitary of a 3D unitary loop. We will define a robust topological invariant for this class of unitaries and use it to infer the existence of equivalence classes corresponding to distinct topological phases.

The boundary is now a 2D Bravais lattice, which we take without loss of generality to be square so that $\Gamma = \mathbb{Z}^2$. On each lattice site there is a d -dimensional Hilbert space $\mathcal{H}_{x,y}$, and the complete Hilbert space is the tensor product over these,

$$\mathcal{H} = \bigotimes_{x,y \in \Gamma} \mathcal{H}_{x,y}.$$

As before, we may assume for concreteness that each on-site Hilbert space describes a single spin- $(d-1)/2$ particle. In general, the boundary behavior is described by an effective edge unitary U_{eff} which acts on this Hilbert space and which is local and two dimensional. However, U_{eff} is not necessarily generated by a local 2D Hamiltonian that acts only within the boundary region. In the language of Ref. 36, U_{eff} is again a cellular automaton (although we note that this reference only studies cellular automata in 1D).

Without translational invariance, there is a large set of distinct effective edge unitaries that could be constructed—for example, we could stack shift operators σ_p with different p in parallel rows in uncountably many ways. In this paper we restrict the discussion to the manageable translationally invariant case, and leave a more general study to future work.

In order to import some of the results from the 1D case, we will treat the infinite 2D boundary as the limiting case of a sequence of quasi-1D cylindrical systems. First, since U is local, it has some Lieb-Robinson length λ_{LR} [49], and we assume for simplicity that the action of U is strictly zero for separations greater than this length.

Then, given a lattice vector $\mathbf{r}$ and sufficiently large integer N such that $|N\mathbf{r}| \gg \lambda_{\text{LR}}$, we define a finite periodic system by identifying all lattice sites that are separated by an integer multiple of $N\mathbf{r}$. This periodic system can be thought of as having a compact dimension along the $\mathbf{r}$ -direction with period $N\mathbf{r}$ and an extended dimension along any primitive lattice vector $\mathbf{r}'$ which is linearly independent to $\mathbf{r}$. We denote the restriction of a unitary operator U to this periodic system as $U_{N,\mathbf{r}}$; since U is translationally invariant and local, this restriction is always well defined.

We may now compute the GNVW index for this semi-infinite system by treating it as essentially 1D. Explicitly, we define a cut along $\mathbf{r}$, which divides the system in two halves (L and R) as illustrated in Fig. 3. The index, $\text{ind}(U_{N,\mathbf{r}}, \mathbf{r})$, associated with this cut can then be calculated by grouping sites in the $\mathbf{r}$ -direction into a single 1D ‘site’ and applying Eq. (7). Physically, the index gives a measure of the flow of quantum information across the cut in the axial direction of the cylinder.

Due to translational invariance, the index $\text{ind}(U_{N,\mathbf{r}}, \mathbf{r})$ does not depend on the location of the cut. The value of $\text{ind}(U_{N,\mathbf{r}}, \mathbf{r})$ will, however, generally depend on the extent of the compact dimension $N\mathbf{r}$: If this dimension is made larger, then more information can flow across the cut. We therefore define a *scaled* additive index

$$\nu(\mathbf{r}) = \lim_{N \rightarrow \infty} \frac{1}{N} \log \text{ind}(U_{N,\mathbf{r}}, \mathbf{r}), \quad (8)$$

where the size of the periodic system is increased by taking the limit $N \rightarrow \infty$ for a fixed lattice vector $\mathbf{r}$. This limit defines a sequence of periodic cylindrical systems which tends towards the infinite plane. The index $\text{ind}(U_{N,\mathbf{r}}, \mathbf{r})$ should scale as a power of N due to translation invariance, and we consequently expect the scaled index $\nu(\mathbf{r})$ to be finite. We can interpret $\nu(\mathbf{r})$ as a measure of the flow of quantum information across a cut along $\mathbf{r}$ due to the action of the unitary, per length $|\mathbf{r}|$. We note for later use that since $\text{ind}(U_{N,\mathbf{r}}, \mathbf{r})$ is always a rational number [36], the scaled index can be equivalently written as a sum over primes p ,

$$\nu(\mathbf{r}) = \sum_p n_p(\mathbf{r}) \log p, \quad (9)$$

with integral coefficients n_p .

In contrast to the 1D case, the flow of quantum information within a 2D boundary depends on the direction of the cut across which it is measured. This is indicated by the fact that the index $\nu(\mathbf{r})$ depends on a lattice vector $\mathbf{r}$. However, due to the properties of unitary evolutions, we would expect different $\nu(\mathbf{r})$ corresponding to different lattice vectors $\mathbf{r}$ to be related. In Appendix A we show that $\nu(\mathbf{r})$ is a linear function of 2D lattice vectors, satisfying the property

$$\nu(\mathbf{r}_1 + \mathbf{r}_2) = \nu(\mathbf{r}_1) + \nu(\mathbf{r}_2). \quad (10)$$

It follows that the coefficients $n_p(\mathbf{r})$ in the sum over primes in Eq. (9) are integer-valued linear functions of

$\mathbf{r}$, and so each may be written as

$$n_p(\mathbf{r}) = \frac{1}{2\pi} \mathbf{G}_p \cdot \mathbf{r}, \quad (11)$$

given as the inner product of $\mathbf{r}$ with some reciprocal lattice vector $\mathbf{G}_p$. The reciprocal lattice vectors $\{\mathbf{G}_p\}$ in this way provide a more fundamental description of an effective edge unitary than a value of $\nu(\mathbf{r})$ alone, as they allow the index $\nu(\mathbf{r})$ to be calculated for *any* value of $\mathbf{r}$.

Overall, translationally invariant unitaries in two dimensions are classified by a set of reciprocal lattice vectors $\{\mathbf{G}_p\}$, indexed by primes p . These determine the scaled additive index $\nu(\mathbf{r})$ along any direction $\mathbf{r}$ through Eq. (9), and consequently quantify the flow of quantum information across any cut in the 2D boundary. Conversely, by ‘measuring’ $\nu(\mathbf{r})$ for a unitary U along some basis $\{\mathbf{r}_1, \mathbf{r}_2\}$ of the lattice, we can uniquely determine the vectors $\{\mathbf{G}_p\}$ using the relation

$$\mathbf{G}_p = n_p(\mathbf{r}_1)\mathbf{b}_1 + n_p(\mathbf{r}_2)\mathbf{b}_2, \quad (12)$$

where $\{\mathbf{b}_1, \mathbf{b}_2\}$ are reciprocal lattice vectors corresponding to $\{\mathbf{r}_1, \mathbf{r}_2\}$ (satisfying $\mathbf{r}_i \cdot \mathbf{b}_j = 2\pi\delta_{ij}$). Since this classification is discrete, it partitions the set of 2D translationally invariant unitaries into discrete equivalence classes.

We can define a representative unitary $V_{\{\mathbf{G}_p\}}$ corresponding to a given set of vectors $\{\mathbf{G}_p\}$ as follows. First, for each value of p with a nonzero reciprocal lattice vector $\mathbf{G}_p$, we define a local Hilbert space with dimension p on each site; the total Hilbert space is the tensor product of these Hilbert spaces over the complete 2D lattice. We then define a translation vector $\mathbf{r}_{\text{tr},p}$ corresponding to each nonzero $\mathbf{G}_p$ through

$$\mathbf{r}_{\text{tr},p} = \frac{1}{2\pi} [(\mathbf{r}_1 \times \mathbf{r}_2) \times \mathbf{G}_p], \quad (13)$$

where it may be verified that $\mathbf{r}_{\text{tr},p}$ lies in the direct lattice with basis $\{\mathbf{r}_1, \mathbf{r}_2\}$.

The representative unitary $V_{\{\mathbf{G}_p\}}$ acts independently on each p -dimensional factor of this Hilbert space as a translation with direction and magnitude $\mathbf{r}_{\text{tr},p}$. In terms of the one-dimensional shift operators defined in Sec. IIIB, we can write this as

$$V_{\{\mathbf{G}_p\}} = \bigotimes_p \sigma_{\mathbf{r}_{\text{tr},p}}, \quad (14)$$

where each shift $\sigma_{\mathbf{r},p}$ now has both a direction $\mathbf{r}$ and a Hilbert space dimension p associated with it, and where the product is taken over all primes p . In this way, the shift for each Hilbert space in the tensor product can have a different magnitude and direction. By expressing a given vector $\mathbf{r}$ in the basis $\{\mathbf{r}_1, \mathbf{r}_2\}$ and exploiting the linearity of $\nu(\mathbf{r})$, it may be verified that this representative unitary $V_{\{\mathbf{G}_p\}}$ generates the expected value of the chiral unitary index $\nu(\mathbf{r})$ for any choice of cut $\mathbf{r}$. An example of a unitary operator with non-zero $\mathbf{G}_2$ and $\mathbf{G}_3$ is illustrated in Fig. 4.

