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From Bosonic Topological Transition to Symmetric Fermion Mass Generation

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A bosonic topological transition (BTT) is a quantum critical point between the bosonic symmetry protected topological phase and the trivial phase. In this work, we investigate such a transition in a (2+1)D lattice model with the maximal microscopic symmetry: an internal SO(4) symmetry. We derive a description for this transition in terms of compact quantum electrodynamics (QED) with four fermion flavors ($N_f = 4$). Within a systematic renormalization group analysis, we identify the critical point with the desired O(4) emergent symmetry and all expected deformations. By lowering the microscopic symmetry we recover the previous $N_f = 2$ non-compact QED description of the BTT. Finally, by merging two BTTs we recover a previously discussed theory of symmetric mass generation, as an SU(2) quantum chromodynamics-Higgs theory with $N_f = 4$ flavors of SU(2) fundamental fermions and one SU(2) fundamental Higgs boson. This provides a consistency check on both theories.

I. INTRODUCTION

Over the past decade, much progress has been made in understanding topological phases,[1] especially the symmetry protected topological (SPT) phases.[2–6] Concepts and methods developed in the study of SPT phases also help to deepen our understanding of some *gapless* states,[7–11] which exist either on the boundary of SPT states or as the quantum critical points between different SPT phases. Within all these novel quantum critical points, the *bosonic topological transition* (BTT)[12–19] between the (2+1)D bosonic symmetry protected topological (BSPT) state and the trivial state has attracted considerable attentions both theoretically and numerically. It is believed that this transition is described by a $N_f = 2$ *non-compact* QED_3, and it can have an as large as O(4) emergent symmetry, due to its self-duality[20–22]. Also, it has been shown recently that this theory is dual to the easy-plane deconfined quantum critical point.[9,11,23]

Another novel transition that has been discussed in recent literatures, in both condensed matter and high energy physics communities, is the *symmetric mass generation* (SMG) transition[14,24–32]. This SMG quantum critical point was observed in various lattice models, using very different numerical techniques. In the condensed matter community, the SMG was found in a fermion model on a double layer honeycomb lattice,[14–16] which we will review later. Despite its different lattice realization, the essence of the SMG is that, eight flavors of (two-component) Dirac fermions in (2+1)D can generate a gap through short range interaction without acquiring a nonzero expectation value of any fermion bilinear operator. This transition is novel and unexpected in the sense that it is clearly beyond the standard Gross-Neveu mechanism of spontaneous generation of Dirac fermion mass, which always corresponds to condensing fermion bilinear operators.

In this paper, we will present a unified framework that captures both novel phase transitions mentioned above. We first show that the BTT between the (2+1)D SO(4) BSPT phases can be understood as a deconfined quantum critical point (DQCP),[33–36] where the low-energy bosonic collective modes (below the physical fermion gap) are fractionalized into $N_f = 4$ flavors of fermionic partons ψ (four Dirac fermions) coupled with a compact U(1) gauge field. The construction is based on the idea of realizing BSPT states from interacting fermionic SPT states,[16,37,38] where the bosonic (spin and charge) freedom $\bar{\psi}\boldsymbol{\Gamma}\psi$ can be treated as fermion bilinear order parameters. The BTT theory can be described by the following Lagrangian,

$$\mathcal{L}_{\mathrm{BTT}} = \bar{\psi}\big(\gamma\cdot(\partial - \mathrm{i}a) + m\sigma^z\big)\psi + \mathcal{L}_{\mathrm{int}}[\psi], \qquad (1)$$

where $\mathcal{L}_{\mathrm{int}}[\psi]$ contains the short-range four-fermion interaction that explicitly breaks the SU(4) fermion flavor symmetry of the $N_f = 4$ QED_3 theory down to its SO(4) subgroup. The BTT is driven by the fermionic parton mass m. At the critical point ($m = 0$), we perform a large-N_f renormalization group (RG) analysis[39–41] to show that the short-range interaction $\mathcal{L}_{\mathrm{int}}[\psi]$ can become relevant at the quantum electrodynamics (QED) fixed point, which drives the theory to a new stable fixed point with the desired O(4) symmetry of the BSPT state, and the only relevant symmetry allowed perturbation is the fermion mass in Eq. (1). Away from the critical point ($m \neq 0$), the fermionic parton opens a gap and becomes a band insulator (coupled to gauge field). The non-trivial (or trivial) BSPT phase corresponds to placing the fermionic parton in the corresponding topological (or trivial) band structure. In this sense, the BTT can be understood as a gauged version of the fermionic parton SPT transition.

To make connection with the BTT, we propose that the SMG is a DQCP as well, where the physical fermion is fractionalized into bosonic ϕ and fermionic ψ partons coupled together by a non-Abelian gauge field $a^a\tau^a$.[66] It can be described by the following Lagrangian,

$$\begin{aligned}\mathcal{L}_{\mathrm{SMG}} = \bar{\psi}\gamma\cdot(\partial - \mathrm{i}a^a\tau^a)\psi + |(\partial - \mathrm{i}a^a\tau^a)\phi|^2 \\ + r|\phi|^2 + u|\phi|^4 + \cdots .\end{aligned} \qquad (2)$$

For the specific bilayer honeycomb model studied extensively, it turns out that the most natural emergent gauge group is SU(2), and the gauge field couples to *four* SU(2) fundamental fermions ψ (totally eight two-component Dirac fermions) and *one* SU(2) fundamental boson ϕ. So this SMG theory is an SU(2) quantum chromodynamics (QCD) with Higgs field ϕ, resembling the Standard Model in some aspects. Driven by the boson mass r, if the bosonic parton ϕ condenses ($r < 0$), the SU(2) gauge field will be gapped out completely through the Higgs mechanism, such that the fermionic parton ψ effectively becomes the physical fermion, which describes the semimetal phase with eight gapless Dirac fermions at low energy. If the bosonic parton ϕ is gapped ($r > 0$), we are left with a QCD theory decoupled from the Higgs field ϕ. Its fate in the IR limit is not fully understood yet. But we assume that it is unstable towards confinement by considering a spontaneous generation of a SU(2) gauge triplet mass. This mass will gap out the fermionic parton and Higgs the gauge group down to U(1). The remaining U(1) gauge field is compact and will then lead to the confined phase. Thus all excitations in the theory are gapped out and the system enters the featureless insulator phase. The similar mechanism was also sketched in Ref. 42. If the same mass were introduced to the physical fermion, it must break some symmetry. However, for the fermionic parton, this triplet mass is not gauge invariant, so the symmetry can be repaired by gauge transformation in the form of the projective symmetry group (PSG).[43] In this way, the featureless Mott insulator can preserve the full symmetry of the Dirac semimetal, therefore the transition between them is indeed a *symmetric* generation of the fermion mass (or more precisely, the fermion gap). The similar Higgs-confine dichotomy in the SU(2) QCD-Higgs theory is also discussed in Ref. 44 recently in the context of cuprates high-T_c superconductor.

It turns out that both BTT and SMG transitions can be realized in the phase diagram of a single lattice model (see Sec. II E for more detailed discusssion). The two transitions are actually closely related. We show that the BTT theory can be obtained from the SMG theory by gapping out the bosonic parton and half of the fermionic partons (four out of the eight Dirac fermions). The remaining half of the fermionic partons in the SMG theory actually become the fermionic partons in the BTT theory. This consistency check lends more confidence to the proposed SMG mechanism, as the assumed SU(2) gauge triplet mass generation also plays a crucial role in the connecting BTT to SMG.

