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Phys. Rev. B **97**, 245133 — Published 21 June 2018
DOI: 10.1103/PhysRevB.97.245133

Bosonic topological phases of matter: bulk-boundary correspondence, SPT invariants and gauging

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(ΩDated: May 29, 2018)

We analyze $2+1d$ and $3+1d$ Bosonic Symmetry Protected Topological (SPT) phases of matter protected by onsite symmetry group G by using dual bulk and boundary approaches. In the bulk we study an effective field theory which upon coupling to a background flat G gauge field furnishes a purely topological response theory. The response action evaluated on certain manifolds, with appropriate choice of background gauge field, defines a set of SPT topological invariants. Further, SPTs can be gauged by summing over all isomorphism classes of flat G gauge fields to obtain Dijkgraaf-Witten topological G gauge theories. These topological gauge theories can be ungauged by first introducing and then proliferating defects that spoil the gauge symmetry. This mechanism is related to anyon condensation in $2+1d$ and condensing bosonic gauge charges in $3+1d$. In the dual boundary approach, we study $1+1d$ and $2+1d$ quantum field theories that have G 't-Hooft anomalies that can be precisely cancelled by (the response theory of) the corresponding bulk SPT. We show how to construct/compute topological invariants for the bulk SPTs directly from the boundary theories. Further we sum over boundary partition functions with different background gauge fields to construct G-characters that generate topological data for the bulk topological gauge theory. Finally, we study a $2+1d$ quantum field theory with a mixed $\mathbb{Z}_2^{T/R}\times U(1)$ anomaly where $\mathbb{Z}_2^{T/R}$ is time-reversal/reflection symmetry, and the $U(1)$ could be a 0-form or 1-form symmetry depending on the choice of time reversal/reflection action. We briefly discuss the bulk effective action and topological response for a theory in $3+1d$ that cancels this anomaly. This signals the existence of SPTs in $3+1d$ protected by 0,1-form $U(1)\times\mathbb{Z}_2^{T,R}$.

CONTENTS

I. INTRODUCTION

Over the last several years the classification and characterization of gapped quantum phases of matter have become an important pursuit in the field of condensed matter physics. The rather vast landscape of gapped phases of matter can be organized according to (i) the type of microscopic matter, i.e., fermionic or bosonic; (ii) global symmetries which could act onsite or on spacetime indices or on both; (iii) gauge symmetries, i.e., manifestation of constraints or conserved charges; (iv) global symmetries (v) entanglement patterns, i.e., broadly speaking short-ranged or long-range entangled matter.[1–4]

A sub-class of the above gapped quantum phases that have gained importance due to both theoretical and experimental reasons in the recent years are short-range entangled phases of matter with global symmetries, also known as symmetry protected topological phases of matter or SPTs.[5,6] Such phases of matter cannot be connected to the trivial product state (trivial insulator) (or to one another) by a symmetric adiabatic deformation that preserves the gap. Equivalence classes of Bosonic SPTs have been classified using group cohomology[7] and the equivariant cobordism group[8,9]. Non-interacting fermionic phases of matter have been classified using tools in homotopy theory[10,11]. Interacting fermionic phases have been studied using super group cohomology[12] and spin cobordism[13,14] (see also[15–19]) respectively.

a. Bulk response theories, etc. For this work we limit ourselves to bosonic SPT phases. Except for Sec. V, we only consider the simplest case of phases protected by discrete abelian global symmetry G. In $d+1$ dimensions, such phases of matter are classified by group cohomology. Each distinct phase can be labelled by a group cocycle[7]

$$\omega \in H^{d+1}_{\text{group}}(G, U(1)). \tag{1}$$

It is expected that the low-energy and long-wavelength physics of each phase may be captured by an invertible topological quantum field theory (TQFT)[8,14] whose Euclidean partition function we will denote by $\mathcal{Z}^q[N]$ where q is representative of ω and N is a compact and oriented $d+1$-dimensional manifold. A device one uses in these classification approaches is to probe the phase of matter by coupling it to a background flat G gauge field. In the presence of background G field A, the partition function takes the form

$$\mathcal{Z}^q[N, A] = e^{iI^q[N,A]}. \tag{2}$$

When G is a discrete group as the case will be for much of this work, and we are working with a triangulated manifold, a background G field is described by a 1-cocycle on N valued in G i.e a coloring of 1-simplices of a given triangulation of N that satisfies a cocycle condition imposing that the group elements compose to the identity along any contractible 1-cycle. When $G = \mathbb{Z}_n$ this may be equivalently modeled as a flat $U(1)$ gauge field whose holonomies along non-contractible cycles are restricted to be valued in $2\pi\mathbb{Z}/n$.

When the correlation length of the system is much shorter than the system size, $I^q[N, A]$ is expected to be almost insensitive to smooth deformations of the background configuration A and manifold N. In fact in the zero correlation length limit we expect $I^q[N, A]$ to be a topological term. Dijkgraaf and Witten in their seminal paper[20] classified topological actions built for discrete gauge fields and showed that distinct topological actions are labelled by group cocycles . More recently it was shown in the context of SPTs[8] that the response theory $I^q[N, A]$ only depends on the cobordism class of $[N, A] \in \Omega^{SO}_{d+1}(BG)$, where $\Omega^{SO}_{d+1}(BG)$ is the oriented cobordism group. More precisely, (N_1, A_1) and (N_2, A_2) are said to be cobordant if there exists an oriented $d+2$ manifold X with a G-bundle that can interpolate between (N_1, A_1) and (N_2, A_2). This provides an equivalence relation on the set of tuples $\{(N, A)\}$, equivalence classes of which can be given the structure of an abelian group. The group operation is simply disjoint union. If we restrict to N being an oriented $d+1$-manifold and A a principle-G-bundle, then this group is $\Omega^{SO}_{d+1}(BG)$. For purposes of classification, one is interested in only the torsion subgroup of $\Omega^{SO}_{d+1}(BG)$. This is finitely generated. Let the generators be labelled $[N_i, A_i]$ where $i \in \mathcal{I}$, a finite index set. Since SPT phases are short-range entangled they have a unique ground state. Consequently, the modulus of the partition function is unity i.e $\mathcal{Z}^q[N, A] \in U(1)$. The topological invariants for SPTs are provided by the set $\{\mathcal{Z}^q[N_i, A_i]\}_{i\in\mathcal{I}}$[8,9,21–23].

In addition to probing an SPT phase with a background G gauge field, one could further sum over all flat G-fields which is known as 'orbifolding' or gauging -G[24,25]. Upon gauging, different SPTs map to distinct topological gauge theories known as Dijkgraaf-Witten theories[20] or their spin analogues[15,19]. The partition function can be computed as

$$\mathcal{Z}^q_{\text{DW}}[N] \propto \sum_{A\in H^1(N,G)} e^{iI^q[N,A]}. \tag{3}$$

Clearly, different $d+1$-cocycles furnish distinct Dijkgraaf-Witten theories. These can be distinguished by the partition functions they furnish on topologically non-trivial manifolds. For example the manifolds generating the cobordism group described above could be used as theoretical devices to distinguish different theories. Alternatively, it is useful to consider Dijkgraaf-Witten theory in the presence of background defects/sources such as

$$\mathcal{Z}^q_{\text{DW}}[N, J_{qp}] \propto \sum_{A\in H^1(N,G)} e^{iI^q[N,A]+\int_N J_{qp}\cup A} \tag{4}$$

where the quasiparticle current J_{qp} is a d-cochain valued in $\widehat{G} \simeq \hom(G, \mathbb{R}/2\pi\mathbb{Z})$. Further we may also introduce quasivortices 'J_{qv}' that introduce non-contractible cycles in N and impose constraints on the holonomy of A along those cycles. Distinct Dijkgraaf-Witten theories assign different topological invariants to linked configurations of multiple quasi-vortices. Hence after gauging, these topological invariants may also be used to distinguish the parent SPT phases.[26–30]

The G-symmetry can be 'ungauged' within Dijkgraaf-Witten theory by gauging a dual symmetry $\widehat{G}$ which is generated by the quasiparticle configurations. Physically this implies proliferating worldlines of quasiparticles and destroying the gauge symmetry. Practically ungauging involves summing over different configurations of J_{qp} with an appropriate weight. As the name suggests, ungauging G gets us back to what we had before gauging G which was a G-SPT labelled by 'q':[17,31–33]

$$\sum_{J_{qp}} \mathcal{Z}^q_{\text{DW}}[N, J_{qp}] e^{-i\int_N J_{qp}\cup A} \propto e^{iI^q[N,A]}. \tag{5}$$

b. Anomalous boundary theories Besides being distinguished by bulk response to flat G-bundles, SPTs have interesting boundary (surface) theories. It is known that d-dimensional surfaces of $d+1$-dimensional SPTs protected by G symmetry support a quantum field theory with a G-'t-Hooft anomaly,[8,34–39] i.e., a quantum field theory with a global G symmetry that cannot be promoted to a gauge symmetry at the quantum level[40] on an intrinsically d-dimensional manifold. More precisely, let M be a d-manifold and A a flat G-bundle, then the partition function of a theory with a possible 't-Hooft anomaly is non gauge-invariant

$$Z^q[M,A] \neq Z^q[M,A+\Delta A]. \tag{6}$$

Here, ΔA is a gauge transformation of A. Usually the strategy when confronted with such ambiguities in quantum field theory is to look for local counter terms that make the partition function unambiguous, i.e., to look for a functional $\mathcal{L}^q_{c.t.}(A)$ built from local G-bundle data such that

$$Z^q_{\mathrm{reg}}[M,A] := Z^q[M,A]e^{i\int_M \mathcal{L}^q_{c.t.}(A)} \tag{7}$$

is gauge invariant. For theories with 't-Hooft anomalies, no such local counter-term can be constructed. In fact one needs a $d+1$-manifold N ($\partial N = M$) which houses the SPT to construct a well-defined partition function which takes the form

$$\mathcal{Z}^q[N,A] = Z^q[M,A]e^{iI^q[N,A]}. \tag{8}$$

Somewhat imprecisely, we use 'A' both for the lifted G-bundle on N as well as its restriction to $\partial N = M$.

An alternate diagnostic of the 't-Hooft anomaly and the one we will consider in this paper is an obstruction to gauging or orbifolding G. We will show that it is impossible to find any local gauge-invariant counterterm $\mathcal{L}_{c.t.}[A]$ such that

$$Z_{\mathrm{orb}}[M] := \sum_{[A]} Z_{\mathrm{reg}}[M,A] = \sum_{[A]} Z[M,A]e^{i\int_M \mathcal{L}_{c.t.}(A)}$$

is invariant under the group of diffeomorphisms of M. In particular we will be interested in the large diffeomorphisms of M.[34–36,41]

c. Bulk-boundary correspondence We note that an 't-Hooft anomaly is a strong non-perturbative constraint in the sense that 't-Hooft anomalies are conserved along the renormalization group flows. Although this is a strong constraint, it by no means uniquely specifies the surface theory on M. Broadly speaking there are three distinct possibilities that can saturate the 't-Hooft anomaly. The anomaly may be saturated by a quantum field theory that (i) spontaneously breaks G symmetry; (ii) is gapless with a non-local action of G; (iii) is gapped and supports non-trivial (fractionalized) excitations that cannot be realized on an intrinsically d dimensional manifold with G symmetry.[42–46]

Using the anomaly matching criteria, once we establish that a certain quantum field theory with partition function $Z^q[\partial N = M]$ is a suitable candidate for the surface/edge theory for an SPT $\mathcal{Z}^q[N]$, we proceed to explore the bulk-boundary correspondence. We do so in two related but distinct ways. (i) We construct SPT topological invariants directly from a surface/edge computation using the recently studied[22] cut and glue approach, and (ii) We construct topological data corresponding to the Dijkgraaf-Witten topological gauge theory directly from the surface/edge theories. The latter is done by first constructing twisted partition functions $Z^q[M,A]$ and then summing them up into G-invariant characters also known as orbifold characters that are representative of bulk excitations. These methods have been well known for $2+1d$ topological phases and their $1+1d$ boundaries[47–54] and were recently generalized to $3+1d$ topological phases and their $2+1d$ surfaces[30,55]. Here we provide a procedure to construct such G- orbifold chacters or G-characters for short by directly implementing cohomological twists instead of explicitly computing twisted partition functions. G-characters are defined in such a way that they transform projectively under large diffeomorphisms (modular transformations) of M, and the projective phases encode the relevant topological data.

Finally we switch directions and consider a bosonic quantum field theory in $2+1$-dimensions with $\mathbb{Z}_2^{T,R} \times U(1)_p$ symmetry. Here $\mathbb{Z}_2^{T,R}$ refers to time reversal or $\mathbb{Z}_2$-reflection symmetry and by $U(1)_p$ we mean a p-form $U(1)$ symmetry that may be gauged by coupling to $p+1$-form flat $U(1)$ gauge field. We specifically consider the cases $p = 0, 1$ and show that for certain action of $\mathbb{Z}_2^{T,R} \times U(1)_p$, there is an 't-Hooft anomaly that can be cancelled by a $3+1d$ invertible topological field theory. This signals the existence of bosonic SPTs in $3+1$-dimensions protected by $\mathbb{Z}_2^{T,R} \times U(1)_p$. We propose bulk candidate effective field theories for these phases of matter.

A. Plan for the paper

Before getting into the details, let us briefly describe the plan for the rest of the paper.

In Sec. II and III, we study bosonic topological phases of matter with global discrete abelian symmetry G in $2+1$ and $3+1$-dimensions respectively. We study these phases and their gauged versions by analyzing the bulk directly and from a complimentary viewpoint, by analyzing their gapless boundary theories. In Sec. IV, we briefly comment on how this generalizes to $d+1$-dimensions.

Bulk analysis

We begin with an invertible TQFT that can describe bosonic G-SPT phases with topologically distinct realizations of G symmetry labelled by 'q'. We carry out the following steps:

- Couple to a background G gauge field A on a closed, oriented $d+1$-dimensional manifold to compute distinct topological response theories
$$\mathcal{Z}^q[N,A] = e^{iI^q[N,A]}. \qquad (9)$$
- In general $I^q[N,A] \in \mathbb{R}/2\pi\mathbb{Z}$ and the set $\left\{e^{iI^q[N_i,A_i]}\right\}_{i\in\mathcal{I}}$ of $U(1)$ phases for all $[N_i, A_i]$ that generate the torsion subgroup of $\Omega_{d+1}(BG)$ form the set of SPT topological invariants, i.e., they differentiate different SPT phases. For a discrete abelian group G which is always isomorphic to $\prod_{i=1}^{k}\mathbb{Z}_{n_i}$, the topological invariants turn out to be a combination of partition functions on lens spaces and three-torus with appropriate flat G bundles in $2+1$-dimensions and (lens space $\times$ a one-sphere) and the four-torus with appropriate G-bundles in $3+1$-dimensions. We compute these topological invariants.
- Gauge G by summing over flat G bundles to obtain the partition function for a G-topological gauge theory, i.e., Dijkgraaf-Witten theory.
- Introduce quasi-particle sources within Dijkgraaf-Witten theory that generate a dual symmetry $\widehat{G}$ and finally ungauge G by gauging $\widehat{G}$ to return to the SPT phase.

Boundary analysis

To compliment the bulk analysis we study a class of simple models that describe possible edges/surfaces for G-bosonic SPTs. We support our analysis with the following computations:

- We couple the boundary theory to a background G gauge field and compute 'twisted partition functions' $Z^q[M,A]$.
- Take the aforementioned approach and try to gauge G. We treat gauge-ability of G as a diagnostic for a trivial/non-trivial bulk and show that the 't-Hooft anomaly matches with the gauge anomaly of the SPT response theory on an open $d+1$ manifold confirming that this model indeed describes the surface of an SPT.
- Once it is established that the theory describes the boundary of an SPT, the SPT invariants can be constructed directly from the surface theory following a cut and glue construction whose calculation essentially restricts to the boundary theory computation.
- Furthermore G-orbifold characters can be constructed from the 'twisted partition functions'. Modular transformations of these characters reproduce the topological data corresponding to the bulk topological gauge theory obtained by gauging the bulk SPT.

SPT protected by $\mathbb{Z}_2^{T,R}\times U(1)$ symmetry in $3+1d$

In Sec. V we study surface theory for $3+1d$ SPTs protected by $\mathbb{Z}_2^{T,R}\times U(1)_p$ for the case $p=0,1$. We show that for a certain action of $\mathbb{Z}_2^{T,R}\times U(1)_p$, there is a 't-Hooft anomaly for the surface theory. We construct bulk effective field theories that cancel this anomaly and discuss the corresponding symmetry protected phases.

Notations

Before getting to the main text we briefly summarize the notations we use. We will be working with topological phases on a $d+1$-dimensional bulk manifold N which is always compact and oriented. When we discuss purely bulk physics then we often consider N to be closed. However when we consider edge/surface physics we consider N to be an open manifold such that $\partial N = M$. We will denote background G-gauge fields by A. These may be both in the bulk or on the boundary. When discussing situations involving only discrete gauge fields on triangulated manifolds, we find it more convenient and rigorous to treat A as a 1-cocycle valued in G. Such a G-valued cocycle is defined on a simplicial triangulation of the manifold N (or M). Alternatively when the situation involves treating fields which take values in a continuous space we treat A as a $U(1)$ gauge field and impose flatness as well as restrict its holonomies to G by means of a Lagrange multiplier field if A is dynamical and by hand if A is a background field. In the former case where A is thought of as a cocycle we use cup products and lattice codifferential operators whereas when A is thought of as a $U(1)$ valued field we use wedge products and exterior derivatives respectively.

When mentioned (for e.g., during the gauging procedure) we will promote A to be dynamical. By $\widehat{G}$, we imply the group Pontrjagin dual to G, i.e., $\widehat{G} = \{\mu : G \to U(1)\}$. For discrete abelian groups, $\widehat{G}\simeq G$.

Bulk notations	
Notation	Description and comments
$\mathcal{Z}^q[N,A]$	SPT partition function on N with background G bundle A. 'q' labels a $d+1$-cocycle $\omega \in H^{d+1}_{\text{group}}(G,U(1))$.
$I^q[N,A]$	SPT response theory valued in $\mathbb{R}/2\pi\mathbb{Z}$.
$\mathcal{Z}^q_{\text{DW}}[N]$	Dijkgraaf-Witten partition function for $q\in H^{d+1}_{\text{group}}(G,U(1))$ obtained by gauging q-SPT.

Boundary notations	
Notation	Description and comments
$Z^q[M,A]$	Partition function for QFT describing surface of q-SPT on d-manifold M in the presence of background G bundle A.
$Z^{q,\epsilon}[M,A]$	Partition function with discrete torsion phase $\epsilon \in H^d(G,U(1))$. Physically ϵ labels a d-dimensional -SPT.
$Z^q_{\text{orb}}[M]$	Partition function obtained by starting from $Z^q[M,A]$ and orbifolding-G.
$\chi^q_{\mu,\lambda_1,\ldots,\lambda_{d-1}}$	Orbifold characters constructed by summing twisted partition functions $Z^q[M,A]$. These can be used to compute topological data for bulk Dijkgraaf-Witten theory.

