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Motivated by neutral excitations in disordered electronic materials and systems of trapped ultracold particles with long-range interactions, we study energy-level statistics of quasiparticles with the power-law hopping Hamiltonian $\propto 1/r^\alpha$ in a strong random potential. In solid-state systems such quasiparticles, which are exemplified by neutral dipolar excitations, lead to long-range correlations of local observables and may dominate energy transport. Focussing on the excitations in disordered electronic systems, we compute the energy-level correlation function $R_2(\omega)$ in a finite system in the limit of sufficiently strong disorder. At small energy differences the correlations exhibit Wigner-Dyson statistics. In particular, in the limit of very strong disorder the energy-level correlation function is given by $R_2(\omega, V) = A_3 \frac{\omega}{\omega_V}$ for small frequencies $\omega \ll \omega_V$ and $R_2(\omega, V) = 1 - (\alpha - d)A_1 \left(\frac{\omega_V}{\omega}\right)^{\frac{d}{\alpha}} - A_2 \left(\frac{\omega_V}{\omega}\right)^2$ for large frequencies $\omega \gg \omega_V$, where $\omega_V \propto V^{-\frac{\alpha}{d}}$ is the characteristic matrix element of excitation hopping in a system of volume V, and A_1, A_2 and A_3 are coefficient of order unity which depend on the shape of the system. The energy-level correlation function, which we study, allows for a direct experimental observation, for example, by measuring the correlations of the ac conductance of the system at different frequencies.

In a strongly disordered electronic system, the properties of charged excitations, such as electrons and holes, are correlated on short length scales of order of the localisation length and decay exponentially with distance. By contrast, neutral excitations, such as localised electron-hole pairs, allow for long-range hops via virtual processes of annihilating a neutral excitation at one location and creating it elsewhere. It has been demonstrated[1], for example, that dipole excitations in 3D can hop virtually with the distance dependence $\propto 1/r^3$. Such power-law hops may lead to long-range correlations between physical observables, such as ac conductivity, even when charged excitations remain localised. While neutral excitations do not carry charge, they are involved in energy transport and, thus, may dominate heat conductivity. Moreover, if certain neutral excitations are delocalised due to the power-law hops they may serve as a bath for other excitations[2,3] and thus lead to the variable-range hopping of charged excitations.

The dynamics of neutral excitations, therefore, plays a fundamental role in transport and phase diagrams of granulated materials, superconducting films in the insulating state[4], systems of defects in insulators[5] and other disordered systems, which has motivated recent studies of conductivity[6] and wavefunctions[6–9] in systems with power-law hopping. Excitations with a generic power-law hopping $\propto 1/r^\alpha$ with tunable α have also been realised recently in 1D[10–13] and 2D[14] arrays of trapped ultracold ions. Such systems may be used to simulate disordered electronic materials, yet serve as a platform for observing novel fundamental phenomena, for example, many-body-localisation transitions[15] or high-dimensional disorder-driven effects[16]. In this paper we study analytically the energy-level statistics (ELS) of excitations with power-law hopping Hamiltonians in strongly disordered systems.

Energy-level statistics in a disordered system reflects fundamental symmetries and is often used to diagnose conducting and insulating phases at different disorder strengths. Abundant numerical data (see Ref. 17 for a review) suggests also that the ELS is linked to the chaotic properties of a system; systems such as chaotic billiards and disordered metals display chaotic or non-chaotic behaviour depending on whether their statistics is Wigner-Dyson[18] or Poisson. Recently, ELS has also received much renewed attention in the context of many-body-localisation transitions[15]; interacting disordered systems are expected[19] to display Poisson or Wigner-Dyson statistics of the *many-body* levels of the system in many-body-delocalised and many-body-localised states, respectively[19–21] . Similarly, ELS has been demonstrated numerically[22] to distinguish between chaotic and non-chaotic behaviour in a generalised Sachdev-Ye-Kitaev model[23], which is often used as a toy model in the studies of quantum chaos. In such studies the many-body ELS at the transition is used as a numerical tool for detecting a phase transition, although, unlike the single-(quasi)particle level statistics, cannot be straightforwardly measured in condensed-matter experiments. It has also been conjectured recently[24] that the ELS of single quasiparticles reflects an interaction-driven transition between chaotic and non-chaotic behaviour.

The relation between phase transitions and excitation statistics can be explored further by analysing the ELS in the respective phases. Although the ELS of neutral excitations in insulating materials also determines the heat transport and correlations between local ac responses, e.g. the ac conductivity, it has largely avoided researchers' attention, in contrast with the statistics of charged excitations[17,25,26]. In this paper, focussing on neutral excitations in disordered solids ($\alpha = 3$), we compute microscopically the correlation functions of energy

levels of excitations with the power-law hopping $\propto 1/r^\alpha$.

This paper is organised as follows. Our main results for the energy-level correlation functions are summarised in Sec. I. In Sec. II we discuss the model of dipole excitations, the simplest type of neutral excitations in a disordered system. Sec. III deals with the statistics of dipole excitations in strongly disordered systems in dimensions $d < 3$ and sufficiently small disordered 3D systems. The case of a 3D system, which requires a special consideration, is addressed in Sec. IV. Section V is devoted to the energy-level correlation functions in systems with a power-law hopping $\propto 1/r^\alpha$ with an arbitrary power α, which have recently been realised in experiments with trapped ultracold particles. We conclude in Sec. VI.

I. SUMMARY OF RESULTS

In this paper, we characterise the statistics of the energy levels of excitation in a disordered system of a finite volume V by the correlation function

$$R_2(\omega) = \left\langle \nu\left(E+\frac{\omega}{2}\right)\nu\left(E-\frac{\omega}{2}\right)\right\rangle_{\rm dis} / \langle \nu(E)\rangle^2_{\rm dis}, \quad (1.1)$$

where $\nu(E)$ is the density of states (DoS) of the excitations for a specific disorder realisation, and $\langle \ldots \rangle_{\rm dis}$ is our convention for disorder averaging.

In the limit of very strong disorder, spatial and energy correlations between observables in electronic systems are dominated by dipole excitations, i.e. pairs of electron and hole excitations located close to each other. Such dipoles allow for long-range hops with the distance dependence[1] $\propto 1/r^3$. In principle, the hopping of more complicated excitations, consisting of multiple electrons and holes, has in general the same distance dependence, but is suppressed due to smaller matrix elements of recombination of those excitations at strong disorder.

The density of states of dipole excitations is proportional to the ac conductance[27] of the system. This allows one to observe the correlation function (1.1) in experiment, e.g., by measuring the correlations of ac conductance $G(\tilde\omega)$ of the system as a function of frequency $\tilde\omega$ and computing the correlator $R_2(\omega) \propto \langle G(\Omega)G(\Omega+\omega)\rangle_\Omega$, where $\langle \ldots \rangle_\Omega$ is the averaging with respect to the frequency Ω in a sufficiently large interval of energies.

Very strong disorder in an electronic system. In the case of very strong disorder in a solid-state system, we find

$$R_2(\omega, V) \approx \begin{cases} 1-(3-d)C_1\left(\frac{\omega_V}{\omega}\right)^{\frac{d}{3}} - C_2\left(\frac{\omega_V}{\omega}\right)^2, & \omega \gg \omega_V \\ C_3\frac{\omega}{\omega_V}, & \omega \ll \omega_V \end{cases} \quad (1.2)$$

where the characteristic frequency ω_V scales with the volume V of the system as $\omega_V \propto V^{-\frac{3}{d}}$; and the coefficients C_1, C_2 and C_3 are independent of the volume V, but the coefficients C_2 and C_3 depend on the shape of the system. This result applies to all strongly disordered systems in dimensions $d < 3$ and to sufficiently small 3D systems. In such systems, the correlations between energy levels come from rare resonances between pairs of excitation states which are located far from each other but have close energies.

3D systems. In 3D electronic systems, unlike the case of lower dimensions, excitation states involve resonances on multiple sites[28], which is why the 3D case requires a special consideration. We find that in 3D systems the energy-level correlation function $R_2(\omega)$ is still given by Eq. (1.2) with the coefficients C_i independent of volume V only in the limits of small and large volumes. However, for a 3D system of arbitrary size, these coefficients have an explicit V dependency, and, thus, the scaling of the correlation function with system size is different.

Arbitrary power-law hopping. The results for the energy-level correlations in a system of neutral electronic excitations with the $\propto 1/r^3$ hopping may be generalised to the case of an arbitrary power-law hopping $\propto 1/r^\alpha$. Such hopping with arbitrary α my be realised, for example, in arrays of trapped ultracold ions[10–14] in optical or magnetic traps. For such hopping we obtain

$$R_2(\omega, V) \approx \begin{cases} 1-(\alpha-d)A_1\left(\frac{\omega_V}{\omega}\right)^{\frac{d}{\alpha}} - A_2\left(\frac{\omega_V}{\omega}\right)^2, & \omega \gg \omega_V \\ A_3\frac{\omega}{\omega_V}, & \omega \ll \omega_V \end{cases}, \quad (1.3)$$

where $\omega_V \propto V^{-\frac{\alpha}{d}}$ is the characteristic interaction energy of dipoles on the system size. In some sense, the scale ω_V is similar to the Thouless energy of quasiparticles in a disordered metal; this quantity gives the inverse characteristic time required for a perturbation created in the middle of the system to reach its boundary.