II. BOSONIC TOPOLOGICAL TRANSITION

A. Field Theory of BTT

The (2+1)D SO(4) BSPT can be described as the disordered phase of the O(4) non-linear sigma model (NLSM) with a Θ-term at $\Theta = 2\pi$. The BTT between the BSPT and the trivial state should have a field theory description with an explicit O(4) symmetry. One possible field theory preserving the $\mathrm{SO}(4) = (\mathrm{SU}(2)_\uparrow \times \mathrm{SU}(2)_\downarrow)/\mathbb{Z}_2$ microscopic symmetry is the compact quantum electrodynamics (QED) with fermion flavor number $N_f = 4$. In the bilayer honeycomb lattice model (to be introduced in Eq. (33) later) where this BTT is realized, the four fermion flavors can be arranged into two spin sectors

$$\psi = (\psi_\uparrow, \psi_\downarrow)^\intercal = (\psi_{\uparrow 1}, \psi_{\uparrow 2}, \psi_{\downarrow 1}, \psi_{\downarrow 2})^\intercal. \tag{3}$$

Within each spin sector (labelled by $\sigma = \uparrow, \downarrow$), the fermion field ψ_σ transforms as the fundamental representation of $\mathrm{SU}(2)_\sigma$,

$$U_\sigma \in \mathrm{SU}(2)_\sigma : \psi_\sigma \to U_\sigma \psi_\sigma, \tag{4}$$

so altogether the field ψ contains four Dirac fermions transforming as the SO(4) spinor.[67] The theory can be considered as a parton construction for the prototype BSPT state with SO(4) symmetry, where the bosonic degree of freedom $\boldsymbol{N}$ (the O(4) vector in NLSM) is fractionalized into the fermionic parton ψ_σ as

$$\boldsymbol{N} = \bar{\psi}_\uparrow (\mu^0, \mathrm{i}\mu^1, \mathrm{i}\mu^2, \mathrm{i}\mu^3) \psi_\downarrow + \text{h.c.}, \tag{5}$$

with an emergent U(1) gauge field a. The BTT can be described by the following compact $N_f = 4$ QED theory

$$\begin{aligned} \mathcal{L}_{\mathrm{BTT}} = &\sum_\sigma \bar{\psi}_\sigma \big(\gamma \cdot (\partial - \mathrm{i}a - \mathrm{i}A^a_\sigma \mu^a) + m(-)^\sigma\big)\psi_\sigma \\ &+ \frac{\mathrm{i}}{8\pi}(\mathsf{CS}[A_\uparrow] - \mathsf{CS}[A_\downarrow]) + \mathcal{L}_{\mathrm{int}}[\psi], \end{aligned} \tag{6}$$

with short-range interactions $\mathcal{L}_{\mathrm{int}}[\psi]$ that explicitly break the SU(4) flavor symmetry[68] down to SO(4). The shorthand notation $\gamma \cdot D$ denotes $\gamma^\mu D_\mu$, where the gamma matrices are chosen as $(\gamma^0, \gamma^1, \gamma^2) = (\sigma^2, \sigma^1, \sigma^3)$ and $\bar{\psi}_\sigma = \psi^\dagger_\sigma \gamma^0$. The fermionic partons couple to the *compact* U(1) gauge field a. The probe fields $A_\sigma = A^a_\sigma \mu^a$ ($\sigma = \uparrow, \downarrow$) are introduced to keep track of the $\mathrm{SU}(2)_\sigma$ symmetries, with μ^a ($a = 1, 2, 3$) being SU(2) generators (i.e. Pauli matrices). The background SU(2) Chern-Simons term $\mathsf{CS}[A_\sigma]$ of the probe field A_σ originates from the UV regularization of the Dirac fermion.

The BTT is driven by the fermionic parton mass m. For $m \neq 0$, integrating out the fermion field ψ in Eq. (6) generates the following response theory

$$\begin{aligned} \mathcal{L}_A &= \frac{\mathrm{i}\nu}{4\pi}(\mathsf{CS}[A_\uparrow] - \mathsf{CS}[A_\downarrow]), \\ \mathsf{CS}[A_\sigma] &= \mathrm{Tr}\left(A_\sigma \wedge \mathrm{d}A_\sigma - \tfrac{2\mathrm{i}}{3} A_\sigma \wedge A_\sigma \wedge A_\sigma\right), \end{aligned} \tag{7}$$

where $\mathsf{CS}[A_\sigma]$ is SU(2) Chern-Simons term for the symmetry probe field $A_\sigma = A^a_\sigma \mu^a$ (in terms of Hermitian gauge connections). The topological index ν is given by $\nu = \frac{1}{2}(1 + \mathrm{sgn}\, m)$. So $m < 0$ corresponds to the featureless Mott phase ($\nu = 0$) and $m > 0$ corresponds to the BSPT phase ($\nu = 1$). Therefore the BTT transition should happen at $m = 0$.

On general grounds, any interaction that is gauge invariant and respects all physical symmetries could appear in $\mathcal{L}_{\rm int}[\psi]$. Because the fermionic parton only respects the physical symmetry SO(4), the interactions will explicitly break the SU(4) flavor symmetry down to its SO(4) subgroup (at least at UV). We need to show that such interactions are relevant so that the SU(4) symmetry will not be restored as an emergent symmetry at IR (this is because the other theory describing this transition, the $N_f = 2$ non-compact QED$_3$, can at most host O(4) symmetry). Following the approach in Ref. 39, we perform a large-N_f controlled one-loop RG analysis and find an IR fixed point at finite coupling strength with an emergent $\mathrm{O}(4) = \mathrm{SO}(4) \rtimes \mathbb{Z}_2$ symmetry (where the $\mathbb{Z}_2 : \psi_\uparrow \leftrightarrow \psi_\downarrow$ transformation exchanges spin sectors and corresponds to the improper O(4) rotation). We will postpone the details of the RG analysis to Sec. II B and briefly summarize our main findings in the following. We found that the most relevant interaction takes the form of

$$\mathcal{L}_{\rm int}[\psi] = \frac{g_2}{4\Lambda} \sum_\sigma \epsilon_{ac}\epsilon_{bd} (\bar\psi_{\sigma a}\psi_{\sigma b})(\bar\psi_{\sigma c}\psi_{\sigma d}), \tag{8}$$

which corresponds to the pair-pair interaction of $\mathrm{SU}(2)_\sigma$ singlets. Λ denotes the UV cut-off. The RG equation for g_2 reads (see Sec. II B Eq. (14))

$$\frac{\mathrm{d}g_2}{\mathrm{d}\ell} = -\Big(1 - \frac{16}{\pi^2}\Big) g_2 - \frac{1}{6\pi^2} g_2^2, \tag{9}$$

which has an stable fixed point at $g_2^* = 6\pi^2(16/\pi^2 - 1)$. It will be verified in Sec. II B that the $\mathbb{Z}_2$ symmetry breaking four-fermion interactions are irrelevant around the fixed point (though the fermion mass m in Eq. (6) that drives the BTT is still a $\mathbb{Z}_2$ breaking relevant perturbation). So the fixed point has the desired emergent O(4) symmetry.

B. Renormalization Group Analysis

In this section, we present the renormalization group (RG) analysis of the SO(4) invariant four-fermion interactions in the (2+1)D QED theory following Ref. 39. To control the RG calculation, we consider the $1/N_f$ expansion, where N_f is the fermion flavor number of the QED theory. In the end, we want to apply the result to the physically relevant $N_f = 4$ case.

First, we need to introduce a systematic large-N_f generalization of the QED theory. One option is to start from the $\mathrm{SO}(4) = (\mathrm{Sp}(1)_\uparrow \times \mathrm{Sp}(1)_\downarrow)/\mathbb{Z}_2$ symmetry (at $N_f = 4$) and promote the symmetry to $(\mathrm{Sp}(N)_\uparrow \times \mathrm{Sp}(N)_\downarrow)/\mathbb{Z}_2$ with $N_f = 4N$. Then the QED theory in Eq. (6) can be generalized to

$$\mathcal{L}_{\rm QED} = \sum_\sigma \bar\psi_\sigma \gamma\cdot(\partial - \mathrm{i}a - \mathrm{i}A_\sigma^a\mu^a)\psi_\sigma + \mathcal{L}_{\rm int}[\psi], \tag{10}$$

where $\psi_\sigma = (\psi_{\sigma 1}, \psi_{\sigma 2}, \cdots, \psi_{\sigma 2N})^\intercal$ is now a $2N$ component complex fermion field in each spin $\sigma = \uparrow, \downarrow$ sector. A_σ^a are the source fields to keep track of the $\mathrm{Sp}(N)_\uparrow \times \mathrm{Sp}(N)_\downarrow$ symmetry and μ^a denote the $\mathrm{Sp}(N)$ generators. The background response of A_σ is omitted.