II. $2+1d$ TOPOLOGICAL PHASES AND THEIR $1+1d$ EDGES

A. Bulk physics

SPT effective field theories: It is known that SPTs with unitary onsite symmetry can be modeled by BF theories with distinct symmetry actions.[38,57,58] For example $G=\mathbb{Z}_n^k$-SPTs in $2+1d$ may be modeled by k-copies of BF theory at 'level' one:

$$\mathcal{S}[a,b]=\int_N \sum_{I,J=1}^{k} \frac{\delta_{IJ}}{2\pi} b^I\wedge da^J+\cdots\,, \tag{10}$$

where a^I and b^I are $U(1)$-connections subject to the flux quantization conditions $\oint_S da, \oint_S db \in 2\pi\mathbb{Z}$ for $S\in Z_2(N;\mathbb{Z})$. By '$\cdots$' we imply other non-topological symmetry preserving terms that we ignore in the limit of zero correlation length. This theory is trivial, in the sense its partition function $\mathcal{Z}[N]=1$[59] on any closed 3-manifold N. However it can be coupled to a flat background G gauge field A^I in topologically distinct ways which correspond to various SPT actions

$$\mathcal{S}^q[a,b,A]=\int_N \frac{\delta_{IJ}}{2\pi} b^I\wedge da^J+\mathcal{S}^q_{\text{cpl}}[a,b,A]. \tag{11}$$

Here, $\mathcal{S}^q_{\text{cpl}}[a,b,A]$ is the part of the action involving coupling to sources A^I. Flat G gauge fields are characterized by their holonomies, or equivalently, $A\in H^1(N,G)$. G-SPTs are classified by group cohomology and can be labelled by a 3-cocycle $\omega\in H^3_{\text{group}}(G,U(1))$. Here '$q$' is meant to be a representative of ω. For finite abelian groups, there are three classes of group 3-cocycles. For $G=(\mathbb{Z}_n)^k$ these take the form

$$\begin{aligned}
\omega_{\text{type-I}}(\mathbf{a},\mathbf{b},\mathbf{c})&=\exp\left\{\frac{2\pi i q_I}{n^2}a^I\left(b^I+c^I-[b^I+c^I]\right)\right\},\\
\omega_{\text{type-II}}(\mathbf{a},\mathbf{b},\mathbf{c})&=\exp\left\{\frac{2\pi i q_{IJ}}{n^2}a^I\left(b^J+c^J-[b^J+c^J]\right)\right\},\\
\omega_{\text{type-III}}(\mathbf{a},\mathbf{b},\mathbf{c})&=\exp\left\{\frac{2\pi i q_{IJK}}{n}a^I b^J c^K\right\},
\end{aligned} \tag{12}$$

where $\mathbf{a}=(a^1,a^2,\ldots,a^k)$, etc., $\mathbf{a},\mathbf{b},\mathbf{c}\in\mathbb{Z}_n^k$ and $[a^I+b^I]:=a^I+b^I \bmod n$. These different families of cocycles are called type-I,II,III respectively[60]. The parameters q_I,q_{IJ},q_{IJK} take values in $\mathbb{Z}$ mod $n\mathbb{Z}$, hence

$$H^3_{\text{group}}[(\mathbb{Z}_n)^k,U(1)]=(\mathbb{Z}_n)^{\left[\binom{k}{1}+\binom{k}{2}+\binom{k}{3}\right]}. \tag{13}$$

Any G SPT is prescribed by the set of $\mathbb{Z}_n$ parameters $q=\{q_I,q_{IJ},q_{IJK}\}\in H^3_{\text{group}}(G,U(1))$. Different coupling terms corresponding to different families of 3-cocycles take the form

$$\begin{aligned}
\mathcal{S}^{q_I}_{\text{cpl}}[a,b,A]&=-\frac{1}{2\pi}\int_N A^I\wedge(db^I+q_I da^I),\\
\mathcal{S}^{q_{IJ}}_{\text{cpl}}[a,b,A]&=-\frac{1}{2\pi}\int_N A^I\wedge(db^I+q_{IJ} da^J),\\
\mathcal{S}^{q_{IJK}}_{\text{cpl}}[a,b,A]&=-\frac{1}{2\pi}\int_N A^I\wedge(db^I+\frac{n^2 q_{IJK}}{2\pi}a^J\wedge a^K),
\end{aligned} \tag{14}$$

where I,J,K are not summed over. Integrating over a^I,b^I one obtains a response theory in terms of background G-bundle:

$$\mathcal{Z}^q[N,A]=\int\mathcal{D}[a,b]e^{i\mathcal{S}^q[a,b,N,A]}=:e^{iI^q[N,A]}. \tag{15}$$

The response theories $I^q[N,A]$ take the form

$$\begin{aligned}
I^{q_I}[N,A]&=-\frac{q_I}{2\pi}\int_N A^I\wedge dA^I,\\
I^{q_{IJ}}[N,A]&=-\frac{q_{IJ}}{2\pi}\int_N A^I\wedge dA^J,\\
I^{q_{IJK}}[N,A]&=-\frac{q_{IJK}n^2}{4\pi^2}\int_N A^I\wedge A^J\wedge A^K.
\end{aligned} \tag{16}$$

The relation between SPT response theories (16) and the respective cocycles (12) can be seen most clearly within a simplicial construction. (See App. C.)

Topological invariants for SPTs: SPT topological invariants are a set of $U(1)$-valued quantities that can distinguish different phases. These are supplied by the partition functions $\{\mathcal{Z}^q[N_i,A_i]\}_{i\in\mathcal{I}}$ which are pure $U(1)$ phases $e^{iI^q[N_i,A_i]}$. Here, $[N_i,A_i]$ are the generators of the torsion subgroup of $\Omega_3^{SO}(BG)$, the oriented G-equivariant cobordism group. For $G=\mathbb{Z}_n^k$, we will confirm that the lens space $L(n,1)$ and three-torus T^3 with appropriate flat G-bundles are sufficient to detect and classify G-SPTs. Lens space is a three-dimensional topological space with the properties that its 1st and 2nd homology groups are pure torsion i.e $H_{1,2}(L(n,1),\mathbb{Z})=\mathbb{Z}_n$. Such a space can be constructed by taking a quotient of the three sphere $|z_1|^2+|z_2|^2=1$ where $z_{1,2}\in\mathbb{C}$ by the natural $\mathbb{Z}_n$ action i.e $(z_1,z_2)\sim(e^{\frac{2\pi i}{n}}z_1, e^{\frac{2\pi i}{n}}z_2)$. Let us compute the partition functions on these manifolds.

- **Type-I and type-II cocycles:** SPTs with type-I and type-II symmetry action can be distinguished by their partition functions on lens space $(L(n,1))$ with an appropriate background G-bundle. The topology of Lens space is captured by the torsion part of its homology groups

$$H_1(L(n,1),\mathbb{Z})=H^2(L(n,1),\mathbb{Z})=\mathbb{Z}_n. \tag{17}$$

Then $[A]\in \mathrm{Tor}(H^2(L(n,1),\mathbb{Z}))$. The Chern-Simons term which appears in the type-I response theory $I^{q_I}[N,A]$ evaluates to

$$\begin{aligned} e^{I^{q_I}[L(n,1),[A]]} &= \exp\left\{\frac{-iq_I}{2\pi}\int_{L(n,1)} A^I\wedge dA^I\right\} \\ &= \exp\left\{-iq_I\oint_{C_A} A\right\} \\ &= \exp\left\{-iq_I a_I\oint_{C_1} A\right\} \\ &= \exp\left\{-\frac{2\pi i q_I a_I^2}{n}\right\}, \end{aligned} \tag{18}$$

where $C_A\in H_1(L(n,1),\mathbb{Z})$ is Poincare dual to $[A]\in\mathrm{Tor}\left(H^2(L(n,1),\mathbb{Z})\right)$. Further we have chosen the configuration $[A]$ such that $C_A=a_IC_1$ where C_1 is the generator of $H_1(L(n,1),\mathbb{Z})$. Hence the SPT invariant is

$$e^{iI^{q_I}[L(n,1),[A]]}=e^{-\frac{2\pi i q_I a_I^2}{n}}. \tag{19}$$

The SPT invariant with type-II response theory (16) can be computed similarly.

$$\begin{aligned} e^{iI^{q_{IJ}}[L(n,1),[A]]} &= \exp\left\{-\frac{iq_{IJ}}{2\pi}\int_{L(n,1)} A^I\wedge dA^J\right\} \\ &= \exp\left\{-iq_{IJ}\oint_{C_A} A^I\right\} \\ &= \exp\left\{-iq_{IJ}a_J\oint_{C_1} A^I\right\} \\ &= \exp\left\{-\frac{2\pi i q_{IJ}a_Ia_J}{n}\right\}. \end{aligned} \tag{20}$$

- **Type-III cocycles:** SPTs with type-III response theories can be detected on T^3 with a background G bundle

$$\begin{aligned} e^{iI^{q_{IJK}}[T^3,A]} &= \exp\left\{-\frac{in^2q_{IJK}}{4\pi^2}\int_{T^3} A^I\wedge A^J\wedge A^K\right\} \\ &= \exp\left\{-\frac{2\pi i q_{IJK}}{n}\epsilon^{ijk}a_{I,i}b_{J,j}c_{K,k}\right\} \end{aligned} \tag{21}$$

where $\mathbf{a}_I=(a_{I,1},a_{I,2},a_{I,3})$ are the holonomies around the three cycles of T^3.

Summarizing, the complete set of invariants for bosonic SPTs protected by $G=\mathbb{Z}_n^k$ are

$$\begin{aligned} &\left\{e^{-iI^q[L(n,1),A]}, e^{-iI^q[T^3,A]}\right\} \\ &= \left\{e^{\frac{2\pi i}{n}(q_Ia_I{}^2+q_{IJ}a_Ia_J)}, e^{2\pi i\frac{q_{IJK}}{n}\epsilon^{ijk}a_{I,i}b_{J,j}c_{K,k}}\right\} \end{aligned} \tag{22}$$

More generally, if $G=\prod_{I=1}^k\mathbb{Z}_{n_I}$, then the SPTs classified by parameters $\{q_I,q_{IJ},q_{IJK}\}$ parametrizing type-I,II,III kind of responses respectively can be detected on $\left\{L(n_I,1),L(\gcd(n_I,n_J),1),T^3\right\}$ respectively.[61–63] In the above computation we have treated A^I as a flat $U(1)$ bundle with holonomies restricted to $\mathbb{Z}_n$, equivalently we could have treated A^I as a $\mathbb{Z}_n$-cocycle on a triangulation of the three torus/ lens space.

Topological gauge theories from gauging SPTs : Gauging of SPTs can be carried out by first computing the response to flat G-bundles (15) and then summing over all flat bundles with the appropriate normalization. By this procedure, one obtains the well known Dijkgraaf-Witten topological gauge theory labelled by $q\in H^3_{\text{group}}(G,\mathbb{R}/2\pi\mathbb{Z})$:

$$\begin{aligned} \mathcal{Z}^q_{\text{DW}}[N] &= \frac{1}{|H^0(N,G)|}\sum_{A\in H^1(N,G)} e^{iI^q[N,A]} \\ &= \int\prod_{I=1}^k\mathcal{D}[A^I,B^I]e^{i\int_N\frac{n\delta_{IJ}}{2\pi}B^I\wedge dA^J+iI^q[N,A]} \end{aligned} \tag{23}$$

where in the second line we have specialized to $G=\mathbb{Z}_n^k$ and written the gauged SPT action in the familiar continuum form as a 'twisted' multicomponent BF theory. A^I,B^I are 1-form $U(1)$ connections. Integrating over B^I imposes that A is a flat G-bundle and takes us back to the original expression. Since $(1/2\pi)dB^I$ is a 2-form with integral periods we can write

$$\frac{1}{2\pi}dB^I=d\beta^I+\sum_{j\in\text{Free}(H^2(N,\mathbb{Z}))}m_j^I\lambda_j \tag{24}$$

where $m_j\in\mathbb{Z}$ and λ_j is a basis on the space of integral harmonic 2-forms. Then, integrating over B^I, we get

$$\begin{aligned}\mathcal{Z}^q_{\rm DW} &= \int \prod_{I=1}^{k} \mathcal{D}[A^I,\beta^I] e^{\frac{in\delta_{IJ}}{2\pi}\int_N \beta^I\wedge F^J_A} \\ &\times \prod_j \left[\sum_{m^I_j\in\mathbb{Z}} e^{in\delta_{IJ}m^I_j\int \lambda_j\wedge A^J}\right] e^{iI^q[N,A]} \\ &= \frac{1}{\mathcal{N}}\int \prod_{I=1}^{k}\mathcal{D}[A^I]\delta(nF^I_A)\prod_j\left[\sum_{m^I_j\in\mathbb{Z}} e^{inm^I_j\int\lambda_j\wedge A^I}\right]e^{iI^q[A,N]} \\ &= \frac{1}{\mathcal{N}}\int\prod_{I=1}^{k}\mathcal{D}[A^I]\delta(nF^I_A)\delta(\oint_{L_j}A^I\in\frac{2\pi}{n}\mathbb{Z})e^{iI^q[N,A]} \\ &= \frac{1}{|H_0(N,\mathbb{Z}^k_n)|}\sum_{A\in H^1(N,\mathbb{Z}^k_n)} e^{iI^q[N,A]}. \end{aligned} \tag{25}$$

The sum over β^I fixes $nF^I_A = 0$ which implies that $F^I_A = 0$ unless $\mathrm{Tor}(H^2(N,\mathbb{Z}))\neq 0$. The sum over m_j sets the holonomy of A^I to be a multiple of $2\pi/n$ along L_j the 1-cycle poincare dual to λ_j. In other words $[A]\in H^1(M,\mathbb{Z}^k_n)$, a flat $\mathbb{Z}^k_n$-gauge field. The factor $\mathcal{N}$ counts the number of gauge transformations of A^I as $\mathbb{Z}_n$-valued field. Let us take a look at few examples:

- **Type-I and type-II cocycles:** Consider a 3-manifold N with vanishing torsion. Then since $dA^I = 0$, we get $I^{q_I}[N,A] = I^{q_{IJ}}[N,A] = 0$. Therefore

$$\begin{aligned}\mathcal{Z}^q_{\rm DW}[N] &= \frac{1}{|G|}\sum_{[A]\in H^1(N,G)} 1 \\ &= |G|^{b_1(N)-1},\end{aligned} \tag{26}$$

where $b_1(N)$ refers to the 1st Betti number of N. If $N = S^1\times M$, the partition function evaluates to

$$\mathcal{Z}^q_{\rm DW}[M\times S^1] \equiv \mathrm{GSD}[M] = |G|^{b_1(M)} \tag{27}$$

where $\mathrm{GSD}[M]$ denotes the groundstate degeneracy on M. Similarly, the gauged partition function for type-I and type-II cocycle on for $G=\mathbb{Z}_n$ and $G=\mathbb{Z}^2_n$ respectively can be evaluated on $L(n,1)$ using (18) and (20)

$$\begin{aligned}\mathcal{Z}^{q_I}_{\rm DW}[L(n,1)] &= \frac{1}{n}\sum_{a_I=0}^{n-1} e^{\frac{2\pi i q_I a_I^2}{n}}, \\ \mathcal{Z}^{q_{IJ}}_{\rm DW}[L(n,1)] &= \frac{1}{n^2}\sum_{a_I,a_J=0}^{n-1} e^{\frac{2\pi i q_{IJ}a_I a_J}{n}},\end{aligned} \tag{28}$$

- **Type-III cocycles:** The partition function on T^3 for type-III cocycle can be computed using (21)

$$\begin{aligned}\mathcal{Z}^q_{\rm DW}[T^3] &= \frac{1}{|G|}\sum_{\mathbf{a},\mathbf{b},\mathbf{c}\in\mathbb{Z}^3_n} e^{\frac{2\pi i q_{IJK}}{n}\epsilon^{ijk}a_{I,i}b_{J,j}c_{K,k}} \\ &=: \mathrm{GSD}[T^2] < |G|^2\end{aligned} \tag{29}$$

For $G=\mathbb{Z}^3_2$, $q_{123}=1$, (29) evaluates to $\mathcal{Z}^q_{\rm DW}[T^3] = 22 = \mathrm{GSD}[T^2]$[60]. Groundstates on a torus can be labelled by the spectrum of Wilson operators in a topological gauge theory, therefore this implies that there are 22 independent Wilson operators. The total quantum dimension is the same for different Dijkgraaf-Witten theories corresponding to the same G, hence we obtain

$$|G|^2 = \sum_{i=1}^{\mathrm{GSD}[T^2]} d_i^2. \tag{30}$$

If $\mathrm{GSD}[T^2] < |G|^2$ there must be at least a single Wilson operator with quantum dimension greater than 1. This is a way to see that type-III theory has non-abelian excitations even though G is an abelian group[60]. A dual approach based on analyzing Wilson operators directly in the continuum theory may also be used to compute this groundstate degeneracy.[64]

Ungauging and anyon condensation: Let us consider the continuum formulation of Dijkgraaf-Witten theory (23) in the presence of quasiparticle sources J_{qp}

$$\begin{aligned}\mathcal{Z}^q_{\rm DW}[N,J_{qp}] = \int\prod_{I=1}^{k}\mathcal{D}[A^I,B^I]\exp\Big\{\int_N \frac{in}{2\pi}B^I\wedge dA^I \\ +iI^q[N,A] + i\int_N J^I_{qp}\wedge A^I\Big\}\end{aligned} \tag{31}$$

where the background fields J^I_{qp} are 2-form fields with integral periods[65]. Upon integrating out B^I, $\oint A^I\in(2\pi\mathbb{Z})/n$, the periods of J_{qp} only make sense modulo n, more precisely $J_{qp}\in H^2(N,\widehat{G})$ where $\widehat{G} = \mathrm{Rep}(G)\simeq G$. There is a perfect pairing

$$\int_N : H^1(N,G)\times H^2(N,\widehat{G})\to\mathbb{R}/2\pi\mathbb{Z} \tag{32}$$

that is realized by wedge product followed by integration. For a simplicial definition of this pairing, consider a 3-simplex as in Fig. 1 $\int_\Delta J_{qp}\cup A = J_{qp}[012](A[23]) = m(a) = \frac{2\pi m a}{n}$.

J_{qp} generates a 1-form $\widehat{G}$ symmetry. To see this, we follow the procedure standard in Hamiltonian quantization of gauge systems. Let $N = M\times S^1$. We define a charge operator $\mathcal{Q}^I(\lambda^I)$ corresponding to $\widehat{G}$ symmetry

$$\delta_{J^I_{qp}}S = \int_N \delta J^I_{qp}\wedge A^I \Rightarrow \mathcal{Q}^I(\lambda^I) := \frac{1}{2\pi}\int_M \lambda^I\wedge A^I \tag{33}$$

where $\mathcal{Q}^I(\lambda^I)$ is the charge operator that generates the 1-form gauge transformation and $\lambda\in\Omega^1_{\mathbb{Z}}(M)$ parametrizes the transformation. Then the 1-form symmetry acts as

$$\begin{aligned}\mathcal{Q}^I(\lambda^I) &: J^I_{qv}\mapsto J^I_{qv} + d\lambda^I; \\ &: B^I\mapsto B^I - \lambda^I.\end{aligned} \tag{34}$$

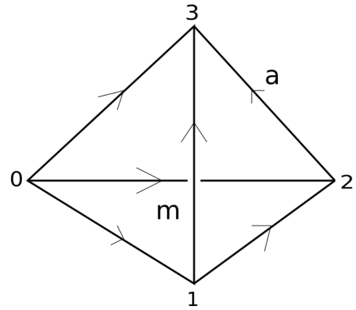

FIG. 1. Triangulation of a three-torus containing one 0-simplex, three 1-simplices, three 2-simplices and six 3-simplices.

Gauging this dual 1-form symmetry means summing over $J^I_{qp} \in H^2(M, \widehat{G})$. Let us call the partition function after gauging the 1 -form $\widehat{G}$ symmetry $\mathcal{Z}^q_{\mathrm{DW}/\widehat{G}}$. Then

$$\begin{aligned}\mathcal{Z}^q_{\mathrm{DW}/\widehat{G}}[N, \hat{A}] &= \sum_{J_{qp}} e^{-i\int_N J^I_{qp}\wedge \hat{A}^I}\, \mathcal{Z}^q_{\mathrm{DW}}[N, J_{qp}] \\ &= \sum_{J_{qp}} \sum_{A} e^{i\int_N J^I_{qp}\wedge (A^I - \hat{A}^I) + iI^q[N,A]} \\ &= e^{iI^q[N,\hat{A}]}. \end{aligned} \quad (35)$$

Hence gauging the dual $\widehat{G}$ 1-form global symmetry is equivalent to un-gauging G. The symmetry is generated by the world-line of A, therefore gauging is synonymous with proliferating the A-lines freely and may be understood as anyon condensation.[66–70]

B. Edge physics

Consider the $1 + 1d$ bosonic conformal field theory on a two-dimensional spacetime manifold M described by the action

$$S[M] = \int_M \sum_{I=1}^{k} \left[\frac{1}{4\pi}\partial_x\phi^{1,I}\partial_t\phi^{2,I} - \mathcal{H}(\phi^{1,I}, \phi^{2,I})\right] \quad (36)$$

where $\phi^{1,I}, \phi^{2,I} : M \mapsto \mathbb{R}/2\pi\mathbb{Z}$. $\mathcal{H}$ denotes the Hamiltonian which we shall set to $\mathcal{H} = 1/4\pi \sum_{I,i}(\partial_x\phi^{i,I})^2$. The action (36) is invariant under different realizations of global 0-form $\mathbb{Z}_n^k$-symmetry. It is well-known that the edge theory for a G-SPT suffers from a G 't-Hooft anomaly, i.e., there is an obstruction to promoting the global G-symmetry to a gauge symmetry. A diagnostic of this anomaly that we will use is modular invariance which is a consistency criteria for a healthy quantum field theory. The idea is as follows: consider putting a quantum field theory on a manifold M. Then we require that the partition function be invariant under large diffeomorphisms of M.[56] We will be particularly interested in $M = T^2$ for which $MCG(T^2) = SL(2,\mathbb{Z})$ which has two generators denoted S and T with the action

$$S : \begin{pmatrix} t \\ x \end{pmatrix} \mapsto \begin{pmatrix} -x \\ t \end{pmatrix}, \qquad T : \begin{pmatrix} t \\ x \end{pmatrix} \mapsto \begin{pmatrix} t+x \\ x \end{pmatrix}. \quad (37)$$

A modular invariant partition function is one for which

$$Z[UM] = Z[M]; \quad U \in MCG(M) \quad (38)$$

A diagnostic for a theory with a global or 't-Hooft anomaly is the inexistence of a modular invariant partition function for the gauged (or orbifolded) theory[34–36,41]. To be more precise the partition function of the gauged theory takes the form

$$Z_{\mathrm{orb}}[M] = \frac{1}{|H^0(M,G)|}\sum_{A\in H^1(M,G)} \theta(A) Z[M,A] \quad (39)$$

where $Z[M,A]$ is the 'twisted' partition function computed in the presence of background flat G gauge field $A \in H^1(M,G)$. In case a theory admits distinct G actions we will denote by 'q' a specific realization of G-symmetry. We label a 'twisted' partition function with this choice of symmetry action by $Z^q[M,A]$. In (39), the different twisted sectors are weighted by $\theta(A)$ where θ is a function $\theta : H^1(M,G) \to U(1)$ as a set. More precisely we must think of $\theta(A)$ as a counterterm built from local gauge data A paired with the manifold, $\theta(A) = \exp\left\{i\int_M \mathcal{L}_{c.t.}(A)\right\}$. Generally there might be inequivalent choices of θ that furnish modular invariant partition functions. More precisely $\theta(A)$ as well as $\theta(A)\epsilon(A)$ may be used to construct modular invariants. Here ϵ is the discrete torsion phase classified by $H^2_{\mathrm{group}}(G, \mathbb{R}/2\pi\mathbb{Z})$ (see App. B for details).