FIG. 4. The action of the representative unitary in the equivalence class characterized by $\mathbf{G}_2 = -2\pi\hat{\mathbf{y}}$ and $\mathbf{G}_3 = 2\pi\hat{\mathbf{x}} - 2\pi\hat{\mathbf{y}}$, with all other $\mathbf{G}_p = 0$, on a 2D lattice spanned by $\{\mathbf{r}_1 = \hat{\mathbf{x}}, \mathbf{r}_2 = \hat{\mathbf{y}}\}$. The on-site Hilbert space is the tensor product of a dimension-2 factor and a dimension-3 factor. The unitary acts as a shift of the dimension-2 factor by translation vector $\hat{\mathbf{y}}$ and a shift of the dimension-3 factor by translation vector $\hat{\mathbf{x}} + \hat{\mathbf{y}}$.

The set of reciprocal lattice vectors $\{\mathbf{G}_p\}$ characterizing a particular equivalence class of unitaries inherits a group structure under two products within the space of unitaries from the group structure of the GNVW index [36]. Under the sequential action of two unitaries $U_3 = U_2 \circ U_1$, the reciprocal lattice vectors add term-wise, $\{\mathbf{G}_{p,3} = \mathbf{G}_{p,1} + \mathbf{G}_{p,2}\}$. Similarly, if we consider the site-wise tensor product of two systems, with unitary $U_3 = U_1 \otimes U_2$, the reciprocal lattice vectors again add term-wise according to $\{\mathbf{G}_{p,3} = \mathbf{G}_{p,1} + \mathbf{G}_{p,2}\}$. In Appendix B we show that an arbitrary set of shifts can always be characterized by a set of reciprocal lattice vectors $\{\mathbf{G}_p\}$ with p prime. In Appendix C, we show that edge behavior described by different $\{\mathbf{G}_p\}$ is stable under locally generated (in 2D) unitary deformations at the edge.

D. 2D boundaries of 3D unitary loops

We are ultimately interested in 3D bulk drives, whose boundaries may be more complicated than the infinite 2D planes considered above. For this reason, we now extend our discussion to 2D systems embedded in 3D. We take some translationally invariant 3D unitary loop drive U_{3D} , defined on a 3D lattice $\Gamma_{3D} = \mathbb{Z}^3$, which may be used to generate a 2D effective edge unitary at any 2D boundary. If the boundary is a 2D plane, then the surface behavior falls into equivalence classes exactly as described above. To describe the behavior at more general boundary surfaces, however, we consider two 2D Bravais lattices $\mathcal{L}_1$

FIG. 5. Illustration of the interface between two periodic systems with shared compact dimension along $\mathbf{r}$, and extended dimensions along $\mathbf{r}'_1$ and $\mathbf{r}'_2$. The chiral unitary index may be calculated by grouping sites along the $\mathbf{r}$ direction and dividing the resulting 1D system into two halves, L and R . The dashed red lines show two possible cuts for dividing the system. The chiral unitary index is independent of the location of the cut. See main text for details.

and $\mathcal{L}_2$, which intersect at a common 1D sublattice with primitive lattice vector $\mathbf{r}$. Each lattice $\mathcal{L}_{1/2}$ is spanned by the basis $\{\mathbf{r}, \mathbf{r}'_{1/2}\}$. We define the complete boundary system to consist of sites belonging to $\mathcal{L}_1$ on one side of the common sublattice, and sites belonging to $\mathcal{L}_2$ on the other. The underlying bulk drive U_{3D} is a translationally invariant unitary loop, and so this procedure defines an effective edge unitary U_{eff} that acts on the quasi-2D boundary system.

Since $\mathbf{r}$ is a vector in both $\mathcal{L}_1$ and $\mathcal{L}_2$, we can still define a periodic system by identifying the Hilbert spaces of sites displaced by $N\mathbf{r}$, as illustrated in Fig. 5. We can therefore again compute $\text{ind}(U_{N,\mathbf{r}})$ by dividing the system along $\mathbf{r}$ into two halves, L and R . However, the GNVW index is a local invariant [36], and so the value of $\text{ind}(U_{N,\mathbf{r}})$ is independent of the location of the dividing cut. In particular, far from the interface (where the axial direction is either $\mathbf{r}'_1$ or $\mathbf{r}'_2$), a computation of $\text{ind}(U_{N,\mathbf{r}})$ will yield the same result. By taking the limit $N \rightarrow \infty$, we see that the scaled index $\nu(\mathbf{r})$ is consistent across the entire boundary.

The arguments above apply to any pair of 2D planar boundaries which intersect at a line. For a 3D bulk unitary U_{3D} , we can find three pairwise-intersecting planar boundaries, in which the interface between each pair is a 1D sublattice spanned by a basis vector of the original 3D lattice Γ_{3D} . This is illustrated in Fig. 6. Since the scaled additive index is a locally computed quantity, the values of $\nu(\mathbf{r})$ computed within different 2D planar boundaries must be consistent with each other and with the linearity described in Eq. (10) (where $\mathbf{r}$ is now promoted to a lattice vector in 3D).

Overall, this means that the effective edge behavior of a translationally invariant 3D loop drive is classified

FIG. 6. Three pairwise-intersecting planar boundaries of a 3D system. The intersection between each pair of planes is spanned by a basis vector of the 3D lattice. Values of the chiral unitary index computed within different boundary planes must be consistent with each other and with linearity.

by a set of *three-dimensional* reciprocal lattice vectors $\{\mathbf{G}_p\}$, indexed by primes p . Given the set $\{\mathbf{G}_p\}$, the scaled index $\nu(\mathbf{r})$ is specified for any 2D boundary and any 1D cut within this boundary (defined by the three-dimensional lattice vector $\mathbf{r}$). Due to the discreteness of this classification, effective edge unitaries arising from 3D unitary loop drives may be put into equivalence classes, each labeled by a different set of reciprocal lattice vectors $\{\mathbf{G}_p\}$. In turn, each 3D unitary loop must have an edge behavior belonging to one of these classes, and the space of locally generated 3D loops inherits the classification.

Just as in the 2D case, we can define a representative effective edge unitary $V_{\{\mathbf{G}_p\}}$ on a particular boundary which corresponds to a given set of vectors $\{\mathbf{G}_p\}$. As before, we define a set of translation vectors $\{\mathbf{r}_{\text{tr},p}\}$ corresponding to $\{\mathbf{G}_p\}$ through

$$\mathbf{r}_{\text{tr},p} = \frac{1}{2\pi} [(\mathbf{r}_1 \times \mathbf{r}_2) \times \mathbf{G}_p], \quad (15)$$

but where $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{G}_p$ are now 3D vectors. The representative unitary $V_{\{\mathbf{G}_p\}}$ acts as a translation with vector $\mathbf{r}_{\text{tr},p}$ on a p -dimensional Hilbert space factor on each site, and may be written as a tensor product of shift operators as

$$V_{\{\mathbf{G}_p\}} = \bigotimes_p \sigma_{\mathbf{r}_{\text{tr},p},p}. \quad (16)$$

Other effective edge unitaries within the same class are related to the representative edge unitary by a locally generated 2D unitary evolution.

For a given equivalence class and boundary surface, the flow of information per unit cell across a cut in the direction of $\mathbf{r}$ is characterized by the index $\nu(\mathbf{r})$. Rewriting Eq. (9), this can be expressed as

$$\nu(\mathbf{r}) = \frac{1}{2\pi} \sum_p (\mathbf{G}_p \cdot \mathbf{r}) \log p. \quad (17)$$

As an example, Fig. 7 shows the action of a simple effective edge unitary and gives the associated vectors $\mathbf{r}_{\text{tr},p}$

FIG. 7. The action of a simple effective edge unitary characterized by reciprocal lattice vector $\mathbf{G}_2 = -2\pi\hat{\mathbf{y}}$ (with all other $\mathbf{G}_p$ zero) in a surface with basis $\mathbf{r}_1 = (1, 0, -1)$ and $\mathbf{r}_2 = (0, 1, 0)$ (note: on-site Hilbert space is not shown). Within this surface, the unitary acts as a translation by vector $\mathbf{r}_{tr,2} = (1, 0, -1)$, indicated by red arrows. The blue dashed line indicates a 1D sublattice of this surface, with primitive lattice vector $\mathbf{r} = (1, -3, -1)$ indicated by the blue arrow. The flow of information across this cut per sublattice unit cell is quantified by the index $\nu(\mathbf{r}) = 1/(2\pi) (\mathbf{G}_2 \cdot \mathbf{r}) \log 2 = 3 \log 2$. See main text for details.

and $\{\mathbf{G}_p\}$, and the index $\nu(\mathbf{r})$ for a choice of cut $\mathbf{r}$.

In the 1D case, each equivalence class of 1D edge behaviors has a representative effective edge unitary which is generated by an exactly solvable 2D bulk exchange drive [32, 33]. We will demonstrate in the next section that the representative edge unitary of each two-dimensional equivalence class may similarly be generated by an exactly solvable 3D bulk exchange drive.

IV. 2D BULK EXCHANGE DRIVES

In the previous section, we obtained a classification of local 2D unitary operators with translational invariance, and argued that this provides an equivalent classification of *bulk* Floquet phases in 3D. We showed that each equivalence class is characterized by an infinite set of reciprocal lattice vectors $\{\mathbf{G}_p\}$, and that each class has a representative effective edge unitary $V_{\{\mathbf{G}_p\}}$ that is a product of shift operators (or translations) by vectors given in Eq. (15). The next aim of this paper is to obtain a set of exactly solvable 3D bulk drives, known as ‘exchange drives’, which may be used to generate these different representative edge behaviors. To aid the discussion, we first review exchange drives in two dimensions and show how they can be used to generate all possible 1D boundary behaviors. In Sec. V, we will naturally extend these ideas to exchange drives in 3D.