Without the interaction $\mathcal{L}_{\rm int}[\psi]$, the $N_f = 4N$ QED theory in Eq. (10) has the SU($4N$) flavor symmetry. The interaction will break the symmetry down to $\mathrm{Sp}(N)_\uparrow \times \mathrm{Sp}(N)_\downarrow$. We will first impose an additional $\mathbb{Z}_2$ symmetry that exchanges the two spin sectors,

$$\mathbb{Z}_2 : \psi_\uparrow \leftrightarrow \psi_\downarrow, \tag{11}$$

which reduces, in the $N = 1$ case, to the improper $\mathbb{Z}_2$ transformation of the O(4) vector. The effect of breaking the $\mathbb{Z}_2$ symmetry will be analyzed later. There are altogether four independent types of the $\mathrm{Sp}(N)_\uparrow \times \mathrm{Sp}(N)_\downarrow \rtimes \mathbb{Z}_2$ symmetric interactions

$$\mathcal{L}_{\rm int} = \frac{1}{N_f\Lambda}(g_1 V_1 + g_1' V_1' + g_2 V_2 + g_3 V_3), \tag{12}$$

where g_1, g_1', g_2, g_3 are dimensionless coupling constants and

$$\begin{aligned} V_1 &= (\textstyle\sum_\sigma \bar\psi_\sigma\psi_\sigma)^2, \\ V_1' &= (\textstyle\sum_\sigma \bar\psi_\sigma\gamma^\mu\psi_\sigma)^2, \\ V_2 &= \textstyle\sum_\sigma J_{ac}J_{bd}(\bar\psi_{\sigma a}\psi_{\sigma b})(\bar\psi_{\sigma c}\psi_{\sigma d}), \\ V_3 &= \textstyle\sum_\sigma J_{ac}J_{bd}(\bar\psi_{\sigma a}\psi_{\bar\sigma b})(\bar\psi_{\sigma c}\psi_{\bar\sigma d}), \end{aligned} \tag{13}$$

where $\sigma = \uparrow, \downarrow$ labels the spin and $\bar\sigma$ denotes the opposite spin of σ. J is the metric of the Sp(N) symplectic structure, such that $J^\intercal = -J$, $J^2 = -1$ and $\forall a : \mu^{a\intercal} J + J\mu^a = 0$. g_1 and g_1' are actually SU($4N$) symmetric and irrelevant even for small N, see for example Ref. 40. We will focus on the SU($4N$) breaking interactions g_2 and g_3, which are all in the form of the Sp(N) pair-pair coupling.

Following the derivation in Ref. 39–41, we obtain the RG equation for g_2 and g_3 as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\ell} g_2 &= -\Big(1 - \frac{64}{\pi^2 N_f}\Big) g_2 - \frac{1}{6\pi^2}(g_2^2 + g_3^2), \\ \frac{\mathrm{d}}{\mathrm{d}\ell} g_3 &= -\Big(1 - \frac{64}{\pi^2 N_f}\Big) g_3 - \frac{1}{3\pi^2} g_2 g_3. \end{aligned} \tag{14}$$

At the QED ($g_i = 0$) fixed point, their scaling dimensions are degenerated

$$\Delta = -1 + \frac{64}{\pi^2 N_f} + \mathcal{O}(1/N_f^2). \tag{15}$$

Pushing this result to $N_f = 4$, we obtain $\Delta \approx 0.6 > 0$, meaning that the g_2 and g_3 interactions are relevant, which will drive the theory alway from the QED fixed point and break the SU(4) flavor symmetry down to $\mathrm{O}(4) = \mathrm{SO}(4) \rtimes \mathbb{Z}_2$. The g_2 interaction reduces to the interaction term in Eq. (8) in the $N_f = 4$ case.

We can track the RG flow away from the QED fixed point. Fig. 1 shows the RG flow diagram for the $N_f = 4$

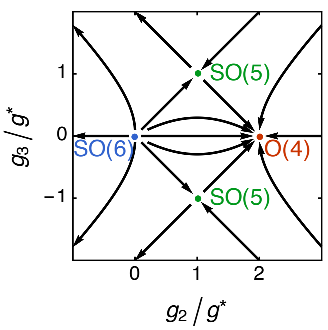

FIG. 1: RG flow diagram for $N_f = 4$ ($N = 1$) in the $\mathbb{Z}_2$ symmetric plane.

($N = 1$) case. Several fixed points are found at finite couplings of the order

$$g^* = 3\pi^2\Big(\frac{64}{\pi^2 N_f} - 1\Big). \tag{16}$$

First, there are two $\mathrm{SO}(5) = \mathrm{Sp}(2)/\mathbb{Z}_2$ invariant fixed points at $(g_2^*, g_3^*) = (g^*, \pm g^*)$. They are related by an SO(6) rotation. They both correspond to the SO(5) fixed point discussed in Ref. 40, with different SO(5) subgroups of the SO(6). Second, there is one stable O(4) symmetric fixed point at $(g_2^*, g_3^*) = (2g^*, 0)$.

Let us consider perturbing this fixed point by $\mathbb{Z}_2$ breaking interactions $(g_4V_4 + g_5V_5)/(N_f\Lambda)$ with

$$\begin{aligned} V_4 &= \textstyle\sum_\sigma (-)^\sigma J_{ac}J_{bd}(\bar\psi_{\sigma a}\psi_{\sigma b})(\bar\psi_{\sigma c}\psi_{\sigma d}),\\ V_5 &= \textstyle\sum_\sigma \mathrm{i}(-)^\sigma J_{ac}J_{bd}(\bar\psi_{\sigma a}\psi_{\bar\sigma b})(\bar\psi_{\sigma c}\psi_{\bar\sigma d}), \end{aligned} \tag{17}$$

where $(-)^\sigma = \pm 1$ is the spin-dependent sign that discriminates between $\sigma = \uparrow$ and $\downarrow$. It is found that both g_4 and g_5 are irrelevant at the O(4) fixed point. Their RG equations are the same as g_3,

$$\frac{\mathrm{d}}{\mathrm{d}\ell}g_i = -\Big(1 - \frac{64}{\pi^2 N_f}\Big)g_i - \frac{1}{3\pi^2}g_2 g_i \quad (i = 4, 5). \tag{18}$$

The scaling dimensions of g_4 and g_5 at the O(4) fixed point $g_2^* = 2g^*$ can be read off from the RG equation, which are given by $1 - 64/\pi^2 N_f \approx -0.6 < 0$ and are irrelevant. So even if we start with the interaction that has only SO(4) symmetry (with $\mathbb{Z}_2$ broken terms g_4 and g_5), the QED theory will flow to a stable fixed point with emergent $\mathrm{O}(4) = \mathrm{SO}(4) \rtimes \mathbb{Z}_2$ symmetry. There is only one relevant $\mathbb{Z}_2$ breaking perturbation at this fixed point, i.e. the mass term $m(-)^\sigma\bar\psi_\sigma\psi_\sigma$ in Eq. (6), which drives the topological transition between the BSPT phase and the featureless Mott phase.

One potential concern is that the fixed point may even have an additional $\mathrm{SO}(2) : \psi_\sigma \to e^{(-)^\sigma \mathrm{i}\theta}\psi_\sigma$ symmetry. However, this SO(2) symmetry is broken by the monopole effect as pointed out in Ref. 11. There it was argued that the monopole operator $\mathcal{M}_a$ (which annihilates the 2π flux of the a gauge field) transforms as an $\mathrm{SO}(6) = \mathrm{SU}(4)/\mathbb{Z}_2$ vector. Required by the microscopic SO(4) symmetry, four out of the six components of the SO(6) vector that transform under SO(4) are not allowed in the path integral. So the SO(6) vector can only be aligned in the remaining two component subspace, which rotates under SO(2). This assigns the SO(2) symmetry charge to the monopole operator $\mathcal{M}_a$. Due to the compactness of the U(1) gauge field, the monopole term $\mathcal{M}_a + \text{h.c.}$ is allowed in the Lagrangian. If we assume that the strength of the monopole term remains finite at the O(4) fixed point, it will break the above mentioned SO(2) symmetry completely. Therefore the O(4) RG fixed point will not have the additional SO(2) symmetry as suspected. The scaling dimension of the monopole operator at this O(4) fixed point requires further analysis. An alternative route to approach this O(4) fixed point is to start with the compact $N_f = 4$ QED without the short-range parton interaction, the monopole argument in Ref. 11 suggests that the theory describes an SO(5) DQCP fixed point. Upon the SO(5) to O(4) symmetry breaking anisotropy (corresponding to adding the parton interaction to the QED theory), the SO(5) fixed point becomes unstable and flows to the O(4) fixed point. Both understandings are consistent with our proposal that the O(4) fixed point can be described by the compact $N_f = 4$ QED theory with parton interaction.