The theory has a 't-Hooft anomaly if there does not exist any gauge invariant $\theta(A)$ such that

$$Z_{\mathrm{orb}}[UT^2] = Z_{\mathrm{orb}}[T^2]; \quad U = S, T \quad (40)$$

We will see that the theory (11) introduced earlier exactly cancels the 't-Hooft anomaly of (36) when $M = \partial N$ and the SPT effective action (11) lives on N. Hence the 't-Hooft anomalies discussed here are prescribed by the same data 'q' $\in H^3_{\mathrm{group}}(G, \mathbb{R}/2\pi\mathbb{Z})$ as $2+1d$ SPTs. Since the anomaly of the $1+1d$ theory is cancelled by the bulk $2+1d$ SPT, together they may be coupled consistently to a background G gauge field and gauged. In other words

$$\mathcal{Z}^q_{\mathrm{DW}}[N] = \frac{1}{|H^0(N,G)|}\sum_{[A]\in H^1(N,G)} Z^q\left[M, A\right] \mathcal{Z}^q\left[N|_{\partial N = M}, A\right] \quad (41)$$

is the partition function for a well-defined or anomaly-free G gauge theory labelled by 3-cocycle 'q' $\in H^3_{\mathrm{group}}(G, \mathbb{R}/2\pi\mathbb{Z})$.

Let us consider the case of $G = \mathbb{Z}_n^2$. We choose the simple case of $\mathbb{Z}_n^2$ to avoid dealing with orbifolding type-III cocycles which appear for $G = \mathbb{Z}_n^k$ when $k \geq 3$. Type-III cocycles are quite subtle for several reasons and we will mostly leave them out of our discussion. For a discussion of 't-Hooft anomalies corresponding to type-III cocycles see references[70,72] Since $H^3_{\text{group}}(\mathbb{Z}_n^2, \mathbb{R}/2\pi\mathbb{Z}) = \mathbb{Z}_n^3 \simeq (q_1, q_2, q_{12})$ there could be three distinct kinds of G actions and combinations thereof. Let us denote these by $\hat{g}_1, \hat{g}_2, \hat{g}_{12}$ respectively. Explicitly their action on (36) is

$$\begin{aligned} \hat{g}_I &: \begin{bmatrix} \phi^{1,I} \\ \phi^{2,I} \end{bmatrix} \mapsto \begin{bmatrix} \phi^{1,I} \\ \phi^{2,I} \end{bmatrix} + \frac{2\pi}{n} \begin{bmatrix} 1 \\ q_I \end{bmatrix} \\ \hat{g}_{IJ} &: \begin{bmatrix} \phi^{1,I} \\ \phi^{2,J} \end{bmatrix} \mapsto \begin{bmatrix} \phi^{1,I} \\ \phi^{2,J} \end{bmatrix} + \frac{2\pi}{n} \begin{bmatrix} 1 \\ q_{IJ} \end{bmatrix} ; I < J \end{aligned} \tag{42}$$

We follow the canonical formalism in order to gauge the global G symmetry. The first step is to compute twisted partition functions $Z^q[M, A]$. Since A is flat it is characterized by holonomies along homology cycles in M i.e $[A] \in \text{Hom}\,[H_1(M,\mathbb{Z}), G]$. Let us fix $M = T^2$, then $[A] \simeq (\mathbf{a}, \mathbf{b})$ where $\mathbf{a}, \mathbf{b} \in G$ are the holonomies along the time and space cycle respectively. The partition functions in the twisted sectors are

$$Z^q[T^2, A] = Z^q_{\mathbf{a},\mathbf{b}} := \text{Tr}_{\mathcal{H}^q_{\mathbf{b}}} \left[\hat{\mathbf{a}}\, e^{2\pi i \tau_1 P - 2\pi\tau_2 H} \right], \tag{43}$$

where $\tau = \tau_1 + i\tau_2$ is the modular parameter of the flat spacetime torus, H, P are the Hamiltonian and the momentum, respectively, and we have defined the twisted Hilbert space $\mathcal{H}^q_{\mathbf{b}}$ which satisfies the boundary conditions

$$\begin{pmatrix} \phi^{1,J} \\ \phi^{2,J} \end{pmatrix} (x+L) = \begin{pmatrix} \phi^{1,J} \\ \phi^{2,J} \end{pmatrix} (x) + \frac{2\pi}{n} \begin{pmatrix} b_J \\ q_J b_J + q_{IJ} b_J \end{pmatrix}. \tag{44}$$

Let us define charge operators

$$Q^{i,I} := \frac{1}{2\pi} \int dx \partial_x \phi^{\bar{i},I}; \quad i, \bar{i} \in 1,2; \quad i \neq \bar{i} \tag{45}$$

which implement $U(1)$ transformations

$$e^{i\lambda Q^{i,I}} : \phi^{i,I} \to \phi^{i,I} + \lambda. \tag{46}$$

Then $\hat{\mathbf{a}}$ appearing in (43) takes the form

$$\hat{\mathbf{a}} := \exp\left\{ \frac{2\pi i}{n} \left[a_I Q^{1,I} + a_1 q_1 Q^{2,1} + (a_2 q_2 + a_1 q_{12}) Q^{2,2} \right] \right\}. \tag{47}$$

These twisted partition functions can be computed using standard methods in conformal field theory (see for example[34,73,74]). We will mainly be interested in modular properties of the twisted partition functions.

$$\begin{aligned} T : Z^q_{\mathbf{a},\mathbf{b}}(\tau) &\mapsto Z^q_{\mathbf{a},\mathbf{b}}(\tau+1) \\ &= T^q_{\mathbf{a},\mathbf{b}} Z^q_{\mathbf{a}+\mathbf{b},\mathbf{b}}(\tau) \\ &= e^{-\frac{2\pi i}{n^2}\left[\sum_I q_I b_I^2 + q_{12} b_1 b_2\right]} Z^q_{\mathbf{a}+\mathbf{b},\mathbf{b}}(\tau), \\ S : Z^q_{\mathbf{a},\mathbf{b}}(\tau) &\mapsto Z^q_{\mathbf{a},\mathbf{b}}(-1/\tau) \\ &= S^q_{\mathbf{a},\mathbf{b}} Z^q_{-\mathbf{b},\mathbf{a}}(\tau) \\ &= e^{\frac{2\pi i}{n^2}\left[2\sum_I q_I a_I b_I + q_{12}(a_1 b_2 + b_1 a_2)\right]} Z^q_{-\mathbf{b},\mathbf{a}}(\tau). \end{aligned} \tag{48}$$

Under large gauge transformations, $Z^q_{\mathbf{a},\mathbf{b}}$ transforms as

$$\begin{aligned} Z^q_{\mathbf{a}+n\mathbf{e}_1,\mathbf{b}}(\tau) &= e^{\frac{2\pi i (q_1 b_1 + q_{12} b_2)}{n}} Z^q_{\mathbf{a},\mathbf{b}}(\tau), \\ Z^q_{\mathbf{a}+n\mathbf{e}_2,\mathbf{b}}(\tau) &= e^{\frac{2\pi i (q_2 b_2 + q_{12} b_1)}{n}} Z^q_{\mathbf{a},\mathbf{b}}(\tau), \\ Z^q_{\mathbf{a},\mathbf{b}+n\mathbf{e}_1}(\tau) &= e^{\frac{2\pi i (q_1 a_1 + q_{12} a_2)}{n}} Z^q_{\mathbf{a},\mathbf{b}}(\tau), \\ Z^q_{\mathbf{a},\mathbf{b}+n\mathbf{e}_2}(\tau) &= e^{\frac{2\pi i (q_2 a_2 + q_{12} a_1)}{n}} Z^q_{\mathbf{a},\mathbf{b}}(\tau), \end{aligned} \tag{49}$$

where $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$.

Gauging trivial symmetry action: Let us first consider the partition functions twisted by trivial symmetry action, i.e., $q = 0$. For this trivial case an equal weight sum over all twisted sectors is modular invariant

$$Z^0_{\text{orb}}(\tau) = \frac{1}{|G|} \sum_{\mathbf{a},\mathbf{b} \in G} Z^0_{\mathbf{a},\mathbf{b}}(\tau). \tag{50}$$

More generally, we may introduce a $U(1)$ valued function $\epsilon : G^2 \to U(1)$ to obtain a partition function

$$Z^{0,\epsilon}(\tau) = \frac{1}{|G|} \sum_{\mathbf{a},\mathbf{b} \in G} \epsilon(\mathbf{a},\mathbf{b}) Z^0_{\mathbf{a},\mathbf{b}}(\tau). \tag{51}$$

Modular invariance and factorizability of the partition function at higher genus impose several constraints on ϵ such that distinct choices of ϵ are classified by $H^2_{\text{group}}(G, U(1))$ as

$$\epsilon(\mathbf{a},\mathbf{b}) = \frac{c(\mathbf{a},\mathbf{b})}{c(\mathbf{b},\mathbf{a})} \tag{52}$$

where $[c] \in H^2_{\text{group}}(G, U(1))$.[75,76] (see App. B for details). Bosonic SPTs in $1{+}1d$ protected by G symmetry are also classified by $H^2_{\text{group}}(G, U(1))$. The partition function for SPT described by $[c] \in H^2_{\text{group}}(G, U(1))$ on a 2-torus with flat G gauge field A evaluates to

$$Z^c_{\mathbf{a},\mathbf{b}} = c(\mathbf{a},\mathbf{b})/c(\mathbf{b},\mathbf{a}) = \epsilon(\mathbf{a},\mathbf{b}). \tag{53}$$

Therefore the freedom of adding a discrete torsion phase while constructing a modular invariant partition function is equivalent to adding a $1+1d$ G-SPT. This is ofcourse expected since a $1+1d$ SPT is perfectly consistent on a closed 2-manifold and therefore should not contribute to the anomaly. Hence the anomaly on the boundary of a $2+1d$ SPT is insensitive to pasting of a $1+1d$ SPT protected by G (or more generally H such that $G \subset H$).

Gauging non-trivial symmetry action: Now let us try to gauge G for the action where $q \neq 0$. We mentioned earlier that this is related to non-trivial $q \in H^3_{\text{group}}(G, \mathbb{R}/2\pi\mathbb{Z})$. Using (48) we obtain the following conditions from requiring modular invariance

$$\begin{aligned}\theta(\mathbf{a},\mathbf{b}) &= e^{\frac{2\pi i}{n^2}\left[\sum_I q_I b_I^2 + q_{12} b_1 b_2\right]}\theta(\mathbf{a}+\mathbf{b},\mathbf{b}),\\ \theta(\mathbf{a},\mathbf{b}) &= e^{-\frac{2\pi i}{n^2}\left[\sum_I 2q_I a_I b_I + q_{12}(a_1 b_2 + b_1 a_2)\right]}\theta(-\mathbf{b},\mathbf{a}). \end{aligned} \tag{54}$$

Using the first equation above, it can be seen that

$$\begin{aligned}\theta(\mathbf{a}+n\mathbf{e}_1,\mathbf{e}_1) &= e^{2\pi i q_1/n}\theta(\mathbf{a},\mathbf{e}_1),\\ \theta(\mathbf{a}+n\mathbf{e}_2,\mathbf{e}_2) &= e^{2\pi i q_2/n}\theta(\mathbf{a},\mathbf{e}_2),\\ \theta(\mathbf{a}+n(\mathbf{e}_1+\mathbf{e}_2),\mathbf{e}_1+\mathbf{e}_2) &= e^{\frac{2\pi i(q_1+q_2+q_{12})}{n}}\theta(\mathbf{a},\mathbf{e}_1+\mathbf{e}_2).\end{aligned} \tag{55}$$

We interpret $\theta(\mathbf{a},\mathbf{b})$ as a local counter-term needed to make the partition function modular invariant. That is $\theta(\mathbf{a},\mathbf{b}) = e^{iS_{c.t.}[\mathbf{a},\mathbf{b}]}$. We learn that requiring modular invariance forces us to choose a counter-term which is not invariant under-large gauge transformations $\mathbf{a}\mapsto\mathbf{a}+n\mathbf{e}_I$ and $\mathbf{a}\mapsto\mathbf{a}+n\mathbf{e}_1+n\mathbf{e}_2$. Hence there is a conflict between gauge invariance and modular invariance which is a diagnostic of a 't-Hooft anomaly. We can however couple the theory to a TQFT in $2+1d$ that cancels the 't-Hooft anomaly of the $1+1d$ theory (36). Above we constructed an invertible TFT (11) that exactly cancels the boundary anomaly. To see this, we compute the following response action

$$I^q[D^2_{\mathbf{a}}\times S^1_{\mathbf{b}}, A] = -\int_{D^2_{\mathbf{a}}\times S^1_{\mathbf{b}}}\left[\frac{q_I}{2\pi}A^I\wedge dA^I + \frac{q_{IJ}}{2\pi}A^I\wedge dA^J\right]. \tag{56}$$

By $D^2_{\mathbf{a}}\times S^1_{\mathbf{b}}$, we denote the configuration where $N = D^2_{xy}\times S^1_t$, and the G gauge field has a symmetry defect puncturing D^2 such that

$$\oint_{S^1_t} A^I = \frac{2\pi}{n}b_I; \qquad \oint_{\partial D^2_{xy}} A^I = \frac{2\pi}{n}a_I \tag{57}$$

Note that this is not a flat field configuration as it is sourced by a extrinsic symmetry defect. Then the partition function for an SPT described by (36) evaluates to

$$\begin{aligned}\mathcal{Z}^q[D^2_{\mathbf{a}}\times S^1_{\mathbf{b}}, A] &= e^{iI^q[D^2_{\mathbf{a}}\times S^1_{\mathbf{b}}, A]}\\ &= e^{-\sum_I \frac{2\pi i q_I a_I b_I}{n^2} - \frac{2\pi i q_{IJ}(a_I b_J + a_J b_I)}{n^2}}\end{aligned} \tag{58}$$

which exactly satisfies the properties (54) and hence cancels the modular anomaly of the $1+1d$ theory. Furthermore it transforms under large gauge transformations in an opposite way to (49). Hence coupled to an invertible TFT in the bulk, (36) is perfectly consistent.

Further, (56) is anomaly-free on a closed manifold and the global G symmetry can be gauged to obtain DW theory with topological order. This topological order is characterized by some data such as braiding phases and topological spin. It has long been known that the topological data of the bulk TQFT can be extracted directly from the $1+1d$ edge theory.[47–54]

G-characters and topological data: In order to obtain bulk topological data such as braiding phases and topological spins of excitations within a topological gauge theory directly from the edge theory, one may exploit the bulk-boundary correspondence. This correspondence establishes a bijection between excitations or line operators in a topological gauge theory and G-characters built from twisted sectors of the edge theory on a spacetime two torus[71]. The complete set of characters may be constructed from the edge theory as

$$\chi_{\mu,\mathbf{a}} = \frac{1}{\sqrt{|G|}}\sum_{\mathbf{b}\in G}\mu(\mathbf{b})Z^q_{\mathbf{b},\mathbf{a}}(\tau) \tag{59}$$

where $\mu\in\text{Rep}(G)$. For example if $G=\mathbb{Z}_n$, then explicitly $\mu(b) = e^{\frac{2\pi i\mu b}{n}}$. Each character constructed from the edge theory corresponds to an excitation within the bulk topological gauge theory. These characters form a projective representation of the mapping class group $SL(2,\mathbb{Z})$ and the S and T matrices of projective phases encode bulk topological data

$$\begin{aligned}\mathcal{S}\chi_{\mu,\mathbf{a}} &= \sum_{\mu',\mathbf{a}'} S_{(\mu,\mathbf{a}),(\mu',\mathbf{a}')}\chi_{\mu',\mathbf{a}'},\\ \mathcal{T}\chi_{\mu,\mathbf{a}} &= \sum_{\mu',\mathbf{a}'} T_{(\mu,\mathbf{a}),(\mu',\mathbf{a}')}\chi_{\mu',\mathbf{a}'}\\ &= \exp\{2\pi i h_{\mu,\mathbf{a}}\}\chi_{\mu,\mathbf{a}}.\end{aligned} \tag{60}$$

Notice the action of $\mathcal{T}$ is diagonal and the eigenvalue of $\chi_{\mu,\mathbf{a}}$, $\exp 2\pi i h_{\mu,\mathbf{a}}$, is the topological spin of the bulk excitation corresponding to $\chi_{\mu,\mathbf{a}}$ via the bulk-boundary correspondence. Instead of directly evaluating the partition function in the twisted sector $Z^q_{\mathbf{b},\mathbf{a}}(\tau)$ (labelled by $\mathbf{b}$, $\mathbf{a}$) and extracting the S and T matrices from it[48,52–54,78], we can construct $\bar{Z}^q_{\mathbf{b},\mathbf{a}}(\tau)$ from $Z^0_{\mathbf{b},\mathbf{a}}(\tau)$ in the following way,

$$\begin{aligned}\bar{Z}^q_{\mathbf{b},\mathbf{a}}(\tau) &:= \gamma^q_{\mathbf{a}}(\mathbf{b})Z^0_{\mathbf{b},\mathbf{a}}(\tau),\\ \bar{\chi}_{\mu,\mathbf{a}} &= \frac{1}{\sqrt{|G|}}\sum_{\mathbf{b}\in G}\mu(\mathbf{b})\bar{Z}^q_{\mathbf{b},\mathbf{a}}(\tau),\end{aligned} \tag{61}$$

where $Z^0_{\mathbf{b},\mathbf{a}}(\tau)$ is the twisted partition function for the trivial SPT phase. In $\bar{Z}^q_{\mathbf{b},\mathbf{a}}(\tau)$, the interesting topological data is encoded in $\gamma^q_{\mathbf{a}}(\mathbf{b})$, which has the important algebraic property

$$\gamma^q_{\mathbf{a}}(\mathbf{b})\gamma^q_{\mathbf{a}}(\mathbf{c}) = \beta^q_{\mathbf{a}}(\mathbf{b},\mathbf{c})\gamma^q_{\mathbf{a}}(\mathbf{b}+\mathbf{c}). \tag{62}$$

The group 2-cocycle $\beta^q_{\mathbf{a}} \in C^2_{\text{group}}(\mathbb{Z}_n, U(1))$ is obtained from $\omega_q(\mathbf{a},\mathbf{b},\mathbf{c})$ [Eq. (12)] by taking a slant product, i.e., $\beta_{\mathbf{a}}(\mathbf{b},\mathbf{c}) = i_{\mathbf{a}}\omega(\mathbf{a},\mathbf{b},\mathbf{c})$ (for details, see App. A). Explicitly, $\beta^q_{\mathbf{a}}$ and $\gamma^q_{\mathbf{a}}$ take the form

$$\begin{aligned}\beta^q_{\mathbf{a}}(\mathbf{b},\mathbf{c}) &= \exp\left\{\frac{2\pi i}{n^2}\sum_I a_I(b_I + c_I - [b_I + c_I])\right\}\\ &\quad\times \exp\left\{\frac{2\pi i q_{IJ}}{n^2} a_I(b_J + c_J - [b_J + c_J])\right\},\\ \gamma^q_{\mathbf{a}}(\mathbf{b}) &= \exp\left\{\frac{2\pi i}{n^2}(\sum_I q_I a_I b_I + q_{IJ} a_I b_J)\right\}.\end{aligned} \tag{63}$$

$\bar{Z}^q_{\mathbf{b},\mathbf{a}}$ in (61) is easier to work with than $Z^q_{\mathbf{b},\mathbf{a}}$ since we do not need to evaluate the twisted partition function $Z^0_{\mathbf{b},\mathbf{a}}$ directly, which may sometimes be tedious. Further, $\bar{Z}^q_{\mathbf{b},\mathbf{a}}(\tau)$ and $Z^q_{\mathbf{b},\mathbf{a}}(\tau)$ have the same properties under modular and large gauge transformation, which is all we require. It is straightforward to check that modular matrices computed from $\bar{\chi}_{\mu,a}$ match up with (60),[34,78]

$$\begin{aligned}\bar{T}_{(\mu,\mathbf{a}),(\mu',\mathbf{a}')} &= \delta_{\mu,\mu'}\delta_{\mathbf{a},\mathbf{a}'}\mu(\mathbf{a})\gamma^q_{\mathbf{a}}(\mathbf{a}),\\ \bar{S}_{(\mu,\mathbf{a}),(\mu',\mathbf{a}')} &= \frac{1}{n}\mu(\mathbf{a}')\mu'^{-1}(-\mathbf{a})\gamma^q_{\mathbf{a}}(\mathbf{a}')\gamma^q_{\mathbf{a}'}(\mathbf{a}).\end{aligned} \tag{64}$$

