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Theory of the supercyclotron resonance and Hall response in anomalous two-dimensional metals
Luca V. Delacrétaz and Sean A. Hartnoll
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Weakly disordered superconducting films can be driven into an anomalous low temperature resistive state upon applying a magnetic field. Recent experiments on weakly disordered amorphous InO$_x$ have established that both the Hall resistivity and the frequency of a cyclotron-like resonance in the anomalous metal are highly suppressed relative to the values expected for a conventional metal. We show that both of these observations can be understood from the flux flow dynamics of vortices in a superconductor with significant vortex pinning. Results for flux flow transport are obtained using a systematic hydrodynamic expansion, controlled by the diluteness of mobile vortices at low temperatures. Hydrodynamic transport coefficients are related to microscopics through Kubo formulae for the longitudinal and Hall vortex conductivities, as well as a ‘vorto-electric’ conductivity.

\[ \rho_{yx} = \left( \sigma_n^H + \sigma_o^H \right) \rho_{xx}^2 + \frac{n_{\text{eff}}^2}{n_s} \frac{\hbar}{e^2}. \]  

(1)

In the remainder we set $\hbar = e^* = 2e = 1$. The three terms in (1) respectively describe a Hall signal arising from currents in the vortex core, currents carried by Bogoliubov quasiparticles in the superfluid, and the co-motion of supercurrent parallel to the vortex current. The Hall conductivity $\sigma_o^H$ of the Bogoliubov quasiparticles is typically negligible due to their approximate particle-hole symmetry. Dominance of the first term, proportional to the Hall conductivity $\sigma_n^H$ of the vortex cores, leads to the relation $\rho_{yx} \sim \rho_{xx}^2$ obtained by Vinokur et al., and observed in some thermally activated flux flow. This scaling arises because — as shown by Bardeen-Stephen and derived below — $\rho_{xx} \sim x$, the fraction of the area occupied by mobile vortex cores, is strongly temperature and field dependent, while $\sigma_n^H$ is not. Dominance of the final ‘collision’ term, on the other hand, is crucial to understand experimental results on free flux flow. There the density of vortices that co-move with the superfluid $\rho_{\text{eff}} \sim H$, the applied field, while $n_s$ is the superfluid density. Thus $\rho_{yx} \sim H \sim x \sim \rho_{xx}$, as is observed, and predicted by Nozières and Vinen.

The general relationship (1) between the Hall and longitudinal resistivities both unifies previous results and establishes their domain of validity. It also allows for regimes in which no single term dominates. The first two terms in (1) lead to $\rho_{yx} \sim \rho_{xx}^2$ while the final term leads to $\rho_{yx} \sim \rho_{xx}$, if the density of co-moving vortices $\rho_{\text{eff}} \sim x$. The full expression, therefore, may well explain the range of scaling relations, $\rho_{yx} \sim \rho_{xx}^\beta$, with $1 \leq \beta \leq 2$, reported in the experimental literature. Competition between effects captured by the first and last terms in (1) has previously been invoked to explain the observed change in sign of the Hall response in some flux-flow regimes. Charging of the vortex cores can lead to a sign reversal of $\sigma_n^H$.

We furthermore obtain expressions for the width $\Omega$ and frequency $\Omega_H$ of a ‘supercyclotron resonance’. This resonance is due to superfluid and vortex flow in a magnetic field.
field. It can coexist with a conventional cyclotron resonance (due to flow of the normal fluid component). We will find

$$\Omega = \frac{2x}{\sigma n} f_s$$ and $$\Omega_H = -\frac{\partial j_x}{\partial u_x} \equiv \frac{n_{\text{eff}}}{m_s}.$$ (2)

The result for $$\Omega$$ in (2) is precisely the Bardeen-Stephen expression for vortex diffusivity, with $$\sigma$$ the conductivity of the vortex cores and $$f_s$$ the superfluid stiffness. In (2), $$\Omega_H$$ is given by a static susceptibility; the second step in the expression defines $$n_{\text{eff}}$$. The mass scale $$m_s$$ is such that $$f_s \equiv n_s/m_s$$. Therefore the phase gradient $$u_\phi \equiv \nabla \phi = m_s v_s$$, with $$v_s$$ the superfluid velocity. We have set $$\hbar = 1$$. Finally, $$j_v$$ is the vortex current. With Galilean invariance, $$n_{\text{eff}} = n_0$$ is the full density of mobile vortices. $$\Omega_H$$ is then precisely the frequency appearing due to the co-motion of vortices and supercurrent in Nozières-Vinen. More generally, pinning can strongly break Galilean invariance, so that the effective number of vortices that co-move with the supercurrent $$n_{\text{eff}} < n_0$$.