A. Model triangular drive

We first describe a simple four-step unitary loop drive in 2D which can be used as a building block for more general drives. This is a modification of the models introduced in Refs. 32 and 33, which in turn build on the noninteracting drive of Ref. 9.

The model may be defined on any Bravais lattice with a two-site basis. For simplicity, however, we will assume the lattice is square (i.e. $\Gamma = \mathbb{Z}^2$), has unit lattice spacing, and has both sites within each unit cell (labeled A and B) coincident.⁴ On each site of each sublattice there is a finite, d -dimensional Hilbert space which, for concreteness, we may again assume describes a spin- $(d-1)/2$. In this way, the state at a particular site may be written $|\mathbf{r}, a, \alpha\rangle$, where $\mathbf{r}$ labels the lattice site, $a \in \{A, B\}$ labels the sublattice, and $\alpha \in \mathcal{H}_{\mathbf{r},a}$ labels the state within the on-site Hilbert space. A basis for many-body states is the tensor product of such states.

Following Ref. 33, we consider exchange operators of the form

$$U_{\mathbf{r},\mathbf{r}'}^{\leftrightarrow} = \sum_{\alpha,\beta} |\mathbf{r}, A, \beta\rangle \otimes |\mathbf{r}', B, \alpha\rangle \langle \mathbf{r}, A, \alpha| \otimes \langle \mathbf{r}', B, \beta|, \quad (18)$$

which exchange the state on site $(\mathbf{r}, A)$ with the state on site $(\mathbf{r}', B)$. Note that the operator $U_{\mathbf{r},\mathbf{r}'}^{\leftrightarrow}$ is strictly local for finite $|\mathbf{r} - \mathbf{r}'|$, having no effect on sites other than $\mathbf{r}$ and $\mathbf{r}'$. It can therefore be generated by a similarly local Hamiltonian.

In terms of this operator, we define the four-step drive $U_4 U_3 U_2 U_1$, where each U_n takes the form

$$U_n = \bigotimes_{\mathbf{r}} U_{\mathbf{r},\mathbf{r}+\mathbf{b}_n}^{\leftrightarrow}, \quad (19)$$

with $\mathbf{b}_1 = -(\hat{\mathbf{x}} + \hat{\mathbf{y}})$, $\mathbf{b}_2 = -\hat{\mathbf{y}}$, $\mathbf{b}_3 = \mathbf{0}$, and $\mathbf{b}_4 = -\hat{\mathbf{x}}$. Each step of the drive is a product of exchange operations over disjoint pairs of sites separated by $\mathbf{b}_n$, as illustrated in Fig. 8(a).

Since the action of the unitary operator is invariant under lattice translations, we can obtain a complete picture of the drive by focusing on the evolution of a particular on-site component of a generic many-body state. We find that a state beginning at an A -site moves in a clockwise loop around the half-plaquette to its lower-left, while a state beginning at a B -site moves in a clockwise loop around the half-plaquette to its upper-right, as illustrated in Fig. 8(b). In this way, each on-site state in the bulk returns to its original position. Since this happens simultaneously for every site, the complete unitary operator acts as the identity on a generic many-body state in the bulk, and is therefore a unitary loop.

⁴ Note that this is in contrast to Refs. 9, 32, and 33, in which the lattice basis is nonzero.

FIG. 8. Illustration of the four-step exchange drive described in the main text. (a) The steps represent exchanges between nearby on-site states. A -sites are depicted in light blue and B -sites are depicted in dark red. (b) On-site states in the bulk follow a triangular loop path around a half-plaquette. On-site states at the edge are transported by an effective translation operator represented by the green arrows.

At the boundary of an open system, however, some exchange operations are forbidden, and the drive generates anomalous chiral transport [32, 33]. For the system in Fig. 8(b), the overall action of the drive is a translation of sublattice states counter-clockwise around the 1D edge: In other words, the effective edge unitary of the drive is a shift σ_d . By current conservation, this edge behavior must be the same along any edge cut, even if the cut is not parallel to a lattice vector. Note that it would be impossible to generate such a chiral translation with a local Hamiltonian in a purely 1D quantum system [33], and so this boundary behavior is anomalous.

B. Bulk characterization of 2D exchange drives

We now construct more general 2D exchange drives from this primitive triangular drive, and show that they may be used to generate all the different 1D edge behaviors (i.e. combinations of shifts $\sigma_p \otimes \sigma_q^{-1}$) described in Sec. III B. In the process, we show that the geometry of a generic 2D exchange drive in the bulk is directly related to its edge behavior.

Assuming the same lattice structure as in Sec. IV A, we consider a general drive with $2N$ steps, $U = U_{2N} \dots U_1$, with individual steps of the drive being exchanges in the form of Eq. (19). We refer to such a drive as an *exchange drive*. Each step is characterized by a Bravais lattice vector $\mathbf{b}_n$, which is the displacement between the exchanged sublattice sites directed from A to B . After n steps, an on-site state beginning at an A -site will be displaced by

$$\mathbf{d}_n = \sum_{m=1}^n (-1)^{m+1} \mathbf{b}_m, \quad (20)$$

where the minus sign arises because each step of the drive moves a state between sublattices. Similarly, a state beginning at a B -site will be displaced by $-\mathbf{d}_n$.

As motivated in Sec. II, we are most interested in many-body loop evolutions, which act as the identity in

FIG. 9. Illustration of the signed area summands in Eq. (22) for a four-step example drive with $\mathbf{b}_1 = \hat{\mathbf{x}} + \hat{\mathbf{y}}$, $\mathbf{b}_2 = \hat{\mathbf{x}}$, $\mathbf{b}_3 = \hat{\mathbf{x}} - \hat{\mathbf{y}}$, $\mathbf{b}_4 = \hat{\mathbf{x}}$. Since $N = 2$ there are $2N - 2 = 2$ terms in the sum. The signed area of the green (red) triangle represents the first (second) term in the sum and is equal to positive (negative) 1, scaled by the primitive triangle area. In total, this four-step exchange drive has $A_s = 0$.

the bulk after a complete driving cycle. The requirement that the drive be a loop enforces the condition

$$\sum_{n=1}^{2N} (-1)^{n+1} \mathbf{b}_n = 0, \quad (21)$$

so that the final displacement vector $\mathbf{d}_{2N}$ is zero. Now, each on-site state follows a closed path during the evolution, and we define the signed area of this path by

$$A_s = \frac{1}{2A_{\text{prim}}} \sum_{n=1}^{2N-2} (-1)^n (\mathbf{d}_n \times \mathbf{b}_{n+1}) \cdot \hat{\mathbf{z}}, \quad (22)$$

where $\hat{\mathbf{z}}$ is a unit vector perpendicular to the system and A_{prim} is the area of a primitive triangle on the lattice ($A_{\text{prim}} = 1/2$ in our convention). Eq. (22) calculates the net oriented area enclosed by an on-site state beginning at a site in the bulk and following the complete evolution of the drive, in units of the primitive triangle area. In general, an arbitrary oriented drive may generate both positively and negatively oriented components, with counter-clockwise loops corresponding to positive areas (see Fig. 9). As defined, the signed area A_s is always an integer, which we will find gives a direct measure of the shift behavior at the edge.

We now introduce operations that we will use to deform an exchange drive while preserving its signed area and (possibly anomalous) edge behavior. Proofs of these statements may be found in Appendix D. First, we define a trivial drive to be an exchange drive in which states follow some exchange path and then exactly retrace this path in reverse, satisfying the condition $\mathbf{b}_n = \mathbf{b}_{2N-(n-1)}$. The signed area of a trivial drive is zero by construction.

Next, given an exchange drive, we note that we may continuously insert trivial drives at any point without affecting its signed area or edge properties. That is, given a general drive $U = U_{2N} \dots U_1$ and a trivial drive T , the drive $U' = U_{2N} \dots U_n T U_{n-1} \dots U_1$ is continuously connected to U and has the same signed area. One may also continuously deform an exchange drive by cyclically permuting its steps. These deformations do not affect the signed area of the drive and leave the transport

at the edge unaffected, results which are proved in Appendix D.⁵

Using the tools above, we can decompose a general loop exchange drive into a sequence of four-step triangular loop drives. To do this, we insert a trivial drive between each pair of steps that does not include the first or final step. The nature of the trivial drive inserted will depend on the parity of the step: After odd steps, we insert the trivial drive $U'_{2n+1}U'_{2n+1}$, where U'_{2n+1} is an exchange step with $\mathbf{b}'_{2n+1} = \mathbf{d}_{2n+1}$. After even steps, we insert the trivial drive $U_{OS}U'_{2n}U'_{2n}U_{OS}$, where U'_{2n} is an exchange step with $\mathbf{b}'_{2n} = \mathbf{d}_{2n}$ and where U_{OS} is an on-site exchange step with $\mathbf{b} = 0$. The extra swap in the even case acts to effectively transform even steps into odd steps.