We would also like to mention that the SO(4) symmetry breaking interactions $(g_6V_6 + g_6'V_6')/(N_f\Lambda)$, in the $(1,1)$ representation of $\mathrm{SU}(2)_\uparrow \times \mathrm{SU}(2)_\downarrow$, is given by

$$\begin{aligned} V_6 &= (\bar\psi_\uparrow\mu^3\psi_\uparrow)(\bar\psi_\downarrow\mu^3\psi_\downarrow),\\ V_6' &= (\bar\psi_\uparrow\gamma^\mu\mu^3\psi_\uparrow)(\bar\psi_\downarrow\gamma^\mu\mu^3\psi_\downarrow), \end{aligned} \tag{19}$$

where μ^3 is one of the $\mathrm{SU}(2) = \mathrm{Sp}(1)$ generators (so that $J\mu^{3\intercal}J = \mu^3$). The RG flow equation of g_6 and g_6' around the O(4) fixed point is given by

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\ell}g_6 &= -\Big(1 - \frac{128}{3\pi^2 N_f} - \frac{6g_2^*}{\pi^2 N_f}\Big)g_6 + \frac{64}{\pi^2 N_f}g_6',\\ \frac{\mathrm{d}}{\mathrm{d}\ell}g_6' &= -\Big(1 + \frac{2g_2^*}{3\pi^2 N_f}\Big)g_6' + \frac{64}{3\pi^2 N_f}g_6, \end{aligned} \tag{20}$$

where $g_2^* = 2g^*$. One can show there is a relevant channel along $(g_6, g_6') \propto (1, 0.07)$, dominated by the g_6V_6 interaction. This interaction will drive the spontaneous generation of the bilinear masses $m_\sigma\bar\psi_\sigma\mu^3\psi_\sigma$. Depending on the sign of g_6, either one of the $m_\uparrow = \pm m_\downarrow$ choices will be favored, which leads to different spontaneous symmetry breaking (SSB) phases that will be elaborated in Sec. II C.

In conclusion, the RG analysis indicates that the interacting compact $N_f = 4$ QED theory in Eq. (6) has a stable O(4) fixed point, which has the emergent $\mathrm{O}(4) = \mathrm{SO}(4) \rtimes \mathbb{Z}_2$ symmetry with only one $\mathbb{Z}_2$ breaking relevant perturbation and one SO(4) breaking relevant perturbation in the $(1,1)$ representation. These properties are all consistent with the known properties of BTT in other versions of field theories.[11,20] So we propose that the SO(4) symmetric BTT can be equally described by the interacting compact $N_f = 4$ QED theory. In the following, we will make connections to another field theory description of BTT with a lower microscopic symmetry, which allows us to access the adjacent SSB phases.

C. Instability of BTT to SSB Phases

In a series of recent theoretical[11,45] and numerical[46,47] works, it was pointed out that the O(4) fixed point describes both the critical point of SO(4) BTT and the deconfined quantum critical point (DQCP)[33–35] of the easy-plane Néel to valence bond solid (VBS) transition. The Néel-VBS transition is driven by a relevant perturbation which is in the $(1,1)$ representation of the SO(4) symmetry. However, in the context of our lattice model (to be reviewed later in Sec. II D), the physical meaning of the O(4) vector $\boldsymbol{N} = (N_0, N_1, N_2, N_3)$ is interpreted differently. The Néel and the VBS order are interpreted as the *in-plane* spin density wave (SDW) order $S^+ \sim N_0 + \mathrm{i}N_3$ and the superconducting (SC) order $\Delta^\dagger \sim N_2 + \mathrm{i}N_1$ respectively. As analyzed in Sec. II B previously, the symmetry breaking perturbation in the $(1,1)$ representation corresponds to the following fermionic parton interaction in the $N_f = 4$ QED theory,

$$\mathcal{L}_{\text{int}} = \frac{g_6}{4\Lambda}(\bar{\psi}_\uparrow \mu^3 \psi_\uparrow)(\bar{\psi}_\downarrow \mu^3 \psi_\downarrow). \tag{21}$$

In terms of the O(4) order parameters, it can be also written as $g_6(N_1^2 + N_2^2 - N_0^2 - N_3^2)$. This interaction discriminates between the SDW and the SC order. On the mean-field level, one could already see that $g_6 > 0$ (or $g_6 < 0$) favors the SDW (or SC) order.

With this interaction, the SO(4) symmetry is explicitly broken down to its $(\mathrm{U}(1)_\uparrow \times \mathrm{U}(1)_\downarrow) \rtimes \mathbb{Z}_2^\updownarrow$ subgroup,

$$\begin{aligned}
\mathrm{U}(1)_\uparrow \times \mathrm{U}(1)_\downarrow : S^+ &\to e^{\mathrm{i}(\theta_\uparrow - \theta_\downarrow)} S^+,\\
\Delta^\dagger &\to e^{-\mathrm{i}(\theta_\uparrow + \theta_\downarrow)} \Delta^\dagger,\\
\mathbb{Z}_2^\updownarrow : N_1 &\to -N_1, N_3 \to -N_3,
\end{aligned} \tag{22}$$

where $\mathrm{U}(1)_\uparrow \times \mathrm{U}(1)_\downarrow$ is a combination of the spin and the charge U(1) symmetries and the $\mathbb{Z}_2^\updownarrow$ conjugates both $\mathrm{U}(1)_\uparrow$ and $\mathrm{U}(1)_\downarrow$ charges. According to the fractionalization scheme in Eq. (5), the symmetry action on the fermionic partons are given by

$$\begin{aligned}
\mathrm{U}(1)_\uparrow &: \psi_\uparrow \to e^{\frac{\mathrm{i}}{2}\theta_\uparrow \mu^3} \psi_\uparrow,\\
\mathrm{U}(1)_\downarrow &: \psi_\downarrow \to e^{\frac{\mathrm{i}}{2}\theta_\downarrow \mu^3} \psi_\downarrow,\\
\mathbb{Z}_2^\updownarrow &: \psi_\sigma \to \mathrm{i}\mu^2 \psi_\sigma.
\end{aligned} \tag{23}$$

The RG analysis in Eq. (20) shows that the g_6 interaction is relevant at the O(4) fixed point, meaning that under the explicit symmetry breaking to $(\mathrm{U}(1)_\uparrow \times \mathrm{U}(1)_\downarrow) \rtimes \mathbb{Z}_2^\updownarrow$, the SO(4) BTT critical point is unstable towards the SDW or the SC phase that further breaks the $\mathrm{U}(1)_\uparrow \times \mathrm{U}(1)_\downarrow$ symmetry spontaneously. These *spontaneous symmetry breaking* (SSB) phases will set in between the BSPT phase and the featureless Mott phase, splitting the BTT into two XY transitions. In the field theory, as g_6 flows to the strong coupling limit, the interaction will drive the spontaneous generation of the mass terms $m_\sigma \bar{\psi}_\sigma \mu^3 \psi_\sigma$ (for both $\sigma = \uparrow, \downarrow$). Depending on the sign of g_6, the interaction will favor one of the $m_\uparrow \sim \pm m_\downarrow$ choices, which corresponds to one of the U(1) SSB phases.

