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# Deconfined Quantum Critical Point on the Triangular Lattice

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In this work we propose a theory for the deconfined quantum critical point (dQCP) for spin-1/2 systems on a triangular lattice, which is a direct unfinetuned quantum phase transition between the standard “ $\sqrt{3}\times\sqrt{3}$ ” noncollinear antiferromagnetic order (or the so-called 120° state) and the “ $\sqrt{12}\times\sqrt{12}$ ” valence solid bond (VBS) order, both of which are very standard ordered phases often observed in numerical simulations. This novel transition is beyond the standard Landau-Ginzburg paradigm, and is also fundamentally different from the original dQCP theory on the square lattice, due to the very different structure of both the magnetic and VBS order on frustrated lattices. We first propose a topological term in the effective field theory that captures the “intertwinement” between the  $\sqrt{3}\times\sqrt{3}$  antiferromagnetic order and the  $\sqrt{12}\times\sqrt{12}$  VBS order. Then using a controlled renormalization group calculation, we demonstrate that there exists an unfinetuned direct continuous deconfined quantum critical point (dQCP) between the two ordered phases mentioned above. This dQCP is described by the  $N_f = 4$  quantum electrodynamics (QED) with an emergent  $\text{PSU}(4)=\text{SU}(4)/Z_4$  symmetry only at the critical point. The topological term aforementioned is also naturally derived from the  $N_f = 4$  QED. We also point out that physics around this dQCP is analogous to the boundary of a 3d bosonic symmetry protected topological state with on-site symmetries only.

## I. INTRODUCTION

The deconfined quantum critical point (dQCP) [1, 2] was proposed as the first explicit example of a direct unfinetuned quantum critical point [56] beyond the standard Landau’s paradigm, because the dQCP is sandwiched between two very different ordered phases with completely unrelated broken symmetries [1]. More precisely, the symmetry that is spontaneously broken on one side of the transition is completely independent from the symmetry that is broken on the other side. This scenario was forbidden in the standard Landau’s paradigm, but was proposed to be possible in quantum spin systems [1, 2]. A lot of numerical work has been devoted to investigating the dQCP with a full spin rotation symmetry [3–14], as well as spin models with only in-plane spin symmetry [15–18]. Recently developed duality between strongly interacting QCPs in  $(2+1)d$  have further improved our understanding of the dQCP [19–24], and the predictions made by duality has received numerical supports [25, 26].

Let us first summarize the key ingredients of the original dQCP on the square lattice [1, 2]:

(1) This is a quantum phase transition sandwiched between the standard antiferromagnetic Néel state and the valence bond solid (VBS) state. The Néel state has ground state manifold (GSM) equivalent to a two dimensional sphere ( $S^2$ ), i.e. all the configurations of the Néel vector form a manifold  $S^2$ . Although the VBS only has four fold degeneracy on the square lattice, there is a strong evidence that the four fold rotation symmetry of the square lattice is enlarged to a  $U(1)$  rotation symmetry right at the dQCP, and the VBS state has an approximate GSM  $S^1$  (one dimensional ring), which is *not* a submanifold of the GSM of the Néel state on the other side of the dQCP. Thus we can view the dQCP on the square lattice as a “ $S^2$ -to- $S^1$ ” transition.

In another proposed realization of the dQCP [27], the Néel order and the VBS order are replaced by the quantum spin Hall order parameter and the  $s$ -wave superconductor, thus in this realization the dQCP is literally a transition between  $S^2$  and  $S^1$ .

(2) The vortex of the VBS order parameter carries a bosonic spinor (spin-1/2) of the spin symmetry, and the Skyrmion of the Néel order carries lattice momentum. This physics can be described by the NCCP<sup>1</sup> model [1, 2]:  $\mathcal{L} = \sum_{\alpha} |(\partial_{\mu} - ia_{\mu})z_{\alpha}|^2 + r|z_{\alpha}|^2 + \dots$ , where the Néel order parameter is  $\vec{N} = z^{\dagger}\vec{\sigma}z$ , the flux of  $a_{\mu}$  is the Skyrmion density of  $\vec{N}$ , and the flux condensate (which is dual to the photon phase of  $a_{\mu}$  [28–30] based on the standard photon-superfluid duality) is the VBS order. Thus there is an “intertwinement” between the Néel and VBS order: the defect of one order parameter is decorated with the quantum number of the symmetry that defines the other order, thus the condensation of the defect leads to the other order. This unusual quantum phase transition is considered “deconfined” because the field theory above is not formulated in terms of the standard Landau order parameter, but in terms of “fractionalized” degrees of freedom such as the spinon field  $z_{\alpha}$ .

(3) If we treat the Néel and the VBS orders on equal footing, we can introduce a five component unit vector  $\vec{n} \sim (N_x, N_y, N_z, V_x, V_y)$ , and the “intertwinement” between the two order parameters is precisely captured by a

topological Wess-Zumino-Witten (WZW) term of the nonlinear sigma model defined in the target space  $S^4$  (four dimensional sphere) where  $\vec{n}$  lives [27, 31].

All the previous works on dQCP have focused on the example proposed in Ref. 1, 2, which is a theory specially designed for the square lattice. In this work we propose a possible dQCP on the triangular lattice (and the Kagome lattice) for spin-1/2 systems with a full  $SU(2)$  spin rotation symmetry. Soon we will see that due to the fundamentally different structure of the magnetic order and VBS order from the square lattice, the dQCP on frustrated lattices demands a completely different formalism, with a very different universality class, and an unexpected emergent symmetry.

Let us first summarize the standard phases for spin-1/2 systems with a full spin rotation symmetry on the triangular lattice. On the triangular lattice, the standard antiferromagnetic order is no longer a collinear Néel order, it is the  $\sqrt{3} \times \sqrt{3}$  noncollinear spin order (or the so-called  $120^\circ$  order) with ground state manifold (GSM)  $SO(3)$ , which is fundamentally different from the GSM  $S^2$  of the collinear magnetic order.