SPT invariants from edge theory: Next we show that the SPT invariants for type-I and type-II SPTs can be computed directly from the edge theory (36). Let us consider an SPT protected by $G = \mathbb{Z}_n^k$ with symmetry action described by some combination of type-I and type-II 3-cocycles 'q'. Then such SPTs can be distinguished by their partition functions on lens space. In[22], it was shown that the Lens space partition function may be simulated by an expectation value of a non-local partial rotation operation on the groundstate on S^2. Let the theory (11) be defined on $N = S^2 \times S^1$, where S^2 is the spatial manifold. The theory has a unique groundstate $|GS^q_{S^2}\rangle$. The partition function on lens space may be simulated as

$$\mathcal{Z}^q[L(n,1),A] = \langle GS^q_{S^2}|\hat{C}_{n,D}(\mathbf{a})|GS^q_{S^2}\rangle \tag{65}$$

where $\hat{C}_{n,D}(\mathbf{a})$ is an operator that implements a partial n-fold rotation on a disc like subregion $D \subset S^2$ followed by the symmetry operation $\hat{\mathbf{a}}$. To motivate this definition, we recall the fact that lens space may be constructed from the surgery[79]

$$L(n,1) = [D^2 \times S^1] \sqcup_\varphi [D^2 \times S^1] \tag{66}$$

where $\sqcup_\varphi$ denotes gluing the boundaries of the two solid tori $\partial[D^2 \times S^1] = T^2$ via the large diffeomorphism $\varphi = ST^nS$. In[22], it was shown that $\hat{C}_{n,D}$ corresponds to the same diffeomorphism φ. Then the lens space partition function with background field holonomy $\mathbf{a} \in G$ around the torsion cycle may be computed as

$$\begin{aligned}\mathcal{Z}^q[L(n,1),A] &= \langle GS^q|\hat{C}_{n,D}(\mathbf{a})|GS^q\rangle\\ &= \frac{\mathrm{Tr}_{\mathcal{H}^q(D)}\left[\hat{C}_{n,D}(\mathbf{a})\rho_D\right]}{\mathrm{Tr}_{\mathcal{H}^q(D)}\left[\rho_D\right]}\end{aligned} \tag{67}$$

where we have traced out the disc-like region $\bar{D}$ compliment to D. We denote the Hilbert space on D (respectively ∂D) for the SPT described by 3-cocycle 'q'$\in H^3_{\text{group}}(G,U(1))$ as $\mathcal{H}^q(D)$ (respectively $\mathcal{H}^q(\partial D)$). The reduced density matrix on ρ_D is given by the thermal density matrix on ∂D at inverse temperature ξ, which is related to the bulk correlation length[80,81]. We note that ∂D is not a physical boundary but rather the boundary of region D where the partial rotation operator acts.

$$\rho_D = \frac{e^{-\xi\hat{H}_{\partial D}}}{\mathrm{Tr}_{\mathcal{H}^q(\partial D)}\left[e^{-\xi\hat{H}_{\partial D}}\right]}. \tag{68}$$

Then the lens space partition function may be evaluated as

$$\begin{aligned}\mathcal{Z}^q[L(n,1),A] &= \frac{\mathrm{Tr}_{\mathcal{H}^q(\partial D)}\left[\hat{C}_{n,\partial D}(\mathbf{a})e^{-\xi\hat{H}_{\partial D}}\right]}{\mathrm{Tr}_{\mathcal{H}^q(\partial D)}\left[e^{-\xi\hat{H}_{\partial D}}\right]}\\ &= \frac{\mathrm{Tr}_{\mathcal{H}^q(\partial D)}\left[\hat{\mathbf{a}}e^{-\frac{i\hat{P}L}{n}-\xi\hat{H}_{\partial D}}\right]}{\mathrm{Tr}_{\mathcal{H}^q(\partial D)}\left[e^{-\xi\hat{H}_{\partial D}}\right]}\\ &= \frac{Z^q_{(\mathbf{a},0)}\left(\frac{i\xi}{L}-\frac{1}{n}\right)}{Z^q_{(0,0)}\left(\frac{i\xi}{L}\right)}\\ &= \frac{\sum_{\mathbf{b}}(ST^nS)^{(\mathbf{b}_\tau,\mathbf{b}_x)}_{(\mathbf{a},0)}\,Z^q_{(\mathbf{b}_\tau,\mathbf{b}_x)}\left(-\frac{1}{n}+\frac{iL}{\xi n^2}\right)}{\sum_{\mathbf{b}}S^{(\mathbf{b}_\tau,\mathbf{b}_x)}_{(0,0)}Z^q_{(\mathbf{b}_\tau,\mathbf{b}_x)}\left(\frac{iL}{\xi}\right)}\\ &= e^{\frac{2\pi i(q_I a_I^2 + q_{IJ}a_I a_J)}{n}}\frac{Z^q_{(-\mathbf{a},0)}\left(-\frac{1}{n}+\frac{iL}{\xi n^2}\right)}{Z^q_{(0,0)}\left(\frac{iL}{\xi}\right)}\\ &= e^{\frac{2\pi i(q_I a_I^2 + q_{IJ}a_I a_J)}{n}}\left(1+\mathcal{O}(e^{-L/\xi})\right).\end{aligned} \tag{69}$$

In the last line we have taken the limit where the inverse temperature ξ is much smaller than L, the circumference of ∂D ($\xi/L \to 0$). Hence we can read off the SPT invariant

$$\mathcal{Z}^q[L(n,1),A] = e^{\frac{2\pi i}{n}\left(q_I a_I^2 + q_{IJ}a_I a_J\right)}. \tag{70}$$

III. $3+1d$ TOPOLOGICAL PHASES AND THEIR $2+1d$ GAPLESS SURFACES

A. Bulk physics

SPT effective actions: Similar to the 2+1-dimensional case, $3+1d$ SPTs can be modeled by multiple copies of level 1 BF theories with topologically distinct coupling to a flat background G bundle. SPT phases with G symmetry are classified by $H^4_{\text{group}}(G,U(1))$. For example consider $G=\mathbb{Z}_n^k$ bosonic SPTs which can be modeled by the following effective field theories[7,37,58,62]

$$\mathcal{S}^q(a,b,A)=\int_N \frac{\delta_{IJ}}{2\pi} b^I\wedge da^J+\mathcal{S}^q_{\text{cpl}}(a,b,A) \qquad (71)$$

where a and b are 1-form and 2-form $U(1)$ gauge field, $I,J=1,\ldots,k$, and q denotes the representative $\omega\in H^4_{\text{group}}(G,U(1))$, For $G=(\mathbb{Z}_n)^k$,

$$H^4_{\text{group}}[(\mathbb{Z}_n)^k,U(1)]=(\mathbb{Z}_n)^{\left[2\times\binom{k}{2}+\binom{k}{3}+\binom{k}{4}\right]} \qquad (72)$$

Different 4-cocycles $[\omega]\in H^4(G,U(1))$ are of three kinds named 'type-II,III,IV' which explicitly take the form

$$\begin{aligned}
\omega_{\text{type-II}}(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})&=e^{\frac{2\pi i q_{IJ}}{n^2}a^Ib^J\left(c^J+d^J-[c^J+d^J]\right)},\\
\omega_{\text{type-III}}(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})&=e^{\frac{2\pi i q_{IJK}}{n^2}a^Ib^J\left(c^K+d^K-[c^K+d^K]\right)},\\
\omega_{\text{type-IV}}(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})&=e^{\frac{2\pi i q_{IJKL}}{n}a^Ib^Jc^Kd^L},
\end{aligned} \qquad (73)$$

where $[a^I+b^I]$ denotes addition modulo n. Here $q=\{q_{IJ},q_{IJK},q_{IJKL}\}$ are a set of parameters valued in $\mathbb{Z}$ mod $n\mathbb{Z}$ that label different SPTs. Distinct SPT effective field theories differ in how they couple to the background flat G gauge field. The coupling terms corresponding to different cocycle types take the form

$$\begin{aligned}
\mathcal{S}^{q_{IJ}}_{\text{cpl}}(a,b,A)&=-\frac{1}{2\pi}\int_N A^I\wedge\left(b^I+\frac{nq_{IJ}}{2\pi}a^J\wedge da^J\right),\\
\mathcal{S}^{q_{IJK}}_{\text{cpl}}(a,b,A)&=-\frac{1}{2\pi}\int_N A^I\wedge\left(b^I+\frac{nq_{IJK}}{2\pi}a^J\wedge da^K\right),\\
\mathcal{S}^{q_{IJK}}_{\text{cpl}}(a,b,A)&=-\frac{1}{2\pi}\int_N A^I\wedge\left(b^I+\frac{n^3q_{IJKL}}{4\pi^2}a^J\wedge a^K\wedge a^L\right).
\end{aligned} \qquad (74)$$

Generally, the coupling to background field A^I may involve a combination of type-II,III,IV terms for some choice of 'q'. For simplicity we will treat these terms separately. The response theory can be obtained by integrating over the matter fields a,b.

$$e^{iI^q[N,A]}=\int\mathcal{D}[\{a,b\}]e^{i\mathcal{S}^q(a,b,N,A)}. \qquad (75)$$

The different response theories are

$$\begin{aligned}
e^{iI^{q_{IJ}}[N,A]}&=\exp\left\{-\frac{inq_{IJ}}{4\pi^2}\int_N A^I\wedge A^J\wedge dA^J\right\},\\
e^{iI^{q_{IJK}}[N,A]}&=\exp\left\{-\frac{inq_{IJK}}{4\pi^2}\int_N A^I\wedge A^J\wedge dA^K\right\},\\
e^{iI^{q_{IJKL}}[N,A]}&=\exp\left\{-\frac{in^3q_{IJKL}}{8\pi^3}\int_N A^I\wedge A^J\wedge A^K\wedge A^L\right\}.
\end{aligned} \qquad (76)$$

In the above, I,J,K,L are not summed over.

Topological invariants for SPTs: Following our previous strategy we evaluate these topological response theories on set of backgrounds $\{(N_i,A_i)\}_{i\in\mathcal{I}}$ which are the generators of the torsion subgroup of the equivariant cobordism group $\Omega_4^{SO}(BG)$[8,9]. The $U(1)$ phases $\left\{e^{-iI^q[N_i,A_i]}\right\}_{i\in\mathcal{I}}$ are the SPT invariants. For $G=\prod_I\mathbb{Z}_{n_I}$, the generating manifolds for type-II,III,IV terms parametrized by $\{q_{IJ},q_{IJK},q_{IJKL}\}$ are $\left\{L(\gcd(n_I,n_J),1)\times S^1, L(\gcd(n_I,n_J,n_K),1)\times S^1,T^4\right\}$ respectively, equipped with some appropriate G-bundle[63]. Here we compute invariants for $G=\mathbb{Z}_n^k$ for which evaluating the partition functions on $L(n,1)\times S^1$ and T^4 suffices. Generalization to other discrete abelian groups is straightforward.

- **Type-II and type-III cocycles:** Type-II and type-III cocycles can be detected on $N=L(n,1)\times S^1$. Let $S\in\text{Tor}\left(H_2(N,\mathbb{Z})\right)$ be Poincare dual to the generator of $A^J\in\text{Tor}\left(H^2(N,\mathbb{Z})\right)$. Then we obtain

$$\begin{aligned}
e^{iI^{q_{IJ}}[N,A]}&=\exp\left\{-\frac{inq_{IJ}}{4\pi^2}\int_N A^I\wedge A^J\wedge dA^J\right\}\\
&=\exp\left\{-\frac{inq_{IJ}}{2\pi}\int_{S=S^1\times C_{A^J}} A^I\wedge A^J\right\}\\
&=\exp\left\{-\frac{inq_{IJ}a_J}{2\pi}\int_{S=S^1\times C_1} A^I\wedge A^J\right\}\\
&=\exp\left\{-\frac{2\pi iq_{IJ}}{n}a_J(b_Ia_J-a_Ib_J)\right\}
\end{aligned} \qquad (77)$$

where we have decomposed the $S=S^1\times C$ where C is the torsion 1-cycle in N. (a_I,b_I) are the $\mathbb{Z}_n$ holonomies along C_1 and S^1 for the Ith flavor of $\mathbb{Z}_n$. The calculation for type-III follows very similarly.

$$\begin{aligned}
e^{iI^{q_{IJK}}[N,A]}&=\exp\left\{-\frac{inq_{IJK}}{4\pi^2}\int_N A^I\wedge A^J\wedge dA^K\right\}\\
&=\exp\left\{-\frac{inq_{IJK}}{2\pi}\int_{S=S^1\times C_{A^K}} A^I\wedge A^J\right\}\\
&=\exp\left\{-\frac{inq_{IJK}a_K}{2\pi}\int_{S=S^1\times C_1} A^I\wedge A^J\right\}\\
&=\exp\left\{-\frac{i2\pi q_{IJK}}{n}a_K(b_Ia_J-a_Ib_J)\right\}.
\end{aligned} \qquad (78)$$

- **Type-IV cocycles:** Type-IV topological term can be detected on T^4 with appropriate background flat G-bundle. The response theory evaluates to

$$\begin{aligned}
e^{iI^{q_{IJKL}}}&=\exp\left\{-\frac{in^3q_{IJKL}}{8\pi^3}\int_{T^4} A^I\wedge A^J\wedge A^K\wedge A^L\right\}\\
&=\exp\left\{-\frac{2\pi iq_{IJKL}}{n}\epsilon^{ijkl}a_{I,i}b_{J,j}c_{K,k}d_{L,l}\right\}
\end{aligned} \qquad (79)$$

where $I, J, K, L = 1, 2, 3, 4$, $\mathbf{a} = (a_1, a_2, a_3, a_4)$, and $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{Z}_n^4$ are the holonomies around the three cycles of T^4.

The complete set of topological invariants for bosonic SPTs protected by $G = \mathbb{Z}_n^k$ then is

$$\left\{ e^{-iI^q[L(n,1)\times S^1, A]}, e^{-iI^q[T^4, A]} \right\} \tag{80}$$

Topological gauge theories from Gauging SPTs: $3+1d$ SPTs can be gauged by first coupling to a flat bundle as we have done above and then summing over all possible flat bundles. The gauged partition function on a manifold N takes the form

$$\mathcal{Z}^q_{\text{DW}}[N] = \frac{1}{|H^0(N,G)|} \sum_{[A]} \mathcal{Z}^q[N, A]. \tag{81}$$

The gauged theory is the well-known Dijkgraaf-Witten theory which has topological order. The ground-state degeneracy on any 3-manifold M can be computed as $\mathcal{Z}^q_{\text{DW}}[M \times S^1] = \text{GSD}^q[M]$. These theories can be differentiated by the phases they assign to multi-linked configurations of vortices.[27,29,30,82–84]. These loop braiding statistics may be computed in the bulk by performing modular transformations on the basis of ground-states on a three-torus[82] and reading off the projective phases in the modular matrices. Alternately they may be computed from the Wilson operator algebra of the Dijkgraaf-Witten theories[29] or by directly computing partition functions on manifolds with multi-link vortex defects embedded. Type-II and type-III Dijkgraaf-Witten theories in $3+1d$ assign non-trivial braiding phases to linked three-loop configurations in spacetime or three-loop braiding processes whereas type-IV theory assigns non-trivial phases to linked four-loop configurations. Let us consider a few specific examples

- **Type-II and type-III Dijkgraaf-Witten theories:** Consider putting type-II or type-III theory on a manifold $N = M \times S^1$ and gauging. Suppose $\text{Tor}(H_1(M), \mathbb{Z}) = 0$. Then $I^q[N, A] = 1$, therefore we get

$$\mathcal{Z}^q_{\text{DW}}[M \times S^1] = \frac{1}{|G|} \sum_{[A]\in H^1(N,G)} 1 = |G|^{b_1(M)} =: \text{GSD}^q\left[M\right]. \tag{82}$$

Next if M has torsion, for example if $N = L(n,1) \times S^1$, for type-II cocycle with $G = \mathbb{Z}_n^2$ we get

$$\mathcal{Z}^{q_{IJ}}_{\text{DW}}[N] = \frac{1}{|n|^2} \sum_{a_I, b_I \in \mathbb{Z}_n} e^{-\frac{2\pi i q_{IJ}}{n} a_J (b_I a_J - a_I b_J)} =: \text{GSD}^{q_{IJ}}\left[L(n,1)\right] \tag{83}$$

Similarly for type-III cocycle with $G = \mathbb{Z}_n^3$ we get

$$\mathcal{Z}^{q_{IJK}}_{\text{DW}}[N] = \frac{1}{|n|^3} \sum_{a_I, b_I \in \mathbb{Z}_n} e^{-\frac{2\pi i q_{IJK}}{n} a_K (b_I a_J - a_I b_J)} =: \text{GSD}^{q_{IJK}}\left[L(n,1)\right] \tag{84}$$

- Unlike type-II and type-III Dijkgraaf-Witten theories in $3+1d$, for Type-IV cocycle, $\text{GSD}^q[T^3] < |G|^3$. Similar to type-III cocycle in $2+1d$ [Eq. (29)], this is related to the fact that type-IV DW theory actually has non-abelian excitations. In other words the quantum dimension of some of the quasivortices is greater than one. The partition function for $G = \mathbb{Z}_n^4$ on the four-torus is

$$\mathcal{Z}^{q_{IJKL}}_{\text{DW}}\left[T^4\right] = \frac{1}{n^4} \sum_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\in\mathbb{Z}_n^4} e^{-\frac{2\pi i q_{IJKL}}{n} \epsilon^{ijkl} a_{I,i} b_{J,j} c_{K,k} d_{L,l}} = \text{GSD}^{q_{IJKL}}\left[T^3\right] \tag{85}$$

where $\mathbf{a} = (a_1, a_2, a_3, a_4)$, and $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{Z}_n^4$ are the holonomies around the three cycles of T^4.

For some purposes it is convenient to formulate the G-gauged theory in the continuum as a coupled BF theory (see for example[29])

$$\mathcal{Z}^q_{\text{DW}}[N] = \frac{1}{|G|} \sum_{[A]\in H^1(N,G)} e^{iI^q[A]} \xrightarrow{G=\mathbb{Z}_n^k} \int \prod_{I=1}^{k} \mathcal{D}[A^I, B^I] e^{\frac{in\delta_{IJ}}{2\pi}\int B^I\wedge dA^J + iI^q[A]} \tag{86}$$

where A and B are 1-form and 2-form $U(1)$ connections with standard quantization conditions. Since $(1/2\pi)dB^I \in \Omega^3_{\mathbb{Z}}(N)$, we can integrate them out to impose that A^I are flat $\mathbb{Z}_n$ gauge fields. The calculation is very similar to (25).

Ungauging in the $3+1d$ bulk: More generally one can gauge G in the presence of background quasiparticle sources $J_{qp} \in H^3(N, \hat{G})$. The gauged partition function takes the form

$$\mathcal{Z}^q_{\text{DW}}[N, J_{qp}] = \int \mathcal{D}[A, B] e^{\frac{in\delta_{IJ}}{2\pi}\int B^I\wedge dA^J + iI^q[A] + i\int_N J^I_{qp}\wedge A^I} \tag{87}$$

where the background fields J^I_{qp} are 3-form fields with integral periods[65]. Since upon integrating out B^I, $\oint A^I \in (2\pi\mathbb{Z})/n$, the periods of J_{qp} are only physically distinguishable modulo n, more precisely $J_{qp} \in H^3(N, \widehat{G})$ where $\widehat{G} = \text{Rep}(G) \simeq G$. There is a perfect pairing

$$\int_N : H^1(N, G) \times H^3(N, \widehat{G}) \to \mathbb{R}/2\pi\mathbb{Z} \tag{88}$$

that is realized by wedge product followed by integration. J_{qp} generates a 2-form $\widehat{G}$ symmetry implemented by the charge operator $\mathcal{Q}^I(\lambda^I)$ corresponding to $\widehat{G}$ symmetry.

$$\mathcal{Q}^I(\lambda^I) := \frac{1}{2\pi} \int_M \lambda^I \wedge A^I \tag{89}$$

where $\lambda^I \in \Omega^2_{\mathbb{Z}}(M)$. Then the 2-form symmetry acts

$$\begin{aligned}\mathcal{Q}^I(\lambda^I) : J^I_{qv} &\mapsto J^I_{qv} + d\lambda^I;\\ : B^I &\mapsto B^I - \lambda^I\end{aligned} \tag{90}$$

Gauging this dual 2-form symmetry means summing over $J^I_{qp} \in H^3(M, \widehat{G})$. Let us call the partition function after gauging the 2 -form $\widehat{G}$ symmetry $\mathcal{Z}^q_{\mathrm{DW}/\widehat{G}}$, then

$$\begin{aligned}\mathcal{Z}^q_{\mathrm{DW}/\widehat{G}}[N, \hat{A}] &= \sum_{J_{qp}} e^{-i\int_N J_{qp}\wedge\hat{A}} \mathcal{Z}^q_{\mathrm{DW}}[N, J_{qp}]\\ &= \sum_{J_{qp}}\sum_{A} e^{i\int_N J_{qp}\wedge(A-\hat{A})+iI^q[N,A]}\\ &= e^{iI^q[N,\hat{A}]}.\end{aligned} \tag{91}$$

Hence gauging the dual $\widehat{G}$ 1-form global symmetry is equivalent to un-gauging. The symmetry is generated by the world-line of A and may be understood as physically as proliferating or condensing the gauge charge $\sim dB$ which is always bosonic since $[B, B] = 0$. Hence this procedure works for all bosonic SPTs protected by onsite symmetry.