Comparison with the quasiparticle statistics in a metallic grain. For small frequencies ω, the correlation functions (1.2) and (1.3) are linear in the energy difference ω, i.e. are described by the Wigner-Dyson statistics[18] in the orthogonal symmetry class, similarly to, e.g., quasiparticles in sufficiently small metallic grains. In the latter system, the correlation function $R_2(\omega)$ is universal[17,29], with the mean level spacing δ in the grain being the characteristic energy scale, so long as $\delta \ll E_{\rm Th}$, where $E_{\rm Th} = DV^{-\frac{2}{d}}$ is the Thouless energy and D is the diffusion coefficient. In the opposite case, $\delta \gtrsim E_{\rm Th}$, the correlations become non-universal for energy differences $\omega \gtrsim E_{\rm Th}$. In contrast to the case of quasiparticles, the correlations of neutral excitations have the characteristic scale ω_V, the interaction energy on the size of the system and are described by functions (1.2) and (1.3) with non-universal coefficients. The power-law dependency of the correlation function $R_2(\omega)$ on the energy difference ω may signal of a possible chaotic behaviour of the dynamics of the excitations (as we discuss in Sec. VI), which may be defined and probed, e.g., via out-of-time-order correlators[30] of operators characterising transport of neutral excitations.

II. MODEL FOR NEUTRAL EXCITATIONS IN SOLIDS

In what immediately follows we describe the effective Hamiltonian of electron-hole dipoles, the simplest type of neutral excitations in an electronic system. The Hamiltonian of the dipole excitations in a disordered medium is given by

$$\hat{\mathcal{H}}_0 = \sum_{\mathbf{r},\mathbf{d}} E_{\mathbf{rd}} \hat{b}^\dagger_{\mathbf{rd}} \hat{b}_{\mathbf{rd}} - \sum_{\mathbf{r},\mathbf{d}} J_{\mathbf{rd}} \left(\hat{b}^\dagger_{\mathbf{rd}} + \hat{b}_{\mathbf{rd}} \right) + \sum_{\boldsymbol{rd},\boldsymbol{r'd'}} E_{\rm int} \left(\mathbf{rd}, \mathbf{r'd'} \right) \hat{b}^\dagger_{\mathbf{rd}} \hat{b}^\dagger_{\mathbf{r'd'}} \hat{b}_{\mathbf{r'd'}} \hat{b}_{\mathbf{rd}}, \tag{2.1}$$

where $\hat{b}^{(\dagger)}_{\mathbf{rd}}$ is the annihilation (creation) operator of a dipole with polarisation $\mathbf{d}$ at location $\mathbf{r}$ (e.g., the location of the positive charge in the dipole); $E_{\mathbf{rd}}$ is the energy of the respective dipole state, which strongly fluctuates from site to site due to the presence of quenched disorder; $J_{\mathbf{rd}}$ is the matrix element of the recombination of the dipole (electron hopping to the location of the hole or vice a versa), which may be assumed real without loss of generality; we have also introduced the interaction energy

$$E_{\rm int} \left(\mathbf{rd}, \mathbf{r'd'} \right) = Q(\hat{\mathbf{d}}, \hat{\mathbf{d}}', \boldsymbol{n}) \frac{|\mathbf{d}||\mathbf{d}'|}{|\mathbf{r} - \boldsymbol{r'}|^3}, \tag{2.2}$$

$$Q(\hat{\mathbf{d}}, \hat{\mathbf{d}}', \boldsymbol{n}) = \hat{\mathbf{d}} \cdot \hat{\mathbf{d}}' - 3(\hat{\mathbf{d}} \cdot \boldsymbol{n})(\hat{\mathbf{d}}' \cdot \boldsymbol{n}) \tag{2.3}$$

between dipoles at locations $\mathbf{r}$ and $\mathbf{r}'$ with polarisations $\mathbf{d}$ and $\mathbf{d}'$, respectively, where $\mathbf{n} = (\mathbf{r} - \mathbf{r}')/|\mathbf{r} - \mathbf{r}'|$ and $\hat{\mathbf{d}} = \mathbf{d}/|\mathbf{d}|$ are unit vectors parallel, respectively, to $\mathbf{r} - \mathbf{r}'$ and $\mathbf{d}$.

Long-range hopping. While strong disorder prevents dipole hopping on short distances, dipole excitations allow for long-range virtual hops[1] between remote sites with close energies via annihilating a dipole at the initial location and then creating it at the final location or vice a versa. The amplitude of such a hop is given, to the leading order in the recombination elements $J_{\mathbf{rd}}$, by

$$T_{\mathbf{rd},\mathbf{r'd'}} \approx \frac{J_{\mathbf{rd}} J_{\mathbf{r'd'}}}{E^2_{\mathbf{rd}}} \frac{|\mathbf{d}||\mathbf{d}'| Q(\hat{\mathbf{d}}, \hat{\mathbf{d}}', \boldsymbol{n})}{|\mathbf{r} - \mathbf{r}'|^3}. \tag{2.4}$$

Thus, in the limit of a strongly disordered system (small recombination elements $J_{\mathbf{rd}}$ compared to the typical fluctuations of the energies $E_{\mathbf{rd}}$), the dynamics of the dipoles is effectively a single-particle problem.

In this paper we do not consider many-dipole processes, such as one dipole decomposing into two dipoles of the same total energy. For the case of sufficiently strong disorder, considered in Sec. III, we assume that such processes are suppressed either by a small size of the system or by the suppressed density of states of the dipoles into which an excitation may decompose (the dipole density of states may, e.g., have a gap at sufficiently low energies.) As discussed in Sec. IV, in very large systems it may be assumed that effective excitations are already renormalised by many-dipole processes.

Cotunnelling through excitation states with high energies. When computing energy-level correlations, we assume that the energies of all sites $E_{\mathbf{rd}}$ lie sufficiently close to each other and do not consider sites with high excitation energies $E_{\mathbf{rd}} \gg \omega$. In principle, cotunnelling through such high-energy sites in a disordered system may lead to ultraviolet divergencies in physical observables[31]. These divergencies may be treated by means of the renormalisation-group (RG) procedure described in Ref. 31, which repeatedly removes highest-energy excitation states from the system while renormalising transition amplitudes between all other states, with energies closer to ω. In this paper, we assume that the system we consider is already renormalised following this procedure and the recombination elements $J_{\mathbf{rd}}$ already include virtual cotunnelling through excitation states with energies far from E.

III. STRONG DISORDER IN SOLIDS

In this section we consider the correlations of dipole energy levels and ac conductances in a strongly disordered system, in which most dipole states are localised almost entirely on single sites $(\mathbf{r}, \mathbf{d})$ (with given polarisations) and are weakly perturbed by the tunnelling to other sites. A dipole is localised almost entirely on one site $(\mathbf{r}, \mathbf{d})$ if there are no other "resonant" sites around it[1,28,32,33] with close energies, $|E_{\mathbf{rd}} - E_{\mathbf{r'd'}}| \lesssim |T_{\mathbf{rd},\mathbf{r'd'}}|$.

The number of resonant sited around a given site $\mathbf{r}$ may be estimated as $N_{\mathbf{rd}} \sim \nu_0 n \sum_{\mathbf{d}'} \int_{\mathbf{r}'} |T_{\mathbf{rd},\mathbf{r'd'}}| \, d\mathbf{r}'$, where n is the concentration of sites and ν_0 is the density of dipole states at an isolated site. For the hopping element $|T_{\mathbf{rd},\mathbf{r'd'}}| \propto 1/|\mathbf{r} - \mathbf{r}'|^3$, given by Eq. (2.4) and for strong disorder, the average number of resonant sites is significantly smaller than unity near each given site in dimensions $d < 3$. In this regime, most of the dipoles are strongly localised and their states may be considered unaffected by the resonant sites. In 3D, however, the number of resonant sites diverges[1,28,32,33] $\propto n\nu_0 \langle \mathbf{d}^2 \rangle J^2 E^{-2}_{\mathbf{rd}} \ln \left(L n^{\frac{1}{3}} \right)$ in the limit of an infinite system, $L \to \infty$, where J is the characteristic recombination matrix element.

In this section, we assume that the disorder is strong and that most dipoles are strongly localised, either due to a low spatial dimension $d < 3$ or due to a sufficiently small size in 3D, $L \ll n^{-\frac{1}{3}} \exp \left(\frac{E^2}{n\nu_0 J^2 d^2} \right)$.

A. Generic expressions for dipolar energy-level correlations

In what immediately follows we derive a generic expression for the correlator $R_2(\omega, \mathbf{r}, \mathbf{r}')$ of the energy levels of a dipole excitation on two sites and then, using it,

compute the correlator $R_2(\omega)$ of the energy levels in a strongly disordered system of volume V.