The VBS order most often discussed and observed in numerical simulations is the so-called  $\sqrt{12} \times \sqrt{12}$  VBS pattern [32–34]. This VBS order is the most natural pattern that can be obtained from the condensate of the vison (or the  $m$  excitation) of a  $Z_2$  spin liquid on the triangular lattice. The dynamics of visons on the triangular lattice is equivalent to a fully frustrated Ising model on the dual honeycomb lattice [35], and it has been shown that with nearest neighbor hopping on the dual honeycomb lattice, there are four symmetry protected degenerate minima of the vison band structure in the Brillouin zone, and that the GSM of the VBS order can be most naturally embedded into manifold  $SO(3)$  (just like the VBS order on the square lattice can be embedded in  $S^1$ ) [35]. Thus the  $\sqrt{3} \times \sqrt{3}$  noncollinear spin order and the  $\sqrt{12} \times \sqrt{12}$  VBS order have a “self-dual” structure, *i.e.* the magnetic order and the VBS order are dual to each other. Conversely on the square lattice, the self-duality between the Néel and VBS order only happens in the easy-plane limit [36].

The self-duality structure on the triangular lattice was noticed in Ref. 37 and captured by a mutual Chern-Simons (CS) theory:

$$\mathcal{L} = |(\partial - ia)z|^2 + r_z|z|^2 + |(\partial - ib)v|^2 + r_v|v|^2 + \frac{i}{\pi}a \wedge db + \dots \quad (1)$$

$z_\alpha$  and  $v_\beta$  carry a spinor representation of  $SO(3)_e$  and  $SO(3)_m$  groups respectively, and when they are both gapped ( $r_z, r_v > 0$ ), they are the  $e$  and  $m$  excitations of a symmetric  $Z_2$  spin liquid on the triangular lattice, with a mutual semion statistics enforced by the mutual Chern-Simons (CS) term [37]. Physically  $z_\alpha$  is the Schwinger boson of the standard construction of spin liquids on the triangular lattice [38–40], while  $v_\beta$  is the low energy effective modes of the vison.

Eq. 1 already unifies much of the physics for spin-1/2 systems on the triangular lattice [37]. For example, when both  $z_\alpha$  and  $v_\beta$  are gapped, the system is in the  $Z_2$  spin liquid mentioned above. The  $\sqrt{3} \times \sqrt{3}$  noncollinear spin order, and the VBS order can be obtained from the self-dual  $Z_2$  spin liquid by condensing  $z_\alpha$  and  $v_\beta$  respectively, and both transitions have an emergent  $O(4)$  symmetry [35, 41].

The problem of finding a dQCP on the triangular lattice between the noncollinear magnetic order and the VBS order, is equivalent to finding a *direct unfine-tuned transition between two different orders each with GSM  $SO(3)$* , or in our notation an “ $SO(3)$ -to- $SO(3)$  transition”.

## II. TOPOLOGICAL TERM OF EFFECTIVE FIELD THEORY

As we discussed in the introduction, the physical picture of the dQCP is the “intertwinement” between the two ordered phases, namely the defect of one order is decorated with the quantum number of the other order, hence once we “melt” one ordered phase by proliferating its defects, the system will automatically be driven into the other order. On the square lattice, if we treat the Néel and the VBS orders on equal footing, we can introduce a five component unit vector  $\vec{n} \sim (N_x, N_y, N_z, V_x, V_y)$ , then the “intertwinement” between the two order parameters is precisely captured by a topological Wess-Zumino-Witten (WZW) term of the nonlinear sigma model defined in the target space  $S^4$  (four dimensional sphere) where  $\vec{n}$  lives [27, 31]:

$$\mathcal{L}_{wzw} = \int d^3x \int_0^1 du \frac{2\pi i}{\Omega_4} \epsilon_{abcde} n^a \partial_x n^b \partial_y n^c \partial_\tau n^d \partial_u n^e, \quad (2)$$

where  $\Omega_4$  is the volume of  $S^4$ .  $\vec{n}(x, \tau, u)$  is any smooth extension of  $\vec{n}(x, \tau)$  such that  $\vec{n}(x, \tau, 0) = (1, 0, 0, 0, 0)$  and  $\vec{n}(x, \tau, 1) = \vec{n}(x, \tau)$ .

In Eq. 1,  $v_\beta$  is the vison of the spin liquid, and it carries a  $\pi$ -flux of  $a_\mu$  due to the mutual CS term in Eq. 1. The  $\pi$ -flux of  $a_\mu$  is bound with the  $Z_2$  vortex of the  $SO(3)_e$  GSM of the  $\sqrt{3} \times \sqrt{3}$  spin order. Due to the homotopy group

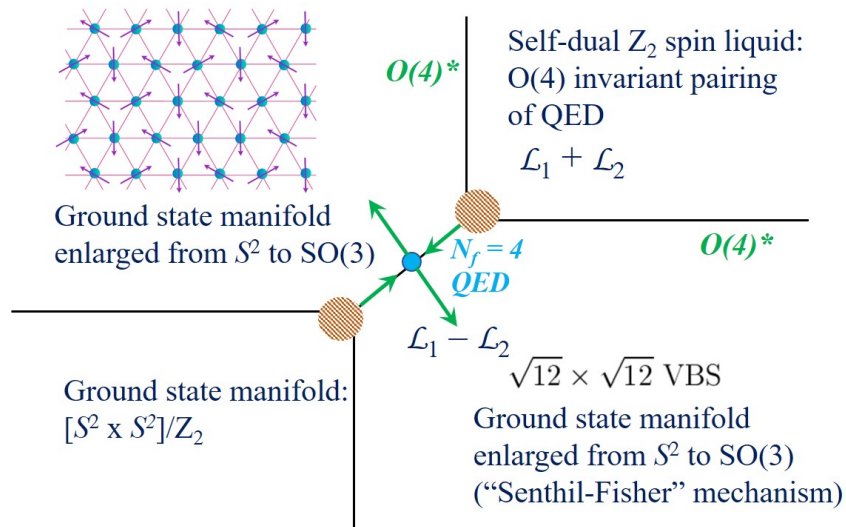


FIG. 1: The global phase diagram of spin-1/2 systems on the triangular lattice. The intertwinement between the order parameters is captured by the WZW term Eq. 3. Our RG analysis concludes that there is a direct unfine-tuned SO(3)-to-SO(3) transition, which is a direct unfine-tuned transition between the noncollinear magnetic order and the VBS order. The detailed structure of the shaded areas demands further studies

$\pi_1[\text{SO}(3)] = Z_2$ , any ordered phase with GSM SO(3) has  $Z_2$  vortex excitations, namely two vortices can annihilate each other. Similarly  $z_\alpha$  is also the  $Z_2$  vortex of the  $\text{SO}(3)_m$  GSM of the VBS order, analogous to the vortex of the VBS order on the square lattice. This mutual “decoration” of topological defects means that there is also an “intertwinement” between the noncollinear  $\sqrt{3} \times \sqrt{3}$  magnetic order and the  $\sqrt{12} \times \sqrt{12}$  VBS orders on the triangular lattice.