To analyze the effect of m_σ mass terms in more details, let us included them in the $N_f = 4$ QED theory given in Eq. (6)

$$\begin{aligned}
\mathcal{L}_{N_f=4} = \sum_\sigma \bar{\psi}_\sigma \big(&\gamma \cdot (\partial - \mathrm{i}a - \mathrm{i}A_\sigma^3 \mu^3) + m(-)^\sigma\\
&+ m_\sigma \mu^3\big)\psi_\sigma + \mathcal{L}_{\text{bg}}[A] + \mathcal{L}_{\text{int}}[\psi].
\end{aligned} \tag{24}$$

The m_σ masses explicitly lowers the microscopic symmetry from SO(4) to $\mathrm{U}(1)_\uparrow \times \mathrm{U}(1)_\downarrow$. Correspondingly, the symmetry probe fields are reduced from the non-Abelian field $A_\sigma^a \mu^a$ to the Abelian field $A_\sigma^3 \mu^3$, compared to Eq. (6). The background response is also reduced from the SU(2) Chern-Simons term in Eq. (6) to its U(1) version

$$\mathcal{L}_{\text{bg}}[A] = \sum_\sigma \frac{\mathrm{i}}{4\pi}(-)^\sigma A_\sigma^3 \wedge \mathrm{d}A_\sigma^3. \tag{25}$$

Let us first investigate the possible phases that can be accessed by tuning the mass terms m and m_σ. Suppose the fermionic parton is fully gapped by these mass terms, the resulting Chern-Simons theory should take the following form

$$\mathcal{L}_{\text{CS}} = \frac{\mathrm{i}}{4\pi} K_{IJ} \mathcal{A}^I \wedge \mathrm{d}\mathcal{A}^J, \tag{26}$$

where $\mathcal{A} = (a, A_\uparrow^3, A_\downarrow^3)$ is a collection of the gauge field a and the symmetry probe fields A_σ^3. The K matrix in this basis is given by

$$\begin{aligned}
K =& \frac{1}{2}\sum_{\sigma,\mu} \operatorname{sgn} m_{\sigma\mu} \begin{bmatrix} 1 & (-)^\mu \delta_{\sigma\uparrow} & (-)^\mu \delta_{\sigma\downarrow} \\ (-)^\mu \delta_{\sigma\uparrow} & \delta_{\sigma\uparrow} & 0 \\ (-)^\mu \delta_{\sigma\downarrow} & 0 & \delta_{\sigma\downarrow} \end{bmatrix}\\
&+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},
\end{aligned} \tag{27}$$

where $m_{\sigma\mu} = m(-)^\sigma + m_\sigma(-)^\mu$ and $\operatorname{sgn} m_{\sigma\mu}$ denotes the sign of $m_{\sigma\mu}$. The delta symbol $\delta_{\sigma\sigma'} = 1$ if $\sigma = \sigma'$ and $\delta_{\sigma\sigma'} = 0$ otherwise. The first term Eq. (27) is obtained by integrating out the fermionic parton, and the second term is from the background $\mathcal{L}_{\text{bg}}$.

There is a rich variety of K matrices in the parameter space as shown in Fig. 2. But in the end, there are only four phases, since different K matrices could describe the same phase. For example, the following two K matrices both correspond to the BSPT phase

$$K_{\text{BSPT}}^{\text{CS}} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \sim K_{\text{BSPT}}^{\text{conf.}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \tag{28}$$

In both cases, the gauge field a is fully gapped, either due to the Chern-Simons effect in $K^{\text{CS}}_{\text{BSPT}}$ or due to the confinement in $K^{\text{conf.}}_{\text{BSPT}}$, leaving no excitations at low energy. Their resulting response theories are also identical. So there should be no phase transition between them. Across this "fake transition" only one flavor of the fermionic parton becomes gapless,

$$\bar{\psi}\gamma\cdot(\partial-\mathrm{i}(a-A^3_\uparrow))\psi-\frac{\mathrm{i}}{8\pi}a\wedge\mathrm{d}a+\frac{\mathrm{i}}{4\pi}a\wedge\mathrm{d}A^3_\uparrow+\cdots. \quad (29)$$

By redefining $a\to a+A^3_\uparrow$, the Chern-Simons term $a\wedge\mathrm{d}A^3_\uparrow$ can be canceled exactly, leading to a *compact* $N_f=1$ QED theory with a level-1/2 $a\wedge\mathrm{d}a$ term, which is completely decoupled from all the symmetry probe fields A^3_σ. Tuning the mass $\bar{\psi}\psi$ in this theory appears to drive a "transition" between a confined phase of a and another phase with massive fermion coupled with a level-1 $a\wedge\mathrm{d}a$ term. Both sides correspond to trivial gapped phases of gauge invariant degrees of freedom, thus this "transition" should not exist. Similar argument applies to other "fake transitions" (dashed lines) in the phase diagram in Fig. 2. On the other hand, the physical transitions (solid lines) in Fig. 2 are all described by *non-compact* $N_f=1$ QED theories with level-1/2 $a\wedge\mathrm{d}a$ terms, which are dual to 3D XY (Wilson-Fisher) transitions according to the fermion-boson duality[10,23] as expected. From the duality point of view, switching from the non-compact to the compact QED theory corresponds to explicitly breaking the U(1) symmetry that defines the XY transition in the dual theory, such that the transition should be lifted along the dashed lines in Fig. 2. The similar four-quadrant phase diagram among SPT and SSB phases was also discussed in other (1+1)D[19] and (2+1)D[46,48] systems.

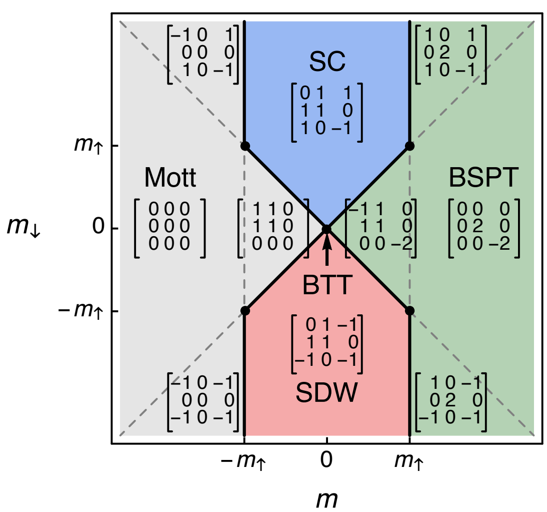

FIG. 2: Phase diagram of the model Eq. (24) and K matrices in different parameter regimes. The K matrices are given in the basis of $\mathcal{A}=(a,A^3_\uparrow,A^3_\downarrow)$. We assume $m_\uparrow>0$, and choose it as the mass scale. The solid lines are physical phase transitions, while the dash lines are not.

Starting from the O(4) fixed point at the center of the phase diagram, the mass m drives the BTT along the horizontal direction. In the BSPT (or featureless Mott) phase, the response theory can be obtained by integrating out the gauge field a, which is found to be

$$\mathcal{L}_A=\sum_\sigma\frac{\mathrm{i}\nu}{2\pi}(-)^\sigma A^3_\sigma\wedge\mathrm{d}A^3_\sigma=\frac{\mathrm{i}\nu}{2\pi}A_{\text{c}}\wedge\mathrm{d}A_{\text{s}}, \quad (30)$$

where $\nu=\frac{1}{2}(1+\operatorname{sgn}m)$. It is consistent with the SO(4) version of the response theory in Eq. (7).

On the other hand, g_6 interaction in Eq. (21) drives the O(4) fixed point into the SSB phases. As a relevant interaction, a strong g_6 leads to the spontaneous generation of the masses m_σ, which puts the system into the SSB phase. From Eq. (23), one can see that the masses m_σ are odd under $\mathbb{Z}_2^\updownarrow: m_\sigma\to -m_\sigma$, so the $\mathbb{Z}_2^\updownarrow$ symmetry is broken. Moreover, in the SSB phase, the Chern-Simons theory in Eq. (26) is reduced to

$$\mathcal{L}_{\text{CS}}=\frac{\mathrm{i}}{2\pi}\sum_\sigma(\operatorname{sgn}m_\sigma)A^3_\sigma\wedge\mathrm{d}a+\mathcal{L}_{\text{bg}}[A]. \quad (31)$$

For $m_\uparrow\sim\pm m_\downarrow$, the gauge field a is coupled to the probe field $A^3_\uparrow\pm A^3_\downarrow$ by the Chern-Simons term, which renders the gauge field a non-compact. The gapless photon mode of the a field will be dual to the Goldstone mode in the SSB phase. If $m_\uparrow$ and $m_\downarrow$ are of the opposite (or same) sign, the gauge flux $\mathrm{d}a$ will carry the spin (or charge) quantum number and the gauge theory will describe the SDW (or SC) phase.[45] If we fix $m=0$ and tune g_6 interaction across zero, the $N_f=4$ QED theory will go through the DQCP of the SDW-SC transition, which has the SO(4) microscopic symmetry and the O(4) emergent symmetry.