B. Surface physics

We model the gapless surface of $3 + 1d$ bosonic SPTs described by (71) by the following quantum field theory[30,85–87]

$$S = \int_M \sum_{I=1}^{k} \left[\frac{1}{2\pi} d\zeta^I \wedge d\phi^I - \mathcal{H}(\zeta^I, \phi^I)\right]. \tag{92}$$

Here, $\phi^I : M \to \mathbb{R}/2\pi\mathbb{Z}$ and ζ^I are 1-form $U(1)$ connections which satisfy the Dirac quantization condition

$$\oint_{Z_1(M,\mathbb{Z})} \frac{d\phi^I}{2\pi} \in \mathbb{Z}; \qquad \oint_{Z_2(M,\mathbb{Z})} \frac{d\zeta^I}{2\pi} \in \mathbb{Z}. \tag{93}$$

This model has a global 0-form (and 1-form) $U(1)^k$ symmetry. We will however be interested in the discrete subgroup $G = \mathbb{Z}^k_n \subset U(1)^k$. Similar to $1 + 1d$, we probe the theory by coupling to a flat G gauge field $A \in H^1(M, G)$ and use modular invariance of the orbifolded partition function as a diagnostic for whether the model with a specific action of G has a 't-Hooft anomaly. In other words we put the theory on $M = T^3$ and check whether it is possible to construct a partition function upon summing all twisted sectors (flat G bundles) such that the summed partition function is invariant under large diffeomorphisms of M as well as large gauge transformations. The group of large diffeomorphisms on $M = T^3$, i.e., $MCG(T^3) = SL(3, \mathbb{Z})$ which is generated by U_1, U_2 with the action

$$\begin{aligned}U_1 : \begin{pmatrix} t \\ x \\ y \end{pmatrix} &\mapsto \begin{pmatrix} y \\ t \\ x \end{pmatrix},\\ U_2 : \begin{pmatrix} t \\ x \\ y \end{pmatrix} &\mapsto \begin{pmatrix} t+x \\ x \\ y \end{pmatrix}.\end{aligned} \tag{94}$$

A modular invariant partition function is one for which

$$Z[UM] = Z[M]; \quad U \in MCG(T^3) \tag{95}$$

The diagnostic for a theory with a global or 't-Hooft anomaly will be the inexistence of a modular invariant partition for the gauged (or orbifolded theory). For a review of quantization of (92), see App. D.

As with the $1 + 1d$ case (36), we expect (92) to accommodate distinct realizations of $G = \mathbb{Z}^k_n$ which we label by 'q'. We will denote partition functions of these models in the presence of a background G bundle A as $Z^q[M, A]$. By anomaly matching one can learn that these quantum field theories require a bulk which cancels the anomaly. Such bulk theories would be provided by SPT effective actions (71). As a warm-up let us consider the simplest G action which is non-anomalous and hence does not require a bulk to support it.

Non-anomalous 0-form $\mathbb{Z}_n$ symmetry: A single copy of (92) is invariant under a global 0-form $U(1)$ symmetry

$$\phi(x) \mapsto \phi(x) + \alpha \tag{96}$$

where α is a constant valued in $\mathbb{R}/2\pi\mathbb{Z}$. Gauging this $U(1)$ symmetry implies introducing a flat 1-form $U(1)$ gauge field A and replacing the differential

$$d\phi \mapsto D_A\phi := d\phi + A \tag{97}$$

with the gauge transformation

$$\begin{aligned}\phi(x) &\mapsto \phi(x) + \alpha(x),\\ A(x) &\mapsto A(x) - d\alpha(x).\end{aligned} \tag{98}$$

Here we gauge a subgroup $\mathbb{Z}_n \subset U(1)$ by restricting the holonomies of A to $\mathbb{Z}_n$. Then defining $d\tilde{\phi} := D_A\phi$ which obeys the twisted quantization condition

$$\oint_L \frac{d\tilde{\phi}}{2\pi} \in \mathbb{Z} + \oint_L \frac{A}{2\pi} \tag{99}$$

i.e., quantizing in the presence of background A implies imposing twisted boundary condition. Then the gauging procedure is the same as before; First we compute the partition functions in the twisted sectors $Z^0[M, A]$ and then sum over them

$$Z^0_{\mathrm{orb}}[M] = \frac{1}{|H^0(M, G)|} \sum_{[A]\in H^1(M,G)} \theta(A) Z^0[M, A]. \tag{100}$$

We compute $Z^0_{\rm orb}[M,A]$ within the canonical formalism. Following (99) we impose twisted boundary conditions. Let us set $M=T^3$ and the holonomies of A along the x,y cycles be $\lambda_{1,2}$ respectively, then the twisted Hilbert space is defined as

$$\mathcal{H}_{\lambda_1,\lambda_2}=\left\{\phi(x,y),\zeta(x,y)\Big|\oint_{L_{1,2}}d\phi=\frac{2\pi}{n}\lambda_{1,2}\right\}. \quad (101)$$

Similarly, we can also twist in the time direction, in the path integral picture, this means coupling to a background $\mathbb{Z}_n$ field with non-trivial holonomy in the time-cycle. In the canonical formalism, this is implemented via a global $\mathbb{Z}_n$ symmetry operator

$$\begin{aligned}\mathcal{G}(\lambda_0):=&\ \exp\left\{\frac{2\pi i\lambda_0}{n}\mathcal{Q}\right\}\\ =&\ \exp\left\{\frac{i\lambda_0}{n}\int_{T^2}d\zeta\right\}=\exp\left\{\frac{2\pi i\lambda_0\beta_0}{n}\right\}\end{aligned} \quad (102)$$

where β_0 is defined in (D3). $\mathcal{G}(\lambda_0)$ implements the transformation $\phi\mapsto\phi+2\pi\lambda_0/n$.

$$\mathcal{G}(\lambda_0):\phi\mapsto\phi+\frac{2\pi\lambda_0}{n}. \quad (103)$$

Then the partition function in the twisted sectors are computed as[30]

$$\begin{aligned}Z^0_{\lambda_0,\lambda_1,\lambda_2}&={\rm Tr}_{\mathcal{H}_{\lambda_1,\lambda_2}}\left[\mathcal{G}(\lambda_0)e^{2\pi iR_0H'}\right]\\ &=Z_{osc}\sum_{N_{0,1,2}\in\mathbb{Z}}\exp\left\{-\frac{\pi\tau_2}{2R_2}N_0^2\right.\\ &-2\pi R_2\tau_2\left(N_1+\frac{\lambda_1}{n}\right)^2-\frac{2\pi R_0R_1}{R_2}\left(N_2+\frac{\lambda_2}{n}\right)^2\\ &\left.+2\pi i\tau_1N_0\left(N_1+\frac{\lambda_1}{n}\right)+\frac{2\pi iN_0\lambda_0}{n}\right\}\end{aligned} \quad (104)$$

As we will mostly be working on T^3, we simply label the partition functions with $\lambda_{0,1,2}$, the G holonomies on T^3. Under $SL(3,\mathbb{Z})$ modular transformations, the twisted sectors transform as

$$\begin{aligned}U_2Z^0_{\lambda_0,\lambda_1,\lambda_2}&=Z^0_{\lambda_0-\lambda_1,\lambda_1,\lambda_2},\\ MZ^0_{\lambda_0,\lambda_1,\lambda_2}&=Z^0_{\lambda_0,-\lambda_2,\lambda_1},\\ U_1'Z^0_{\lambda_0,\lambda_1,\lambda_2}&=Z^0_{\lambda_1,\lambda_0,\lambda_2}.\end{aligned} \quad (105)$$

A modular invariant partition function may be constructed by taking an equal weight sum, i.e., $\theta(A)=1$ in (100)

$$Z_{\rm orb}=\frac{1}{n}\sum_{\lambda_0,\lambda_1,\lambda_2\in\mathbb{Z}_n}Z^0_{\lambda_0,\lambda_1,\lambda_2} \quad (106)$$

In fact, we need not choose $\theta(A)=1$. We saw in (51), there was a freedom worth $H^2_{\rm group}(G,U(1))$ in constructing a modular invariant partition function which corresponded to pasting a $1+1d$ G SPT onto (36) and then gauging. Similarly in $2+1d$, given a modular invariant partition function, we can always find a new one by picking a $[\beta]\in H^3(BG,\mathbb{R}/2\pi\mathbb{Z})$ and orbifolding with phase-factors

$$Z^\beta_{\rm orb}[M]=\frac{1}{|G|}\sum_{[\mathcal{A}]\in{\rm Map}[M,BG]}e^{i\int_M\mathcal{A}^*\beta}Z[M,\mathcal{A}^*EG] \quad (107)$$

where BG is the classifying space of G which is a space whose homotopy groups satisfy the property $\pi_i(BG)=\delta_{i,1}G$. Furthermore BG admits a certain G bundle EG known as the universal. Together these have the nice properties that isomorphism classes of G bundles a manifold M are equivalent to homotopy classes of maps from M to BG. Furthermore one may obtain any G bundle $E\to M$ as a pullback of the universal bundle onto M. Above, $\exp\{i\int_M\mathcal{A}^*\beta\}$ is the partition function for an $2+1$-dimensional G SPT with background flux $\mathcal{A}^*EG$, hence the freedom of adding a phase corresponds to pasting a $2+1d$ SPT onto (92).

Anomalous symmetry action: Let us consider orbifolding G action corresponding to type-II or type-III cocycle. The minimum case where such a symmetry can be implemented is for $G=\mathbb{Z}_n^3$ on three copies of (92).

$$S=\int_M\left[\frac{\delta_{IJ}}{2\pi}d\phi^I\wedge d\zeta^J-\mathcal{H}(\phi^I,\zeta^I)\right] \quad (108)$$

where $I,J=1,2,3$. The simplest G action acts independently on the three copies as (96) as described above. Other G-actions couple the multiple copies in a non-trivial way and may be labelled by $q=\{q_{IJ},q_{IJK}\}$. Let us consider the coupling to background G field A and consider the action

$$\begin{aligned}S=\int_M\sum_{I,J=1,2}&\left[\frac{\delta_{IJ}}{2\pi}d\phi^I\wedge d\zeta^J-\mathcal{H}(\phi^I,\zeta^I)\right.\\ &+\frac{1}{2\pi}A^I\wedge\Big(d\zeta^I+\frac{n}{2\pi}q^{IJ}d\phi^I\wedge d\phi^J\\ &\left.+\frac{n}{2\pi}q_{IJK}d\phi^J\wedge d\phi^K\Big)\right]\end{aligned} \quad (109)$$

where $q_{IJ},q_{IJK}\in[0,\dots,n-1]$ are $\mathbb{Z}_n$ valued parameters that parametrize distinct couplings to the background field. By inspecting the equations of motion we learn that the fields ϕ^I and ζ^I satisfy twisted boundary conditions

$$\begin{aligned}&\frac{1}{2\pi}\oint_L d\phi^I=\frac{1}{2\pi}\oint_L A^I,\\ &\frac{1}{2\pi}\oint_S d\zeta^I=\frac{q_{IJ}n}{4\pi^2}\oint_S d\phi^I\wedge A^J+\frac{q_{IJK}n}{4\pi^2}\oint_S d\phi^J\wedge A^K.\end{aligned} \quad (110)$$

Upon fixing background A such that

$$\oint_{L_i\in H_1(T^3,Z)}A^I=\frac{2\pi\lambda_i^I}{n} \quad (111)$$

we define twisted Hilbert spaces as

$$\mathcal{H}^q_{\lambda_1^I,\lambda_2^J} = \Big\{\phi^I(x,y),\zeta^I(x,y)\Big| \oint_{L_i} d\phi^I = \frac{2\pi\lambda_i^I}{n}, \\ \oint_{T^2} d\zeta^I = \frac{2\pi}{n}\left(q_{IJ}\epsilon^{ij}\lambda_i^I\lambda_j^J + q_{IJK}\epsilon^{ij}\lambda_i^J\lambda_j^K\right)\Big\} \tag{112}$$

The symmetry operators take the form

$$\begin{aligned}\mathcal{G}^q_I(\lambda_0^I) &= \exp\left\{\frac{2\pi i\lambda_0^I}{n}\mathcal{Q}^I\right\}; \quad \text{where } \mathcal{Q}^I := \int_\Sigma \frac{\delta\mathcal{L}}{\delta A_0^I} \\ &= \exp\left\{\frac{2\pi i\lambda_0^I}{n}\int_{T^2}\left(d\zeta^I + \frac{nq_{IJ}}{2\pi}d\phi^I\wedge d\phi^J + \frac{nq_{IJK}}{2\pi}d\phi^J\wedge d\phi^K\right)\right\} \\ &= \exp\left\{\frac{2\pi i\lambda_0^I}{n}\left[\beta_0^I + \frac{nq_{IJ}}{2\pi}\epsilon^{ij}\beta_i^I\beta_j^J + \frac{nq_{IJK}}{2\pi}\epsilon^{ij}\beta_i^J\beta_j^K\right]\right\}\end{aligned} \tag{113}$$

Using the twisted Hilbert space (112) and the symmetry operator (113), the twisted partition functions can be computed

$$\begin{aligned}Z^q_{\lambda_0^I,\lambda_1^I,\lambda_2^I} &= \mathrm{Tr}_{\mathcal{H}^q_{\lambda_i^I}}\left[\mathcal{G}^q_I(\lambda_0^I)e^{2\pi i R_0 H'}\right] \\ &= Z_{osc}\sum_{N^I_{0,1,2}\in\mathbb{Z}}\exp\sum_{I=1,2,3}\Bigg\{-\frac{\pi\tau_2}{2R_2}\left[N_0^I + \epsilon^{ij}q_{IJ}\left(N_i^I+\frac{\lambda_i^I}{n}\right)\lambda_j^J + \epsilon^{ij}q_{IJK}\left(N_i^J+\frac{\lambda_i^J}{n}\right)\lambda_j^K\right]^2 \\ &\quad -2\pi R_2\tau_2\left(N_1^I+\frac{\lambda_1^I}{n}\right)^2 - \frac{2\pi R_0R_1}{R_2}\left(N_2^I+\frac{\lambda_2^I}{n}\right)^2 \\ &\quad + 2\pi i\tau_1\left(N_1^I+\frac{\lambda_1^I}{n}\right)\left[N_0^I + \epsilon^{ij}q_{IJ}\left(N_i^I+\frac{\lambda_i^I}{n}\right)\lambda_j^J + \epsilon^{ij}q_{IJK}\left(N_i^J+\frac{\lambda_i^J}{n}\right)\lambda_j^K\right] \\ &\quad + \frac{2\pi i\lambda_0^I}{n}\left[N_0^I + 2\epsilon^{ij}q_{IJ}\left(N_i^I+\frac{\lambda_i^I}{n}\right)\lambda_j^J + 2\epsilon^{ij}q_{IJK}\left(N_i^J+\frac{\lambda_i^J}{n}\right)\lambda_j^K\right]\Bigg\}.\end{aligned} \tag{114}$$

Under large gauge transformations, the partition functions in the different sectors transform as

$$Z^q_{\lambda_0^I+n\mathbf{e}_I,\lambda_1^I,\lambda_2^I} = e^{\frac{2\pi i\epsilon_{ij}}{n}\left(q_{IJ}\lambda_i^I\lambda_j^J + q_{IJK}\lambda_i^J\lambda_j^K\right)}Z^q_{\lambda_0^I,\lambda_1^I,\lambda_2^I}. \tag{115}$$

On the other hand, under $SL(3,\mathbb{Z})$ modular transformations, the partition functions in the different sectors transforms as

$$\begin{aligned}U_2 Z^q_{\lambda_0^I,\lambda_1^I,\lambda_2^I} &= e^{-\sum_I\frac{2\pi i\lambda_1^I\epsilon^{ij}}{n^2}\left(q_{IJ}\lambda_i^I\lambda_j^J+q_{IJK}\lambda_i^J\lambda_j^K\right)}Z^q_{\lambda_0^I-\lambda_1^I,\lambda_1^I,\lambda_2^I}, \\ MZ^q_{\lambda_0^I,\lambda_1^I,\lambda_2^I} &= Z^q_{\lambda_0^I,-\lambda_2^I,\lambda_1^I}, \\ U_1' Z^q_{\lambda_0^I,\lambda_1^I,\lambda_2^I} &= e^{\sum_I\frac{4\pi i\lambda_0^I\epsilon^{ij}}{n^2}\left(q_{IJ}\lambda_i^I\lambda_j^J+q_{IJK}\lambda_i^J\lambda_j^K\right)}Z^q_{-\lambda_1^I,\lambda_0^I,\lambda_2^I}.\end{aligned} \tag{116}$$

Let us try to construct a modular invariant partition function

$$Z^q_{\text{orb}} = \frac{1}{|G|}\sum_{\lambda_0^I,\lambda_1^I,\lambda_2^I\in G}\theta^q(\lambda_0^I,\lambda_1^I,\lambda_2^I)Z^q_{\lambda_0^I,\lambda_1^I,\lambda_2^I}. \tag{117}$$

Imposing invariance under U_2 transformation, we obtain

$$\frac{\theta^q(\lambda_0^I,\lambda_1^I,\lambda_2^I)}{\theta^q(\lambda_0^I-\lambda_1^I,\lambda_1^I,\lambda_2^I)} = e^{\sum_I\frac{2\pi i}{n^2}\left(q_{IJ}(\lambda_1^I)^2\lambda_2^J + q_{IJK}(\lambda_1^J)^2\lambda_2^K\right)}. \tag{118}$$

Inspecting the U_2 transformation property of θ^q, we find the following constraints under large gauge transformations

$$\begin{aligned}\theta^q\left(n(\mathbf{e}_I+\mathbf{e}_J),\mathbf{e}_I+\mathbf{e}_J,\mathbf{e}_J\right) &= e^{\frac{2\pi i q_{IJ}}{n}}\theta^q\left(0,\mathbf{e}_I+\mathbf{e}_J,\mathbf{e}_J\right), \\ \theta^q\left(n(\mathbf{e}_I+\mathbf{e}_J),\mathbf{e}_I+\mathbf{e}_J,\mathbf{e}_K\right) &= e^{\frac{2\pi i q_{IJK}}{n}}\theta^q(0,\mathbf{e}_I+\mathbf{e}_J,\mathbf{e}_K).\end{aligned} \tag{119}$$

This shows that there is a conflict between gauge invariance and modular invariance when $q\neq 0$ indicating a 't-Hooft anomaly.

To show that this 't-Hooft anomaly for $Z^q[M,A]$ is cancelled by a bulk SPT, consider the following combination of type-II and type-III response theories (76):

$$I^q[N,A] = -\frac{n}{4\pi^2}\int_N \Big\{ q_{IJ} A^J \wedge A^I \wedge dA^I + q_{IJK} A^J \wedge A^K \wedge dA^I \Big\}. \quad (120)$$

Let $N = D^2 \times S^1 \times S^1$ with a G configuration such that the holonomies around the first and second S^1 are λ_1^I and λ_2^I respectively. Further consider a puncture on D^2 such that the holonomy of the gauge field around ∂D^2 is λ_0^I. We denote this configuration $[N,A] \equiv D^2_{\lambda_0^I} \times S^1_{\lambda_1^I} \times S^1_{\lambda_2^I}$. The response theory for this background configuration evaluates to

$$e^{iI^q[D^2_{\lambda_0^I}\times S^1_{\lambda_1^I}\times S^1_{\lambda_2^I},A]} = e^{-\frac{2\pi i\epsilon^{ij}}{n^2}\left[q_{IJ}\lambda_0^I\lambda_i^I\lambda_j^J + q_{IJ}\lambda_0^I\lambda_i^J\lambda_j^K\right]} \quad (121)$$

which has the same properties as those required from $\theta^q(\lambda_0^I,\lambda_1^I,\lambda_2^I)$ in order to make the gauged theory consistent.

G-characters and topological data: Similar to the 1+1-dimensional case one can construct G characters from the $2+1d$ surface theory which encode topological data of the bulk topological gauge theory labelled by $[\omega] \in H^4_{\text{group}}(G,U(1))$. The characters are constructed as

$$\chi^q_{\mu^I,\lambda_1^I,\lambda_2^I} = \frac{1}{\sqrt{|G|}}\sum_{\lambda_0^I\in G}\mu^I(\lambda_0^I) Z^q_{\lambda_0^I,\lambda_1^I,\lambda_2^I} \quad (122)$$

where $\mu \in \text{Rep}(G)$, for $G = \mathbb{Z}_n^k$, $\mu^I(\lambda_0^I) = \exp\left\{\frac{2\pi i \delta_{IJ}\mu^I\lambda_0^J}{n}\right\}$. Instead of working with $Z^q_{\lambda_0^I,\lambda_1^I,\lambda_2^I}$, we find it convenient and illustrative to work with $\bar{Z}^q_{\lambda_0^I,\lambda_1^I,\lambda_2^I}$, where

$$\bar{Z}^q_{\lambda_0^I,\lambda_1^I,\lambda_2^I} := \gamma^q_{\lambda_1^I,\lambda_2^I}(\lambda_0^I) Z^0_{\lambda_0^I,\lambda_1^I,\lambda_2^I} \quad (123)$$

where $\gamma^q_{\lambda_1^I,\lambda_2^I}$ is a projective G representation which satisfies

$$\gamma^q_{\lambda_1^I,\lambda_2^I}(\lambda_0^I)\gamma^q_{\lambda_1^I,\lambda_2^I}(\lambda_0^{I'}) = \beta^q_{\lambda_1^I,\lambda_2^I}(\lambda_0^I,\lambda_0^{I'})\gamma^q_{\lambda_1^I,\lambda_2^I}(\lambda_0^I+\lambda_0^{I'})$$

where $\quad \beta^q_{\lambda_1^I,\lambda_2^I}(\lambda_0^I,\lambda_0^{I'}) = i_{\lambda_1^I} i_{\lambda_2^I}\omega^q(\lambda_1^I,\lambda_2^I,\lambda_0^I,\lambda_0^{I'})$

where $i_\lambda\omega$ denotes the slant product with respect to λ, for details, please see App. A. We note that $\bar{Z}^q_{\lambda_0^I,\lambda_1^I,\lambda_2^I}$ and $Z^q_{\lambda_0^I,\lambda_1^I,\lambda_2^I}$ have the same properties under modular and large gauge transformations, hence it will suffice for our purposes to use $\bar{Z}^q$ instead of Z^q. Then we may write

$$\begin{aligned}\bar{\chi}^q_{\mu^I,\lambda_1^I,\lambda_2^I} &= \frac{1}{\sqrt{|G|}}\sum_{\lambda_0^I\in G}\mu^I(\lambda_0^I)\bar{Z}^q_{\lambda_0^I,\lambda_1^I,\lambda_2^I} \\ &=: \frac{1}{\sqrt{|G|}}\sum_{\lambda_0^I\in G}\Gamma^q_{\mu^I,\lambda_1^I,\lambda_2^I}(\lambda_0^I) Z^0_{\lambda_0^I,\lambda_1^I,\lambda_2^I} \quad (124)\end{aligned}$$

where $\Gamma^q_{\mu^I,\lambda_1^I,\lambda_2^I}(\lambda_0^I) := \mu^I(\lambda_0^I)\gamma^q_{\lambda_1^I,\lambda_2^I}(\lambda_0^I)$. For the specific case of type-II and type-III cocycle, $\gamma^q_{\lambda_1^I,\lambda_2^I}(\lambda_0^I)$ takes the form

$$\gamma^q_{\lambda_1^I,\lambda_2^I}(\lambda_0^I) = \exp\left\{\frac{2\pi i}{n^2}\left(q_{IJ}\lambda_0^I\lambda_1^I\lambda_2^J + q_{IJK}\lambda_0^I\lambda_1^J\lambda_2^K\right)\right\} \quad (125)$$

By the bulk boundary correspondence, the character $\bar{\chi}^q_{\mu^I,\lambda_1^I,\lambda_2^I}$ corresponds to a bulk excitation with linked fluxes λ_1^I and λ_2^I and charge μ^I.[30,88,77] The dimension of the representation $\dim(\Gamma^q_{\mu^I,\lambda_1^I,\lambda_2^I})$ is the quantum dimension of the excitation corresponding to $\bar{Z}^q_{\lambda_0^I,\lambda_1^I,\lambda_2^I}$.