To capture the “intertwinement” of the two phases both with GSM SO(3), *i.e.* to capture the mutual decoration of topological defects, we need to design a topological term for these order parameters, just like the O(5) WZW term for the dQCP on the square lattice [31]. The topological term we design is as follows:

$$\mathcal{L}_{wzw} = \int d^3x \int_0^1 du \frac{2\pi i}{256\pi^2} \epsilon_{\mu\nu\rho\lambda} \text{tr}[\mathcal{P}\partial_\mu\mathcal{P}\partial_\nu\mathcal{P}\partial_\rho\mathcal{P}\partial_\lambda\mathcal{P}]. \quad (3)$$

Here  $\mathcal{P}$  is a  $4 \times 4$  Hermitian matrix field:

$$\mathcal{P} = \sum_{a,b=1}^3 N_e^a N_m^b \sigma^{ab} + \sum_{a=1}^3 M_e^a \sigma^{a0} + \sum_{b=1}^3 M_m^b \sigma^{0b}, \quad (4)$$

where  $\sigma^{ab} = \sigma^a \otimes \sigma^b$ , and  $\sigma^0 = \mathbf{1}_{2 \times 2}$ . All vectors  $\vec{N}_e$ ,  $\vec{N}_m$ ,  $\vec{M}_e$  and  $\vec{M}_m$  transform as vectors under  $\text{SO}(3)_e$  and  $\text{SO}(3)_m$  depending on their subscripts. And we need to also impose some extra constraints:

$$\mathcal{P}^2 = \mathbf{1}_{4 \times 4}, \quad \vec{N}_e \cdot \vec{M}_e = \vec{N}_m \cdot \vec{M}_m = 0. \quad (5)$$

Then  $\vec{N}_e$  and  $\vec{M}_e$  together will form a SO(3) “tetrad”, which is equivalent to the SO(3) manifold.  $\vec{N}_m$  and  $\vec{M}_m$  form another SO(3) manifold. With the constraints in Eq. 5, the matrix field  $\mathcal{P}$  is embedded in the manifold

$$\mathcal{M} = \frac{U(4)}{U(2) \times U(2)}. \quad (6)$$

The maximal symmetry of the WZW term Eq. 3 is  $\text{PSU}(4) = \text{SU}(4)/Z_4$  (which contains both  $\text{SO}(3)_e$  and  $\text{SO}(3)_m$  as subgroups), as the WZW term is invariant under a SU(4) transformation:  $\mathcal{P} \rightarrow U^\dagger \mathcal{P} U$  with  $U \in \text{SU}(4)$ , while the  $Z_4$  center of SU(4) does not change any configuration of  $\mathcal{P}$ . The WZW term Eq. 3 is well-defined based on its homotopy group  $\pi_4[\mathcal{M}] = \mathbb{Z}$ , just like  $\pi_4[S^4] = \mathbb{Z}$ . Obviously the SU(4) symmetry contains both  $\text{SO}(3)_e$  and  $\text{SO}(3)_m$  as subgroups.

The topological WZW term in Eq. 3 is precisely the boundary theory of a  $3d$  symmetry protected topological (SPT) state with a PSU(4) symmetry [42]. We will discuss this further later.

Let us test that this topological term captures the correct intertwining. *i.e.* it must capture the physics that the  $Z_2$  vortex of  $\text{SO}(3)_e$  carries spinor of  $\text{SO}(3)_m$ , and vice versa. This effect is most conveniently visualized after breaking  $\text{SO}(3)_m$  down to  $\text{SO}(2)_m$ , and the  $Z_2$  vortex of the  $\text{SO}(3)_m$  manifold becomes the ordinary vortex of an  $\text{SO}(2)$  order parameter. This symmetry breaking allows us to take  $\vec{N}_m = (0, 0, 1)$ , *i.e.*  $N_m^1 = N_m^2 = 0$ ,  $N_m^3 = 1$ . Because  $\vec{N}_m \cdot \vec{M}_m = 0$  (Eq. 5),  $\vec{M}_m = (M_m^1, M_m^2, 0)$ . Then one allowed configuration of  $\mathcal{P}$  is

$$\mathcal{P} = \sum_{a=1}^3 N_e^a \sigma^{a3} + \sum_{b=1}^2 M_m^b \sigma^{0b} = \vec{n} \cdot \vec{\Gamma}, \quad (7)$$

where  $\vec{n}$  is a five component vector and  $|\vec{n}| = 1$  due to the constraint  $\mathcal{P}^2 = \mathbf{1}_{4 \times 4}$ .  $\vec{\Gamma}$  are five anticommuting Gamma matrices. Now the WZW term Eq. 3 reduces *precisely* to the standard O(5) WZW at level-1 in  $(2+1)d$ , and it becomes manifest that the vortex of  $(M_m^1, M_m^2)$  (the descendant of the  $Z_2$  vortex of  $\text{SO}(3)_m$  under the assumed symmetry breaking) is decorated with a spinor of  $\text{SO}(3)_e$ . To explicitly visualize the effect of the “decorated vortex”, one can follow the procedure of Ref. 27, and create a vortex of  $(n_4, n_5)$ . Then the physics in the vortex core becomes a zero-dimensional quantum mechanics problem, whose exact solution reveals that there is a spin-1/2 carried by each vortex.

### III. FIELD THEORY AND RENORMALIZATION GROUP ANALYSIS

Eq. 3 is a topological term in the low energy effective field theory that describes the physics of the ordered phases. But a complete field theory which reduces to the WZW term in the infrared is still demanded. For example, the O(5) nonlinear sigma model with a WZW term at level-1 can be derived as the low energy effective field theory of the  $N = 2$  QCD with SU(2) gauge field, which has an explicit SO(5) global symmetry [24].

The WZW term in Eq. 3 can be derived in the same manner, by coupling the matrix field  $\mathcal{P}$  to the Dirac fermions of the  $N_f = 4$  QED:

$$\mathcal{L} = \sum_{j=1}^4 \bar{\psi}_j \gamma \cdot (\partial - ia) \psi_j + m \sum_{i,j} \bar{\psi}_i \psi_j \mathcal{P}_{ij}. \quad (8)$$

The WZW term of  $\mathcal{P}$  is generated after integrating out the fermions using the same method as Ref. 43, and the PSU(4) global symmetry becomes explicit in  $N_f = 4$  QED [57].