Experiments on anomalous metals.— We can return now to the measurements on InO2. The first observation is that $$\Omega_H \lesssim 10^{-3} \Omega^3$$. The consequences of this fact follow from (2), whereby $$\Omega_H/\Omega \approx \frac{1}{2} \sigma n n_{\text{eff}}/n_{s} v_s$$. The area occupied by mobile vortex cores is $$x \sim n_0 v_0^2 / n_s$$, where $$\xi$$ is the superconducting correlation length. Here $$x \propto n_s$$ because in a magnetic field we expect the flux through all the different vortices to be aligned. The above together with $$\sigma_n \sim 10 e^2/h \sim 0.4 e^2/h^{1.3}$$ then implies

$$n_{\text{eff}}/n_0 \lesssim 10^{-5}.$$ (3)

It follows that there is essentially vanishing parallel co-motion of vortices and supercurrent, as quantified by the dissipationless susceptibility $$\nabla \phi / \nabla \phi$$ in (2). Indeed, strong pinning in InO2 causes $$\rho_{zx} \sim x$$ to vary by more than an order of magnitude as a function of applied field in the anomalous metal. InO2 is therefore far from the ‘Nozières-Vinen’ flow regime.

Secondly, to a good approximation $$\rho_{yx} \sim \rho_{xx}^2$$ where the Hall signal is detectable. This requires the final term in (1) to be negligible, so that $$n_{\text{eff}}/n_0 \lesssim \rho_{yx}$$. From here we can obtain $$n_{\text{eff}}/n_0 \sim n_{\text{eff}}/(n_s x) \lesssim \rho_{yx}/x \sim \rho_{yx}/(\rho_{xx} \sigma_n) = \tan \theta_H/\sigma_n$$. We used the Bardeen-Stephen result $$\rho_{xx} \sim x/\sigma_n$$, recovered below. The measured $$\tan \theta_H$$ becomes as small as $$10^{-4}$$, leading to $$n_{\text{eff}}/n_0 \lesssim 10^{-4}$$, consistent with (3). The conclusion (3) is therefore reached from two independent experiments. Furthermore, the data shows that for the anomalous metal, $$\rho_{yx}/\rho_{xx}^2 = \sigma_n^H \sim 2 \times 10^{-6} \Omega^{-1}$$ in (1) — recall that $$\sigma_n^H \sim 0$$ due to particle-hole symmetry of the Bogoliubov excitations. This is of the same magnitude as the Hall conductivity of the high temperature normal state, and is consistent with the interpretation of $$\sigma_n^H$$ as the Hall conductivity of the vortex cores. It follows that $$\rho_{yx}/\rho_{xx}^4 \sim (\rho_{xx}/\rho_{xx}^4)^2$$, and hence the observed $$\rho_{yx} \sim 0.01 \Omega$$ at a field of 5T, suppressed by almost three orders of magnitude relative to the high temperature value, follows from the suppression of $$\rho_{xx}$$ in the anomalous metal.

A condition analogous to (3) must also hold for the systems mentioned above where a $$\rho_{yx} \sim \rho_{xx}^2$$ scaling was previously observed. The supercyclotron resonance will be easiest to observe in materials that instead exhibit free flux flow, with negligible pinning, so that $$\rho_{yx} \sim \rho_{xx}$$.