While most dipoles are strongly localised on single sites, there exist rare pairs of sites with close energies, on which dipole states get strongly hybridised due to the tunnelling. This hybridisation of pairs of sites leads to correlations between dipole states on arbitrarily long distances, which lead to correlations between energy levels and various observables. In the limit of strong disorder (small system size) under consideration, one may neglect resonances between clusters of three or more sites.

Correlation function in a system of two sites. The hybridisation of two dipole states with close energies $E_{\mathbf{rd}}$ and $E_{\boldsymbol{r'd'}}$ at locations $\mathbf{r}$ and $\mathbf{r}'$ and with polarisations $\mathbf{d}$ and $\mathbf{d}'$ leads to the creation of two hybridised states with energies

$$E_{\pm}=\frac{1}{2}\left(E_{\boldsymbol{r'd'}}+E_{\mathbf{rd}}\right)\pm\frac{1}{2}\left[\left(E_{\mathbf{rd}}-E_{\mathbf{r'd'}}\right)^2+4\left|T_{\mathbf{rd},\mathbf{r'd'}}\right|^2\right]^{\frac{1}{2}}, \tag{3.1}$$

where the hopping amplitude $T_{\mathbf{rd},\mathbf{r'd'}}$ is given by Eq. 2.4. The density of states on such a pair of sites is given by $\nu(E)=\delta\left(E-E_+\right)+\delta\left(E-E_-\right)$. The correlation function $R_2(\omega,\mathbf{r},\mathbf{r}')$ of dipole states on two sites $\mathbf{r}$ and $\mathbf{r}'$ is given by

$$\begin{aligned}R_2\left(\omega,\mathbf{r},\mathbf{r}'\right)=&\frac{1}{4\nu_0^2}\int d\mathbf{d}_1 d\mathbf{d}_2\ f\left(\mathbf{d}_1\right)f\left(\mathbf{d}_2\right)\\&\left\langle\left[\delta\left(E+\frac{\omega}{2}-E_+\right)+\delta\left(E+\frac{\omega}{2}-E_-\right)\right]\right.\\&\left.\left[\delta\left(E-\frac{\omega}{2}-E_+\right)+\delta\left(E-\frac{\omega}{2}-E_-\right)\right]\right\rangle_{\text{dis}}\end{aligned} \tag{3.2}$$

where $f(\mathbf{d})$ is the distribution function of the dipole moments, assumed independent of the on-site energy fluctuations; $\langle\ldots\rangle_{\text{dis}}$ is the averaging with respect to the realisations of disorder, which affects both the energies $E_{\mathbf{rd}}$ and $E_{\mathbf{r}'d'}$ and the hopping $T_{\mathbf{rdr}'d'}$ via the recombination elements $J_{\mathbf{rd}}$; we have also used that the density of dipole states ν_0 at each site may be considered constant close to the energy E under considerations, so long as $\omega\ll E,\nu_0^{-1}$.

To make further progress, we assume that the recombination elements $J_{\mathbf{rd}}$ and the on-site energies $E_{\mathbf{rd}}$ fluctuate independently. Introducing a variable τ, such that $2T_{\mathbf{rd},\mathbf{r'd'}}\tau=E_{\mathbf{rd}}-E_{\boldsymbol{r'd'}}$, and the distribution function $P(J_{\mathbf{rd}})$ of the recombination elements $J_{\mathbf{rd}}$, the two-site correlator (3.2) is reduced to

$$\begin{aligned}R_2\left(\omega,\mathbf{r},\mathbf{r}'\right)=&\frac{1}{2}\int dJ_{\mathbf{rd}}\ dJ_{\mathbf{r'd'}}\ P(J_{\mathbf{rd}})P(J_{\mathbf{r'd'}})\\&\int d\mathbf{d}_1 d\mathbf{d}_2\ f\left(\mathbf{d}_1\right)f\left(\mathbf{d}_2\right)\int d\tau\ \left|T_{\mathbf{rd},\mathbf{r'd'}}\right|\\&\delta\left(\omega-2\left|T_{\mathbf{rd},\mathbf{r'd'}}\right|\sqrt{\tau^2+1}\right).\end{aligned} \tag{3.3}$$

Correlation function on multiple sites. Eq. (3.3) describes dipole energy correlations in a system of two sites. In what immediately follows, we derive the energy-level correlation function for a system of $N\gg1$ sites.

In the absence of dipole tunnelling between sites ($T_{\mathbf{rd},\mathbf{r'd'}}=0$), it is given by

$$\begin{aligned}R_2^{\text{uncorr}}(\omega)=&\\\frac{\sum_{\mathbf{rd},\mathbf{r'd'}}\int dE_{\mathbf{rd}}dE_{\mathbf{r'd'}}\nu_0^2\delta(E-E_{\mathbf{rd}})\delta(E+\omega-E_{\mathbf{r'd'}})}{(\nu_0 N)^2}&\\=\frac{N(N-1)}{N^2}\overset{N\to\infty}{\longrightarrow}1&\end{aligned} \tag{3.4}$$

For finite tunnelling, dipole states on different sites get hybridised, and the correlation function $R_2(\omega)$ deviates from unity. In the limit of strong disorder, which we consider in this section, pairs of resonant sites are rare, only a small fraction of dipole states get hybridised due to dipole hopping and resonances of three or more sites may be neglected.

The correlation function in such a system of multiple sites may be found by hybridising dipole states on all pairs of sited with close energies and computing (see Appendix A for details) the modification of the correlation function similarly to Eq. (3.3). The full correlation function in a system of many sites with rare resonances is given by

$$R_2\left(\omega\right)=\frac{1}{V^2}\int dJ_{\mathbf{rd}}\ dJ_{\mathbf{r'd'}}\ P(J_{\mathbf{rd}})P(J_{\mathbf{r'd'}})\int d\mathbf{r}d\mathbf{r}'\int d\mathbf{d}\ d\mathbf{d}'\ f\left(\mathbf{d}\right)f\left(\mathbf{d}'\right)\ \Theta\left[1-\left(2T_{\mathbf{rd},\mathbf{r'd'}}/\omega\right)^2\right]/\left[1-\left(2T_{\mathbf{rd},\mathbf{r'd'}}/\omega\right)^2\right]^{\frac{1}{2}}, \tag{3.5}$$

where the hopping element $T_{\mathbf{rd},\mathbf{r'd'}}$ is given by Eq. (2.4) and $\Theta(\ldots)$ is the theta-function.

A rigorous evaluation of the correlation function $R_2(\omega)$, given by Eq. (3.5), requires making an assumption about the distribution $P(J_{\mathbf{rd}})$ of the electron-hole recombination matrix elements $J_{\mathbf{rd}}$. Since the exact form of the distribution will affect only numerical coefficient, we assume, for simplicity, that they are sharply peaked

near certain value J,

$$P(J_{\mathbf{rd}}) = \delta(J - J_{\mathbf{rd}}). \tag{3.6}$$

We will also assume a uniformly random orientation of the dipole moments in the d-dimensional space,

$$f(\mathbf{d}) = \delta\left(|\mathbf{d}| - d_0\right) / \left(\Omega_d d_0^{d-1}\right), \tag{3.7}$$

where $\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}$ is the area of a unit sphere in d dimensions.

We note that in general the recombination elements $J_{\mathbf{rd}}$ may have arbitrary signs, in contrast to our choice of their distribution (3.6). Indeed, in a system of electrons in a disordered system such elements are determined by overlap integrals of oscillating wavefunctions. However, such fluctuations of the sign do not affect qualitatively the correlation function $R_2(\omega)$ in the limit of strong disorder considered in this section.

By switching to the integration with respect to the centre-of-mass $\frac{\mathbf{r}+\mathbf{r}'}{2}$ and relative $\tilde{\mathbf{r}} = \mathbf{r} - \mathbf{r}'$ coordinates, Eq. (3.5) may be simplified as

$$R_2(\omega) = \frac{1}{V}\int d\mathbf{d}\; d\mathbf{d}'\; f(\mathbf{d})\, f(\mathbf{d}') \int\limits_{\tilde{r}>\tilde{r}_\omega} d\tilde{\boldsymbol{r}} \left[1 - \left(\frac{\tilde{r}_\omega(\mathbf{d}_1, \mathbf{d}_2, \tilde{\boldsymbol{r}}/\tilde{r})}{\tilde{r}}\right)^6\right]^{-\frac{1}{2}} \tag{3.8}$$

where we have introduced $\tilde{r}_\omega(\mathbf{d}_1, \mathbf{d}_2, \boldsymbol{n}) = \left[\frac{2J^2 d_0^2}{E^2}\left(\frac{|Q(\mathbf{d}_1,\mathbf{d}_2,\boldsymbol{n})|}{\omega}\right)\right]^{\frac{1}{3}}$, the characteristic distance at which the the dipole interaction energy is of order ω. The integral in Eq. (3.8) cannot be evaluated exactly for arbitrary parameters. Below we compute the asymptotic behaviour of the correlation function $R_2(\omega)$ in the limit of large $\omega \gg \omega_V$ and small $\omega \ll \omega_V$ frequencies, where

$$\omega_V = \frac{2J^2 d_0^2}{E^2 V^{\frac{3}{d}}} \tag{3.9}$$

is the characteristic interaction energy between dipoles on the length of the system $L \sim V^{-\frac{1}{d}}$.