Our goal is to demonstrate that the  $N_f = 4$  QED corresponds to an unfine-tuned dQCP between the noncollinear magnetic order and the VBS order, or in our notation a “SO(3)-to-SO(3)” transition (as the dQCP is sandwiched between two ordered phases both with GSM SO(3)). The PSU(4) global symmetry of  $N_f = 4$  QED must be explicitly broken down to the physical symmetry. The most natural terms that break this PSU(4) global symmetry down to  $\text{SO}(3)_e \times \text{SO}(3)_m$  are four-fermion interaction terms. Under the assumption of an emergent Lorentz invariance, which often happens at quantum critical points and algebraic spin liquids (such as the original dQCP on the square lattice), there are *only two* such linearly independent terms that break the PSU(4) global symmetry down to  $\text{SO}(3)_e \times \text{SO}(3)_m$ :

$$\mathcal{L}_1 = (\bar{\psi} \vec{\sigma} \psi) \cdot (\bar{\psi} \vec{\sigma} \psi), \quad \mathcal{L}_2 = (\bar{\psi} \vec{\tau} \psi) \cdot (\bar{\psi} \vec{\tau} \psi), \quad (9)$$

where  $\psi$  carries both indices from the Pauli matrices  $\vec{\sigma}$  and  $\vec{\tau}$ , so that  $\psi$  is a vector representation  $(\frac{1}{2}, \frac{1}{2})$  of  $\text{SO}(4) \sim \text{SO}(3)_e \times \text{SO}(3)_m$ .

One can think of some other four fermion terms, for example  $\mathcal{L}' = \sum_{\mu} (\bar{\psi} \vec{\sigma} \gamma_{\mu} \psi) \cdot (\bar{\psi} \vec{\sigma} \gamma_{\mu} \psi)$ , but we can repeatedly use the Fierz identity, and reduce these terms to a linear combination of  $\mathcal{L}_1$  and  $\mathcal{L}_2$ , as well as SU(4) invariant terms:  $\mathcal{L}' = -2\mathcal{L}_2 - \mathcal{L}_1 + \dots$  (for more details please refer to the appendix). The ellipses are SU(4) invariant terms, which according to Ref. 44–46 are irrelevant at the  $N_f = 4$  QED.

The renormalization group (RG) flow of  $\mathcal{L}_1$  and  $\mathcal{L}_2$  can be most conveniently calculated by generalizing the two dimensional space of Pauli matrices  $\vec{\tau}$  to an  $N$ -dimensional space, *i.e.* we generalize the QED<sub>3</sub> to an  $N_f = 2N$  QED<sub>3</sub> with SU(2)  $\times$  SU( $N$ ) symmetry. And we consider the following two independent four fermion terms:

$$g\mathcal{L} = g(\bar{\psi} \vec{\sigma} \psi) \cdot (\bar{\psi} \vec{\sigma} \psi), \quad g'\mathcal{L}' = g'(\bar{\psi} \vec{\sigma} \gamma_{\mu} \psi) \cdot (\bar{\psi} \vec{\sigma} \gamma_{\mu} \psi). \quad (10)$$

One can check that all SU(2)  $\times$  SU( $N$ ) four fermion interactions in this QED<sub>3</sub> can be written in terms of the linear combinations of these two terms above up to SU( $2N$ ) invariant terms which according to Ref. 44–46 are irrelevant

under RG even for small  $N$ . At the first order of  $1/N$  expansion, the RG equation reads

$$\begin{aligned}\beta(g) &= \left(-1 + \frac{128}{3(2N)\pi^2}\right)g + \frac{64}{(2N)\pi^2}g', \\ \beta(g') &= -g' + \frac{64}{3(2N)\pi^2}g.\end{aligned}\tag{11}$$

There are two RG flow eigenvectors:  $(1, -1)$  with RG flow eigenvalue  $-1 - 64/(3(2N)\pi^2)$ , and  $(3, 1)$  with eigenvalue  $-1 + 64/((2N)\pi^2)$ . This means that when  $N = 2$  there is one irrelevant eigenvector with

$$\mathcal{L} - \mathcal{L}' = 2(\mathcal{L}_1 + \mathcal{L}_2) + \dots,\tag{12}$$

and a relevant eigenvector with

$$3\mathcal{L} + \mathcal{L}' = 2(\mathcal{L}_1 - \mathcal{L}_2) + \dots.\tag{13}$$

Again the ellipses are SU(4) invariant terms that are irrelevant. In fact,  $\mathcal{L}_1 + \mathcal{L}_2$  preserves the exchange symmetry (duality) between  $\text{SO}(3)_e$  and  $\text{SO}(3)_m$ , in other words  $\mathcal{L}_1 + \mathcal{L}_2$  preserves the O(4) symmetry that contains an extra improper rotation in addition to SO(4), while  $\mathcal{L}_1 - \mathcal{L}_2$  breaks the O(4) symmetry down to SO(4). Thus  $\mathcal{L}_1 + \mathcal{L}_2$  and  $\mathcal{L}_1 - \mathcal{L}_2$  both *must be* eigenvectors under RG. The RG flow is sketched in Fig. 1.

To make a complete story, we should also discuss other perturbations on the  $N_f = 4$  QED. The fermion bilinear terms are forbidden either by the flavor symmetry, or discrete space-time symmetries, while higher order fermion interactions (such as eight fermion interactions) are very likely to be irrelevant. The monopoles of  $a_\mu$  were ignored in this RG calculation. According to Ref. 47, monopoles of QED carry nontrivial quantum numbers. A multiple-monopole could be a singlet under the global symmetry, and hence allowed in the action, but it will have a higher scaling dimension than the single monopole. It is known that with large- $N_f$  all the monopoles are irrelevant, but the scaling dimension of the multiple-monopole for the current case with  $N_f = 4$  needs further study.