Finally, we would like to mention that there is a related but different theory of BTT with a lower microscopic symmetry $(\mathrm{U}(1)\rtimes\mathbb{Z}_2)\times\mathrm{SU}(2)$, where one of the $\mathrm{SU}(2)_\sigma$ ($\sigma=\uparrow,\downarrow$) symmetry is broken explicitly to its $\mathrm{U}(1)_\sigma\rtimes\mathbb{Z}_2^\sigma$ subgroup. Without lost of generality, let us choose the microscopic symmetry to be $(\mathrm{U}(1)_\uparrow\rtimes\mathbb{Z}_2^\uparrow)\times\mathrm{SU}(2)_\downarrow$, then the $m_\uparrow\bar{\psi}_\uparrow\mu^3\psi_\uparrow$ mass is allowed. Naively, a finite $m_\uparrow$ seems to break the $\mathbb{Z}_2^\uparrow$ symmetry by picking one direction along the μ^3-axis, but we will see that the $\mathbb{Z}_2^\uparrow$ symmetry persists in the low-energy effective theory as a particle-hole symmetry. Fixing a finite mass $m_\uparrow>0$, two Dirac fermions $\psi_\uparrow$ in the $N_f=4$ QED theory will be gapped, leaving an $N_f=2$ QED thoery for the fermion $\psi_\downarrow$ at the BTT.[13] The effective theory in Eq. (24) is thus reduced to

$$\begin{aligned}\mathcal{L}_{N_f=2}=&\ \bar{\psi}_\downarrow\big(\gamma\cdot(\partial-\mathrm{i}a-\mathrm{i}A^a_\downarrow\mu^a)-m+m_\downarrow\mu^3\big)\psi_\downarrow\\&+\frac{\mathrm{i}}{2\pi}A^3_\uparrow\wedge\mathrm{d}a+\mathcal{L}_{\text{bg}}[A]+\mathcal{L}_{\text{int}}[\psi].\end{aligned} \quad (32)$$

The gauge field a is non-compact in this theory, and the conserved gauge flux $\mathrm{d}a$ corresponds to the $\mathrm{U}(1)_\uparrow$ symmetry charge. The $\mathbb{Z}_2^\uparrow:\psi_\downarrow\to\psi_\downarrow^\dagger, a\to -a$ symmetry is realized as the particle-hole symmetry.

Driven by m and $m_\downarrow$, all four phases in the phase diagram Fig. 2 can be realized within the framework of the $N_f = 2$ QED theory as well.[20,45,46] They are separated by quantum phase transitions. These phase transitions are of 3D XY universality class, described by the non-compact $N_f = 1$ QED theory coupled to the "level-1/2 Chern-Simons term".[10,23] The four XY transition lines join at the BTT multi-critical point, described by the non-compact $N_f = 2$ QED theory in Eq. (32), which also has the emergent $\mathrm{O}(4) = \mathrm{SO}(4) \rtimes \mathbb{Z}_2$ symmetry at low energy,[36] where the improper $\mathbb{Z}_2 : \psi_\uparrow \leftrightarrow \psi_\downarrow$ symmetry is realized as the self-duality.[9,20,21] The self-dual $N_f = 2$ QED is also dual to the non-compact CP^1 theory via the fermion-boson duality.[10,11,23] These dual theories all describe the BTT multi-critical point, which has two relevant perturbations: one leads to the transition between the featureless Mott and the BSPT phases, and the other leads to the transition between two SSB phases. The scenarios of the BTT under different microscopic symmetries is concluded in Tab. I.

TABLE I: Effective descriptions of the BTT with different microscopic symmetries.

microscopic symmetry	effective theory
SO(4)	critical, compact $N_f = 4$ QED
$\mathrm{SU}(2)_\uparrow \times (\mathrm{U}(1)_\downarrow \rtimes \mathbb{Z}_2^\downarrow)$ $(\mathrm{U}(1)_\uparrow \rtimes \mathbb{Z}_2^\uparrow) \times \mathrm{SU}(2)_\downarrow$	critical, non-compact $N_f = 2$ QED
$\mathrm{U}(1)_\uparrow \times \mathrm{U}(1)_\downarrow \rtimes \mathbb{Z}_2^\updownarrow$	not critical, SSB phases set in

D. Lattice Model and Symmetries

To be concrete, let us briefly review the lattice model that realizes the above mentioned SO(4) BTT. The model is defined on the double layer honeycomb lattice as shown in Fig. 3, where the two sites from different layers sit on top of each other (like the AA stacking bilayer graphene[49]) and will be treated as a combined site. On each site i of the bilayer honeycomb lattice, there are four complex fermion modes $c_{i\sigma\tau}$, where $\sigma = \uparrow, \downarrow$ labels the *spin* and $\tau = 1, 2$ labels the *layer*. The fermion operators can be arranged into the vector form $c_i = (c_{i\uparrow 1}, c_{i\uparrow 2}, c_{i\downarrow 1}, c_{i\downarrow 2})^\intercal$. The model Hamiltonian reads,[16]

$$\begin{split} H &= H_{\text{band}} + H_{\text{int}}, \\ H_{\text{band}} &= -t \sum_{\langle ij \rangle} c_i^\dagger c_j + \lambda \sum_{\langle\langle ij \rangle\rangle} \mathrm{i}\nu_{ij} c_i^\dagger \sigma^3 c_j + \text{h.c.}, \\ H_{\text{int}} &= J \sum_i \big(\boldsymbol{S}_{i1} \cdot \boldsymbol{S}_{i2} + S_{i1}^z S_{i2}^z + \tfrac{1}{8}\rho_i^2 \big). \end{split} \tag{33}$$

where $\boldsymbol{S}_{i\tau} = \frac{1}{2} c_{i\tau}^\dagger \boldsymbol{\sigma} c_{i\tau}$ is the spin operator with Pauli matrices $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ acting in the *spin* sector, and $\rho_i = (\sum_{\sigma\tau} c_{i\sigma\tau}^\dagger c_{i\sigma\tau} - 2)$ is the on-site total charge density (measured with respect to the half-filling). The Kane-Mele spin-orbit coupling λ is defined on the 2nd neighbor bonds with the sign factor ν_{ij} being $+1$ (-1) for hopping along (against) the bond direction specified in Fig. 3.

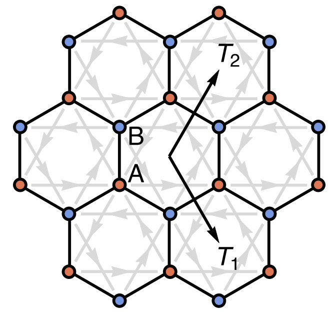

FIG. 3: Honeycomb lattice and space group symmetries. The lattice can be partitioned into A (red) and B (blue) sublattices. The sublattice sign $(-)^i$ is $+1$ on A and -1 on B. The black arrows mark the $T_{1,2}$ translation vectors. The background arrow (light gray) shows Haldane's 2nd neighbor hopping direction ν_{ij}.