1. Large-frequency limit

For $\omega \gg \omega_V$, the value of the integral (3.8) in dimensions $d < 3$ comes from distances $\tilde{r}$ of order $\tilde{r}_\omega$, which in this limit are significantly shorter than the characteristic system size $V^{\frac{1}{d}}$. The upper limit of integration with respect to $\tilde{r}$ in Eq. (3.8) may be extended to infinity, giving

$$R_2(\omega \gg \omega_V) \approx 1 - \frac{2\pi^{\frac{d+1}{2}}\Gamma\left(1-\frac{d}{6}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{1}{2}-\frac{d}{6}\right)d}\left\langle |Q|^{\frac{d}{3}}\right\rangle_{\mathbf{d}}\left(\frac{\omega_V}{\omega}\right)^{\frac{d}{3}} \tag{3.10}$$

where $\left\langle |Q|^{\frac{d}{3}}\right\rangle_{\mathbf{d}} = \int d\hat{\mathbf{d}}_1 d\hat{\mathbf{d}}_2 d\boldsymbol{n}\left|Q\left(\hat{\mathbf{d}}_1, \hat{\mathbf{d}}_2, \boldsymbol{n}\right)\right|^{\frac{d}{3}}$ is our convention for the function $|Q|^{\frac{d}{3}}$ averaged with respect to the directions $\hat{\mathbf{d}}_1$ and $\hat{\mathbf{d}}_2$ of the dipole moments $\mathbf{d}_1$ and $\mathbf{d}_2$ and of the vector $\mathbf{n}$, where Q describes the angular dependence of the dipolar interactions [cf. Eq. (2.3)].

Equation (3.10) is our main result for the energy-level correlation function in a generic strongly disordered system in d spatial dimensions in the limit of large frequencies energy differences ω. For $d = 2$ it gives

$$R_2^{d=2}(\omega \gg \omega_V) = 1 - 1.28\left(\frac{\omega_V}{\omega}\right)^{\frac{2}{3}}. \tag{3.11}$$

where we used $\left\langle |Q|^{\frac{2}{3}}\right\rangle \approx 0.95$ in two dimensions. Let us note that for $d = 3$ the coefficient before the last term in Eq. (3.10) vanishes, and, in order to obtain the frequency dependency of $R_2(\omega)$, it is necessary to evaluate a correction of higher order in $1/\omega$. Direct integration in Eq. (3.8) yields

$$R_2^{d=3}(\omega \gg \omega_V) = 1 - C_2\left(\frac{\omega_V}{\omega}\right)^2, \tag{3.12}$$

where C_2 is a coefficient of order unity which depends on the shape of the sample.

Eq. (3.10), which accurately describes the large-frequency behaviour of correlation function in dimensions $d \neq 3$, and Eq. (3.12), which applies for $d = 3$, may be combined into the interpolation formula

$$R_2(\omega) \approx 1 - (3-d)\, C_1\left(\frac{\omega_V}{\omega}\right)^{\frac{d}{3}} - C_2\left(\frac{\omega_V}{\omega}\right)^2. \tag{3.13}$$

We emphasise that in general in dimensions $d \neq 3$ the correlation function $R_2(\omega)$ contains contributions $\propto 1/\omega^\beta$ with $d/3 < \beta < 2$, which exceed the last term in Eq. (3.13). However, in these dimensions the leading-order large-frequency behaviour of the energy-level correlations is determined by the first term in the right-hand side of Eq. (3.13). In 3D, the leading order correlations are described by the last term of (3.13). Thus, Eq. (3.13) accurately describes the large-frequency asymptotics of the of the correlation function $R_2(\omega)$ in all dimensions.

2. Small-frequency limit

In the limit $\omega \ll \omega_V$ Eq. (3.8) may be rewritten as

$$R_2(\omega) = C_3\frac{\omega}{\omega_V} \tag{3.14}$$

where the coefficient C_3 is given by $C_3 = \frac{\pi}{V^{1+\frac{3}{d}}}\int d\tilde{\boldsymbol{r}}\,\tilde{r}^3 \int d\mathbf{d}_1 d\mathbf{d}_2 f(\mathbf{d}_1) f(\mathbf{d}_2)\,\delta(Q)$.

Equation (3.14) is our main result for the dipole energy-level correlation function in a strongly disordered electronic system in arbitrary dimensions in the limit of small frequencies. It demonstrates that the correlation

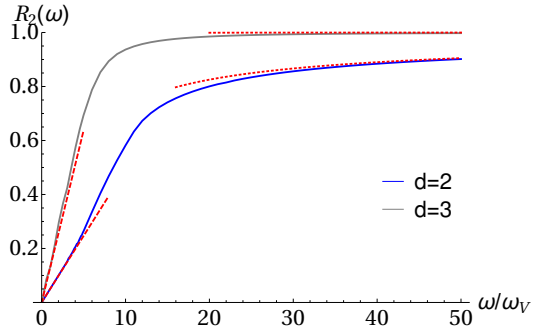

FIG. 1. (Colour online) The correlation function $R_2(\omega)$ of the dipole energy levels in an electronic system as a function of frequency ω [in units ω_V], obtained from a numerical integration of Eq. (3.8). The small- and large- frequency asymptotic behaviour given by Eqs. (3.11), (3.12) and 3.14 are shown in red.

function is linear in frequency ω for such strongly disordered systems, similarly to the case of a weakly disordered metal in the orthogonal symmetry class[17]. Such a linear dependency may also be expected from a phenomenological random-matrix-theory argument[25], based on considering a two-level system with a random Hamiltonian.

The whole frequency dependency of the correlation function $R_2(\omega)$ for dimensions $d = 2$ and $d = 3$, obtained from numerical integration of Eq. (3.8), is shown in Fig. 1. The dashed lines in Fig. (1) show the low-frequency behaviour described by the linear dependence (3.14); the dotted lines show the high-frequency asymptotics described by Eqs. (3.11) and (3.12).

IV. ARBITRARY DISORDER STRENGTH IN A 3D ELECTRONIC SYSTEM

In Sec. III we considered energy-level correlations in a system, which is either sufficiently strongly disordered or sufficiently small, and only sparse resonances between pairs of dipole sites are essential for correlations of energy levels, while higher-order resonances may be neglected. As discussed in Sec. III and as first pointed out in Refs. 28, 32, and 33, excitations with the hopping amplitude $\propto 1/r^3$ have infinitely many resonances in 3D systems at arbitrarily strong disorder, unlike systems in lower dimensions $d < 3$. Therefore, sufficiently large 3D electronic systems may host rather complicated dipole states, which involve resonances on multiple sites.

The correlation functions may then be still found by replacing the dipole states by effective hybridised states (which may effectively be many-dipole states) on the scale of the volume V of interest and investigating the hopping matrix elements for such states. The energies of the effective hybridised states may be assumed to have a uniform probability distribution due to the uniform energy distributions of dipoles on constituent sites and their independence of the hopping elements. Assuming that the hybridised states have dipole moments, one may expect the results of Sec. III for the frequency dependency of the correlator $R_2(\omega)$ to carry over directly to the case of weaker disorder, which allows for complicated hybridised states. Namely, one may expect that at low frequencies $R_2(\omega) \propto \omega$, while for $\omega \to \infty$ $R_2(\omega) = 1 - \frac{\text{const}}{\omega^2}$. We emphasise, however, that the dependencies on the system size (volume V) may be different from those found in Sec. III because the effective dipole moments of hybridised states of multiple sites in general depend on the volume V.

Three decades ago, a renormalisation procedure was developed in Ref. 28 for constructing hybridised dipole-like states in 3D by repeatedly hybridising pairs of dipole-like states with closest energies within a given distance while increasing the system size or the interaction radius. A more rigorous recent study[6] for a similar 2D problem with $\propto 1/r^2$ dipole hopping established the existence of fixed points in a system on the orthogonal symmetry class (which is also the focus of this paper) with critical wavefunctions of the dipole states.

In what immediately follows we construct a renormalisation procedure at sufficiently strong disorder, similar to that of Ref. 28, to explore qualitatively the correlations of complex multi-site excitations with effective renormalised dipole moments and recombination matrix elements $J_{\mathbf{rd}} \equiv \langle 0|\hat{\mathcal{H}}_0|\mathbf{rd}\rangle$ (the matrix elements of the Hamiltonian between the excitation state and the ground state of a non-interacting system).