Since  $u(\mathcal{L}_1 - \mathcal{L}_2)$  is relevant, then when the coefficient  $u > 0$ , a simple mean field theory implies that this term leads to a nonzero expectation value for  $\langle \bar{\psi}\vec{\sigma}\psi \rangle$ . It appears that this order parameter is a three component vector, and so the GSM should be  $S^2$ . However, using the ‘‘Senthil-Fisher’’ mechanism of Ref. 31, the actual GSM is enlarged to SO(3) due to the gauge fluctuation of  $a_\mu$  (for a review of the ‘‘Senthil-Fisher’’ mechanism, please refer to the appendix). When  $u < 0$ , the condensed order parameter is  $\langle \bar{\psi}\vec{\tau}\psi \rangle$ , and the ‘‘Senthil-Fisher’’ mechanism again enlarges the GSM to SO(3). Based on our calculation, because  $u(\mathcal{L}_1 - \mathcal{L}_2)$  is the *only* relevant perturbation allowed by symmetry,  $u$  drives a direct unfine-tuned continuous SO(3)-to-SO(3) transition, which is consistent with a transition between the  $\sqrt{3} \times \sqrt{3}$  noncollinear magnetic order and the  $\sqrt{12} \times \sqrt{12}$  VBS order. And our theory predicts that at the critical point, there is an emergent PSU(4) symmetry.

Now let us investigate the perturbation  $\mathcal{L}_1 + \mathcal{L}_2$ . First of all, let us think of a seemingly different term:  $\mathcal{L}_3 = \sum_{a,b} (\bar{\psi}\sigma^a\tau^b\psi) (\bar{\psi}\sigma^a\tau^b\psi)$ . This term also preserves the O(4) symmetry, and after some algebra we can show that  $\mathcal{L}_3 = -(\mathcal{L}_1 + \mathcal{L}_2) + \dots$ . Another very useful way to rewrite  $\mathcal{L}_3$  is that:

$$\mathcal{L}_3 = -(\bar{\psi}^t J \epsilon \bar{\psi}) (\psi^t J \epsilon \psi) + \dots = -\hat{\Delta}^\dagger \hat{\Delta} + \dots\tag{14}$$

where  $\hat{\Delta} = \psi^t J \epsilon \psi$ ,  $J = \sigma^2 \otimes \tau^2$ .  $\epsilon$  is the antisymmetric tensor acting on the Dirac indices.

Thus although the O(4) invariant deformation in our system (at low energy it corresponds to  $\mathcal{L}_1 + \mathcal{L}_2$ ) is perturbatively irrelevant at the  $N_f = 4$  QED fixed point, when it is strong and nonperturbative, the standard Hubbard-Stratonovich transformation and mean field theory suggests that, depending on its sign, it may lead to either a condensate of  $\hat{\Delta}$ , or condensate of  $(\bar{\psi}\sigma^a\tau^b\psi)$  over certain critical strength of  $\mathcal{L}_3$ . The condensate of  $(\bar{\psi}\sigma^a\tau^b\psi)$  has GSM  $[S^2 \times S^2]/Z_2$ , and is identical to the submanifold of  $\mathcal{P}$  when  $\vec{M}_e = \vec{M}_m = 0$  in Eq. 4. The  $Z_2$  in the quotient is due to the fact that  $\mathcal{P}$  is unaffected when both  $\vec{N}_e$  and  $\vec{N}_m$  change sign simultaneously.

Now we show that the condensate of  $\hat{\Delta}$  is precisely the self-dual  $Z_2$  topological order described by Eq. 1. First of all, in the superconductor phase with  $\hat{\Delta}$  condensate, there will obviously be a Bogoliubov fermion. This Bogoliubov fermion carries the  $(1/2, 1/2)$  representation under  $\text{SO}(3)_e \times \text{SO}(3)_m$ . The deconfined  $\pi$ -flux of the gauge field  $a_\mu$  is bound to a  $2\pi$ -vortex of the complex order parameter  $\hat{\Delta}$ , which then traps 4 Majorana zero modes. The 4 Majorana zero modes transform as a vector under the SO(4) action that acts on the flavor indices. The 4 Majorana zero modes define 4 different states that can be separated into two groups of states depending on their fermion parities. In fact, the two groups should be identified as the  $(1/2, 0)$  doublet and the  $(0, 1/2)$  doublet of  $\text{SO}(3)_e \times \text{SO}(3)_m$ . Therefore, the  $\pi$ -flux with two different types of doublets should be viewed as two different topological excitations. Let us denote the  $(1/2, 0)$  doublet as  $e$  and the  $(0, 1/2)$  doublet as  $m$ . Both  $e$  and  $m$  have bosonic topological spins. And they differ by a Bogoliubov fermion. Therefore, their mutual statistics is semionic (which rises from the braiding between the fermion and the  $\pi$ -flux). At this point, we can identify the topological order of the  $\hat{\Delta}$  condensate as the  $Z_2$  topological order described by Eq. 1.

#### IV. INTERPRETATION OF THE DQCP AS THE BOUNDARY OF A $3d$ SYSTEM

Decorating quantum numbers to topological defects is also a key physical picture of constructing symmetry protected topological (SPT) states. The analogy between the dQCP on the square lattice and a  $3d$  bulk SPT state with an  $\text{SO}(5)$  symmetry was discussed in Ref. 24. Many  $3d$  SPT states can be constructed by decorating the defects in the system with a lower dimensional SPT state, and then proliferating the defects [48, 49].

The physics around the dQCP discussed in this work is equivalent to the boundary state of a  $3d$  bosonic SPT state with  $\text{SO}(3)_e \times \text{SO}(3)_m$  symmetry, once we view both  $\text{SO}(3)$  groups as onsite symmetries. We have already mentioned that the topological WZW term Eq. 3 is identical to the boundary theory of a  $3d$  SPT state with  $\text{PSU}(4)$  symmetry [42], which comes from a  $\Theta$ -term in the  $3d$  bulk. And by breaking the symmetry down to either  $\text{SO}(3)_e \times \text{SO}(2)_m$  or  $\text{SO}(2)_e \times \text{SO}(3)_m$ , the bulk SPT state is reduced to a  $\text{SO}(3) \times \text{SO}(2)$  SPT state, which can be interpreted as the decorated vortex line construction [49], namely one can decorate the  $\text{SO}(2)$  vortex line with the Haldane phase with the  $\text{SO}(3)$  symmetry, and then proliferate the vortex lines. In our case, the bulk SPT state with  $\text{SO}(3)_e \times \text{SO}(3)_m$  symmetry can be interpreted as a similar decorated vortex line construction, *i.e.* we can decorate the  $Z_2$  vortex line of one of the  $\text{SO}(3)$  manifolds with the Haldane phase of the other  $\text{SO}(3)$  symmetry, then proliferating the vortex lines. The  $Z_2$  classification of the Haldane phase is perfectly compatible with the  $Z_2$  nature of the vortex line of a  $\text{SO}(3)$  manifold. Using the method in Ref. 24, one can also derive that the  $(3+1)d$  bulk SPT state must have a topological response action  $\mathcal{S} = i\pi \int w_2[\mathcal{A}_e] \cup w_2[\mathcal{A}_m]$  in the presence of background  $\text{SO}(3)_e$  gauge field  $\mathcal{A}_e$  and  $\text{SO}(3)_m$  gauge field  $\mathcal{A}_m$  ( $w_2$  represents the second Stiefel–Whitney class). This topological response theory also matches exactly with the decorated vortex line construction.