The Hamiltonian in Eq. (33) has a pretty high symmetry $\mathrm{SO}(4) \times \mathrm{SO}(3)_\ell$, where $\mathrm{SO}(4) = (\mathrm{SU}(2)_\uparrow \times \mathrm{SU}(2)_\downarrow)/\mathbb{Z}_2$ and $\mathrm{SO}(3)_\ell = \mathrm{SU}(2)_\ell/\mathbb{Z}_2$. It is sometimes more convenient to lift the symmetry to $\mathrm{SU}(2)_\uparrow \times \mathrm{SU}(2)_\downarrow \times \mathrm{SU}(2)_\ell$, which can be defined by first rearranging the fermion operators $c_{i\sigma\tau}$ on each site into the following matrix form

$$C_{i\sigma} = \begin{bmatrix} 1 & \\ & (-)^\sigma \end{bmatrix} \begin{bmatrix} c_{i\sigma 1} & -c_{i\sigma 2}^\dagger \\ c_{i\sigma 2} & c_{i\sigma 1}^\dagger \end{bmatrix} \begin{bmatrix} 1 & \\ & (-)^i \end{bmatrix}, \tag{34}$$

where $(-)^\sigma$ is a staggered sign between spins,

$$(-)^\sigma = \begin{cases} +1 & \text{if } \sigma = \uparrow, \\ -1 & \text{if } \sigma = \downarrow, \end{cases} \tag{35}$$

and $(-)^i$ is a staggered sign between sublattices (see Fig. 3).

$$(-)^i = \begin{cases} +1 & \text{if } i \in A \text{ sublattice}, \\ -1 & \text{if } i \in B \text{ sublattice}. \end{cases} \tag{36}$$

Then the $\mathrm{SU}(2)_\uparrow \times \mathrm{SU}(2)_\downarrow \times \mathrm{SU}(2)_\ell$ symmetry acts on the matrix-form fermion operator $C_{i\sigma}$ (respectively for $\sigma = \uparrow, \downarrow$) as follows

$$C_{i\sigma} \to V C_{i\sigma} U_\sigma^\dagger, \tag{37}$$

for $U_\sigma \in \mathrm{SU}(2)_\sigma$ and $V \in \mathrm{SU}(2)_\ell$. Another way to understand the symmetry is to view the four complex fermion modes on each site as eight Majorana fermion modes, which form the eight-dimensional real spinor representation of an SO(7) group, in which the symmetry group $\mathrm{SO}(4) \times \mathrm{SO}(3)_\ell$ can be naturally embedded.

The action of these symmetries is most transparent by writing down the fermion bilinear operators that transform as vectors under $\mathrm{SO}(4)\times\mathrm{SO}(3)_\ell$. To this purpose, we define the O(4) vector $\boldsymbol{N}_i$ and the O(3) vector $\boldsymbol{M}_i$ in terms of the matrix-form fermion $C_{i\sigma}$ introduced in Eq. (34),

$$\begin{aligned}\boldsymbol{N}_i &= (-)^i \operatorname{Tr}\big(C_{i\downarrow}^\dagger C_{i\uparrow}(\mu^0, \mathrm{i}\mu^1, \mathrm{i}\mu^2, \mathrm{i}\mu^3)\big),\\ \boldsymbol{M}_i &= (-)^i \sum_\sigma (-)^\sigma \tfrac{1}{2}\operatorname{Tr}(C_{i\sigma}^\dagger \boldsymbol{\tau} C_{i\sigma}).\end{aligned} \tag{38}$$

where μ^a and τ^a ($a = 0,1,2,3$) are Pauli matrices acting in the *particle-hole* and the *layer* sectors respectively. Under the $\mathrm{SU}(2)_\uparrow\times\mathrm{SU}(2)_\downarrow\times\mathrm{SU}(2)_\ell$ symmetry action defined in Eq. (37), $\boldsymbol{N}_i$ rotates as the vector representation of $\mathrm{SO}(4) = (\mathrm{SU}(2)_\uparrow\times\mathrm{SU}(2)_\downarrow)/\mathbb{Z}_2$ and $\boldsymbol{M}_i$ rotates as the vector representation of $\mathrm{SO}(3)_\ell = \mathrm{SU}(2)_\ell/\mathbb{Z}_2$. It is instructive to label the operators by the spin quantum numbers $(s_\uparrow, s_\downarrow, s_\ell)$ of the $\mathrm{SU}(2)_\uparrow\times\mathrm{SU}(2)_\downarrow\times\mathrm{SU}(2)_\ell$ symmetry, as summarized in Tab. II.

TABLE II: The $\mathrm{SU}(2)_\uparrow\times\mathrm{SU}(2)_\downarrow\times\mathrm{SU}(2)_\ell$ symmetry charge $(s_\uparrow, s_\downarrow, s_\ell)$ of various operators.

operator	symmetry charge
physical fermion c_i	$(\frac{1}{2},0,\frac{1}{2})\oplus(0,\frac{1}{2},\frac{1}{2})$
O(4) vector $\boldsymbol{N}_i$	$(\frac{1}{2},\frac{1}{2},0)$
O(3) vector $\boldsymbol{M}_i$	$(0,0,1)$

It can be checked that the model Hamiltonian in Eq. (33) respects the $\mathrm{SO}(4)\times\mathrm{SO}(3)_\ell$ symmetry. In particular, the complicated-looking interaction H_{int} is such chosen to preserve the symmetry. To make the symmetry property manifest, the interaction can be rewritten as

$$H_{\text{int}} = -\frac{J}{8}\sum_i \boldsymbol{M}_i\cdot\boldsymbol{M}_i, \tag{39}$$

which is just the inner product of the $\mathrm{SO}(3)_\ell$ vector on each site and is obviously symmetric. The Hamiltonian also preserves some lattice symmetries and the time-reversal symmetry, but we will defer the discussion of those discrete symmetries when needed.

E. Phase Diagram

The phase diagram of the lattice model Eq. (33) has been explored in several recent numerical works[14,15,18,30,46,50]. The phase diagram contains a featureless Mott phase and two (non-trivial) BSPT phases separated from each other by continuous quantum phase transitions. In the free fermion limit ($J = 0$), the spin-orbit coupling λ gaps out the fermion and drives the system to a quantum spin Hall (QSH) insulator with spin Hall conductance $\sigma_{\text{sH}} = 2\operatorname{sgn}\lambda$, as illustrated in Fig. 4. With weak interaction J, the QSH phase becomes equivalent to the BSPT phase at low-energy, as the fermionic edge modes are gapped out by the interaction and the spin Hall current is now carried by collective bosonic edge modes,[16,18,38] where the low-energy bosonic freedom corresponds to the fermion bilinear order parameter $\boldsymbol{N}$ defined in Eq. (38). The BSPT phases protected by the $\mathrm{SO}(4) = (\mathrm{SU}(2)_\uparrow\times\mathrm{SU}(2)_\downarrow)/\mathbb{Z}_2$ symmetry are $\mathbb{Z}$ classified in (2+1)D. The topological index $\nu\in\mathbb{Z}$ can be defined as the level of the topological response theory in Eq. (7). We label the BSPT phases by their topological index $\nu = \operatorname{sgn}\lambda$ in Fig. 4.

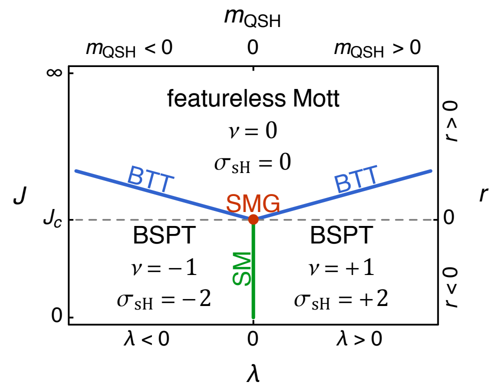

FIG. 4: Schematic phase diagram of both the bilayer honeycomb model Eq. (33) tuned by λ, J and the field theory Eq. (55) tuned by m_{QSH}, r. There is a featureless Mott phase and two bosonic symmetry protected topological (BSPT) phases, labeled by the topological index ν or equivalently the quantum spin Hall conductance σ_{sH}. Different phases are separated by quantum phase transitions: a fermionic transition (in green) corresponding to the Dirac semimetal (SM) and two bosonic topological transitions (BTTs) (in blue). The three transition lines meet at the symmetric mass generation (SMG) tricritical point (in red). The dashed line is a "faked transition" in the field theory that does not correspond to any physical transition.