After these insertions, the modified drive can be partitioned into a sequence of $(2N - 2)$ four-step loop drives,

$$U' = \dots U'_4 U_{OS} U_4 U'_3 \cdot U'_3 U_3 U_{OS} U'_2 \cdot U'_2 U_{OS} U_2 U_1, \quad (23)$$

$$= \dots L_3 \cdot L_2 \cdot L_1,$$

a process which is illustrated in Fig. 10. It is simple to verify that each four-step loop drive in the partition has a minimum of one on-site swap step, and thus forms either a triangular drive or a trivial drive. Since the operations used to modify the drive preserve the signed area, the signed area of the complete drive may be written in terms of its components as

$$A_s(U) = A_s(U') = \sum_n A_s(L_n), \quad (24)$$

where we have written $A_s(U)$ for the signed area of loop drive U , etc. This property is illustrated in Fig. 10.

C. Bulk-edge correspondence of 2D exchange drives

Using the loop decomposition of the previous section, the signed area of an arbitrary exchange drive may be related to its chiral transport (of quantum information) at the edge. We define a primitive drive to be a four-step drive in which bulk states follow the path of a primitive triangle, such as the drive described in Sec. IV A. Since a primitive drive is triangular, one of its steps must be an on-site swap with $\mathbf{b}_n = 0$. However, as cyclic permutations of loops are equivalent (see Appendix D), we may assume without loss of generality that the on-site swap occurs on the third step. Therefore, we may equivalently define a primitive drive as a four-step loop drive in which $\{\mathbf{b}_2, \mathbf{b}_4\}$ form a basis for the Bravais lattice and $\mathbf{b}_3 = 0$.

Now, every primitive drive has an effective edge action equivalent either to the model drive in Sec. IV A

FIG. 10. Illustration of the triangular decomposition in Eq. (23) for an example drive, with steps given by U_1 through U_6 . Since $N = 3$ there are $2N - 2 = 4$ loops in the decomposition but the fourth loop is a trivial drive and we do not depict it here. For clarity, sublattice sites which are not reached by the state localized in the bottom left have been omitted from the figure. The complete drive has signed area $A_s = 4$ (in units of the primitive triangle), while the loops in the decomposition have areas (from left to right) of $A_s = 1/2$, $A_s = 1/2$ and $A_s = 1$.

or to its inverse—in other words, its edge action is a shift σ_d or a shift σ_d^{-1} . To see this, we perform an invertible orientation- and area-preserving transformation which maps the generic primitive drive (characterized by the basis $\{\mathbf{b}_2, \mathbf{b}_4\}$) onto the model primitive drive presented in Sec. IV A or its inverse (characterized by the basis $\{-\hat{\mathbf{y}}, -\hat{\mathbf{x}}\}$ or $\{-\hat{\mathbf{x}}, -\hat{\mathbf{y}}\}$, respectively). The chosen transformation preserves the orientation of sites at the edge, and will map the edge behavior of the generic primitive drive directly onto that of the model primitive drive (or its inverse).

The decomposition of an exchange drive into triangular drives given in Eq. (23) does not generally reduce the original drive to *primitive* drives (as some of the constituent triangles will have areas larger than A_{prim}). However, we can use what we know about primitive triangles to deduce the effective edge behavior of a general (nonprimitive) triangular drive, U_Δ . To see this, note that a drive of this form *is* primitive on some number of *sublattices* of the original lattice. This can be shown by considering the sublattice formed from the span of the vectors $\{\mathbf{b}_2, \mathbf{b}_4\}$ defining U_Δ , on which the drive is clearly primitive. Other sublattices on which U_Δ is primitive can be obtained by translating the first sublattice by the basis vectors of the original lattice. This is illustrated in Appendix E, where it is also demonstrated that states on different sublattices do not interact during the drive.

We claim, and prove in Appendix E, that the number of Bravais sublattices N on which a four-step triangular drive is primitive is given by $N = |A_s|$, where A_s is the signed area of that triangular drive. In this way, a

⁵ Note that these properties can be demonstrated without appealing to the edge classification discussed in Sec. III.

four-step triangular drive acts on $|A_s|$ separate sublattices as either the model drive (if $\text{sgn}(A_s) = -1$) or its inverse (if $\text{sgn}(A_s) = 1$). Since the edge behaviors of the model drive and its inverse are shifts of the d -dimensional Hilbert spaces in opposite directions, the overall edge behavior of a general triangular drive is A_s copies of this shift with the appropriate chirality.

Combining the discussions above, we find that the edge behavior of a general 2D translation-invariant exchange drive U is characterized by its signed area in the bulk, $A_s(U)$, and is equivalent to A_s copies of a chiral shift of a d -dimensional on-site Hilbert space. Since the bulk motion of a primitive drive has the opposite chirality to its edge motion, a (negative) positive signed bulk area corresponds to (counter-)clockwise translation at the edge. By forming tensor products of exchange drives, each corresponding to a different on-site Hilbert space, all possible 1D boundary behaviors (with general form $\sigma_p \otimes \sigma_q^{-1}$) can be realized.

V. BULK AND EDGE BEHAVIOR OF 3D EXCHANGE DRIVES

A. Bulk-edge correspondence for 3D exchange drives

We now extend the ideas of the previous section to translation-invariant exchange drives in 3D. As in the 2D case, an exchange drive may be defined on any 3D lattice with a two-site basis $\{A, B\}$. For concreteness, we can assume the lattice is cubic (i.e. $\Gamma_{3D} = \mathbb{Z}^3$) and has two coincident sublattices. A boundary of such a system may then be obtained by taking a planar slice through Γ to expose some surface containing a 2D Bravais sublattice. More general boundaries can be obtained by taking several intersecting slices. As discussed in Sec. III, the general edge behavior is described by a set of reciprocal lattice vectors $\{\mathbf{G}_p\}$, while the quantum information flow within a boundary can be characterized by the scaled index $\nu(\mathbf{r})$, defined across a cut in the direction of $\mathbf{r}$.

As before, we consider bulk exchange drives comprising $2N$ steps in the form of Eq. (19), with each $\mathbf{b}_n \in \Gamma_{3D}$ now a 3D lattice vector. We recall that these exchange drives are loops, and that they involve local exchange operations that occur throughout the lattice simultaneously (due to translational invariance). Generalizing the signed area of Eq. (22), we claim that the bulk characterization of a 3D exchange drive is given by the reciprocal lattice vector

$$\mathbf{G} = \frac{2\pi}{V_r} \sum_{n=1}^{2N-1} (-1)^n (\mathbf{d}_n \times \mathbf{b}_{n+1}), \quad (25)$$

where V_r is the volume of the direct lattice unit cell. We will show that this bulk invariant $\mathbf{G}$ is directly related to the set of reciprocal lattice vectors $\{\mathbf{G}_p\}$ (introduced in Sec. III D) which characterize the edge behavior.

As in the 2D case, the bulk characterization may be justified by decomposing a general exchange drive into four-step triangular drives. While the decomposition in Eq. (23) continues to hold, the triangular components are now generally not coplanar. Nevertheless, it follows from the arguments of the previous section that the vector $\mathbf{G}$ for a general drive is the sum of the $\mathbf{G}$ for each triangular drive in its decomposition. The decomposition therefore preserves the value of $\mathbf{G}$, and we can understand the edge behavior of a general exchange drive by focusing on its triangular components.

As in 2D, a triangular drive may be defined by the vectors $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$, where a cyclic permutation has been chosen so that $\mathbf{b}_3 = \mathbf{0}$. In this setup, the triangular drive lies in a plane we call the ‘triangle plane’, which includes the vectors $\mathbf{b}_2$ and $\mathbf{b}_4$. We consider the action of this drive on some 2D boundary lattice, spanned by the basis $\{\mathbf{r}_1, \mathbf{r}_2\}$, which defines a ‘surface plane’. Neglecting the case where the surface plane and triangle plane are parallel (where the edge behavior is trivial), the intersection of these planes is a 1D Bravais sublattice generated by a primitive vector $\mathbf{a}_1 \in \Gamma_{3D}$. We can therefore choose an ordered basis $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ for Γ_{3D} , where $\{\mathbf{a}_1, \mathbf{a}_2\}$ span the triangle plane (and $\mathbf{a}_3$ is any linearly independent primitive vector). Note that $\mathbf{b}_2$ and $\mathbf{b}_4$ are not necessarily primitive vectors, and in general $(\mathbf{b}_2 \times \mathbf{b}_4) = A_s(\mathbf{a}_1 \times \mathbf{a}_2)$, where A_s is the signed area discussed previously. According to Eq. (25), this triangular drive will have the characteristic reciprocal lattice vector

$$\mathbf{G} = \frac{2\pi}{V_r} (\mathbf{b}_2 \times \mathbf{b}_4) = \frac{2\pi}{V_r} A_s (\mathbf{a}_1 \times \mathbf{a}_2). \quad (26)$$