When two dipole states ($\mathbf{rd}$) and ($\mathbf{r'd'}$) are hybridised, they are being replaced by two other states with the annihilation operators $\hat{b}_{\mathbf{r}_+\mathbf{d}_+}$ and $\hat{b}_{\mathbf{r}_-\mathbf{d}_-}$

$$\begin{pmatrix} \hat{b}_{\mathbf{r}_+\mathbf{d}_+} \\ \hat{b}_{\mathbf{r}_-\mathbf{d}_-} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{b}_{\mathbf{rd}} \\ \hat{b}_{\mathbf{r'd'}} \end{pmatrix}, \tag{4.1}$$

where $\cot(2\theta) \equiv \tau = (E_{\mathbf{rd}} - E_{\mathbf{r'd'}})/2T_{\mathbf{rd},\mathbf{r'd'}}$. The recombination matrix elements of the hybridised states and the elements of hopping to remote sites $\tilde{\mathbf{r}}\tilde{\mathbf{d}}$ with $|\tilde{\mathbf{r}} - \mathbf{r}|, |\tilde{\mathbf{r}} - \mathbf{r'}| \gg |\mathbf{r} - \mathbf{r'}|$ are given by

$$(J_{\mathbf{r}_+\mathbf{d}_+}, J_{\mathbf{r}_-\mathbf{d}_-})^T = U(J_{\mathbf{rd}}, J_{\mathbf{r'd'}})^T, \tag{4.2}$$

$$(T_{\tilde{\mathbf{r}}\tilde{\mathbf{d}},\mathbf{r}_+\mathbf{d}_+}, T_{\tilde{\mathbf{r}}\tilde{\mathbf{d}},\mathbf{r}_-\mathbf{d}_-})^T = U(T_{\tilde{\mathbf{r}}\tilde{\mathbf{d}},\mathbf{rd}}, T_{\tilde{\mathbf{r}}\tilde{\mathbf{d}},\mathbf{r'd'}})^T. \tag{4.3}$$

Because the hopping of dipole excitations depends on the product $J_{\mathbf{rd}} \equiv \mathbf{p}_{\mathbf{rd}}\mathbf{d}$ of its recombination element $J_{\mathbf{rd}}$ and the dipole moment $\mathbf{d}$, it is convenient to introduce a new variable

$$\mathbf{p}_{\mathbf{rd}} \equiv J_{\mathbf{rd}}\mathbf{d} \tag{4.4}$$

and describe the evolution of its distribution function $F(\mathbf{p}_{\mathbf{rd}})$ when repeatedly hybridising dipole states. Assuming that the initial distributions of the dipole moments $\mathbf{d}$ and the recombination elements are isotropic, the distribution $F(\mathbf{p}_{\mathbf{rd}})$ also remains isotropic under renormalisation and depends only on the absolute value of $\mathbf{p}_{\mathbf{rd}}$.

The renormalisation procedure involves repeated hybridisation of pairs of dipole states with close energies. When increasing the system size L (or the interaction radius), new states are formed out of previously hybridised states. In the spirit of Ref. 28, we neglect resonances of three or more sites, which is justified in the limit of sufficiently strong disorder under consideration, with $\nu_0\langle \mathbf{d}^2\rangle J^2 E_{\mathbf{rd}}^{-2} n \ll 1$, where $n = N/V$ is the density of the dipoles. Introducing variable $\ell = \log L$, where L is the system size (or the interaction cutoff radius), we arrive at the RG flow equation for the distribution function $F(\mathbf{p_{rd}})$:

$$\frac{\partial F\left(\mathbf{p_{rd}}\right)}{\partial \ell} = \frac{n\nu_0}{E^2}\int d\mathbf{n}\int d\mathbf{p}_{\mathbf{r}_1\mathbf{d}_1} d\mathbf{p}_{\mathbf{r}_2\mathbf{d}_2} F\left(\mathbf{p}_{\mathbf{r}_1\mathbf{d}_1}\right) F\left(\mathbf{p}_{\mathbf{r}_2\mathbf{d}_2}\right) \int d\tau \left|Q\left(\mathbf{p}_{\mathbf{r}_1\mathbf{d}_1}, \mathbf{p}_{\mathbf{r}_2\mathbf{d}_2}, \mathbf{n}\right)\right| \left[\delta\left(\mathbf{p}-\mathbf{p}_+\right) + \delta\left(\mathbf{p}-\mathbf{p}_-\right) - \delta\left(\mathbf{p}-\mathbf{p}_{\mathbf{r}_1\mathbf{d}_1}\right) - \delta\left(\mathbf{p}-\mathbf{p}_{\mathbf{r}_2\mathbf{d}_2}\right)\right]. \tag{4.5}$$

Similarly, when increasing the size of the system (or the interaction cutoff radius), the energy-level correlation function $R_2(\omega)$ gets renormalised according to the equation (see Appendix for details)

$$\frac{\partial R_2(\omega,\ell)}{\partial \ell} = \frac{1}{V}\int d\mathbf{n}\int d\mathbf{p}_{\mathbf{r}_1\mathbf{d}_1}\, d\mathbf{p}_{\mathbf{r}_2\mathbf{d}_2} F\left(\mathbf{p}_{\mathbf{r}_1\mathbf{d}_1}\right) F\left(\mathbf{p}_{\mathbf{r}_2\mathbf{d}_2}\right) \int d\tau \left|Q\left(\mathbf{p}_{\mathbf{r}_1\mathbf{d}_1}, \mathbf{p}_{\mathbf{r}_2\mathbf{d}_2}, \mathbf{n}\right)\right| / E^2 \left[\delta\left(\omega - 2\left|T_{\mathbf{r}_1\mathbf{d}_1,\mathbf{r}_2\mathbf{d}_2}\right|\sqrt{\tau^2+1}\right) - 2\delta\left(\omega - 2T_{\mathbf{r}_1\mathbf{d}_1,\mathbf{r}_2\mathbf{d}_2}\tau\right)\right]. \tag{4.6}$$

The solutions of the RG Eqs. (4.5) and (4.6) describe the distributions of the parameters of dipole states and energy-level correlation functions in a 3D system of arbitrary size.

Level correlations at intermediate lengths. Because the hybridised states are very close in structure to large-size dipoles, it is possible to apply immediately the results of Sec. III to their energy-level corrections, which gives

$$R_2(\omega) = \begin{cases} \tilde{C}_3(V)\omega, & \omega \to 0 \\ 1 - \tilde{C}_2(V)/\omega^2, & \omega \to \infty. \end{cases} \tag{4.7}$$

We note, however, that the dependencies of the coefficients $\tilde{C}_3(V)$ and $\tilde{C}_2(V)$ on the system of volume V are in general different from the dependencies $\tilde{C}_3(V) \propto V$ and $\tilde{C}_2(V) \propto 1/V^2$ in the case of strong disorder [cf. Eqs. (3.14) and (3.12)], because the parameters of the hybridised composite dipoles depend on the system size. These dependencies will depend on the details of the initial dipole distributions.

Fixed point. The procedure of the hybridisation described above leaves invariant the quantities $J^2_{\mathbf{r}_1\mathbf{d}_1} + J^2_{\mathbf{r}_2\mathbf{d}_2}$ and $\mathbf{p}^2_{\mathbf{r}_1\mathbf{d}_1} + \mathbf{p}^2_{\mathbf{r}_2\mathbf{d}_2}$ for pairs of states. As a result, the quantity $\mathbf{p_{rd}}$ and the dipole states remains bounded for the typical renormalised composite dipole states. It is natural to assume then that the distributions of the parameters $p_{\mathbf{rd}}$ of the renormalised dipole states approach a fixed point $F^*(\mathbf{p_{rd}})$. Such type of a fixed point has been obtain, under certain approximations, in Ref. 28 for a similar model for dipoles with the power-law hopping $\propto 1/r^3$ in 3D. Assuming a similar fixed point exists here, the typical value of the parameter $\mathbf{p_{rd}}$ saturates to a constant value in sufficiently large sample, and the results of Sec. (III) for both the energy and size dependencies of the energy-level correlator $R_2(\omega)$ may be carried over directly. Thus, the energy-level statistics in very large 3D systems are given by Eq. (4.7) with $\tilde{C}_3(V) \propto V$ and $\tilde{C}_2(V) \propto 1/V^2$.

V. GENERIC POWER-LAW HOPPING

Excitations with power-law hopping are often simulated by means of ultracold particles in magnetic or optical traps. Recently, excitations with power-law hopping $\propto 1/r^\alpha$ with tunable $\alpha = 0\ldots3$ have been realised in 1D[10–13] and 2D[14] arrays of trapped ultracold ions. Excitations with power-law hopping also exist in systems of Rydberg atoms[34] ($\alpha = 6$ or $\alpha = 3$) and polar molecules[35,36] ($\alpha = 3$).