We have shown that the physics around the critical point has the same effective field theory as the boundary of a  $3d$  SPT state [42]. The anomaly (once we view all the symmetries as onsite symmetries) of the large- $N$  generalizations of our theory will be analyzed in the future, and a Lieb-Shultz-Mattis theorem for  $\text{SU}(N)$  and  $\text{SO}(N)$  spin systems on the triangular and Kagome lattice can potentially be developed in the same way as Ref. 50, 51.

#### V. SUMMARY

In summary, we proposed a theory for a potentially direct unfine-tuned continuous quantum phase transition between the noncollinear magnetic order and VBS order on the triangular lattice, and at the critical point the system has an emergent  $\text{PSU}(4)$  global symmetry. Our proposed dQCP is fundamentally different from the original example on the square lattice due to the very different structure of both the magnetic and VBS orders compared with the unfrustrated square lattice. Our conclusion is based on a controlled RG calculation, and an effective nonlinear sigma model with a topological WZW term.

Similar structure of noncollinear magnetic order and VBS orders can be found on the Kagome lattice. For example, it was shown in Ref. 52 that the vison band structure could have symmetry protected four degenerate minima just like the triangular lattice (although the emergence of  $\text{O}(4)$  symmetry in the infrared is less likely). Indeed, algebraic spin liquids with  $N_f = 4$  QED as their low energy description have been discussed extensively on both the triangular and the Kagome lattice [40, 53–55]. Ref. 40 also made the observation that the noncollinear magnetic order, the VBS order, and the  $Z_2$  spin liquid are all nearby a  $N_f = 4$  QED (the so-called  $\pi$ -flux state from microscopic construction). The  $Z_2$  spin liquid was shown to be equivalent to the one constructed from Schwinger boson [39], which can evolve into the  $\sqrt{3} \times \sqrt{3}$  magnetic order, and the  $\sqrt{12} \times \sqrt{12}$  VBS order through an  $\text{O}(4)^*$  transition. But we should stress that in this work we only focus on the field theory for the “ $\text{SO}(3)$ -to- $\text{SO}(3)$ ” dQCP, without fully determining the relation between the field theory and the microscopic degrees of freedom.

It is a challenge to find an antiferromagnetic spin model on a frustrated lattice without sign problem. But we note that in Ref. 34 spin nematic phases with GSM  $S^N/Z_2$  (analogous to the spin-1/2  $\sqrt{3} \times \sqrt{3}$  state with GSM  $\text{SO}(3) = S^3/Z_2$ ) and the  $\sqrt{12} \times \sqrt{12}$  VBS order are found in a series of sign-problem free models on the triangular lattice. Thus it is possible to design a modified version of the models discussed in Ref. 34 to bring together the spin nematic order and VBS order, and then access the dQCP that we are proposing.

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## Appendix A: The ordered phases and the ‘‘Senthil-Fisher’’ mechanism

Here we reproduce the discussion in Ref. 31, and demonstrate how the GSM of the order of  $\bar{\psi}\vec{\sigma}\psi$  (and similarly  $\bar{\psi}\vec{\tau}\psi$ ) is enlarged from  $S^2$  to  $\text{SO}(3)$ . First we couple the  $N_f = 4$  QED to a three component dynamical unit vector field  $\mathbf{N}(x, \tau)$ :

$$\mathcal{L} = \bar{\psi}\gamma_\mu(\partial_\mu - ia_\mu)\psi + m\bar{\psi}\boldsymbol{\sigma}\psi \cdot \mathbf{N}. \quad (\text{A1})$$

The flavor indices are hidden in the equation above for simplicity. Now following the standard  $1/m$  expansion of Ref. 43, we obtain the following action after integrating out the fermion  $\psi_j$ :

$$\mathcal{L}_{eff} = \frac{1}{g}(\partial_\mu \mathbf{N})^2 + i2\pi \text{Hopf}[\mathbf{N}] + i2a_\mu J_\mu^T + \frac{1}{e^2} f_{\mu\nu}^2, \quad (\text{A2})$$

where  $1/g \sim m$ .  $J_0^T = \frac{1}{4\pi}\epsilon_{abc}N^a\partial_x N^b\partial_y N^c$  is the Skyrmion density of  $\mathbf{N}$ , thus  $J_\mu^T$  is the Skyrmion current. The second term of Eq. A2 is the Hopf term of  $\mathbf{N}$  which comes from the fact that  $\pi_3[S^2] = \mathbb{Z}$ .

Now if we introduce the  $\text{CP}^1$  field  $z_\alpha = (z_1, z_2)^t = (n_1 + in_2, n_3 + in_4)^t$  for  $\mathbf{N}$  as  $\mathbf{N} = z^\dagger \boldsymbol{\sigma} z$ , the Hopf term becomes precisely the  $\Theta$ -term for the  $\text{O}(4)$  unit vector  $\mathbf{n}$  with  $\Theta = 2\pi$ :

$$i2\pi \text{Hopf}[\mathbf{N}] = \frac{i2\pi}{2\pi^2} \epsilon_{abcd} n^a \partial_x n^b \partial_y n^c \partial_\tau n^d. \quad (\text{A3})$$

In the  $\text{CP}^1$  formalism, the Skyrmion current  $J_\mu^T = \frac{1}{2\pi}\epsilon_{\mu\nu\rho}\partial_\nu\alpha_\rho$ , where  $\alpha_\mu$  is the gauge field that the  $\text{CP}^1$  field  $z_\alpha$  couples to. The coupling between  $a_\mu$  and  $\alpha_\mu$

$$2ia_\mu J_\mu^T = \frac{i2}{2\pi}\epsilon_{\mu\nu\rho}a_\mu\partial_\nu\alpha_\rho \quad (\text{A4})$$

takes precisely the form of the mutual CS theory of a  $Z_2$  topological order, and it implies that the gauge charge  $z_\alpha$  is an anyon of a  $Z_2$  topological order, and the condensate of  $z_\alpha$  (equivalently the order of  $\mathbf{N}$ ) has a  $\text{GSM} = \text{SO}(3) = S^3/Z_2$ , where  $S^3$  is the manifold of the unit vector  $\vec{n}$ .