We now consider the edge behavior of this drive in the surface plane. We can write the ordered basis for the surface plane $\{\mathbf{r}_1, \mathbf{r}_2\}$ in terms of the basis of the 3D lattice as $\mathbf{r}_1 = \mathbf{a}_1$ and $\mathbf{r}_2 = D\mathbf{a}_2 + E\mathbf{a}_3$ (where $D, E \in \mathbb{Z}$ are coprime). This surface is equivalently characterized by the outward-pointing reciprocal lattice vector

$$\mathbf{k}_s = \frac{2\pi}{V_r} (\mathbf{r}_1 \times \mathbf{r}_2). \quad (27)$$

We claim that the edge behavior of the bulk triangular drive described above is a shift (or translation) within the surface lattice given by the direct lattice vector

$$\mathbf{r}_{\text{tr}} = \frac{2\pi}{V_k} (\mathbf{k}_s \times \mathbf{G}) = \frac{1}{2\pi} [(\mathbf{r}_1 \times \mathbf{r}_2) \times \mathbf{G}], \quad (28)$$

where V_k is the volume of the 3D reciprocal lattice unit cell. For the triangular drive above this reduces to

$$\mathbf{r}_{\text{tr}} = \frac{1}{V_r} (\mathbf{r}_1 \times \mathbf{r}_2) \times (\mathbf{b}_2 \times \mathbf{b}_4) = -A_s E \mathbf{a}_1. \quad (29)$$

The fact that this is the correct edge behavior can be justified as follows: Since a triangular drive in 3D acts on a stack of parallel decoupled planes, the edge surface will host a 1D shift (or translation) for each triangle plane that terminates on it. The number of triangle planes

terminating per unit cell of the 2D boundary sublattice is exactly E , and the factor of A_s accounts for the fact that the triangular drive may not be primitive. The overall minus sign arises because the chirality of the bulk motion is opposite that of the edge motion. Thus, $\mathbf{r}_{\text{tr}}$ gives the effective edge translation correctly for a triangular drive and an arbitrary edge surface.

Since $\mathbf{G}$ for a general exchange drive is given by the sum of $\mathbf{G}$ over its triangular components, it follows that Eq. (28) holds for *any* 3D exchange drive. In this way, Eqs. (25) and (28) completely characterize the bulk and edge behavior of a generic 3D translation-invariant exchange drive.

B. Tensor Products of 3D exchange drives

In Sec. III we found that 2D boundary behaviors form equivalence classes characterized by a set of reciprocal lattice vectors $\{\mathbf{G}_p\}$. The representative edge behavior of given class is a tensor product of shifts $\sigma_{\mathbf{r}_{\text{tr},p}}$ by vectors $\mathbf{r}_{\text{tr},p}$ (defined in Eq. (15)), each acting on an on-site Hilbert space with prime dimension p . In order to generate the edge behavior of a general equivalence class, we should take a tensor product of the bulk exchange drives described in the previous section.

For the equivalence class with reciprocal lattice vectors $\{\mathbf{G}_p\}$, we take a tensor product Hilbert space which has an on-site factor of dimension p for each non-zero $\mathbf{G}_p$. For each p -dimensional subspace, we choose a bulk exchange drive that is characterized by the reciprocal lattice vector $\mathbf{G} = \mathbf{G}_p$, as defined in Eq. (25). Any bulk exchange drive with this property is suitable, but for simplicity we can always choose a four-step triangular drive with the appropriate area. Then, by the reasoning above, the complete product drive will produce the required translation by lattice vector $\mathbf{r}_{\text{tr},p}$ for each p -dimensional subspace on an exposed surface. In other words, a product drive of this form in the bulk will reproduce the representative effective edge unitary of the equivalence class $V_{\{\mathbf{G}_p\}}$ on an exposed boundary. In this way, 3D product drives of this form are representatives of the different equivalence classes of 3D dynamical Floquet phases.

C. Bulk-edge correspondence for general Floquet drives

Finally, we summarize the interpretation of this bulk-edge correspondence for unitary evolutions in general. An arbitrary Floquet drive will not generally be a unitary loop but, as motivated in Sec. II, can often be interpreted as a unitary loop composed with an evolution with a constant Hamiltonian. In many cases the unitary loop component can be extracted from a given Floquet drive directly, as discussed in Refs. 24 and 32. Since the unitary loop component captures the inherently *dynamical* part of an evolution, a classification of unitary loops is equiv-

alent to a classification of Floquet topological phases. In this way, the conclusions above are applicable beyond the somewhat restrictive case of unitary loop evolutions.

However, a unitary loop will not in general be a pure exchange drive. Nevertheless, from the arguments above, an arbitrary unitary loop will have boundary behavior which is topologically equivalent to that of a pure exchange drive. Specifically, the effective edge unitary of an arbitrary loop can be brought into a tensor product of shift operators by a locally generated unitary operator acting only within the boundary region. It is in this sense that a generic unitary loop is equivalent to an exchange drive, and in this manner that unitary loops form equivalence classes labeled by a set of reciprocal lattice vectors $\{\mathbf{G}_p\}$.

In order to determine in practice which equivalence class a given unitary loop belongs to, it is first necessary to obtain its corresponding effective edge unitary U_{eff} for a choice of boundary. This can be achieved by removing terms from the generating Hamiltonian which connect sites across the boundary, as discussed in Sec. II. Once this has been obtained, the GNVW index across a given cut within the (necessarily finite) 2D boundary region can be computed using Eq. (7). Note that this requires grouping sites in a direction parallel to the cut to obtain a quasi-1D chain in the manner described in Sec. III C. In order to compute the scaled additive index $\nu(\mathbf{r})$, the GNVW index must be calculated in this manner for a sequence of finite systems as the dimensions are made infinite, and the resulting $\nu(\mathbf{r})$ obtained using Eq. (8). The value of $\nu(\mathbf{r})$ should tend towards its limiting value very quickly once the dimensions of the system are larger than the Lieb-Robinson length of U_{eff} .

However, the value of $\nu(\mathbf{r})$ for a cut in a boundary surface does not on its own determine the equivalence class of the unitary loop. To obtain the complete set of reciprocal lattice vectors $\{\mathbf{G}_p\}$, we must compute $\nu(\mathbf{r})$ for three linearly independent boundary cut directions $\mathbf{r}$. With these, the values of $\{\mathbf{G}_p\}$ can finally be extracted using equation Eq. (17).

In principle, the procedure described above can be used to determine the equivalence class of any 3D unitary loop. While the process is more involved than calculating the topological invariant of a unitary loop in 2D, both methods ultimately rely on computing the GNVW index for a (quasi)-1D system. In Ref. 32, the authors argued that matrix product unitaries (MPUs) [52] offer a particularly efficient representation of unitary operators at a 1D boundary, which enables the GNVW index to be computed in a numerically efficient manner. Specifically, the GNVW classification implies that any 1D local unitary operator is a combination of a shift operation and a locally generated 1D unitary (or finite-depth quantum circuit). Since both shifts and finite-depth quantum circuits admit efficient MPU representations, it follows that any effective edge unitary also admits an efficient MPU representation. Once expressed as an MPU, the GNVW index can be calculated straightforwardly using

the method described in Ref. 32.

Since our procedure for calculating the topological invariant of a 2D effective edge unitary relies on computation of the GNVW index, we might hope to be able to apply the advantages of an MPU representation in this case too. Indeed, we believe that the arguments of Ref. 32 can be applied directly to this higher dimensional case, with two important caveats. First, the boundary region is now 2D, and calculation of $\nu(\mathbf{r})$ formally requires taking the limit of infinite system size. MPUs are applicable only to 1D systems, and so in order to use them we must ‘regroup’ lattice sites along one direction to form a quasi-1D chain. However, this regrouping is already part of the procedure for calculating $\nu(\mathbf{r})$, and so MPUs are compatible with this part of the approach (albeit at the expense of using larger matrices to account for the additional sites). Although taking the infinite system-size limit is not feasible with MPUs, we expect the scaling behavior (and limiting value of $\nu(\mathbf{r})$) to become apparent very quickly once the system size is larger than the Lieb-Robinson length of the effective edge unitary. In addition, since $\nu(\mathbf{r})$ takes discrete (rational) values, small finite-size effects can be identified and ignored.

Secondly, and perhaps more fundamentally, the usefulness of MPUs for describing 1D boundary behavior relies on the fact that translations and finite-depth quantum circuits admit efficient MPU representations. In order to use MPUs to describe (finite) 2D boundaries, there must be no additional types of inherently 2D boundary behavior that cannot be expressed efficiently using MPUs. A main conjecture of this work, which we elaborate on below, is that the edge behavior described in previous sections provides a *complete* classification of 3D unitary loops (with translational symmetry but no other symmetries). Provided this holds, the edge behavior at a 2D boundary can again only be a combination of shifts and locally generated 2D unitary evolutions, both of which admit MPU representations. A rigorous proof of this statement remains an important open question.

VI. DISCUSSION

In summary, we have studied 3D many-body Floquet topological phases with translational invariance but no other symmetry from the perspective of their edge behavior. We found that phases of this form fall into equivalence classes that are somewhat analogous to those of weak topological phases. Members of each class share the same anomalous information transport at a 2D boundary, which is equivalent to a tensor product of shifts (or translations) that arise in lower dimensions. Each equivalence class is described by a set of reciprocal lattice vectors $\{\mathbf{G}_p\}$ corresponding to each prime p , which may be computed directly from the unitary evolution. The representative edge behavior in each equivalence class (corresponding to a tensor product of pure shifts) can be generated by an exactly solvable exchange drive in the

bulk.