The Hall resistivity measurements further reveal a weak field dependence of $$\sigma_{xy} = \rho_{yx}/\rho_{xx}^2$$, with $$\sigma_{xy}$$ possibly vanishing below a field $$H_{M2} > H_c^1$$. A strictly vanishing zero temperature $$\sigma_{xy}$$ over some field range requires that the vortex core contribution $$\sigma_n^H = 0$$ (in 1), in addition to the vanishing of vortex/superfluid co-motion implied by (3). Such particle-hole symmetry is seen away from a flux flow regime in more disordered samples.

Hydrodynamic approach.— Our analysis is anchored in the observation2 of a narrow peak at zero frequency in the optical response $$\sigma(\omega)$$. This peak defines a lifetime that is around $$10^5$$ times longer than that of the electronic quasiparticles in the material. Such a hierarchy of timescales allows a systematic hydrodynamic expansion of the collective response; all non-collective modes have decayed before the timescales of interest. Furthermore, the conductance peak narrows as the magnetic field is reduced towards the onset of superconductivity at $$H_c$$. This strongly suggests that the appropriate low energy description of the anomalous metal is superfluid hydrodynamics with a slow phase-relaxation timescale. Phase relaxation requires the inclusion of vortices in the hydrodynamic description. The hydrodynamic variables are therefore the electrical and vortex currents $$j$$ and $$j_v$$, and the phase gradient $$u_\phi = \nabla \phi$$. The conductance peak in fact survives into the superconducting phase, and at the very end we explain how this can arise from the contribution of pinned vortices to the optical conductivity.

Working within linear response and assuming homogeneous currents, the equations for the hydrodynamic variables in the presence of a uniform electric field $$E$$ are completely fixed. The Josephson relation, allowing for transverse vortex flow, is (with $$\hbar = e^* = 1$$)

$$\hat{u}_\phi = E^i + \epsilon^{ij} j_v^j.$$ (4)

Here $$\epsilon^{ij}$$ is antisymmetric with $$\epsilon^{xy} = 1$$. We must now express the electric and vortex currents in terms of the electric field and superfluid velocity. The most general relation that obeys the Onsager constraint is shown in the supplementary material to be38:

$$\left( \begin{array}{c} j_o^i \\ j_v^i \\ E^i \\ \alpha \end{array} \right) = \left( \begin{array}{cccc} \sigma_{ij}^o & \alpha^{ij} & \epsilon^{ij} & 0 \\ \alpha^{ij} & \sigma_v^i & 0 & 0 \\ \epsilon^{ij} & 0 & \Omega_{ij}/f_s & 0 \\ \alpha & \epsilon & 0 & \alpha_H^i \end{array} \right) \left( \begin{array}{c} j_o^i \\ j_v^i \\ E^i \\ \alpha \end{array} \right).$$ (5)

Here the normal component electric current $$j_o \equiv j - j_v u_\phi$$. This ‘generalized Ohm’s law’ introduces six transport coefficients: $$\sigma_{ij}^o = \sigma_{ij} + \sigma_n^H \epsilon^{ij}$$, $$\Omega_{ij} = \Omega_{ij}^o + \Omega_{ij}^H$$, $$\epsilon^{ij} = \alpha_i \delta^{ij} + \epsilon^{ij}$$, and $$\alpha_H^i = \alpha_i \delta^{ij} + \epsilon^{ij}$$. Their physical meaning is as follows: $$\Omega_{ij}$$ is the vortex conductivity, $$\sigma_{ij}^o$$ is the electrical conductivity of the normal (non-superfluid) component
We see that $\Omega$ determines the superfluid to the poles at similarly from the Green’s function for $J$ activities $\hat{\Omega}$ and mechanism for phase relaxation. Vortex conductivities in terms of $\dot{\sigma}$ and $\hat{\alpha}$ sample area, ensuring slow phase relaxation — before wherein mobile vortices occupy a small fraction of the sample area, ensuring slow phase relaxation — before the hydrodynamic Green’s functions given in the supplementary material. In the remainder we evaluate (9) and (11) for phase relaxation due to vortex flux flow. In the Bardeen-Stephen phase relaxation rate $\Omega$ was recovered in this way. We can now extend that result to obtain $\Omega_H$ and $\alpha_v$.