## Appendix B: Deriving the WZW term

Let us consider a theory of  $\text{QED}_3$  with  $N_f = 4$  flavors of Dirac fermions coupled to a matrix order parameter field  $\mathcal{P}$ :

$$\mathcal{L} = \sum_{i,j} \bar{\psi}_i (\gamma_\mu (\partial_\mu - ia_\mu) \delta_{ij} + m \mathcal{P}_{ij}) \psi_j. \quad (\text{B1})$$

$\mathcal{P}$  takes values in the target manifold  $\mathcal{P} \in \mathcal{M} = \frac{U(4)}{U(2) \times U(2)}$ . We can parametrize the matrix field  $\mathcal{P} = U^\dagger \Omega U$ , where  $U \in \text{SU}(4)$  and  $\Omega = \sigma^z \otimes \mathbf{1}_{2 \times 2}$ .  $\mathcal{P}$  satisfies  $\mathcal{P}^2 = \mathbf{1}_{4 \times 4}$  and  $\text{tr} \mathcal{P} = 0$ .

The effective action after integrating over the fermion fields formally reads

$$\begin{aligned} \mathcal{S}_{eff}[a_\mu, \mathcal{P}] &= -\ln \int D\bar{\psi} D\psi \exp \left[ - \int d^3x \mathcal{L}(\psi, a_\mu, \mathcal{P}) \right] \\ &= -\ln \det[\mathcal{D}(a_\mu, \mathcal{P})] = -\text{Tr} \ln[\mathcal{D}(a_\mu, \mathcal{P})]. \end{aligned} \quad (\text{B2})$$

The expansion of  $\mathcal{S}_{eff}$  has the following structure

$$\mathcal{S}_{eff}[a_\mu, \mathcal{P}] = \mathcal{S}_{eff}[a_\mu = 0, \mathcal{P}] + O(a) \quad (\text{B3})$$

and we will look at the first term in the expansion. In general, all terms that respect the symmetry of the original action will appear in the expansion of the fermion determinant. Here we want to derive the topological term of  $\mathcal{P}$ . One way to obtain the effective action is the perturbative method developed in Ref. 43. Let us vary the action over the matrix field  $\mathcal{P}$

$$\delta \mathcal{S}_{eff} = -\text{Tr}(m \delta \mathcal{P} (\mathcal{D}^\dagger \mathcal{D})^{-1} \mathcal{D}^\dagger) \quad (\text{B4})$$



and then expand  $(\mathcal{D}^\dagger \mathcal{D})^{-1}$  in gradients of  $\mathcal{P}$ .

$$\begin{aligned} (\mathcal{D}^\dagger \mathcal{D})^{-1} &= (-\partial^2 + m^2 - m\gamma_\mu \partial_\mu \mathcal{P})^{-1} \\ &= (-\partial^2 + m^2)^{-1} \\ &\times \left( \sum_{n=0}^{\infty} ((-\partial^2 + m^2)^{-1} m\gamma_\mu \partial_\mu \mathcal{P})^n \right) \end{aligned}$$

Since the coefficient of the WZW term is dimensionless, we will look at the following term in the expansion

$$\begin{aligned} \delta W(\mathcal{P}) &= -\text{Tr}[m^2 \delta \mathcal{P} (-\partial^2 + m^2)^{-1} \\ &\quad ((-\partial^2 + m^2)^{-1} m\gamma_\mu \partial_\mu \mathcal{P})^3 \mathcal{P}] \\ &= -K \int d^3x \text{Tr}[\delta \mathcal{P} (\gamma_\mu \partial_\mu \mathcal{P})^3 \mathcal{P}] \end{aligned}$$

where  $K = \int \frac{d^3p}{(2\pi)^3} \frac{m^5}{(p^2 + m^2)^4} = \frac{1}{64\pi}$  is a dimensionless number, and ‘‘Tr’’ is the trace over the Dirac and flavor indices. After tracing over the Dirac indices,

$$\text{Tr}(\gamma_\mu \gamma_\nu \gamma_\rho) = 2i\epsilon_{\mu\nu\rho} \quad (\text{B5})$$

we obtain the following term for the variation

$$\delta W(\mathcal{P}) = -\frac{2\pi i}{64\pi^2} \epsilon_{\mu\nu\rho} \int d^3x \text{tr}[\delta \mathcal{P} \partial_\mu \mathcal{P} \partial_\nu \mathcal{P} \partial_\rho \mathcal{P}], \quad (\text{B6})$$

where ‘‘tr’’ is the trace for the flavor indices only.

We can restore the topological term of the nonlinear  $\sigma$ -model by the standard method of introducing an auxiliary coordinate  $u$ . The field  $\tilde{\mathcal{P}}(x, u)$  interpolates between  $\tilde{\mathcal{P}}(x, u=0) = \Omega$  and  $\tilde{\mathcal{P}}(x, u=1) = \mathcal{P}(x)$ . The topological term reads

$$W(\tilde{\mathcal{P}}) = -\frac{2\pi i}{256\pi^2} \epsilon_{\mu\nu\rho\delta} \int_0^1 du \int d^3x \text{tr}[\tilde{\mathcal{P}} \partial_\mu \tilde{\mathcal{P}} \partial_\nu \tilde{\mathcal{P}} \partial_\rho \tilde{\mathcal{P}} \partial_\delta \tilde{\mathcal{P}}] \quad (\text{B7})$$

(the extra factor of  $1/4$  comes from the anti-symmetrization of the  $u$  coordinate with other indices).

### Appendix C: Linear Dependence of four-fermion interactions in $N_f = 2N$ QED<sub>3</sub>

In this section, we study all the  $\text{SU}(2) \times \text{SU}(N)$  symmetric four-fermion interactions in the  $N_f = 2N$  QED<sub>3</sub> and their linear dependence up to  $\text{SU}(2N)$ -invariant terms.