These equivalence classes capture all possible topological phases (with no symmetry) whose edge behavior is equivalent to that of a tensor product of lower dimensional phases. For this classification to be *complete*, however, there would need to exist no intrinsically 3D (‘strong’) Floquet topological phases in this class. In the noninteracting case, we know this to be true: Ref. 18 demonstrates that noninteracting Floquet systems in class A host only a trivial 3D phase. Given that exchange drives can be interpreted as the many-body extension of noninteracting phases, this property leads us to conjecture that our classification *is* complete. However, further investigation and proof of this conjecture remain important goals for future work, which could perhaps be pursued using a higher-dimensional extension of the methods of Ref. 36. However, even without this conjecture holding, our work provides a complete classification of weak Floquet topological phases in 3D.

In classifying these phases, we developed a method for determining the effective edge behavior of an arbitrary exchange drive in 2D or 3D using geometric aspects of its action in the bulk. We found that 3D exchange drives may be characterized by an infinite set of reciprocal lattice vectors $\{\mathbf{G}_p\}$, with p indexing prime Hilbert space dimensions, which may be calculated directly from the form of the bulk exchange drive. These bulk quantities are equal to the topological invariants that classify the edge behavior of the exchange drive, providing a form of bulk-edge correspondence. The vectors $\{\mathbf{G}_p\}$ share some similarities to weak invariants of static topological insulators [53–57]. However, in contrast to the static case, these 3D chiral Floquet phases cannot generally be viewed as stacks of decoupled 2D layers. Instead, different Hilbert space factors within a tensor product may stack in different directions, as illustrated in Fig. 4.

We briefly discuss how our work fits into the broader classification schemes that have been introduced to study topological phases. Since the phases we describe are weak and require translational invariance, their formal classification derives from that of strong topological phases in one lower dimension. Specifically, our phases are related to tensor products of layers of 2D Floquet topological phases, as has been motivated throughout this work. Such 2D phases are classified by rational numbers p/q [32, 33], and in principle the reciprocal lattice vectors $\{\mathbf{G}_p\}$ of our classification could be expressed as a set of rational numbers p/q corresponding to each spatial dimension—although this description would obscure much of the symmetry of the phase.

More broadly, bosonic Floquet symmetry-protected topological phases have been classified within the group cohomology framework, in one dimension and beyond [19–22, 24]. In brief, many inherently dynamical phases protected by a (unitary) symmetry group G in d dimensions are believed to be classified by the cohomology group $H^d(G, \text{U}(1))$. However, the cohomology approach is known to be unable to capture static chiral phases [58],

a property which also seems to extend to the Floquet case [32]. Whether or not chiral phases (such as those in this paper) can be brought within the cohomology framework remains an interesting outstanding question.

Our classification suggests a number of interesting directions for future work. A natural follow-up is to ask whether a similar classification can be obtained for 3D Floquet phases of fermions, as well as in systems with additional symmetries. In addition, by combining these phases with topological order, it may be possible to obtain analogues of the Floquet enriched topological phases found in Refs. 34 and 35. Finally, it would be useful to obtain a rigorous proof of the conjecture that there are no inherently 3D Floquet topological phases in systems without symmetry, perhaps by developing an extension of the GNVW index to higher dimensions [36].

ACKNOWLEDGMENTS

We thank X. Liu for useful discussions. D. R., F. H., and R. R. acknowledge support from the NSF under CAREER DMR-1455368 and the Alfred P. Sloan foundation.

Appendix A: Linearity of the Index $\nu(\mathbf{r})$

In this appendix, we study the properties of the index $\nu(\mathbf{r})$ as the direction of the cut $\mathbf{r}$ is varied. As in the main text, we consider a translationally invariant local unitary operator U restricted to a periodic cylindrical system. Specifically, we take three cylindrical systems defined by $(N, \mathbf{r}_1)$, $(N, \mathbf{r}_2)$ and $(N, \mathbf{r}_1 + \mathbf{r}_2)$, and write the action of the unitary restricted to each of these as $U_{N, \mathbf{r}_1}$, $U_{N, \mathbf{r}_2}$ and $U_{N, \mathbf{r}_1 + \mathbf{r}_2}$, respectively.

We now construct a fourth system, as shown in Fig. 11, by cutting the systems $(N, \mathbf{r}_1)$ and $(N, \mathbf{r}_2)$ each along a common extended direction $\mathbf{r}'$ and reconnecting them along this line. The reconnection is carried out by restoring local terms in the unitary such that the final system is compact along the $\mathbf{r}_1 + \mathbf{r}_2$ direction with length $N(\mathbf{r}_1 + \mathbf{r}_2)$. We write the action of the unitary on this composite system as $U'_{N, \mathbf{r}_1 + \mathbf{r}_2}$, and note that further than λ_{LR} away from either cut, the action of $U'_{N, \mathbf{r}_1 + \mathbf{r}_2}$ is identical that of $U_{N, \mathbf{r}_1 + \mathbf{r}_2}$.

We now argue that both of these unitaries correspond to the same index $\nu(\mathbf{r}_1 + \mathbf{r}_2)$ and further, that $\nu(\mathbf{r}_1 + \mathbf{r}_2) = \nu(\mathbf{r}_1) + \nu(\mathbf{r}_2)$. First, since $U'_{N, \mathbf{r}_1 + \mathbf{r}_2}$ and $U_{N, \mathbf{r}_1 + \mathbf{r}_2}$ differ (if at all) only in the vicinity of the two horizontal cuts used in defining the system, we must have

$$\text{ind}(U_{N, \mathbf{r}_1 + \mathbf{r}_2}) = \delta \times \text{ind}(U'_{N, \mathbf{r}_1 + \mathbf{r}_2}), \quad (\text{A1})$$

where δ is the contribution to the index caused by rejoining local terms in the unitary action across the cuts. We discuss the scaling of δ below as N is taken to infinity, and argue that it is negligible in the infinite limit.

We also construct a 1D cell structure for these systems compatible with both a ‘triangular’ slice along the $\mathbf{r}_1$

FIG. 11. The construction of a system from two periodic systems with common extended dimension $\mathbf{r}'$. (a) The lower periodic system has compact dimension along $\mathbf{r}_1$, with sites connected by the dashed green lines identified. Similarly, the upper system has periodic boundary conditions in the $\mathbf{r}_2$ dimension, with sites connected by dashed blue lines identified. (b) By cutting each system along a 1D sublattice in the $\mathbf{r}'$ dimension, and identifying sites between the two systems along the cuts (connected by dashed green/blue lines) we construct a system with periodic dimension along $\mathbf{r}_1 + \mathbf{r}_2$.

direction followed by the $\mathbf{r}_2$ direction, and a ‘linear’ slice along the $\mathbf{r}_1 + \mathbf{r}_2$ direction, as shown in Fig. 12. The index $\text{ind}(U)$ computed for a given unitary must be the same for either of these cuts, from the properties of the GNVW index [36]. Choosing the triangular slice, we see that the unitary $U'_{N, \mathbf{r}_1 + \mathbf{r}_2}$ acts like either $U_{N, \mathbf{r}_1}$ or $U_{N, \mathbf{r}_2}$ away from the horizontal cuts. Overall, this means that

$$\text{ind}(U_{N, \mathbf{r}_1 + \mathbf{r}_2}) = \delta \times \text{ind}(U_{N, \mathbf{r}_1}) \times \text{ind}(U_{N, \mathbf{r}_2}), \quad (\text{A2})$$

where the δ here may differ from that in Eq. (A1), but will have the same scaling.

In both cases, the multiplicative correction to the index δ , which is introduced when rejoining two periodic systems, is bounded above and below by constants which depend only on the Lieb-Robinson length λ_{LR} of the underlying 2D unitary U , and the on-site Hilbert space dimension d . Explicitly, for any 1D system, the largest possible correction is achieved by a unitary whose action is equivalent to translation of the entire Hilbert space by the Lieb-Robinson length λ_{LR} . The upper bound on δ describes the case where the unitary before cutting and rejoining translates a region of dimension λ_{LR} near each cut from L to R by a distance λ_{LR} , but after cutting and rejoining translates the region from R to L by λ_{LR} . The lower bound is obtained by considering the opposite case. These bounds are essentially independent of the system size N , and so δ stays approximately constant as the limit $N \rightarrow \infty$ is taken.

By constructing a sequence of systems with increasing N , and calculating $\nu(\mathbf{r})$ using Eqs. (8) and (A2), we obtain the relations

$$\nu'(\mathbf{r}_1 + \mathbf{r}_2) = \nu(\mathbf{r}_1 + \mathbf{r}_2) = \nu(\mathbf{r}_1) + \nu(\mathbf{r}_2). \quad (\text{A3})$$

FIG. 12. Illustration of a 1D cell structure imposed on a 2D system with a compact dimension. For a system with compact dimension of length $N(\mathbf{r}_1 + \mathbf{r}_2)$ we can group the Hilbert spaces of sites within a blue region and pair of green systems into a single site on a 1D chain. The 1D GNVW index, however, is independent of the choice of location of cut used to define L and R in its computation. Therefore, dividing the 2D system using a ‘triangular’ cut (along $N\mathbf{r}_1$ followed by $N\mathbf{r}_2$) or using a ‘linear’ cut (along $N(\mathbf{r}_1 + \mathbf{r}_2)$) gives the same index $\text{ind}(U)$.