The modular $SL(3,\mathbb{Z})$ matrices can be computed as

$$\begin{aligned}U_2\bar{\chi}^q_{\mu^I,\lambda_1^I,\lambda_2^I} &= \frac{1}{\sqrt{|G|}}\sum_{\lambda_0^I\in G}\Gamma^q_{\mu^I,\lambda_1^I,\lambda_2^I}(\lambda_0^I) Z^0_{\lambda_0^I+\lambda_1^I,\lambda_1^I,\lambda_2^I} \\ &= \frac{1}{\sqrt{|G|}}\sum_{\lambda_0^I}\frac{\Gamma^q_{\mu^I,\lambda_1^I,\lambda_2^I}(\lambda_0^I)}{\Gamma^q_{\mu^I,\lambda_1^I,\lambda_2^I}(\lambda_0^I+\lambda_1^I)}\Gamma^q_{\mu^I,\lambda_1^I,\lambda_2^I}(\lambda_0^I+\lambda_1^I) Z^0_{\lambda_0^I+\lambda_1^I,\lambda_1^I,\lambda_2^I} \\ &= \exp\left\{-\frac{2\pi i\lambda_0^I}{n^2}\left(q_{IJ}\lambda_1^I\lambda_2^J + q_{IJK}\lambda_1^J\lambda_2^K\right) - \frac{2\pi i\delta_{IJ}\mu^I\lambda_1^J}{n}\right\}\bar{\chi}^q_{\mu^I,\lambda_1^I,\lambda_2^I}, \\ U'_1\bar{\chi}^q_{\mu^I,\lambda_1^I,\lambda_2^I} &= \frac{1}{\sqrt{|G|}}\sum_{\lambda_0^I}\Gamma^q_{\mu^I,\lambda_1^I,\lambda_2^I}(\lambda_0^I) Z_{\lambda_1^I,\lambda_0^I,\lambda_2^I} \\ &= \frac{1}{|G|}\sum_{\lambda_1^{I'},\mu^{I'}}\Gamma^q_{\mu^I,\lambda_1^I,\lambda_2^I}(\lambda_1^{I'})\left[\Gamma^q_{\mu^{I'},\lambda_1^{I'},\lambda_2^I}(\lambda_1^I)\right]^{-1}\bar{\chi}^q_{\mu^{I'},\lambda_1^{I'},\lambda_2^I} \quad (126)\end{aligned}$$

These match with modular matrices computed directly from the orbifold partition functions with twisted symmetry action. The U_2 eigenvalues are analogous to topological spin for string operators whereas the projective phases for the U_1' transformation encodes the braiding statistics between string-like and particle like excitations as well braiding of three-strings known as three-loop braiding.[27,29,30,82]

SPT invariants from surface computations: Above we saw that bosonic SPTs protected by $G = \mathbb{Z}_n^k$ and described by type-II and/or type-III 4-cocycles 'q'$\in H^4_{\text{group}}(G, U(1))$ can be detected by their partition functions on $L(n,1)\times S^1$ with appropriate background G-bundle. Now we show that these invariants can be directly computed from the surface theory (92). This computation is based on the fact that the partition function on $L(n,1)\times S^1$ can be simulated by the groundstate expectation value of a partial C_n rotation operation on the spatial manifold $S^2\times S^1$.[22]

Consider putting the theory (71) with type-II and/or type-III coupling to background field A on spatial manifold $M = S^2\times S^1$, since $H_1(S^2\times S^1,\mathbb{Z}) = \mathbb{Z}$, we may introduce a background field with holonomy $\mathbf{b}\in G$ around this spatial S^1. We denote this groundstate as $|GS^q_{S^2\times S^1_{\mathbf{b}}}\rangle$. Let $\hat{C}_{n,D}(a)$ be an operator implementing a non-local partial rotation on a disc-like region $D\subset S^2$ with flux $\mathbf{a}\in G$ inserted. Then we may show that the SPT invariant is given by the phase of

$$\begin{aligned}\mathcal{Z}^q[L(n,1)\times S^1_{\mathbf{b}}, A] &= \langle GS^q_{S^2\times S^1_{\mathbf{b}}}|\hat{C}_{n,D}(\mathbf{a})|GS^q_{S^2\times S^1_{\mathbf{b}}}\rangle \\ &= \frac{\mathrm{Tr}_{\mathcal{H}^q(D\times S^1_{\mathbf{b}})}\left[\rho_{D\times S^1_{\mathbf{b}}}\hat{C}_{n,D}(\mathbf{a})\right]}{\mathrm{Tr}_{\mathcal{H}^q(D\times S^1_{\mathbf{b}})}\left[\rho_{D\times S^1_{\mathbf{b}}}\right]} \\ &= \frac{\mathrm{Tr}_{\mathcal{H}^q(S^1\times S^1_{\mathbf{b}})}\left[e^{-\xi H_{T^2}}\hat{C}_{n,\partial D}(\mathbf{a})\right]}{\mathrm{Tr}_{\mathcal{H}^q(S^1\times S^1_{\mathbf{b}})}\left[e^{-\xi H_{T^2}}\right]}\end{aligned} \tag{127}$$

where we have traced over disc-like region $\bar{D}$ complement to $D\subset S^2$ and used the fact that the reduced density matrix effectively reduces to the thermal density matrix on $\partial[D\times S^1] = T^2$. We note that T^2 is not a physical boundary instead it is simply the boundary of the region on which the partial rotation operator $\hat{C}_{n,D}$ acts.

$$\rho_{D\times S^1_{\mathbf{b}}} \approx \frac{e^{-\xi H_{T^2}}}{\mathrm{Tr}_{\mathcal{H}^q(S^1\times S^1_{\mathbf{b}})}\left[e^{-\xi H_{T^2}}\right]} \tag{128}$$

where $\mathcal{H}^q(S^1\times S^1_{\mathbf{b}})$ is the Hilbert space on the torus with holonomies $0,\mathbf{b}\in G$ along $(\partial D, S^1_{\mathbf{b}})$ respectively. Let $\tau := \tau_1 + i\tau_2$ denote the modular parameter for the $t-x$ two torus on which the modular matrices $U_2, U_1^{'}$ act as $T, S\in SL(2,\mathbb{Z})\subset SL(3,\mathbb{Z})$. Since the $\hat{C}_{n,\partial D}$ acts as a boost along the x direction. The computation is effectively very similar to the $1+1d$ calculation (69), except with holonomy $\mathbf{b}\in G$ inserted along the S^1 cycle in the y-direction

$$\begin{aligned}\mathcal{Z}^q[L(n,1)\times S^1, A] &= \frac{Z^q_{(\mathbf{a},0,\mathbf{b})}\left(\tau = \frac{i\xi}{L} - \frac{1}{n}\right)}{Z^q_{(0,0,0)}\left(\tau = \frac{i\xi}{L}\right)} \\ &= \frac{\sum_{\mathbf{c}}\left(\Gamma\right)^{(\mathbf{c}_\tau,\mathbf{c}_x,\mathbf{c}_y)}_{(\mathbf{a},0,\mathbf{b})} Z^q_{(\mathbf{c}_\tau,\mathbf{c}_x,\mathbf{c}_y)}\left(\tau = -\frac{1}{n} + \frac{iL}{\xi n^2}\right)}{\sum_{\mathbf{c}}\left(U_1'\right)^{(\mathbf{c}_\tau,\mathbf{c}_x,\mathbf{c}_y)}_{(0,0,0)} Z^q_{(\mathbf{c}_\tau,\mathbf{c}_x,\mathbf{c}_y)}\left(\tau = \frac{iL}{\xi}\right)} \\ &= e^{\sum_I \frac{2\pi i}{n}\left(q_{IJ}a^I(a^Ib^J - b^Ia^J) + q_{IJK}a^I(a^Jb^K - b^Ja^K)\right)} \frac{Z^q_{(-\mathbf{a},0,\mathbf{b})}\left(\tau = -\frac{1}{n} + \frac{iL}{\xi n^2}\right)}{Z^q_{(0,0,0)}\left(\tau = \frac{iL}{\xi}\right)} \\ &= e^{\sum_I \frac{2\pi i}{n}\left(q_{IJ}a^I(a^Ib^J - b^Ia^J) + q_{IJK}a^I(a^Jb^K - b^Ja^K)\right)}\left(1 + \mathcal{O}(e^{-L/\xi}) + \cdots\right),\end{aligned} \tag{129}$$

where in the 2nd line we have defined the diffeomorphism $\Gamma = U_1^{'}U_2^{n}U_1^{'}$ Then by taking the limit $\xi/L \to 0$, we can read off the SPT invariant.

IV. $d+1$-DIMENSIONAL TOPOLOGICAL PHASES AND THEIR d-DIMENSIONAL BOUNDARIES

Several features discussed in the previous sections for $2+1d$ and $3+1d$ bosonic SPTs can be generalized to arbitrary dimensions. Let us consider bosonic SPT phases protected by symmetry G in $d+1$-dimensions where G is a discrete abelian group which for simplicity we shall assume to be $\mathbb{Z}_n^k$. and their d-dimensional boundaries. SPT phases with discrete abelian symmetry G are classified by $H^{d+1}_{\text{group}}(G, U(1))$. Then each such SPT phase can be labelled by a group cocycle $[\omega] \in H^{d+1}_{\text{group}}(G, U(1))$. Let us consider a few low dimensional examples of the group cohomology classification

$$
\begin{aligned}
H^2_{\text{group}}[\mathbb{Z}_n^k, U(1)] &= (\mathbb{Z}_n)^{\left[\binom{k}{2}\right]},\\
H^3_{\text{group}}[\mathbb{Z}_n^k, U(1)] &= (\mathbb{Z}_n)^{\left[\binom{k}{1}+\binom{k}{2}+\binom{k}{3}\right]},\\
H^4_{\text{group}}[\mathbb{Z}_n^k, U(1)] &= (\mathbb{Z}_n)^{\left[2\binom{k}{2}+\binom{k}{3}+\binom{k}{4}\right]},\\
H^5_{\text{group}}[\mathbb{Z}_n^k, U(1)] &= (\mathbb{Z}_n)^{\left[\binom{k}{1}+2\binom{k}{2}4\binom{k}{3}+3\binom{k}{4}+\binom{k}{5}\right]}.
\end{aligned}
\tag{130}
$$

We can read-off some pattern, notably in odd-dimensions due to the existence of Chern-Simons terms one can build a topological action with a single $\mathbb{Z}_n$ gauge field. The procedure to build a continuum topological action from a $d+1$-cocycle or vice versa is essentially the same as the lower dimensional analogs. For example in $4+1d$, the Chern-simons like terms $(q_{IJK}/4\pi^2)A^I \wedge dA^J \wedge dA^K$ correspond to the cocycle

$$
\omega_{q_{IJK}}(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e}) = e^{\frac{2\pi q_{IJK}}{n^3} a^I(b^J+c^J-[b^J+c^J])(d^J+e^J-[d^K+e^K])}
\tag{131}
$$

Similarly the topological action of the kind

$$
I^{q_{IJKL}} = -\frac{q_{IJKL}n^2}{8\pi^3}\int_{N_{d+1}} A^I \wedge A^J \wedge A^K \wedge dA^L
\tag{132}
$$

corresponds to the cocycle

$$
\omega_{q_{IJKL}}(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e}) = e^{\frac{2\pi q_{IJKL}}{n^2} a^I b^J c^K(d^L+e^L-[d^L+e^L])}.
\tag{133}
$$

Next, one can design effective actions for SPTs with specific actions of G which imply specific coupling to the flat background G gauge field A. Of course these models must have a unique groundstate, no fractional excitations and most importantly furnish the correct topological response theories. By inspection one can realize that all such models can be modeled simply as multicomponent BF theories at 'level' 1. In $d+1$-dimensions these take the form

$$
\mathcal{S} = \int_{N_{d+1}} \left[\frac{\delta_{IJ}}{2\pi} b^I \wedge da^J + \frac{1}{2\pi} A^I \wedge (db^I + \dots)\right],
\tag{134}
$$

where b^I and a^I are $d-1$-form and 1-form $U(1)$ connections which satisfy the usual Dirac quantization conditions. '$\dots$' refers to piece in the coupling to background gauge field that determines the topological response. For example for the coupling to background A that gives rise to Chern-Simons like term '$AdAdA$' and '$AAAdA$' type term (132) respectively are

$$
\begin{aligned}
\mathcal{S}^q_{cpl} &= -\frac{1}{2\pi}\int A^I \wedge (db^I - \frac{q}{2\pi} da^J \wedge da^K) \quad \text{and}\\
\mathcal{S}^q_{cpl} &= -\frac{1}{2\pi}\int A^I \wedge (db^I - \frac{q}{4\pi^2} a^J \wedge a^K \wedge da^L).
\end{aligned}
\tag{135}
$$

The gauging and ungauging procedures too have straightforward generalizations. The partition function takes the form

$$\mathcal{Z}^q_{\rm DW}[N] = \frac{1}{|G|}\sum_{[A]\in H^1(N,G)} e^{iI^q[N,A]}, \tag{136}$$

where $I^q[N,A]$ is the topological response theory corresponding to an SPT labelled by cocycle $q \in H^{d+1}_{\rm group}(G,U(1))$ that is obtained after integrating out a, b. This can be ungauged as

$$e^{iI^q[N,A]} = \sum_{J_{qp}\in H^d(\widehat{G},U(1))} e^{-i\int_N J_{qp}\wedge A} Z^q_{\rm DW}[N, J_{qv}]. \tag{137}$$

The generalization of boundary physics is more subtle. First we propose a surface theory described by the action

$$S = \frac{1}{2\pi}\int_{M_d} d\zeta\wedge d\varphi - H[\zeta,\phi]. \tag{138}$$

Such a theory may be derived by enforcing the full $U(1)_0 \times U(1)_{d-1}$ symmetry of the bulk BF theory[87], where $U(1)_p$ stands for a p-form $U(1)$ symmetry. Let $M_d = X_{d-1}\times S^1$ where X_{d-1} is a compact oriented manifold without boundary. The twisted Hilbert space $\mathcal{H}_A(X_{d-1})$ on X_{d-1} in the presence of background $\mathbb{Z}_n^k$ gauge field A can be derived as before. For example for the 5-cocycles (131) and (133) given above, ζ is a 2-form $U(1)$ connection and X is a 3-manifold, the twisted Hilbert spaces take the form

$$\begin{aligned}
\mathcal{H}^{q_{IJK}}_A(X_3) &= \left\{\zeta^I(x),\varphi^I(x)\middle|\oint_L\frac{d\varphi^I}{2\pi} = \oint_L\frac{A^I}{2\pi};\quad \oint_V\frac{d\zeta^I}{2\pi} = \frac{q_{IJK}}{4\pi^2}\oint_V A^J\wedge dA^K\right\},\\
\mathcal{H}^{q_{IJKL}}_A(X_3) &= \left\{\zeta^I(x),\varphi^I(x)\middle|\oint_L\frac{d\varphi^I}{2\pi} = \oint_L\frac{A^I}{2\pi};\quad \oint_V\frac{d\zeta^I}{2\pi} = \frac{q_{IJKL}}{8\pi^3}\oint_V A^J\wedge A^K\wedge A^L\right\},
\end{aligned} \tag{139}$$

where $L\in H_1(X_3,\mathbb{Z})$ and $V\in H_3(X_3,\mathbb{Z})$. We note that it is not clear how to implement this procedure for 'type-$d+1$' cocycles $\in H^{d+1}(\mathbb{Z}_n^k, U(1))$. These cocycles take the form

$$\begin{aligned}
&\omega_{q_{I_1I_2I_3\cdots I_{d+1}}}(a_1^{I_1}, a_2^{I_2},\cdots, a_{d+1}^{I_{d+1}})\\
&\quad = e^{\frac{2\pi i q_{I_1I_2\cdots I_{d+1}}}{n} a_1^{I_1}a_2^{I_2}\ldots a_{d+1}^{I_{d+1}}}
\end{aligned} \tag{140}$$

and generally give non-abelian topological order upon gauging in the bulk. A quick way to see this is by the fact that these cocycles reduce to non-abelian topological order upon dimensional reduction. Alternately one can check that this kind of cocycle gives rise to an algebra that does not have any non-trivial one-dimensional representations. Since the charges in Dijkgraaf-Witten theories carry a 'twisted' representation. This leads to the fact that non-trivial fluxes have quantum dimension > 1. They cannot be embedded in $U(1)^k$, hence we need to go beyond effective field theory of the form (138) to model boundary theories for SPTs protected by such group cocyles.

For all other cocycle types the twisted partition function may be computed on $M_d = X_{d-1}\times S^1$ as

$$Z^q[M_d, A] = \mathrm{Tr}_{\mathcal{H}_A(X_{d-1})}\left[\prod_I \mathcal{G}^I\left(\oint_{S^1}\frac{A^I}{2\pi}\right)e^{2\pi i R_0 H}\right] \tag{141}$$

where $\mathcal{G}^I$ is the $\mathbb{Z}_n$ symmetry operator corresponding to I-th $\mathbb{Z}_n$ copy and R_0 is the radius of S^1 along the time direction. Then we expect $Z^q[M_d,A]$ to have a 't-Hooft anomaly that can be cancelled by the response of an SPT on $N_{d+1}|_{\partial N_{d+1}=M_d}$, i.e., together the bulk and boundary partition functions

$$Z^q[M_d,A]e^{iI^q[N_{d+1},A]} \tag{142}$$

are gauge invariant and do not suffer from any 't-Hooft anomaly.

V. $2+1d$ SURFACE WITH $U(1)\times\mathbb{Z}_2^{R,T}$ 'T-HOOFT ANOMALY

In this section we study a mixed $U(1)\times\mathbb{Z}_2^{T,R}$ 't-Hooft anomaly for the following model:

$$S = \int_M\left[\frac{1}{2\pi}d\zeta\wedge d\phi - \mathcal{H}(\zeta,\phi)\right]. \tag{143}$$

Here, $\mathbb{Z}_2^{T,R}$ represents time-reversal or reflection symmetry, which can be combined with unitary on-site symmetry. We show that for different symmetry actions there may be a $\mathbb{Z}_2^{T,R}\times U(1)_0$ or $\mathbb{Z}_2^{T,R}\times U(1)_1$ anomaly where $U(1)_p$ refers to p-form $U(1)$ global symmetry. We show that for such a symmetry action, the $\mathbb{Z}_2^{T,R}$ projected partition function is not invariant under large $U(1)_p$ gauge transformation. In the context of fermionic SPT phases, similar calculations have been carried out for the surface theory (gapless $(2+1)d$ Dirac fermion theory) of $(3+1)d$ time-reversal or CR symmetric topological insulators.[36]

Details of quantization of (143) can be found in App. D. Here we will need the form of the mode expansion which decomposes into oscillator and zero-mode parts as

$$\begin{aligned}\phi(x,y,t) &= \phi^0(x,y,t) + \phi^{osc}(x,y,t),\\ \zeta_j(x,y,t) &= \zeta_j^0(x,y,t) + \zeta_j^{osc}(x,y,t).\end{aligned} \tag{144}$$

The zero-mode part takes the form

$$\begin{aligned}\phi(x,y,t) &= \alpha_0 + \frac{\beta_1 x}{R_1} + \frac{\beta_2 y}{R_2} + \cdots,\\ \zeta_j(x,y,t) &= \frac{\alpha_j}{2\pi R_j} + \frac{\beta_0}{2\pi R_1 R_2} x\delta_{j,2} + \cdots.\end{aligned} \tag{145}$$

The canonical algebra for this theory implies $[\alpha_0, \beta_0] = i$ and $[\alpha_1, \beta_2] = i = -[\alpha_2, \beta_1]$. We will only be interested in the zero mode part of the mode expansion throughout this section as we seek to diagnose mixed $\mathbb{Z}_2^{T,R} \times U(1)_p$ anomaly and $U(1)_p$ only acts on the zeromode part of the mode expansion.