**Supercurrent relaxation due to flux flow.** — The supercurrent operator is given by the gradient of the phase integrated outside of vortex cores, where the phase is well-defined: $J_\phi \equiv \int_{\mathbb{R}^2\text{cores}} \nabla \phi \, d^2x$. This definition holds in the limit of weak phase relaxation with dilute, independent vortices in an otherwise well-defined background phase — corresponding to the $x \ll 1$ limit in the Kubo formulae, taken prior to any low frequency limit. The supercurrent operator is relaxed by charge fluctuations that are described by a ‘self-charging’ term in the Hamiltonian: $H = \frac{1}{\chi} \int n^2 \, d^2x$, where $n$ is the charge density and $\chi$ the charge compressibility. The commutator $[\phi(x), n(y)] = i\delta(x-y)$ and single-valuehood of the density operator $n$ everywhere then leads to the expression

$$
\dot{J}_\phi = \frac{2}{\chi} \int_{\text{cores}} \nabla n \, d^2x.
$$

This operator relation can now be used to obtain the Green’s functions (9) and (11). The factor of 2 in (12) was missed in our previous work, but is physically important. When computing $J_\phi$ one must allow for the fact that the location of the core is time-dependent; in this way only mobile vortices are seen to contribute. See supplementary material for details.

The operator relation (12) is at the heart of our approach. Taking the expectation value of (12) in a state with a single large vortex and using $(\nabla n) = \chi \nabla \mu$ in the core leads to the standard classical relation between the vortex current and the microscopic electric field $-\nabla \mu$ in the core.

If (i) correlations between excitations in distinct vortex cores are neglected and (ii) the vortex cores are assumed to be large compared to the mean free path of the normal state in the core, then the Kubo formulæ can be evaluated explicitly. Using the operator (12), the first contribution to (9) becomes

$$
\frac{1}{\chi} \int_{\omega \to 0} \frac{1}{x} \Im \frac{G_{R_{\Phi}}(\omega)}{G_{R_{\Phi}}(\omega)} \, d^2y,
$$

with $x$ the fraction of the total area covered by mobile vortex cores. The integral is over a single core. The control parameter in this entire computation is $x \ll 1$, so that dilute vortices lead to slow phase relaxation. The large core assumption allowed the Green’s function in the core to be translationally invariant so that $G_R(x, y) = G_R(x-y)$. In the large core limit the charge density diffuses so that $G_{nn}(\omega, k) = \sigma_n k^2/(-i\omega + Dk^2)$. The conductivity of the normal state in the core $\sigma_n = \chi D$, with $D$ the diffusivity. The integral in (13) is then easily evalu-
uated to give
\begin{equation}
\frac{1}{\omega} \text{Im} G^R_{j_x p_x} (\omega) = \frac{2x}{\sigma_n} \delta_{ij}.
\end{equation}

The susceptibility term in (9) can be written
\begin{equation}
\chi_{j_x p_x} = \frac{1}{fs} \frac{\partial^2 \phi}{\partial u_{p_x}^2} = \frac{e^k}{fs} \frac{\partial \phi j_k}{\partial u_{p_x}^2}.
\end{equation}

The first equality uses \( \chi_{AB} = \partial(A)/\partial s_B \). Here \( s_B \) is the source for \( B \), and in the case at hand \( s_B = f_s u_0 \). The second equality uses the Josephson relation (4). The electric field term, which is in fact \( E - \nabla \mu \) in general, drops out because \( E \) is held fixed and \( \chi_{j_x \nabla \mu} = 0 \) at any nonzero temperature (where the response at low wavevector \( k \) is nonsingular, so that \( \chi_{j_x \nabla \mu} \sim k \to 0 \)). Putting (14) and (15) together gives the results for \( \Omega \) and \( \Omega_H \) stated in (2) above. Finally, the inclusion of correlations between distinct vortex cores and finite size corrections to Green’s functions in the cores (i.e. lifting the two assumptions made above) do not lead to additional contributions to \( \Omega_H \), as we note in the supplementary material [37].