In what follows, we compute energy-level statistics in a strongly disordered system with a generic power-law hopping described by the Hamiltonian

$$\mathcal{H} = \sum_{\mathbf{r},\lambda} E_{\mathbf{r}\lambda'} \hat{b}^\dagger_{\mathbf{r}\lambda}\hat{b}_{\mathbf{r}\lambda'} - \sum_{\mathbf{r},\mathbf{r}',\lambda,\lambda'} T_{\mathbf{r}\lambda,\mathbf{r}'\lambda'} \hat{b}^\dagger_{\mathbf{r}\lambda}\hat{b}_{\mathbf{r}'\lambda'} \tag{5.1}$$

where the operators $\hat{b}_{\mathbf{r}\lambda}$ and $\hat{b}^\dagger_{\mathbf{r}\lambda}$ annihilate and create excitations at location $\mathbf{r}$; λ labels discrete degrees of freedom of excitations at a given location, e.g. the spatial orientation of the excitations; and we have also introduced the hopping element

$$T_{\mathbf{r}\lambda,\mathbf{r}'\lambda'} = \frac{2Q_\alpha(\lambda,\lambda')}{|\mathbf{r}-\mathbf{r}'|^\alpha}. \tag{5.2}$$

with the kernel $Q_\alpha(\lambda,\lambda')$ which is independent of the distance but depends on the excitation states λ.

Correlations at strong disorder. In what follows, we compute the energy-level correlation function in the case

of very strong disorder, when correlations come from rare resonances on pairs of sites. Following the same steps as when deriving Eq. (3.8), we arrive at

$$R_2(\omega) = \frac{1}{V}\int d\lambda \int d\lambda' \; P(\lambda) P(\lambda') \int\limits_{\tilde{r}>\tilde{r}_\omega} d\tilde{\boldsymbol{r}} \left[1 - \left(\frac{2Q_\alpha(\lambda,\lambda',\tilde{\boldsymbol{r}}/\tilde{r})}{\omega \tilde{r}^\alpha}\right)^2\right]^{-\frac{1}{2}} \quad (5.3)$$

where $\tilde{r}_\omega = (2Q_\alpha/\omega)^{\frac{1}{\alpha}}$ and $P(\lambda)$ is the probability distribution of the excitation states λ.

The typical splitting between neighbouring energy levels is given by the characteristic matrix element of quasiparticle hopping

$$\omega_V = 2V^{-\frac{\alpha}{d}} \langle |Q(\lambda,\lambda')| \rangle_{\lambda,\lambda'} \quad (5.4)$$

where $\langle \cdots \rangle_\lambda = \int d\lambda P(\lambda) \cdots$. For sufficiently smooth probability distributions of the function $Q(\lambda,\lambda')$ and its moments $Q(\lambda,\lambda')^\beta$, the coefficients

$$\mathcal{A}_\beta = \frac{2^\beta \left\langle |Q(\lambda,\lambda')|^\beta \right\rangle_{\lambda,\lambda'}}{V^{\frac{\alpha\beta}{d}} \omega_V^\beta} \quad (5.5)$$

are of order unity, and ω_V is the only energy scale in the problem. In the limit $\omega \gg \omega_V$ the correlation function is given by

$$R_2(\omega \gg \omega_V) \approx 1 - \frac{2\pi^{\frac{d+1}{2}} \Gamma\left(1-\frac{d}{2\alpha}\right) \mathcal{A}_{\frac{d}{\alpha}}}{\Gamma\left(\frac{d}{2}\right) \Gamma\left(\frac{1}{2}-\frac{d}{2\alpha}\right) d} \left(\frac{\omega_V}{\omega}\right)^{\frac{d}{\alpha}}. \quad (5.6)$$

Due to the divergence of the Gamma function $\Gamma\left(\frac{1}{2}-\frac{d}{2\alpha}\right) \sim \frac{2\alpha}{\alpha-d}$ when the dimension d approaches α, the coefficient in the last term in Eq. (5.6) vanishes for $d = \alpha$. Therefore, the energy-level correlations in the dimensions $d = \alpha$ at $\omega = \omega_V$ are described by the next-leading term in $1/\omega$:

$$R_2^{d=\alpha}(\omega \gg \omega_V) \approx 1 - A_2 \left(\frac{\omega_V}{\omega}\right)^2, \quad (5.7)$$

where A_2 is a coefficient of order unity which depends on the shape of the system. Equations (5.6) and (5.7) may be combined into one interpolation formula

$$R_2(\omega) \approx 1 - (\alpha - d) A_1 \left(\frac{\omega_V}{\omega}\right)^{\frac{d}{\alpha}} - A_2 \left(\frac{\omega_V}{\omega}\right)^2. \quad (5.8)$$

Small-frequency limit. In the limit $\omega \ll \omega_V$ we obtain, similarly to Eq. (3.14),

$$R_2(\omega) = A_3 \frac{\omega}{\omega_V}, \quad (5.9)$$

where the dimensionless coefficient $A_3 = \frac{\pi \left\langle |Q(\lambda,\lambda')| \right\rangle_{\lambda,\lambda'}}{V^{1+\frac{\alpha}{d}}} \int d\tilde{\boldsymbol{r}} \tilde{r}^\alpha \int d\lambda d\lambda' P(\lambda) P(\lambda') \delta(Q_\alpha)$ depends on the shape of the system.

Behaviour in high and low dimensions. Equations (5.8) and (5.9) describe energy-level correlations accurately in low dimensions $d < \alpha$ or in sufficiently small systems in higher dimensions $d \geq \alpha$. In that case, the correlations come from rare resonances between excitation states on pairs of sites. In higher dimensions, $d \geq \alpha$, the number of resonances is infinite in the limit of an infinite system, which may lead to a strong renormalisation of the excitation states. Based on the arguments similar to those of Sec. IV, we expect that in higher-dimensional systems the frequency dependency of the correlation function is still given by Eqs. (5.8) and (5.9), however, the volume dependence is in general different.

VI. CONCLUSION AND OUTLOOK

Motivated by neutral excitations in disordered electronic systems and trapped ultracold particles with power-law interactions, we have computed the energy-level correlations functions for particles with the power-law hopping $\propto r^\alpha$. Our main results for the correlation functions for systems in various dimensions and various energy intervals are summarised by Eqs. (3.10), (3.12), (3.14), (5.6), (5.7) and (5.9). In a disordered electronic systems the correlation function may be observed as a correlator of ac conductances $R_2(\omega) \propto \langle G(\Omega) G(\Omega+\omega) \rangle_\Omega$, where $\langle \ldots \rangle_\Omega$ is the averaging with respect to frequency Ω in a sufficiently large energy window.

At small energy differences, the energy-level correlations displays Wigner-Dyson statistics[18] which hints at the possibility of chaotic dynamics of the excitations involved. This chaotic behaviour could be identified, for example, via out-of-time-order correlators[30] of, e.g., local voltages or charges in a system with excitations which allow for power-law hopping. We do not present such analysis here and leave them for future studies. Also, although we do not expect many-body processes to change the form of the correlation function in very large samples, the role of such processes also deserves a separate investigation.

Another question, which deserves a study, is the relation between the disorder strength and the level statistics in systems with the power-law hopping $\propto 1/r^\alpha$ for sufficiently small α. Indeed, those system support excitations with the dispersion $\varepsilon_{\mathbf{k}} \propto k^{\alpha-d}$. In dimensions $d > \frac{3\alpha}{2}$ they display a plenty of unconventional disorder-driven phenomena, such as disorder-driven transitions or sharp crossovers in non-Anderson universality classes, unconventional Lifshitz tails, etc., (see Ref. 16 for a review) and possibly transitions in the energy-level statics. While we have obtained the strong-disorder asymptotics of the respective level statistics in this paper, we leave further studies of the possibility of such transitions for future work.

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Appendix A: Change to the level correlation function when adding hopping between two sites

In this section we derive the modification of the correlation function $R_2(\omega)$, defined by Eq. 1.1, when two dipole states hybridise. We consider a system of N sites, where two sites $(\mathbf{r}_1, \mathbf{d}_1)$ and $(\mathbf{r}_2, \mathbf{d}_2)$ are initially isolated from each other and from the rest of the system and compute the change of $R_2(\omega)$ when adding hopping $T_{\mathbf{r}_1\mathbf{d}_1,\mathbf{r}_2\mathbf{d}_2}$ between the two sites. The model under consideration applies to the case of very strong disorder, when resonant pairs of sites are rare, as well as to dipole hybridisation during one step of the strong-disorder RG, when composite dipole states may be considered, and the hybridisation between two dipole states being merged and the rest of the system may be neglected.

The modified density of dipole states after adding the hopping

$$\tilde{\nu}(E) = \left(\delta(E - E_+) + \delta(E - E_-) + \sum_\lambda \delta(E - E_\lambda) \right) \tag{A1}$$

consists of the contribution of the hybridised-states' energies E_+ and E_- and that of the rest of the system [the sum in Eq. (A1)]. Because the hopping is small, and the hopping and the on-site dipole energies fluctuate independently, the average density of states is unaltered by the hybridisation $\langle\tilde{\nu}\rangle_{\rm dis} = \langle\nu\rangle_{\rm dis} = N\nu_0$. The change of to the disorder-averaged correlation function $R_2(\omega)$, defined by Eq. (1.1), is given by

$$\begin{aligned}\delta R_2(\omega) = \quad & \frac{1}{N^2\nu_0^2} \int dJ_{\mathbf{r}_1\mathbf{d}_1} P(J_{\mathbf{r}_1\mathbf{d}_1}) \int dJ_{\mathbf{r}_2\mathbf{d}_2} P(J_{\mathbf{r}_2\mathbf{d}_2}) \int d\mathbf{d}_1 \; d\mathbf{d}_2 \; f(\mathbf{d}_1) f(\mathbf{d}_2) \int \nu_0 dE_{\mathbf{r}_1\mathbf{d}_1} \nu_0 dE_{\mathbf{r}_2\mathbf{d}_2} \\ & \left\langle \tilde{\nu}\left(E + \frac{\omega}{2}\right) \tilde{\nu}\left(E - \frac{\omega}{2}\right) - \nu\left(E + \frac{\omega}{2}\right) \nu\left(E - \frac{\omega}{2}\right) \right\rangle_\lambda , \end{aligned} \tag{A2}$$

where $\langle \ldots \rangle_\lambda$ is the averaging with respect to the disorder in the rest of the system, independent of the parameters on sites $(\mathbf{r}_1, \mathbf{d}_1)$ and $(\mathbf{r}_2, \mathbf{d}_2)$.