$U(1)_0$ and $U(1)_1$ symmetry

The action (143) is invariant under a 0-form and 1-form $U(1)$ symmetry. The 0-form symmetry transformation is

$$\mathcal{G}^{(0)}(\theta) : \phi \mapsto \phi + \theta \tag{146}$$

explicitly the symmetry operator is $\mathcal{G}^{(0)}(\theta) = \exp\{i\beta_0\theta\}$. To gauge the 0-form $U(1)$ symmetry we introduce a flat 1-form background gauge field A, the gauge equivalence

$$\begin{aligned}\phi(x) &\mapsto \phi(x) + \theta(x),\\ A(x) &\mapsto A(x) - d\theta(x),\end{aligned} \tag{147}$$

and define the covariant derivative $D_A\phi := d\phi + A$. Then the gauged action is

$$S[\zeta, \phi, A] = \int_M \left[\frac{1}{2\pi} d\zeta \wedge D_A\phi - \mathcal{H}\right]. \tag{148}$$

Notice that ϕ satisfies $U(1)_0$ twisted quantization condition

$$\oint_{L_i \in H_1(T^2,\mathbb{Z})} \frac{d\phi}{2\pi} = \oint_{L_i \in H_1(T^2,\mathbb{Z})} \frac{A}{2\pi} := \lambda_i. \tag{149}$$

Hence we may define the $U(1)_0$ twisted Hilbert space as

$$\mathcal{H}_{\lambda_1,\lambda_2} = \left\{\phi(x), \zeta(x)\Big| \oint_{L_i \in H_1(T^2,\mathbb{Z})} \frac{d\phi}{2\pi} = \lambda_i\right\} \tag{150}$$

Similarly (143) is invariant under a global 1-form $U(1)$ symmetry under which acts as

$$\mathcal{G}^{(1)}(\theta\eta) : \zeta(x) \mapsto \zeta(x) + \theta\eta(x); \qquad \theta \in \mathbb{R}/2\pi\mathbb{Z} \tag{151}$$

where η is a flat bundle. Gauging the 1-form $U(1)$ symmetry implies introducing a flat 2-form background gauge field B, and the gauge equivalence

$$\begin{aligned}\zeta(x) &\mapsto \zeta(x) + \theta(x)\eta(x),\\ B(x) &\mapsto B(x) - d\theta(x) \wedge \eta(x),\end{aligned} \tag{152}$$

with the covariant derivative $D_B\zeta := d\zeta + B$. The gauged action is

$$S[\zeta, \phi, B] = \int_M \left[\frac{1}{2\pi} D_B\zeta \wedge d\phi - \mathcal{H}\right]. \tag{153}$$

The 1-form field ζ satisfies $U(1)_1$ twisted quantization condition

$$\oint_{T^2} \frac{d\zeta}{2\pi} = \oint_{T^2} \frac{B}{2\pi} =: \lambda_0. \tag{154}$$

We may define the $U(1)_1$ twisted Hilbert space as

$$\mathcal{H}_{\lambda_0} = \left\{\phi(x), \zeta(x)\Big| \oint_{T^2} \frac{d\zeta}{2\pi} = \oint_{T^2} \frac{B}{2\pi} = \lambda_0\right\} \tag{155}$$

$\mathbb{Z}_2^{T,R} \times U(1)_0$ anomaly

Let us consider the following choice of $\mathbb{Z}_2^R$ action implemented by P_0 on (92),

$$\begin{aligned}P_0 &: \phi(t,x,y) \to \phi(t,x,-y),\\ &: \zeta_1(t,x,y) \to -\zeta_1(t,x,-y),\\ &: \zeta_2(t,x,y) \to \zeta_2(t,x,-y) + \Delta\zeta_2,\end{aligned} \tag{156}$$

where $\Delta\zeta_2 = 0$ or π. The zero-mode operators transform under $\mathbb{Z}_2^R$ action as

$$\begin{aligned}P_0 &: \alpha_0 \to \alpha_0,\\ &: \alpha_1 \to -\alpha_1,\\ &: \alpha_2 \to \alpha_2 + \Delta\zeta_2 R_2,\\ &: \beta_0 \to \beta_0,\\ &: \beta_1 \to \beta_1,\\ &: \beta_2 \to -\beta_2.\end{aligned} \tag{157}$$

Hence since $\mathcal{G}^{(0)}(\theta) = e^{i\beta_0\theta}$, we find $\left[\mathcal{G}^{(0)}(\theta), P_0\right] = 0$. We postulate the following P_0 action on zeromode vacuum sectors

$$\begin{aligned}P_0|\alpha_0, \alpha_1, \alpha_2\rangle &= e^{iB_0(\alpha_0,\alpha_1,\alpha_2)}|\alpha_0, -\alpha_1, \alpha_2 + \Delta\zeta_2\rangle,\\ P_0|\beta_0, \beta_1, \beta_2\rangle &= e^{iA_0(\beta_0,\beta_1,\beta_2)}|\beta_0, \beta_1, -\beta_2\rangle.\end{aligned} \tag{158}$$

The $U(1)$ phase can be read off from the fourier representation of the zero-mode ket

$$|\beta_0, \beta_1, \beta_2\rangle = \int \prod_\mu d\alpha_\mu e^{\{i(\alpha_0\beta_0 + \alpha_1\beta_2 - \alpha_2\beta_1)\}}|\alpha_0, \alpha_1, \alpha_2\rangle \tag{159}$$

which implies $A_0(\beta_0, \beta_1, \beta_2) = B_0 + \beta_1\Delta\zeta_2$. Writing $\beta_\mu = N_\mu + \lambda_\mu$ where $N_\mu \in \mathbb{Z}$ is the untwisted winding mode and $\lambda_\mu \in \mathbb{R}/\mathbb{Z}$ is the $U(1)$ twist parameters introduced above. We obtain

$$P_0|\beta_0, \beta_1, \beta_2\rangle = P[\lambda_1]e^{iN_1\Delta\zeta_2}|\beta_0, \beta_1, -\beta_2\rangle. \tag{160}$$

If we require that our $\mathbb{Z}_2^R$ action does not depend on the $U(1)$ twist[35], we must impose $P[\lambda_1] = 1$ (i.e., $B = \lambda_1 \Delta\zeta_2$)

$$\langle \beta_0, \beta_1, \beta_2 | P_0 | \beta_0, \beta_1, \beta_2 \rangle = e^{iN_1 \Delta\zeta_2} \delta_{\beta_2,0}. \tag{161}$$

The P_0 twisted partition function in the presence of background $U(1)_0$ flux takes the form

$$\begin{aligned} Z[\mathcal{K} \times S^1, \lambda_1] &= \mathrm{Tr}_{\mathcal{H}_{\lambda_1}} \left[P_0 e^{-2\pi R_0 (H + i\frac{\tau_1}{\tau_2} P_x + (i\frac{\tau_1}{\tau_2}\beta + \gamma) P_y)} \right] \\ &= Z_{osc} \sum_{N_{0,1} \in \mathbb{Z}} \exp\Big\{ - \frac{\pi\tau_2}{2r^2 R_2} N_0^2 \\ &\quad - 2\pi r^2 R_2 \tau_2 (N_1 + \lambda_1)^2 \\ &\quad + 2\pi i \tau_1 N_0 (N_1 + \lambda_1) + i\Delta\zeta_2 N_1 \Big\}. \end{aligned} \tag{162}$$

Note we cannot insert λ_2 flux as it is inconsistent with P_0 projection. For the non-trivial choice of P_0 action, i.e., $\Delta\zeta_2 = \pi$, under a large $U(1)_0$ gauge transformation $\lambda_1 \to \lambda_1 + 1$ the parity twisted partition function changes sign

$$Z[\mathcal{K} \times S^1, \lambda_1] = -Z[\mathcal{K} \times S^1, \lambda_1 + 1]. \tag{163}$$

This is a $\mathbb{Z}_2$ anomaly that signals the existence of a bosonic topological insulator protected by $\mathbb{Z}_2^T \times U(1)_0$ global symmetry.[9,42]

In[9], it was shown that bosonic SPTs protected by $G = U(1)_0 \times \mathbb{Z}_2^T$ (or equivalently $U(1)_0 \times \mathbb{Z}_2^R$) in $3+1d$ are classified by $\mathbb{Z}_2^4$. The only mixed term in the response theory takes the form

$$I[N, w_1, A] = \int_N \frac{n}{2\pi^2} w_1 \cup w_1 \cup F \tag{164}$$

where $n \in \mathbb{Z}_2$ parametrizes different phases and w_1 is the first Stiefel-Whitney class of the tangent bundle of the manifold, i.e., $\oint_L w_1 = 0$ or π for any orientation preserving or reversing cycle respectively. The effective matter theory for such an SPT coupled to background geometry can be modeled as

$$\mathcal{S} = \int_N \left[\frac{1}{2\pi} b \cup \delta a + \frac{1}{2\pi} A \cup \delta b + \frac{n}{2\pi^2} w_1 \cup w_1 \cup \delta a \right]. \tag{165}$$

Upon integrating out the matter fields a, b using the fact that the cup product is supercommutative upto boundary terms and δ is a $\mathbb{Z}_2$ graded derivation, we find the correct response (164).

$\mathbb{Z}_2^{T,R} \times U(1)_1$ anomaly

We may consider another distinct $\mathbb{Z}_2^R$ action given by P_1

$$\begin{aligned} P_1 &: \phi(t,x,y) \to \phi(t,x,-y) + \Delta\phi, \\ &: \zeta_1(t,x,y) \to -\zeta_1(t,x,-y), \\ &: \zeta_2(t,x,y) \to \zeta_2(t,x,-y). \end{aligned} \tag{166}$$

Since $P_1^2 = 1$, $\Delta\phi = 0, \pi$. We choose non-trivial action, i.e., $\Delta\phi = \pi$. The zero mode operators transform under P_1 as

$$\begin{aligned} P_1 &: \alpha_0 \to \alpha_0 + \pi, \\ &: \alpha_1 \to -\alpha_1, \\ &: \alpha_2 \to \alpha_2, \\ &: \beta_0 \to \beta_0, \\ &: \beta_1 \to \beta_1, \\ &: \beta_2 \to -\beta_2. \end{aligned} \tag{167}$$

We postulate the following P_1 action on zeromode vacuum sectors

$$\begin{aligned} P_1 |\alpha_0, \alpha_1, \alpha_2\rangle &= e^{iB_1(\alpha_0,\alpha_1,\alpha_2)} |\alpha_0 + \Delta\phi, -\alpha_1, \alpha_2\rangle, \\ P_1 |\beta_0, \beta_1, \beta_2\rangle &= e^{iA_1(\beta_0,\beta_1,\beta_2)} |\beta_0, \beta_1, -\beta_2\rangle. \end{aligned} \tag{168}$$

Similar to the case above for P_0, the $U(1)$ phase can be read off from the fourier representation of the zero-mode ket. We find

$$\begin{aligned} P_1 |\beta_0, \beta_1, \beta_2\rangle &= \exp\{i(B_1 - \pi\beta_0)\} |\beta_0, \beta_1, -\beta_2\rangle, \\ P_1 |\beta_0, \beta_1, \beta_2\rangle &= \exp\{-i\pi N_0\} |\beta_0, \beta_1, -\beta_2\rangle, \end{aligned} \tag{169}$$

where we have written $\beta_1 = N_1 + \lambda_1$ and imposed that the P_1 eigenvalue does not depend on $U(1)$ twist λ_1. This implies that $B_1 = \pi\lambda_1$. We obtain

$$\langle \beta_0, \beta_1, \beta_2 | P_1 | \beta_0, \beta_1, \beta_2 \rangle = e^{i\pi N_0} \delta_{\beta_2,0}. \tag{170}$$

The P_1 twisted partition function which is the partition function on $\mathcal{K} \times S^1$[35,36] takes the form

$$\begin{aligned} Z[\mathcal{K} \times S^1, \lambda_0] &= \mathrm{Tr}_{\mathcal{H}_{\lambda_0}} \left[P_1 e^{-2\pi R_0 (H + i\frac{\tau_1}{\tau_2} P_x + (i\frac{\tau_1}{\tau_2}\beta + \gamma) P_y)} \right] \\ &+ Z_{osc} \sum_{N_{0,1} \in \mathbb{Z}} \exp\Big\{ - \frac{\pi\tau_2}{2r^2 R_2} (N_0 + \lambda_0)^2 \\ &- 2\pi r^2 R_2 \tau_2 N_1^2 + 2\pi i \tau_1 (N_0 + \lambda_0) N_1 + i\pi N_0 \Big\} \end{aligned} \tag{171}$$

Under a large gauge transformation $\lambda_0 \to \lambda_0 + 1$ the parity twisted partition function changes sign

$$Z[\mathcal{K} \times S^1, \lambda_0] = -Z[\mathcal{K} \times S^1, \lambda_0 + 1] \tag{172}$$

This is a $\mathbb{Z}_2$ anomaly in the sense that it is cancelled if we take two copies of the theory. This signals the existence of a bosonic topological insulator protected by $\mathbb{Z}_2^T \times U(1)_1$ global symmetry.

We propose the response theory might be

$$I[N, B, w_1] = \int_N w_1 \cup \delta B \tag{173}$$

which can be modeled as

$$\mathcal{S} = \int_N \left[\frac{1}{2\pi} b \cup \delta a + \frac{1}{2\pi} B \cup \delta a + \frac{1}{2\pi} w_1 \cup \delta B \right]. \tag{174}$$

VI. CONCLUSION AND OUTLOOK

In conclusion we have studied a class of invertible topological field theories that admit topologically distinct G actions where G is a discrete abelian group. We study these from complimentary bulk and boundary approaches. In the bulk these model bosonic G-SPTs which are labelled by $[\omega] \in H^{d+1}_{\text{group}}(G, U(1))$. Different SPTs furnish distinct responses to background flat gauge field A depending on ω. We explicitly compute these responses on manifolds with field configurations that can distinguish different SPTs. These set of responses supply SPT topological invariants. Next we describe the gauging procedure and confirm that gauging an SPT gives a topological gauge theory which is none other that Dijkgraaf-Witten theory labelled by ω. We show that Dijkgraaf-Witten theories can be ungauged by gauging a dual symmetry $\widehat{G}$. This is synonymous to condensing the Bosonic charge of the gauge theory.

In the dual boundary approach, we study bosonic quantum field theories with global G symmetry which suffer from a G-'t-Hooft anomaly. For the cases we study, it is shown that these 't-Hooft anomalies can be cancelled by a Dijkgraaf-Witten topological action in one dimension higher signaling that these theories are consistent on the surface of SPTs. Further we compute SPT invariants directly from the boundary theory and describe a procedure of constructing G-characters by orbifolding G on the boundary. These characters can be used to generate modular data for the bulk topological gauge theory further confirming the bulk boundary correspondence for these topological gauge theories.

Finally we study a quantum field theory in $2+1d$ that suffers from a mixed anomaly between time reversal/reflection and $U(1)$. Depending on how time reversal/reflection acts the $U(1)$ could be a 0-form or 1-form symmetry. We postulate the topological action of the $3+1d$ bulk that cancels such a 't-Hooft anomaly. For 0-form (resp. 1-form) $U(1)\times\mathbb{Z}_2^T$ this theory could model the surface of the bosonic SPT phase with this symmetry.

We close with a few comments on open issues:

- In this work we only study gapless surfaces of SPT phases however for bulk spatial dimension ≥ 3, the boundary can support a gapped QFT with anomalous topological order[42,90–92]. For onsite symmetry G and in $3+1d$, the SPT invariant can be extracted from the violation of pentagon identity[43] on the $2+1d$ G-equivariant topological order. Moreover the time reversal anomaly can be computed using a recently proposed anomaly indicator by Wang and Levin[93] however it would be interesting to explore how SPT invariants can be extracted for mixed symmetry groups with both anti-unitary symmetries such as time reversal/ mirror reflection as well as onsite unitary symmetry.
- Categorical generalizations of groups have been explored for constructing topological gauge theories. The topological actions of these gauge theories serve as response functions for non-trivial gapped phases of matter protected by the corresponding generalizations of groups. These gapped phases of matter have global symmetries termed as *generalized global symmetries*[94,95] that act on higher dimensional objects within a quantum field theory in addition to point-like objects[96]. There is much to be studied in such theories both on the side of SPTs as well as the corresponding topological gauge theories. There are several open directions such as 't-Hooft anomalies for (higher) generalizations of groups[97], quantization of higher topological gauge theories, spectrum of higher gauge theories etc.
- Floquet SPTs[98–101] or non-trivial dynamical gapped phases of matter as well as several phases of matter protected by certain spacegroup symmetries have not been understood much within the framework of low energy topological field theories. Since TQFT is a robust framework to study phases of matter it is interesting to ask whether such spacetime symmetries can be incorporated within such a framework[102].
- Although we can understand bulk physics for SPTs protected by discrete abelian group G directly by analyzing the boundary. There is a class of cocycles such as Type-III in $2+1d$ and type-IV in $3+1d$ bulk that cannot be captured by our scheme and consequently we cannot study such phases directly from the boundary. This has to do with the fact that upon gauging such SPTs one gets non-abelian topological order that cannot be embedded in $U(1)^k$. In future work we would like to consider a class of models that can admit non-abelian symmetries with the hope that these can model the boundary behavior of type-III or respectively type-IV SPTs.

ACKNOWLEDGMENTS

AT would like to thank Michael Stone, Srinidhi Ramamurthy, Michael V. Pak, Xueda Wen, Lakshya Bhardwaj and Itziar Ochoa de Alaiza for several helpful discussions. XC was supported by a postdoctoral fellowship from the Gordon and Betty Moore Foundation, under the EPiQS initiative, Grant GBMF4304, at the Kavli Institute for Theoretical Physics. KS is supported by RIKEN Special Postdoctoral Researcher Program. This work is supported in part by the NSF under Grant No. DMR-1455296.

Appendix A: Group cohomology for finite abelian groups

Here we collect some facts about the group cohomology of discrete abelian groups. In this paper we use both additive and multiplicative definition of group action. When we use additive definition, we will always work with $\mathbb{R}/2\pi\mathbb{Z}$ coefficients, however, when we work with multiplicative definition, we will work with $U(1)$ coefficients. Here we define $H^n_{\text{group}}(G,\mathbb{R}/2\pi\mathbb{Z})$. The space of n-cochains is defined as the set of homomorphisms

$$C^n_{\text{group}}(G,\mathbb{R}/2\pi\mathbb{Z}) = \{f : G^n \to \mathbb{R}/2\pi\mathbb{Z}\}\,. \tag{A1}$$

C^n_{group} is an abelian group under pointwise addition:

$$\begin{aligned}(f+g)(a_1,a_2,\dots,a_n) &= f(a_1,a_2,\dots,a_n)\\ &\quad + g(a_1,a_2,\dots,a_n),\end{aligned} \tag{A2}$$

where $f,g \in C^n_{\text{group}}$. Then there exists a coboundary operator $\delta : C^n_{\text{group}} \to C^{n+1}_{\text{group}}$ with the action

$$\begin{aligned}(\delta f)(a_1,\dots,a_{n+1}) &= f(a_2,\dots,a_{n+1})\\ &+ (-1)^{n+1} f(a_1,\dots,a_n)\\ &+ \sum_{i=1}^{n}(-1)^i f(a_1,\dots,a_i+a_{i+1},\dots,a_{n+1})\end{aligned} \tag{A3}$$

δ satisfies the properties

$$\begin{aligned}\delta(f+g) &= \delta f + \delta g,\\ \delta^2 &= 0.\end{aligned} \tag{A4}$$

δ naturally defines two subgroups of C^n_{group}-the group of n-cochains these are n-coycles $Z^n_{\text{group}}(G,\mathbb{R}/2\pi\mathbb{Z})$ and n-coboundaries $B^n_{\text{group}}(G,\mathbb{R}/2\pi\mathbb{Z})$ where $B^n_{\text{group}} \subset Z^n_{\text{group}} \subset C^n_{\text{group}}$

$$\begin{aligned}Z^n_{\text{group}} &= \left\{ f \in C^n_{\text{group}} \;\middle|\; \delta f = 0 \right\},\\ B^n_{\text{group}} &= \left\{ f \in C^n_{\text{group}} \;\middle|\; f = \delta h,\ h \in C^{n-1}_{\text{group}} \right\}.\end{aligned} \tag{A5}$$

Then the cohomology is defined as usual as

$$H^n_{\text{group}}(G,\mathbb{R}/2\pi\mathbb{Z}) = \frac{Z^n_{\text{group}}(G,\mathbb{R}/2\pi\mathbb{Z})}{B^n_{\text{group}}(G,\mathbb{R}/2\pi\mathbb{Z})}. \tag{A6}$$

The slant product can be defined, which lowers the degree by 1

$$i_a : C^n_{\text{group}}(G,\mathbb{R}/2\pi\mathbb{Z}) \to C^{n-1}_{\text{group}}(G,\mathbb{R}/2\pi\mathbb{Z}). \tag{A7}$$

Explicitly, this takes the form

$$\begin{aligned}(i_a f)(a_1,\dots,a_{n-1}) &= (-1)^{n-1} f(a,a_1,\dots,a_{n-1})\\ &+ \sum_{i=1}^{n-1}(-1)^{n-1+i} f(a_1,\dots,a_i,a,a_{i+1},\dots,a_{n-1}).\end{aligned} \tag{A8}$$