With the same assumptions, the vorto-electric conductivity similarly gets a contribution from inside the vortex cores given by
\begin{equation}
\alpha_v = -\frac{x}{\chi} \lim_{\omega \to 0} \frac{1}{\omega} \text{Im} G^R_{n e^{i \alpha} j_y} (\omega, y) d^2 y.
\end{equation}

The contribution from outside of the cores turns out to be suppressed by powers of \( x \) compared to the inside-core contribution, as we show in the supplementary material [37]. The Green’s function in the core appearing in (16) again follows from the diffusive normal state dynamics. It is given by \( G^R_{n e^{i \alpha} j_y} (\omega, k) = -i \omega \sigma_n^H k^2 / (-i \omega + Dk^2) \) and is derived in the supplementary material [37]. Here \( \sigma_n^H \) is the Hall conductivity of the normal state in the core. Using this Green’s function we obtain
\begin{equation}
\alpha_v = x \frac{\sigma_n^H}{\sigma_n} = -x \tan \theta_n^H.
\end{equation}

Here \( \theta_n^H \) is the Hall angle of the normal state.

Conductivity and resistivity.—Inserting the flux-flow results (2) and (17) into the hydrodynamic expressions (6) and (7) gives the dc conductivities at small \( x \):
\begin{equation}
\sigma_{xx} = \frac{\sigma_n}{2x},
\end{equation}
\begin{equation}
\sigma_{xy} = \sigma_n^H + \sigma_o^H + \sigma_{xx}^2 \frac{n_{\text{eff}}}{n_q}.
\end{equation}

The final term in (19) is larger than the first two by a factor of \( 1/x \), because \( \sigma_{xx}^2 \sim 1/x^2 \) and \( n_{\text{eff}} \sim x \). We saw in our earlier discussion, however, that the other terms can dominate when \( n_{\text{eff}} \) is suppressed. Assuming \( \sigma_{xy} \ll \sigma_{xx} \) then gives the Hall resistivity (1).

Final remark.—The hydrodynamic theory can be extended into the superconducting phase, and explains how dynamical depinning of vortices leads to the zero frequency conductance peaks observed in [3]. Ignoring the (small) parity-odd terms, the optical conductivity (6) is a simple Lorentzian \( \sigma(\omega) = f_s / (-i \omega + \Omega) \). We have noted that \( \Omega \) is the vortex conductivity. A simple model of vortex pinning is to let \( \Omega \to \Omega(\omega) = \omega \Omega / (\omega + i \omega_o) \). Here \( \omega_o \) is a pinning frequency. This form arises in the limit of strong momentum relaxation from the general hydrodynamics of pinned lattices [41]. The upshot is then the optical conductivity
\begin{equation}
\sigma(\omega) = \frac{f_s}{\Omega + \omega_o \left( \omega_o + \omega - i \omega \right)} = \frac{f_s}{\omega_o \left( \omega_o + \omega - i \omega \right)}.
\end{equation}

A superconducting delta function arises once the pinning frequency \( \omega_o \) becomes nonzero. It is accompanied by a zero frequency Lorentzian peak whose width is continuous across the superconducting-anomalous metal transition (which is driven by \( \omega_o \to 0 \), not \( \Omega \to 0 \)). This is what the data shows [3], further supporting the picture of the anomalous metal as being due to the flux flow of mobile vortices. Indeed, zero field amorphous InO\(_x\) shows a canonical BKT transition as a function of temperature. The conductance peak in the high temperature BKT phase [42] is due to mobile unpaired vortices, and is continuously connected in the phase diagram to the conductance peak seen in the anomalous metal [2,3].

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Spatial inhomogeneity may be an important ingredient of anomalous metals. Hydrodynamics is a powerful framework for inhomogeneous dynamics, but we focus on the simplest, homogeneous situation in this work.

See supplementary material at .... for details..

Here and throughout we have defined $\Omega_H$ with a different sign relative to our previous work. We have also renamed $\rho_v$ in our previous work as $\alpha_v$, because $\rho_v$ carried misleading connotations.

This is the most relevant Hamiltonian for phase relaxation in the large core limit that we will be considering. Other terms can become relevant away from this limit.