In particular, we see that ν is a linear function of 2D lattice vectors.

Appendix B: Further details on the classification of 2D effective edge unitaries

In the main text, we argued that translationally invariant unitary operators in 2D form equivalence classes labeled by a set of reciprocal lattice vectors $\{\mathbf{G}_p\}$ with p prime. In this appendix, we show that generic (site-by-site) tensor products of such unitary operators always reduce to this form.

We first note that we can associate a reciprocal lattice vector $\mathbf{G}_n$ with each term of such a tensor product, using the arguments of Sec. III. We can therefore initially characterize a general product drive by a set of pairs $\{(\mathbf{G}_n, d_n)\}$, where d_n labels the Hilbert space dimension of the n th term (but where the d_n will not generally be prime or unique). To remove any repetition, if any two terms in the product have the same Hilbert space dimension $d_n = d_m$, we may replace the pairs $(\mathbf{G}_n, d_n)$ and $(\mathbf{G}_m, d_m)$ with the single pair $(\mathbf{G}_n + \mathbf{G}_m, d_n = d_m)$. This is because the information transported is equivalent after the replacement, as may be demonstrated by regrouping the sites on the lattice using the methods of Ref. 33.

To reduce all the Hilbert space dimensions to primes, we may view any term for which d_n is not prime as a tensor product of drives, according to its prime factorization. Explicitly, if $d_n = 2^{n_2}3^{n_3}5^{n_5}\dots$, we can replace $(\mathbf{G}_n, d_n)$ with a term for every prime factor $\{(n_2\mathbf{G}_n, 2), (n_3\mathbf{G}_n, 3), (n_5\mathbf{G}_n, 5), \dots\}$. Again, the information transported in the 2D boundary system is equivalent in both cases by the regrouping operations of Ref. 33.

By performing this reduction to prime dimensions and further combining terms of the same dimension, we find

FIG. 13. Boundary behavior on a 2D surface described by different $\mathbf{G}_p$ and $\mathbf{G}'_p$ cannot be deformed into one another by locally generated unitary evolutions within the boundary. (a) Two different boundary behaviors, corresponding to distinct translation vectors $\mathbf{r}_{\text{tr}}$, are indicated (by red and blue arrows) on a 2D boundary. (b) This 2D behavior can be reduced to an effective 1D model by grouping lattice sites in the direction of one of the $\mathbf{r}_{\text{tr}}$. In this effective model, one effective edge unitary becomes a permutation of the on-site Hilbert space (within the red grouping) and the other becomes a translation in the horizontal direction combined with a permutation (blue arrows).

that a general effective edge unitary can always be characterized by a set of reciprocal lattice vectors $\{\mathbf{G}_p\}$, each corresponding to an on-site Hilbert space with prime dimension p . Using Eq. (9), the scaled chiral flow associated with this effective edge unitary can easily be calculated.

Appendix C: Stability of 2D effective edge unitaries

In Ref. 33 it is shown that a shift (translation) operator $(\sigma_p)^n$ acting on a 1D boundary cannot be continuously deformed to a different shift operator $(\sigma_p)^{n'}$ with $n \neq n'$ through a locally generated 1D unitary evolution. This includes the trivial shift operator $(\sigma_p)^0 = \mathbb{I}$. In this appendix we formally show that this stability continues to hold when applied to the more complicated boundary behavior (described by some reciprocal lattice vector $\mathbf{G}_p$) that may act at a 2D boundary.

We consider two 2D boundary systems (which we assume to be identical 2-tori with finite size) with the same on-site Hilbert space dimension p . [If these drives have different on-site Hilbert space dimensions or different sizes then they are trivially inequivalent.] On each system, we take unitaries characterized by inequivalent $\mathbf{G}_p$ and $\mathbf{G}'_p$, leading to distinct behavior. The action of each unitary is characterized by a translation vector within the 2D boundary surface, as argued in Sec. III.

We now create an effective 1D system by grouping the sites on the 2-torus surface as illustrated in Fig. 13. If the translation vectors of the two drives are not parallel, we group together the sites on the 2-torus that lie in the direction of the translation vector of (say) the second drive. If the translations of the two drives are parallel, we group together the sites of the 2-torus that lie along

any chosen direction that is not parallel to the translation vectors. In both cases, we are left with two effective 1D edge behaviors that are topologically distinct [32, 33]. By the arguments of Ref. 33, the two effective edge unitaries cannot be deformed into one another by a local 1D perturbation, which implies that the original 2D effective edge unitaries cannot be deformed into one another either. This argument holds for each step in the sequence of boundary systems if their size is made infinite, and extends straightforwardly to drives labeled by a set of reciprocal lattice vectors $\{\mathbf{G}_p\}$.

Appendix D: Continuous modifications of loop drives

In this appendix we define transformations which may be carried out on a unitary exchange drive, and prove that these transformations leave the effective edge behavior unaltered.

Proposition 1. *Given a 2D unitary loop L which acts trivially in the bulk but nontrivially (i.e. as a shift) at the boundary of an open system and a unitary swap U which interchanges pairs of states separated by a finite distance, we consider the sequence of drives $U^{-1}LU$. We claim that this sequence has the same edge behavior as L .*

Proof. Since U acts as a product over disjoint pairs of sites, we can disentangle its effects in the bulk from its effects on the edge. To do this, we extend the original edge region of L to include sites which are connected to it by the action of U . In this way, we can write the composite unitary as the product of the identity in the bulk and a piece which acts at the edge, as shown in Fig. 14. Now, considering the action restricted to this new edge region, the unitary acts as a product of local unitaries and a shift (translation) operator. However, no local 1D unitary evolution can generate (or destroy) chiral edge behavior [33], and so the conjugation with U can have no effect on the chiral properties of L .

An alternative point of view is that conjugation with U acts as a local basis transformation of the Hilbert space restricted to the edge. A local basis transformation of a quasi-1D system cannot change the global properties of the drive.

□

Note that U is an exchange operator and can be continuously connected to the identity, and so we can define $U(\theta)$ such that $U(0) = \mathbb{I}$ and $U(1) = U$. We therefore see that conjugation with $U(\theta)$ defines a continuous transformation within the space of unitary loops. Further, note that this composite unitary $U^{-1}LU$ is also a loop as it is trivial in the bulk.

Proposition 2. *Given a unitary loop L and a finite sequence of local unitary swaps $\{U_1, \dots, U_N\}$, then the composite unitary operator $(U_1 \dots U_N)^{-1}L(U_1 \dots U_N)$ has the same edge behavior as L .*

FIG. 14. (a) A unitary loop L acts trivially in the bulk but may act nontrivially in a quasi-1D edge region located near a boundary (green shaded region). (b) Conjugating the unitary loop L with a product of disjoint pairwise swaps (thick blue lines) may connect bulk sites to the edge region. We define a new quasi-1D edge region which includes these former bulk sites (green shaded region). See main text for details.

Proof. One repeats the argument in Proposition 1 N times. □

Proposition 3. *Any drive T comprising a sequence of unitary swaps $(U_1 \dots U_N)$ followed by the inverse swaps in reverse order $(U_N^{-1} \dots U_1^{-1})$ has trivial effective edge behavior.*

Proof. This follows directly by Proposition 2 if we take L to be $\mathbb{I}$. □

Note that T above is a general ‘trivial’ drive as defined in Sec. IV. We can therefore continuously append or remove trivial drives from a sequence of loop drives without affecting the effective edge behavior.

Proposition 4. *Given a unitary loop L which is the product of a sequence of local unitary swaps $L = U_1 \dots U_N$, then any cyclic permutation of the steps of L is a loop with the same edge behavior.*

Proof. Consider a cyclic permutation of L , $L' = U_N U_{N+1} \dots U_N U_1 \dots U_{N-1}$. Construct the unitary $V = (U_N U_{N+1} \dots U_N)^{-1}$. Then $V^{-1}L'V$ is the cyclic permutation we are considering and by Proposition 2 has the same edge behavior as L . □

Appendix E: Nonprimitive triangular drives

In this appendix, we show that the number of independent sublattices on which a triangular drive is primitive is equal to the magnitude of its signed area (in units of the primitive triangle area). Consider an arbitrary four-step triangular drive defined by vectors $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$, which we take without loss of generality to have $\mathbf{b}_3 = 0$. If the triangle is not primitive, there are additional Bravais lattice points on the edges or contained within the interior of the triangle, the number of which we denote by e and i respectively.

FIG. 15. A nonprimitive triangular exchange drive is represented by the solid lines, and a parallelogram is formed over a choice of edge. The shading of lattice sites indicates membership of different sublattices spanned by vectors of the parallelogram. The area of the solid triangle is four times the area of a primitive triangle, and there are correspondingly four different sublattices spanned by its component vectors.