Further it can be checked by explicit computation that $\delta(i_a f) = i_a(\delta f)$. Therefore, if $f \in Z^n_{\text{group}}(G,\mathbb{R}/2\pi\mathbb{Z})$, then $i_a f \in Z^{n-1}_{\text{group}}(G,\mathbb{R}/2\pi\mathbb{Z})$, i.e., i_a establishes a homomorphism

$$i_a : H^n_{\text{group}}(G,\mathbb{R}/2\pi\mathbb{Z}) \to H^{n-1}_{\text{group}}(G,\mathbb{R}/2\pi\mathbb{Z}). \tag{A9}$$

Appendix B: Orbifolding with discrete torsion and relation to $1+1d$ SPTs

Consider the following partition function on a torus

$$Z_{\text{orb}}(\tau) = \sum_{\mathbf{a},\mathbf{b}\in G} \epsilon(\mathbf{a},\mathbf{b}) Z_{\mathbf{a},\mathbf{b}}. \tag{B1}$$

Under modular transformations, the twisted sectors transform as

$$\begin{aligned}T &: Z_{\mathbf{a},\mathbf{b}}(\tau) \mapsto Z_{\mathbf{a}+\mathbf{b},\mathbf{b}}(\tau),\\ S &: Z_{\mathbf{a},\mathbf{b}}(\tau) \mapsto \mathcal{Z}_{-\mathbf{b},\mathbf{a}}(\tau).\end{aligned} \tag{B2}$$

Since the mapping class group of a torus is $SL(2,\mathbb{Z})$, a general element may be written as

$$U = \begin{pmatrix} p & q\\ r & s\end{pmatrix}; \quad ps - qr = 1 \tag{B3}$$

Then this implies that

$$\epsilon(\mathbf{a}^p\mathbf{b}^q, \mathbf{a}^r\mathbf{b}^s) = \epsilon(\mathbf{a},\mathbf{b}). \tag{B4}$$

Further, consider putting the theory on Σ^2, a Riemann surface of genus 2. Then $\epsilon : \text{Hom}[\pi_1(\Sigma^2), G] \to U(1)$. By modular invariance we demand[75]

$$\epsilon(\mathbf{a}_1,\mathbf{b}_1;\mathbf{a}_2,\mathbf{b}_2) = \epsilon(\mathbf{a}_1\mathbf{b}_1\mathbf{b}_2^{-1},\mathbf{b}_1;\mathbf{a}_2\mathbf{b}_2\mathbf{b}_1^{-1},\mathbf{b}_2), \tag{B5}$$

where $\mathbf{a}_1,\mathbf{a}_2,\mathbf{b}_1,\mathbf{b}_2$ are the G-fluxes inserted along the non-contractible cycles $L^1_x, L^2_x, L^1_y, L^2_y$ respectively. Further by the factorization property at genus 2,

$$\epsilon(\mathbf{a}_1,\mathbf{b}_1;\mathbf{a}_2,\mathbf{b}_2) = \epsilon(\mathbf{a}_1,\mathbf{b}_1)\epsilon(\mathbf{a}_2,\mathbf{b}_2) \tag{B6}$$

If we normalize $\epsilon(\mathbf{1},\mathbf{1}) = 1$, then by modular invariance (B3),

$$\epsilon(\mathbf{g},\mathbf{1}) = \epsilon(\mathbf{1},\mathbf{g}) = 1. \tag{B7}$$

Using these facts and (B5),(B6) it can be shown that ϵ is a 1-dimensional representation of G

$$\epsilon(\mathbf{a}_1+\mathbf{a}_2,\mathbf{b}) = \epsilon(\mathbf{a}_1,\mathbf{b})\epsilon(\mathbf{a}_2,\mathbf{b}). \tag{B8}$$

It was shown in[75,76] the set of inequivalent ϵ that satisfy (B7) and (B8) are classified by $[c] \in H^2_{\text{group}}(G,U(1))$ and can be written as

$$\epsilon(\mathbf{a},\mathbf{b}) = \frac{c(\mathbf{a},\mathbf{b})}{c(\mathbf{b},\mathbf{a})}. \tag{B9}$$

Since $[c] \in H^2_{\text{group}}(G,U(1))$ it satisfies the cocycle condition

$$c(\mathbf{a},\mathbf{bc})c(\mathbf{b},\mathbf{c}) = c(\mathbf{ab},\mathbf{c})c(\mathbf{a},\mathbf{b}). \tag{B10}$$

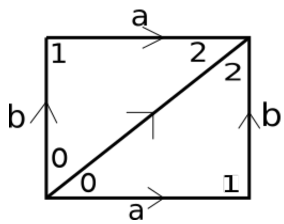

FIG. 2. A triangulation of T^2 with flux $a, b \in G$ along the two cycles. Dijkgraaf Witten theory labelled by $H^2_{\text{group}}(G, U(1))$ associates the $U(1)$ phase $c(a,b)/c(b,a)$ to this assohnment A.

Now using this form of ϵ, we may verify that the above two properties are satisfied. First

$$\begin{aligned}
\frac{\epsilon(\mathbf{a}_1\mathbf{a}_2, \mathbf{a}_3)}{\epsilon(\mathbf{a}_1, \mathbf{a}_3)\epsilon(\mathbf{a}_2, \mathbf{a}_3)} &= \frac{c(\mathbf{a}_1\mathbf{a}_2, \mathbf{a}_3)c(\mathbf{a}_3, \mathbf{a}_1)c(\mathbf{a}_3, \mathbf{a}_2)}{c(\mathbf{a}_3, \mathbf{a}_1\mathbf{a}_2)c(\mathbf{a}_1, \mathbf{a}_3)c(\mathbf{a}_2, \mathbf{a}_3)} \\
&= \frac{c(\mathbf{a}_1\mathbf{a}_2, \mathbf{a}_3)c(\mathbf{a}_3, \mathbf{a}_2)}{c(\mathbf{a}_1\mathbf{a}_3, \mathbf{a}_2)c(\mathbf{a}_1, \mathbf{a}_3)} \\
&= 1, \\
\epsilon(\mathbf{a}^p\mathbf{b}^q, \mathbf{a}^r\mathbf{b}^s) &= \epsilon(\mathbf{a}^p, \mathbf{a}^r\mathbf{b}^s)\epsilon(\mathbf{b}^q, \mathbf{a}^r\mathbf{b}^s) \\
&= \epsilon(\mathbf{a}, \mathbf{a}^r\mathbf{b}^s)^p\epsilon(\mathbf{b}, \mathbf{a}^r\mathbf{b}^s)^q \\
&= \epsilon(\mathbf{a}^r\mathbf{b}^s, \mathbf{a})^{-p}\epsilon(\mathbf{a}^r\mathbf{b}^s, \mathbf{b})^{-q} \\
&= \epsilon(\mathbf{b}^s, \mathbf{a})^{-p}\epsilon(\mathbf{a}^r, \mathbf{b})^{-q} \\
&= \epsilon(\mathbf{a}, \mathbf{b})^{(ps-qr)} \\
&= \epsilon(\mathbf{a}, \mathbf{b})
\end{aligned} \tag{B11}$$

Furthermore the discrete torsion phase $\epsilon(\mathbf{a}, \mathbf{b}) = c(\mathbf{a}, \mathbf{b})/c(\mathbf{a}, \mathbf{b})$ is exactly the response of a $1 + 1d$ SPT protected by G and characterized by 2-cocycle $[c] \in H^2_{\text{group}}(G, U(1))$ in the presence of G-flux $\mathbf{a}, \mathbf{b}$ along the two non-contractible cycles of the torus. (See Fig. 2.) To see this recall that given a triangulation K of manifold M, Dijkgraaf-Witten theory associates to an assignment $A : H_1(K, \mathbb{Z}) \to G$ a $U(1)$ phase i.e the response theory of an SPT classified by $[c]$, explicitly given by

$$e^{iI^c[K,A]} = \prod_{\sigma \in C_2(K)} \langle c(A), \sigma \rangle^{o_\sigma} \tag{B12}$$

where $o_\sigma = \pm 1$, the orientation of simplex σ. For a simplex $\sigma[v_0v_1v_2]$ and an assignment $A(v_0v_1) = \mathbf{a}$, $A(v_1v_2) = \mathbf{b}$, we get $\langle c(A), \sigma \rangle = c(\mathbf{a}, \mathbf{b})$. Then it is easy to check

$$e^{iI^c[T^2,A]} = \frac{c(\mathbf{a}, \mathbf{b})}{c(\mathbf{b}, \mathbf{a})} = \epsilon(\mathbf{a}, \mathbf{b}) \tag{B13}$$

Appendix C: SPT response theory and group cocycles

In this appendix we show the relation between the SPT response theories and the respective group cocycles. We would like to show explicitly that the SPT response theories written in the main text matches the expression for the group cocycle. Consider a triangulation of the manifold N. (See App. A in[9] for an introduction to simplicial calculus.) Then a flat G gauge field $[A] \in C^1(N, G)$ that satisfies the conditions

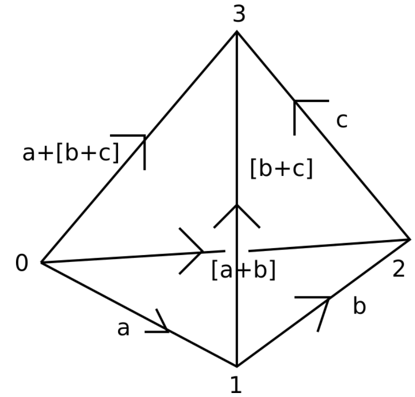

FIG. 3. Configuration of a flat $\mathbb{Z}_n$ gauge field on a 3-simplex. $a, b, c \in \mathbb{Z}_n$.

- $A(\partial f) = 0$ for all $f \in C_2(N, \mathbb{Z})$.
- $A(-e) = A(e)^{-1}$ for all $e \in C_1(N, \mathbb{Z})$ where $-e$ implies reversing the orientation of edge e. .

Let us consider the specific case of $\mathbb{Z}_n$ SPT in $2+1d$. We pick a triangulation for a 3-manifold N. Then a 3-simplex $\sigma^i = [v_0v_1v_2v_3]$ comes with an ordering of vertices $0 < 1 < 2 < 3$ that picks an orientation. A choice of $[A]$ means assigning $A[v_0v_1] = 2\pi a/n$. $A[v_1v_2] = 2\pi b/n$ and $A[v_2v_3] = 2\pi c/n$ where $a, b, c \in [0, 1, .., n-1]$. Then it straightforward to check that for this choice of flat field $[A]$ (see Fig. 3).

$$\left\langle \frac{q}{2\pi} A \cup \delta A, \sigma^i \right\rangle = \frac{2\pi q}{n^2} a(b + c - [b + c]) \tag{C1}$$

The precise meaning of δA should be understood as follows. Let $A \in Z^1(M; \frac{2\pi}{n}\mathbb{Z}/\mathbb{Z})$ be a $\mathbb{Z}_n$ field. The coefficient $\frac{2\pi}{n}\mathbb{Z}/\mathbb{Z}$ means $A(01)$ takes values in $\frac{2\pi a}{n}$ mod 2π with $a \in \mathbb{Z}$, i.e. $A(01) \in \{0, \frac{2\pi}{n}, \ldots, \frac{2\pi(n-1)}{n}\}$. We shall define the topological action like "$A \cup \delta A$". To do so, we introduce a lift

$$A \mapsto \widetilde{A} \in C^1(M; \frac{2\pi}{n}\mathbb{Z}). \tag{C2}$$

The closed condition of A implies that

$$\delta\widetilde{A} \in C^1(M; 2\pi\mathbb{Z}), \tag{C3}$$

i.e. $(\delta\widetilde{A})(012)$ takes values in $2\pi\mathbb{Z}$. A lift $\widetilde{A}$ is not unique: an integer valued 1-cochain $a \in C^1(M; 2\pi\mathbb{Z})$ also gives a lift

$$A \mapsto \widetilde{A} + a, \qquad a \in C^1(M; 2\pi\mathbb{Z}). \tag{C4}$$

We define a topological action $S[A]$ of $\mathbb{Z}_n$ fields by

$$I[A] := \frac{q}{2\pi}\int_M \widetilde{A}\cup\delta\widetilde{A} \in \frac{2\pi\mathbb{Z}}{n}. \tag{C5}$$

This is ill-defined as $\frac{2\pi\mathbb{Z}}{n}$-valued action. However, $I[A]$ mod $2\pi\mathbb{Z}$ is well-defined: Under a change of lift, the action is changed as

$$\begin{aligned}
\frac{1}{2\pi}\widetilde{A}\cup\delta\widetilde{A} &\mapsto \frac{1}{2\pi}(\widetilde{A}+a)\cup(\delta\widetilde{A}+\delta a) &\text{(C6)}\\
&= \frac{1}{2\pi}\left[\widetilde{A}\cup\delta\widetilde{A}+a\cup\delta\widetilde{A}+\widetilde{A}\cup\delta a+a\cup\delta a\right] &\text{(C7)}\\
&= \frac{1}{2\pi}\widetilde{A}\cup\delta\widetilde{A}+\underbrace{\frac{1}{2\pi}a\cup\delta\widetilde{A}}_{2\pi\mathbb{Z}}-\underbrace{\frac{1}{2\pi}\delta(\widetilde{A}\cup a)}_{\text{exact}}+\underbrace{\frac{1}{2\pi}\delta\widetilde{A}\cup a}_{2\pi\mathbb{Z}}+\underbrace{a\cup\delta a}_{2\pi\mathbb{Z}} &\text{(C8)}\\
&= \frac{1}{2\pi}\widetilde{A}\cup\delta\widetilde{A}-\underbrace{\frac{1}{2\pi}\delta(\widetilde{A}\cup a)}_{\text{exact}} \qquad (\text{mod } 2\pi\mathbb{Z}). &\text{(C9)}
\end{aligned}$$

This means $e^{iI[A]}$ serves as a $U(1)$-valued topological action. Similarly for type-II and III cocycle, it is straightforward to check

$$\begin{aligned}
\left\langle \frac{q_{IJ}}{2\pi}A^I\cup\delta A^J,\sigma^i\right\rangle &= \frac{2\pi q_{IJ}}{n^2}a^I(b^J+c^J-[b^J+c^J]),\\
\left\langle \frac{q_{IJK}n^2}{4\pi^2}A^I\cup A^J\cup A_K,\sigma^i\right\rangle &= \frac{2\pi q_{IJK}}{n}a^Ib^Jc^K. \qquad \text{(C10)}
\end{aligned}$$

Consider a triangulation of a three-torus as shown in Fig. 4. The triangulation has six 3-simplices . Then it is easy to check that the partition function takes the form[20,60]

$$\begin{aligned}
\mathcal{Z}[T^3,a,b,c] &= \frac{1}{|G|}\prod_{\sigma\in Z_3}\langle\omega[A],\sigma\rangle^{o_\sigma}\\
&= \frac{\omega(a,b,c)\omega(b,c,a)\omega(c,a,b)}{\omega(a,c,b)\omega(b,a,c),\omega(c,b,a)} \qquad \text{(C11)}
\end{aligned}$$

This matches with field theory calculation in (26) and (21). Furthermore one can compute the SPT or Dijkgraaf Witten theory partition function on lens space $L(n,1)$. This was recently shown in[63] and we do not repeat the calculation here. The field theory calculation (18) matches the result in[63].

Similarly for $3+1d$ SPTs for $G=\mathbb{Z}_n^k$, we consider a 4-simplex $\sigma^i=[v_0v_1v_2v_3v_4]$ and a flat G field $[A]$ with the assignment $A^I(v_0v_1)=2\pi a^I/n$, $A^I(v_1v_2)=2\pi b^I/n$, $A^I(v_2v_3)=2\pi c^I/n$ and $A^I(v_3v_4)=2\pi d^I/n$.

$$\begin{aligned}
\left\langle \frac{2\pi q_{IJ}n}{4\pi^2}A^I\cup A^J\cup\delta A^J,\sigma^i\right\rangle &= \frac{2\pi i q_{IJ}}{n^2}a^Ib^J\left(c^J+d^J-[c^J+d^J]\right),\\
\left\langle \frac{2\pi q_{IJK}n}{4\pi^2}A^I\cup A^J\cup\delta A^K,\sigma^i\right\rangle &= \frac{2\pi i q_{IJK}}{n^2}a^Ib^J\left(c^K+d^K-[c^K+d^K]\right),\\
\left\langle \frac{q_{IJKL}n^3}{8\pi^3}A^I\cup A^J\cup A^K\cup A^L,\sigma^i\right\rangle &= \frac{2\pi i q_{IJKL}}{n}a^Ib^Jc^Kd^L.
\end{aligned}\tag{C12}$$

The computations for partition functions on T^4 and $L(n,1)\times S^1$ are more tedious but quite similar to those in 1-dimension lower on T^3 and $L(n,1)$ as the latter are dimensionally reduced versions of the former.

Simplicial calculus is naturally analogous to differential calculus where p-cochains map to p-forms, cup product maps to wedge product and the differential δ maps to the exterior derivative 'd'. This matches with the response

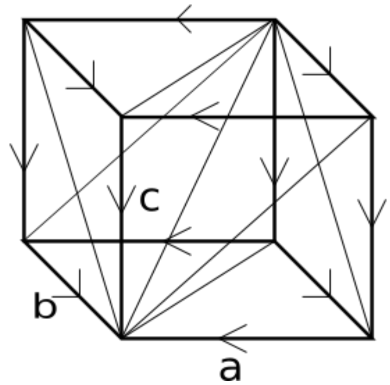

FIG. 4. Triangulation of a three-torus containing one 0-simplex, three 1-simplices, three 2-simplices and six 3-simplices.

theories in Eqs. (77) and (79) for response theories of SPTs.

Appendix D: Quantization of $2+1d$ surface theory

The equal time canonical commutation relations for (92) are

$$\left[\phi(\mathbf{x},t), \epsilon^{ij}\partial_i\zeta_j(\mathbf{x}',t)\right] = 2\pi i\delta(\mathbf{x}-\mathbf{x}') \tag{D1}$$

The mode expansion decomposes into oscillator and zero mode parts

$$\begin{aligned}\phi(x,y,t) &= \phi^0(x,y,t) + \phi^{osc}(x,y,t)\\ \zeta_j(x,y,t) &= \zeta_j^0(x,y,t) + \zeta_j^{osc}(x,y,t)\end{aligned} \tag{D2}$$

The zero-mode part takes the form

$$\begin{aligned}\phi(x,y,t) &= \alpha_0 + \frac{\beta_1 x}{R_1} + \frac{\beta_2 y}{R_2} + ...\\ \zeta_j(x,y,t) &= \frac{\alpha_j}{2\pi R_j} + \frac{\beta_0}{2\pi R_1 R_2}x\delta_{j,2} + ...\end{aligned} \tag{D3}$$

The canonical algebra for this theory implies $[\alpha_0,\beta_0]=i$ and $[\beta_1,\alpha_2]-[\beta_2,\alpha_1]=i$. One possible choice of commutation relations that satisfy this algebra is

$$[\beta_1,\alpha_2]=0; \quad [\beta_2,\alpha_1]=-i \tag{D4}$$

however to quantize β_μ we impose

$$[\alpha_1,\beta_2]=i=-[\alpha_2,\beta_1] \tag{D5}$$

with this $\beta_\mu\in\mathbb{Z}$. The oscillator part of the mode expansions are

$$\begin{aligned}\phi^{osc}(r) &= \frac{1}{\sqrt{R_1R_2}}\sqrt{\frac{1}{2\lambda_1}}\\ &\quad\times\sum_{k\neq 0}\frac{1}{\omega(k)^{1/2}}\left[\hat{a}(\vec{k})e^{-i\vec{k}.\vec{r}} + \hat{a}^\dagger(\vec{k})e^{i\vec{k}.\vec{r}}\right]\\ \zeta_j^{osc}(r) &= \frac{1}{\sqrt{R_1R_2}}\sqrt{\frac{\lambda_1}{8\pi^2}}\\ &\quad\times\sum_{k\neq 0}\frac{-1}{\omega(k)^{3/2}}\epsilon_{jl}k^l\left[\hat{a}(\vec{k})e^{-i\vec{k}.\vec{r}} + \hat{a}^\dagger(\vec{k})e^{i\vec{k}.\vec{r}}\right]\end{aligned} \tag{D6}$$

With the commutator algebra $\left[a(k),a^\dagger(k')\right]=\delta_{k,k'}$. The partition function is given by

$$Z = \mathrm{Tr}_{\mathcal{H}}\left[e^{2\pi i R_0 H'}\right] \tag{D7}$$

where $\mathcal{H}=\otimes_{N_{0,1,2}}\mathcal{H}_{N_0,N_1,N_2}$ and $H'=H+i\alpha R_1P_x/R_0$. The zero-mode part is

$$\begin{aligned}Z_0 = \sum_{N_0,N_1,N_2\in\mathbb{Z}}\exp\Big\{&-\frac{\pi\tau_2}{2r^2R_2}N_0^2 - 2r^2\pi R_2\tau_2 N_1^2\\ &-\frac{2r^2\pi R_0R_1}{R_2}N_2^2 + 2\pi i\tau_1 N_0N_1\Big\}\end{aligned} \tag{D8}$$

The oscillator part of the partition function is the same as that of free boson.

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$$MCG(M) = \pi_0\left[\mathrm{Diff}(M)\right] \tag{D9}$$

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