First of all, we can write down all the  $\text{SU}(2) \times \text{SU}(N)$  symmetric four-fermion terms:

$$(\bar{\psi}\psi)(\bar{\psi}\psi), \quad (\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma^\mu\psi), \quad (\text{C1})$$

$$(\bar{\psi}\vec{\sigma}\psi)(\bar{\psi}\vec{\sigma}\psi), \quad (\bar{\psi}\gamma^\mu\vec{\sigma}\psi)(\bar{\psi}\gamma^\mu\vec{\sigma}\psi), \quad (\text{C2})$$

$$(\bar{\psi}T^a\psi)(\bar{\psi}T^a\psi), \quad (\bar{\psi}\gamma^\mu T^a\psi)(\bar{\psi}\gamma^\mu T^a\psi), \quad (\text{C3})$$

$$(\bar{\psi}\vec{\sigma}T^a\psi)(\bar{\psi}\vec{\sigma}T^a\psi), \quad (\bar{\psi}\gamma^\mu\vec{\sigma}T^a\psi)(\bar{\psi}\gamma^\mu\vec{\sigma}T^a\psi), \quad (\text{C4})$$

where  $\vec{\sigma}$  is the generator of the  $\text{SU}(2)$  symmetry and  $T^a$  (with  $a = 1, 2, \dots, N^2 - 1$ ) is the generator of the  $\text{SU}(N)$  symmetry. Here, we’ve also implicitly assumed the summation over repeated indices in these expressions. The two terms on the second line are exactly the terms introduced in Eq. 9 of the main text.

Since all the  $\text{SU}(2N)$  invariant four-fermion interaction are irrelevant under RG [44–46], we are only concerned with the linear dependence of all the  $\text{SU}(2) \times \text{SU}(N)$  symmetric four-fermion interactions up to  $\text{SU}(2N)$  invariant ones. First, we notice that the terms in Eq. C1 are  $\text{SU}(2N)$  invariant. Therefore, we can ignore them for this

analysis. Notice that we can rewrite the two terms in Eq. C3 as

$$\begin{aligned}
& (\bar{\psi}T^a\psi)(\bar{\psi}T^a\psi) \\
&= -\frac{N}{4}(\bar{\psi}\gamma^\mu\bar{\sigma}\psi)(\bar{\psi}\gamma^\mu\bar{\sigma}\psi) - \frac{N}{4}(\bar{\psi}\bar{\sigma}\psi)(\bar{\psi}\bar{\sigma}\psi), \\
&\quad -\frac{N}{4}(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma^\mu\psi) - \frac{N+4}{4}(\bar{\psi}\psi)(\bar{\psi}\psi)
\end{aligned} \tag{C5}$$

$$\begin{aligned}
& (\bar{\psi}T^a\gamma^\mu\psi)(\bar{\psi}T^a\gamma^\mu\psi) \\
&= \frac{N}{4}(\bar{\psi}\gamma^\mu\bar{\sigma}\psi)(\bar{\psi}\gamma^\mu\bar{\sigma}\psi) - \frac{3N}{4}(\bar{\psi}\bar{\sigma}\psi)(\bar{\psi}\bar{\sigma}\psi), \\
&\quad +\frac{N-4}{4}(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma^\mu\psi) - \frac{3N}{4}(\bar{\psi}\psi)(\bar{\psi}\psi).
\end{aligned} \tag{C6}$$

Therefore, up to  $SU(2N)$  invariant terms, the two terms in Eq. C3 can be written as linear combination of the two terms in Eq. C2. In the rewriting given above, we've used the Fierz identity  $\sum_a T_{ij}^a T_{kl}^a = N\delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl}$  for the  $SU(N)$  group as well as the Fierz identities  $\bar{\sigma}_{ab}\cdot\bar{\sigma}_{cd} = 2\delta_{ad}\delta_{bc} - \delta_{ab}\delta_{cd}$  for the Pauli matrices  $\bar{\sigma}$  and  $\gamma_{\alpha\beta}^\mu\gamma_{\eta\rho}^\mu = 2\delta_{\alpha\rho}\delta_{\beta\eta} - \delta_{\alpha\beta}\delta_{\eta\rho}$  for the Gamma matrices  $\gamma^\mu$ . Similarly, we can rewrite the two terms in Eq. C4 as

$$\begin{aligned}
& (\bar{\psi}\bar{\sigma}T^a\psi)(\bar{\psi}\bar{\sigma}T^a\psi) \\
&= \frac{N}{4}(\bar{\psi}\gamma^\mu\bar{\sigma}\psi)(\bar{\psi}\gamma^\mu\bar{\sigma}\psi) + \frac{N-4}{4}(\bar{\psi}\bar{\sigma}\psi)(\bar{\psi}\bar{\sigma}\psi), \\
&\quad -\frac{3N}{4}(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma^\mu\psi) - \frac{3N}{4}(\bar{\psi}\psi)(\bar{\psi}\psi)
\end{aligned} \tag{C7}$$

$$\begin{aligned}
& (\bar{\psi}\gamma^\mu\bar{\sigma}T^a\psi)(\bar{\psi}\gamma^\mu\bar{\sigma}T^a\psi) \\
&= -\frac{N+4}{4}(\bar{\psi}\gamma^\mu\bar{\sigma}\psi)(\bar{\psi}\gamma^\mu\bar{\sigma}\psi) + \frac{3N}{4}(\bar{\psi}\bar{\sigma}\psi)(\bar{\psi}\bar{\sigma}\psi), \\
&\quad +\frac{3N}{4}(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma^\mu\psi) - \frac{9N}{4}(\bar{\psi}\psi)(\bar{\psi}\psi)
\end{aligned} \tag{C8}$$

Therefore, all the  $SU(2)\times SU(N)$  symmetric four-fermion interactions can be written as linear combinations of  $(\bar{\psi}\bar{\sigma}\psi)(\bar{\psi}\bar{\sigma}\psi)$  and  $(\bar{\psi}\gamma^\mu\bar{\sigma}\psi)(\bar{\psi}\gamma^\mu\bar{\sigma}\psi)$ , namely the two terms in Eq. C2 (as well as Eq. 9 in the main text).

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- [57] the global symmetry of the  $N_f = 4$  QED is PSU(4) instead of SU(4) because the  $Z_4$  center of the SU(4) flavor symmetry group is also part of the U(1) gauge group.