By specifying an edge of the triangle, we may form a parallelogram (over this edge) as illustrated in Fig. 15. This parallelogram may be tessellated to tile a sublattice partitioned by the drive. Each interior point of the original triangle results in two interior points of the parallelogram. Each edge point of the original triangle which lies

on the edge used to construct the parallelogram results in an interior point of the parallelogram. Edge points on the other edges of the original triangle each result in two edge points of the parallelogram; however, these points are separated by a sublattice vector. By tiling the lattice with the same parallelogram but shifting the origin to these edge points and interior points, the total number of distinct sublattices spanned by the drive is found to be $1 + e + 2i$.

Pick's theorem states that the area of a lattice polygon, in terms of the unit cell area, is given by

$$A = v/2 + e/2 + i - 1,$$

where v is the number of vertices. Recalling that the signed area defined in Eq. (22) is given in terms of the primitive triangle area, we obtain

$$\begin{aligned} |A_s| &= 2A \\ &= 1 + e + 2i \end{aligned}$$

for a triangular drive. Hence, the number of independent sublattices is equal to the magnitude of the signed area of the drive. Since each independent sublattice generates its own edge behavior, the edge behavior of a triangular drive is equivalent to a composition of $|A_s|$ primitive drives.

-
- [1] V. Khemani, A. Lazarides, R. Moessner, and S. L. Sondhi, *Physical Review Letters* **116**, 250401 (2016).
 - [2] D. V. Else, B. Bauer, and C. Nayak, *Physical Review X* **7**, 011026 (2017).
 - [3] C. W. von Keyserlingk and S. L. Sondhi, *Physical Review B* **93**, 245146 (2016).
 - [4] C. W. von Keyserlingk, V. Khemani, and S. L. Sondhi, *Physical Review B* **94**, 085112 (2016).
 - [5] D. V. Else, B. Bauer, and C. Nayak, *Physical Review Letters* **117**, 090402 (2016).
 - [6] N. Y. Yao, A. C. Potter, I. D. Potirniche, and A. Vishwanath, *Physical Review Letters* **118**, 030401 (2017).
 - [7] T. Kitagawa, E. Berg, M. Rudner, and E. Demler, *Physical Review B* **82**, 235114 (2010).
 - [8] L. Jiang, T. Kitagawa, J. Alicea, A. R. Akhmerov, D. Pekker, G. Refael, J. I. Cirac, E. Demler, M. D. Lukin, and P. Zoller, *Physical Review Letters* **106**, 220402 (2011).
 - [9] M. S. Rudner, N. H. Lindner, E. Berg, and M. Levin, *Physical Review X* **3**, 031005 (2013).
 - [10] M. Thakurathi, A. A. Patel, D. Sen, and A. Dutta, *Physical Review B* **88**, 155133 (2013).
 - [11] J. K. Asbóth, B. Tarasinski, and P. Delplace, *Physical Review B* **90**, 125143 (2014).
 - [12] F. Nathan and M. S. Rudner, *New Journal of Physics* **17**, 125014 (2015).
 - [13] J. Klinovaja, P. Stano, and D. Loss, *Physical Review Letters* **116**, 176401 (2016).
 - [14] P. Titum, N. H. Lindner, M. C. Rechtsman, and G. Refael, *Physical Review Letters* **114**, 056801 (2015).
 - [15] D. Carpentier, P. Delplace, M. Fruchart, and K. Gawedzki, *Physical Review Letters* **114**, 106806 (2015).
 - [16] M. Fruchart, *Physical Review B* **93**, 115429 (2016).
 - [17] P. Titum, E. Berg, M. S. Rudner, G. Refael, and N. H. Lindner, *Physical Review X* **6**, 021013 (2016).
 - [18] R. Roy and F. Harper, *Physical Review B* **96**, 155118 (2017).
 - [19] C. W. von Keyserlingk and S. L. Sondhi, *Physical Review B* **93**, 245145 (2016).
 - [20] D. V. Else and C. Nayak, *Physical Review B* **93**, 201103 (2016).
 - [21] A. C. Potter, T. Morimoto, and A. Vishwanath, *Physical Review X* **6**, 041001 (2016).
 - [22] R. Roy and F. Harper, *Physical Review B* **94**, 125105 (2016).
 - [23] A. C. Potter and T. Morimoto, *Physical Review B* **95**, 155126 (2017).
 - [24] R. Roy and F. Harper, *Physical Review B* **95**, 195128 (2017).
 - [25] T. Kitagawa, M. A. Broome, A. Fedrizzi, and M. S. Rudner, *Nature Communications* **3**, 882 (2012).
 - [26] M. C. Rechtsman, J. M. Zeuner, Y. Plotnik, Y. Lumer, D. Podolsky, F. Dreisow, S. Nolte, M. Segev, and A. Szameit, *Nature* **496**, 196 (2013).
 - [27] G. Jotzu, M. Messer, R. Desbuquois, and M. Lebrat, *Nature* **515**, 237 (2014).
 - [28] K. Jiménez-García, L. J. LeBlanc, R. A. Williams, M. C. Beeler, C. Qu, M. Gong, C. Zhang, and I. B. Spielman, *Physical Review Letters* **114**, 125301 (2015).
 - [29] L. J. Maczewsky, J. M. Zeuner, S. Nolte, and A. Szameit, *Nature Communications* **8**, 1 (2016).

- [30] J. Zhang, P. W. Hess, A. Kyprianidis, P. Becker, A. Lee, J. Smith, G. Pagano, I. D. Potirniche, A. C. Potter, A. Vishwanath, N. Y. Yao, and C. Monroe, *Nature* **543**, 217 (2017).
- [31] S. Choi, J. Choi, R. Landig, G. Kucsko, H. Zhou, J. Isoya, F. Jelezko, S. Onoda, H. Sumiya, V. Khemani, C. von Keyserlingk, N. Y. Yao, E. Demler, and M. D. Lukin, *Nature* **543**, 221 (2017).
- [32] H. C. Po, L. Fidkowski, T. Morimoto, A. C. Potter, and A. Vishwanath, *Physical Review X* **6**, 041070 (2016).
- [33] F. Harper and R. Roy, *Physical Review Letters* **118**, 115301 (2017).
- [34] H. C. Po, L. Fidkowski, A. Vishwanath, and A. C. Potter, *Physical Review B* **96**, 245116 (2017).
- [35] L. Fidkowski, H. C. Po, A. C. Potter, and A. Vishwanath, *arXiv* (2017), 1703.07360v1.
- [36] D. Gross, V. Nesme, H. Vogts, and R. F. Werner, *Communications in Mathematical Physics* **310**, 419 (2012).
- [37] A. Lazarides, A. Das, and R. Moessner, *Physical Review E* **90**, 012110 (2014).
- [38] L. D'Alessio and M. Rigol, *Physical Review X* **4**, 041048 (2014).
- [39] P. Ponte, A. Chandran, Z. Papic, and D. A. Abanin, *Annals of Physics* **353**, 196 (2015).
- [40] L. D'Alessio and A. Polkovnikov, *Annals of Physics* **333**, 19 (2013).
- [41] P. Ponte, Z. Papic, F. Huveneers, and D. A. Abanin, *Physical Review Letters* **114**, 140401 (2015).
- [42] A. Lazarides, A. Das, and R. Moessner, *Physical Review Letters* **115**, 030402 (2015).
- [43] D. A. Abanin, W. De Roeck, and F. Huveneers, *Annals of Physics* **372**, 1 (2016).
- [44] L. Zhang, V. Khemani, and D. A. Huse, *Physical Review B* **94**, 224202 (2016).
- [45] R. Nandkishore and D. A. Huse, *Annual Review of Condensed Matter Physics* **6**, 15 (2015).
- [46] D. A. Abanin, W. De Roeck, and F. Huveneers, *Physical Review Letters* **115**, 256803 (2015).
- [47] T. Kuwahara, T. Mori, and K. Saito, *Annals of Physics* **367**, 96 (2016).
- [48] D. A. Abanin, W. De Roeck, W. W. Ho, and F. Huveneers, *Communications in Mathematical Physics* **354**, 809 (2017).
- [49] E. H. Lieb and D. W. Robinson, *Communications in Mathematical Physics* **28**, 251 (1972).
- [50] A. Kitaev, *Annals of Physics* **321**, 2 (2006).
- [51] B. R. Duschatko, P. T. Dumitrescu, and A. C. Potter, *arXiv* (2018), 1804.02004v1.
- [52] J. Ignacio Cirac, D. Pérez-García, N. Schuch, and F. Verstraete, *Journal of Statistical Mechanics: Theory and Experiment* **2017**, 083105 (2017).
- [53] B. I. Halperin, *Japanese Journal of Applied Physics* **26**, 1913 (1987).
- [54] L. Fu, C. L. Kane, and E. J. Mele, *Physical Review Letters* **98**, 106803 (2007).
- [55] L. Fu and C. L. Kane, *Physical Review B* **76**, 045302 (2007).
- [56] R. Roy, *Physical Review B* **79**, 195322 (2009).
- [57] R. Roy, *New J. Phys.* **12**, 065009 (2010).
- [58] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, *Physical Review B* **87**, 155114 (2013).