We note that, according to Eq. (A1), products of the modified densities of states $\tilde{\nu}\left(E + \frac{\omega}{2}\right) \tilde{\nu}\left(E - \frac{\omega}{2}\right)$ in Eq. (A2) contain three types of terms involving products of δ-functions, (i) $\delta(E - E_\pm + \omega/2)\delta(E - E_\mp - \omega/2)$, (ii) $\delta(E - E_\pm + \omega/2)\delta(E - E_\lambda - \omega/2)$ and (iii) $\delta(E - E_\lambda + \omega/2)\delta(E - E_{\lambda'} - \omega/2)$. Terms (iii) are cancelled by equivalent contributions from $\nu\left(E + \frac{\omega}{2}\right) \nu\left(E - \frac{\omega}{2}\right)$. Contributions of type (ii) also vanish, due to the identity

$$\int dE_{\mathbf{r}_1\mathbf{d}_1} \, dE_{\mathbf{r}_2\mathbf{d}_2} \left[\delta(E - E_+) + \delta(E - E_-) - \delta(E - E_{\mathbf{r}_1\mathbf{d}_1}) - \delta(E - E_{\mathbf{r}_2\mathbf{d}_2}) \right] = 0. \tag{A3}$$

Equation (A2) may therefore be simplified to include only the averaging with respect to the dipole parameters on sites $(\mathbf{r}_1, \mathbf{d}_1)$ and $(\mathbf{r}_2, \mathbf{d}_2)$:

$$\begin{aligned}\delta R_2(\omega) = \frac{1}{N^2} & \int dJ_{\mathbf{r}_1\mathbf{d}_1} P(J_{\mathbf{r}_1\mathbf{d}_1}) \int dJ_{\mathbf{r}_2\mathbf{d}_2} P(J_{\mathbf{r}_2\mathbf{d}_2}) \int d\mathbf{d}_1 \; d\mathbf{d}_2 \; f(\mathbf{d}_1) f(\mathbf{d}_2) \int dE_{\mathbf{r}_1\mathbf{d}_1} \int dE_{\mathbf{r}_2\mathbf{d}_2} \\ & \left\{ \left[\delta\left(E + \frac{\omega}{2} - E_+\right) + \delta\left(E + \frac{\omega}{2} - E_-\right) \right] \left[\delta\left(E - \frac{\omega}{2} - E_+\right) + \delta\left(E - \frac{\omega}{2} - E_-\right) \right] - \right. \\ & \left. \left[\delta\left(E + \frac{\omega}{2} - E_{\mathbf{r}_1\mathbf{d}_1}\right) + \delta\left(E + \frac{\omega}{2} - E_{\mathbf{r}_2\mathbf{d}_2}\right) \right] \left[\delta\left(E - \frac{\omega}{2} - E_{\mathbf{r}_1\mathbf{d}_1}\right) + \delta\left(E - \frac{\omega}{2} - E_{\mathbf{r}_2\mathbf{d}_2}\right) \right] \right\}. \end{aligned} \tag{A4}$$

Changing variables to $\tau = \frac{1}{2T_{\boldsymbol{r},\boldsymbol{d},\boldsymbol{r'd'}}}\left(E_{\boldsymbol{rd}} - E_{\boldsymbol{r'd'}}\right)$ and integrating out $\frac{1}{2}(E_{\boldsymbol{rd}} + E_{\boldsymbol{r'd'}})$ gives

$$\delta R_2(\omega) = \frac{2}{N^2}\int dJ_{\mathbf{r}_1\mathbf{d}_1} P(J_{\mathbf{r}_1\mathbf{d}_1}) \int dJ_{\mathbf{r}_2\mathbf{d}_2} P(J_{\mathbf{r}_2\mathbf{d}_2}) \int d\mathbf{d}\; d\mathbf{d}_2 f\left(\mathbf{d}_1\right) f\left(\mathbf{d}_2\right) \times$$
$$\times \int d\tau \left|T_{\mathbf{r}_1\mathbf{d}_1,\mathbf{r}_2\mathbf{d}_2}\right| \left[\delta\left(\omega - 2\left|T_{\mathbf{r}_1\mathbf{d}_1,\mathbf{r}_2\mathbf{d}_2}\right|\sqrt{\tau^2+1}\right) - 2\delta\left(\omega - 2\tau T_{\mathbf{r}_1\mathbf{d}_1,\mathbf{r}_2\mathbf{d}_2}\right)\right]. \tag{A5}$$

So far, we have obtained the expression for the correction to the level correlation, following the hybridization of a single pair of dipoles. It is straightforward to obtain the expression for the level correlation given in Eq. 3.5 which takes into account all resonances in a given volume V by integrating over $\frac{1}{2}\frac{N^2}{V^2} d\mathbf{r}_1 d\mathbf{r}_2$ and then integrating out the variable τ.

Application to the strong-disorder renormalisation procedure

The modification of the correlation function $R_2(\omega)$ when hybridising dipole states may be considered as a step of an RG procedure, discussed in Sec. IV and applied in Refs. 28, 32, and 33 (see also Ref. 37). During this procedure, pairs of resonant dipole states are being repeatedly hybridised while increasing the hopping distance $r = |\boldsymbol{r_1} - \boldsymbol{r_2}| = e^{\ell}$ or the system size.

As discussed in Section. IV, the hopping of dipoles depends only on the product $\mathbf{p_{rd}} = J_{\mathbf{rd}}\mathbf{d}$, which is why it is convenient to introduce the distribution function $F(\mathbf{p_{rd}})$ of variable $\mathbf{p_{rd}}$, which flows under the renormalisation procedure. We can now obtain the modification to $R_2(\omega)$ as a result of hybridizing dipoles in the volume element. The number of dipole pairs separated by vectors $\mathbf{r}$ in an infinitesimal element of space, confined by the radii r and $r + dr$ and the spatial angle $d\Omega$, is given by $\frac{1}{2}r^2 dr \int d\Omega \int \frac{N}{V} d\mathbf{r}_1 \frac{N}{V} d\mathbf{r}_2 \delta\left[\mathbf{r} - (\boldsymbol{r_1} - \boldsymbol{r_2})\right]$. Utilizing the expression in Eq. (A5) for hybridization of a single pair of dipoles and multiplying by the number of dipoles in the volume element, we obtain the modification of the energy-level correlation function

$$\delta R_2(\omega,\ell) = r^2 dr \int d\Omega \int \frac{1}{2}\frac{N}{V} d\mathbf{r}_1 \frac{N}{V} d\mathbf{r}_2 \delta\left[\mathbf{r} - (\boldsymbol{r_1} - \boldsymbol{r_2})\right] \cdot \frac{1}{N^2} \int d\mathbf{p}_{\mathbf{r}_1\mathbf{d}_1}\; d\mathbf{p}_{\mathbf{r}_2\mathbf{d}_2} F\left(\mathbf{p}_{\mathbf{r}_1\mathbf{d}_1}\right) F\left(\mathbf{p}_{\mathbf{r}_2\mathbf{d}_2}\right)$$
$$2\int d\tau \left|T_{\mathbf{r}_1\mathbf{d}_1,\mathbf{r}_2\mathbf{d}_2}\right| \left[\delta\left(\omega - 2\left|T_{\mathbf{r}_1\mathbf{d}_1,\mathbf{r}_2\mathbf{d}_2}\right|\sqrt{\tau^2+1}\right) - 2\delta\left(\omega - 2\tau T_{\mathbf{r}_1\mathbf{d}_1,\mathbf{r}_2\mathbf{d}_2}\right)\right] \tag{A6}$$

where the factor of $\frac{1}{2}$ in the right-hand side prevents double counting of dipoles.

To make further progress, we note that the contribution to Eq. (A6), which comes from the second $\delta-$function, may be simplified as

$$\int \frac{d\mathbf{r}}{2} \int \frac{d\mathbf{r}_1\; d\mathbf{r}_2}{V^2} \delta\left[\mathbf{r} - (\boldsymbol{r_1} - \boldsymbol{r_2})\right] \int d\mathbf{p}_{\mathbf{r}_1\mathbf{d}_1}\; d\mathbf{p}_{\mathbf{r}_2\mathbf{d}_2} F\left(\mathbf{p}_{\mathbf{r}_1\mathbf{d}_1}\right) F\left(\mathbf{p}_{\mathbf{r}_2\mathbf{d}_2}\right) \int d\tau \cdot 2 |T_{\mathbf{r}_1\mathbf{d}_1,\mathbf{r}_2\mathbf{d}_2}| \left[-2\delta\left(\omega - 2\tau T_{\mathbf{r}_1\mathbf{d}_1,\mathbf{r}_2\mathbf{d}_2}\right)\right] = -1. \tag{A7}$$

The correlation function $R_2(\omega)$ is obtained by integrating Eq. (A6) with respect to ℓ from $\ell = 0$ to $\ell = \log L$, where L is the size of the system or the interaction cutoff radius. Performing also integration with respect to $(\boldsymbol{r_1} - \boldsymbol{r_2})$ and $\frac{1}{2}(\boldsymbol{r_1} + \boldsymbol{r_2})$ over the volume V, we arrive at

$$\int_{\ell=0}^{\ell=\ln L} \frac{\partial R_2(\omega,\ell)}{\partial \ell} d\ell = -1 + \frac{1}{V}\int d^3 r \int d\mathbf{p}_{\mathbf{r}_1\mathbf{d}_1}\; d\mathbf{p}_{\mathbf{r}_2\mathbf{d}_2} F\left(\mathbf{p}_{\mathbf{r}_1\mathbf{d}_1}\right) F\left(\mathbf{p}_{\mathbf{r}_2\mathbf{d}_2}\right) \int d\tau \left|T_{\mathbf{r}_1\mathbf{d}_1,\mathbf{r}_2\mathbf{d}_2}\right| \delta\left(\omega - 2\left|T_{\mathbf{r}_1\mathbf{d}_1,\mathbf{r}_2\mathbf{d}_2}\right|\sqrt{\tau^2+1}\right). \tag{A8}$$

Equation (A8) together with the initial condition $R_2(\omega, \ell = 0) = 1$ gives

$$R_2(\omega, \ln L) = \frac{1}{V}\int_0^{\ln L} d\ell \int d\Omega \int d\mathbf{p}_{\mathbf{r}_1\mathbf{d}_1}\; d\mathbf{p}_{\mathbf{r}_2\mathbf{d}_2} F\left(\mathbf{p}_{\mathbf{r}_1\mathbf{d}_1}\right) F\left(\mathbf{p}_{\mathbf{r}_2\mathbf{d}_2}\right) \int d\tau \frac{\left|Q\left(\mathbf{p}_{\mathbf{r}_1\mathbf{d}_1}, \mathbf{p}_{\mathbf{r}_2\mathbf{d}_2}\right)\right|}{E^2} \delta\left(\omega - 2\left|T_{\mathbf{r}_1\mathbf{d}_1,\mathbf{r}_2\mathbf{d}_2}\right|\sqrt{\tau^2+1}\right). \tag{A9}$$

Equation (A9) may also be rewritten in the form of the RG flow equation

$$\frac{\partial R_2(\omega,\ell)}{\partial \ell} = \frac{1}{V}\int d\Omega \int d\mathbf{p}_{\mathbf{r}_1\mathbf{d}_1}\; d\mathbf{p}_{\mathbf{r}_2\mathbf{d}_2} F\left(\mathbf{p}_{\mathbf{r}_1\mathbf{d}_1}\right) F\left(\mathbf{p}_{\mathbf{r}_2\mathbf{d}_2}\right) \int d\tau \frac{\left|Q\left(\mathbf{p}_{\mathbf{r}_1\mathbf{d}_1}, \mathbf{p}_{\mathbf{r}_2\mathbf{d}_2}, \Omega\right)\right|}{E^2}$$
$$\left[\delta\left(\omega - 2\left|T_{\mathbf{r}_1\mathbf{d}_1,\mathbf{r}_2\mathbf{d}_2}\right|\sqrt{\tau^2+1}\right) - 2\delta\left(\omega - 2T_{\mathbf{r}_1\mathbf{d}_1,\mathbf{r}_2\mathbf{d}_2}\tau\right)\right]. \tag{A10}$$

Appendix B: Change to the distribution function of dipoles when hybridising two sites

In this section, we derive the RG flow equation for the distribution function $F(\mathbf{p_{rd}})$ of the dipole parameter $\mathbf{p_{rd}} = J_{\mathbf{rd}}\mathbf{d}$, the product of the dipole moment $\mathbf{d}$ and the recombination matrix element $J_{\mathbf{rd}}$, discussed in Sec. IV. When two dipoles with parameters $\mathbf{p}_{\mathbf{r}_1\mathbf{d}_1}$ and $\mathbf{p}_{\mathbf{r}_2\mathbf{d}_2}$ are hybridised, they get replaced by two other dipole states with parameters $\mathbf{p}_+$ and $\mathbf{p}_-$, and the distribution function gets modified according to

$$\delta F\left(\mathbf{p_{rd}}\right) = \int d\mathbf{p}_{\mathbf{r}_1\mathbf{d}_1} d\mathbf{p}_{\mathbf{r}_2\mathbf{d}_2} F\left(\mathbf{p}_{\mathbf{r}_1\mathbf{d}_1}\right) F\left(\mathbf{p}_{\mathbf{r}_2\mathbf{d}_2}\right) \int \nu_0 dE_{\mathbf{r}_1\mathbf{d}_1} \nu_0 dE_{\mathbf{r}_2\mathbf{d}_2} \left[\delta\left(\mathbf{p}-\mathbf{p}_+\right) + \delta\left(\mathbf{p}-\mathbf{p}_-\right) - \delta\left(\mathbf{p}-\mathbf{p}_{\mathbf{r}_1\mathbf{d}_1}\right) - \delta\left(\mathbf{p}-\mathbf{p}_{\mathbf{r}_2\mathbf{d}_2}\right)\right] \quad \text{(B1)}$$

Changing variables to $\tau = \left(E_{\mathbf{r}_1\mathbf{d}_1} - E_{\mathbf{r}_2\mathbf{d}_2}\right)/\left(2T_{\mathbf{r}_1\mathbf{d}_1,\mathbf{r}_2\mathbf{d}_2}\right)$, and considering the effects of all resonances in a spherical shell of radius $r \to r + dr$, with $r = |\mathbf{r}_1 - \mathbf{r}_2|$, gives

$$\delta F\left(\mathbf{p_{rd}}\right) = \frac{N}{V} r^2 dr \int d\Omega \int d\mathbf{p}_{\mathbf{r}_1\mathbf{d}_1} d\mathbf{p}_{\mathbf{r}_2\mathbf{d}_2} F\left(\mathbf{p}_{\mathbf{r}_1\mathbf{d}_1}\right) F\left(\mathbf{p}_{\mathbf{r}_2\mathbf{d}_2}\right) \int \nu_0 \left|T_{\mathbf{r}_1\mathbf{d}_1,\mathbf{r}_2\mathbf{d}_2}\right| d\tau \left[\delta\left(\mathbf{p}-\mathbf{p}_+\right) + \delta\left(\mathbf{p}-\mathbf{p}_-\right) - \delta\left(\mathbf{p}-\mathbf{p}_{\mathbf{r}_1\mathbf{d}_1}\right) - \delta\left(\mathbf{p}-\mathbf{p}_{\mathbf{r}_2\mathbf{d}_2}\right)\right]. \quad \text{(B2)}$$

Introducing the RG parameter $\ell = \log r$ and using that $T_{\mathbf{r}_1\mathbf{d}_1,\mathbf{r}_2\mathbf{d}_2} = Q(\mathbf{p}_{\mathbf{r}_1\mathbf{d}_1}, \mathbf{p}_{\mathbf{r}_2\mathbf{d}_2})/\left(E^2 r^3\right)$, we obtain the RG flow equation for the distribution function $F\left(\mathbf{p_{rd}}\right)$:

$$\frac{\partial F\left(\mathbf{p_{rd}}\right)}{\partial \ell} = \frac{n\nu_0}{E^2} \int d\Omega \int d\mathbf{p}_{\mathbf{r}_1\mathbf{d}_1} d\mathbf{p}_{\mathbf{r}_2\mathbf{d}_2} F\left(\mathbf{p}_{\mathbf{r}_1\mathbf{d}_1}\right) F\left(\mathbf{p}_{\mathbf{r}_2\mathbf{d}_2}\right) \int d\tau \left|Q\left(\mathbf{p}_{\mathbf{r}_1\mathbf{d}_1}, \mathbf{p}_{\mathbf{r}_2\mathbf{d}_2}\right)\right| \left[\delta\left(\mathbf{p}-\mathbf{p}_+\right) + \delta\left(\mathbf{p}-\mathbf{p}_-\right) - \delta\left(\mathbf{p}-\mathbf{p}_{\mathbf{r}_1\mathbf{d}_1}\right) - \delta\left(\mathbf{p}-\mathbf{p}_{\mathbf{r}_2\mathbf{d}_2}\right)\right]. \quad \text{(B3)}$$