The topologically trivial phase ($\nu = 0$) appears in the strong interacting limit $J\to\infty$. In this limit, the Hamiltonian is dominated by H_{int} in Eq. (33) and the ground state of the system is simply a direct product of every on-site ground state,

$$|\Psi\rangle = \prod_i (c_{i\uparrow 1}^\dagger c_{i\downarrow 2}^\dagger - c_{i\downarrow 1}^\dagger c_{i\uparrow 2}^\dagger)|0\rangle_c, \tag{40}$$

where $|0\rangle_c$ denotes the $c_{i\sigma\tau}$ fermion vacuum state. From the on-site interaction spectrum summarized in Tab. III, one can see that the ground state $|\Psi\rangle$ is unique, fully-gapped, and SO(7) symmetric[69] (which is also $\mathrm{SO}(4)\times\mathrm{SO}(3)_\ell$ symmetric). Moreover, as a product state, $|\Psi\rangle$ is topologically trivial by definition, hence the topological index should vanish, i.e. $\nu = 0$. Above the ground state, the excitation gap is of the order J, which can be considered as a Mott gap. Small perturbations should not close the gap, so the $|\Psi\rangle$ state actually represents a stable phase in the large J regime, which is the *featureless Mott phase*[51–53].

TABLE III: The spectrum of on-site interaction. E_n and d_n are respectively the energy and the degeneracy of each level. The ground state energy has been shifted to 0. All the eigenstates are labeled by the representations of the $\mathrm{SO}(4) \times \mathrm{SO}(3)_\ell$ symmetry group (or of the larger SO(7) group).

E_n	d_n	representation	
$3J/2$	$\times 4$	SO(4) vector	
$9J/8$	$\times 8$	SO(7) spinor	
J	$\times 3$	SO(3) vector	
0	$\times 1$	SO(7) scalar	(ground state)

Phases labeled by different topological indices ($\nu \in \mathbb{Z}$) must be separated from each other by quantum phase transitions, as shown in Fig. 4. The topological transition between the $\nu = +1$ and $\nu = -1$ BSPT (or QSH) phases (driven by the spin-orbit coupling λ at weak interaction) is simply a fermion gap closing transition. At this transition, the system becomes a Dirac semimetal with eight gapless Dirac fermions at low-energy. Since the short-range interaction J is irrelevant around the semimetal fixed point, the transition can be understood within the free fermion band theory. A more exotic transition in the phase diagram is the BTT, which is the transition between the featureless Mott ($\nu = 0$) and the non-trivial BSPT ($\nu = \pm 1$) phases. At the transition, the physical fermions are gapped and only their collective bosonic fluctuations become gapless, hence the transition is *bosonic*. As the SPT order changes across the transition, the transition is also *topological*. Several recent numerical simulations[14,15,17] indicate that the SO(4) symmetric BTT is a continuous transition. We propose that it can be described by the compact $N_f = 4$ QED theory, analyzed in the previous discussion.

The three phase boundaries in Fig. 4 join at a tricritical point, known as the SMG critical point.[32] If we focus on the $\lambda = 0$ axis, the SMG can also be viewed as the transition that the eight Dirac fermions in the semimetal phase are simultaneously gapped out by the interaction J without spontaneous symmetry breaking, i.e. without fermion bilinear condensation. A consistent theory of SMG must be compatible with both the BTT and the semimetal theory within the reach of perturbation. We have gained much understanding of the BTT theory form the above discussion. To pin down the SMG theory, we also need the input from the semimetal side, which we will briefly review in the following.

III. SYMMETRIC MASS GENERATION

A. Semimetal: Field Theory and Symmetries

The Dirac semimetal critical line refers to the fermionic transition between the $\nu = \pm 1$ BSPT phases (or more naturally interpreted as weakly interacting QSH phases), which is along the $\lambda = 0$ axis with $J < J_c$ in the phase diagram Fig. 4 of the model Eq. (33). In the semimetal phase, interactions are irrelevant, and the effective theory simply contains eight free Dirac fermions, or equivalently sixteen free Majorana fermions $\mathrm{c} \equiv (\operatorname{Re} c, \operatorname{Im} c)^\intercal$,

$$\mathcal{L}_{\mathrm{SM}} = \frac{1}{2} \sum_{Q,\sigma} \bar{\mathrm{c}}_{Q\sigma} \gamma \cdot (\partial - \mathrm{i} A_\sigma^a \upmu^a - \mathrm{i} A_\ell^a \uptau^a) \mathrm{c}_{Q\sigma}. \quad (41)$$

Hereinafter we will use the upright letters (like c) for the real/Majorana fields, and the italic letters (like c) for the complex/Dirac fields. The Majorana fermion field $\mathrm{c}_{Q\sigma}$ is labeled by the valley index $Q = K_\pm$ and the spin index $\sigma = \uparrow, \downarrow$. To simplify the representation of the $\mathrm{SU}(2)_\uparrow \times \mathrm{SU}(2)_\downarrow \times \mathrm{SU}(2)_\ell$ symmetry in the field theory, the valley modes $|K_\pm\rangle = (|K\rangle \pm \mathrm{i}|K'\rangle)/\sqrt{2}$ are redefined as combinations of the low-energy fermion modes from the K and K' points of the graphene Brillouin zone.[70] For each fixed Q and σ, the Majorana field $\mathrm{c}_{Q\sigma}$ contains eight real components: two for the Lorentz (sublattice A, B) degrees of freedom, two for the layer ($\tau = 1, 2$), and two for the particle-hole ($\operatorname{Re} c, \operatorname{Im} c$). The adjoint Majorana fields $\bar{\mathrm{c}}_{Q\sigma}$ are defined as $\bar{\mathrm{c}}_{Q\sigma} = \mathrm{c}_{Q\sigma}^\intercal \gamma^0$. The external SU(2) gauge fields A_σ and A_ℓ are introduced to keep track of the $\mathrm{SU}(2)_\sigma$ and the $\mathrm{SU}(2)_\ell$ symmetries respectively. Their charges (symmetry group generators) are represented in the layer $\otimes$ particle-hole subspace as

$$\begin{aligned} A_\sigma &= A_\sigma^a \upmu^a : (\upmu^1, \upmu^2, \upmu^3) \equiv (\sigma^{23}, \sigma^{21}, \sigma^{02}), \\ A_\ell &= A_\ell^a \uptau^a : (\uptau^1, \uptau^2, \uptau^3) \equiv (\sigma^{12}, \sigma^{20}, \sigma^{32}), \end{aligned} \quad (42)$$

where $\sigma^{ij} = \sigma^i \otimes \sigma^j$ and the 1st (2nd) Pauli index belongs to the layer (particle-hole) subspace (see Appendix A for derivation). Putting together the valley, spin, layer and particle-hole degrees of freedom, there are in total sixteen Majorana cones in the semimetal phase of the double layer honeycomb model.

Besides the continuous on-site symmetry $\mathrm{SO}(4) \times \mathrm{SO}(3)_\ell$ defined in Eq. (37), the lattice model also possess the space group symmetry of the honeycomb lattice and several anti-unitary symmetries. Among them, we will focus on the translation symmetry and the chiral symmetry. There are two linearly independent lattice translations, denoted by T_1 and T_2 as shown in Fig. 3. The chiral symmetry $\mathbb{Z}_2^{\mathcal{S}}$ (also known as the CT symmetry) is defined as $\mathcal{S} : c_i \to \mathcal{K}(-)^i c_i^\dagger$, where $\mathcal{K}$ denotes the complex conjugation operator and $(-)^i$ is the sublattice sign defined in Eq. (36). In the momentum space (see Appendix A), both lattice translations T_1 and T_2 are implemented as three-fold rotations in the valley subspace,

$$T_{1,2}:\begin{bmatrix}\mathrm{c}_{K_+\sigma}\\ \mathrm{c}_{K_-\sigma}\end{bmatrix}\to\frac{1}{2}\begin{bmatrix}-1 & \sqrt{3}\\ -\sqrt{3} & -1\end{bmatrix}\begin{bmatrix}\mathrm{c}_{K_+\sigma}\\ \mathrm{c}_{K_-\sigma}\end{bmatrix}. \tag{43}$$

Note that $|K_\pm\rangle=(|K\rangle\pm\mathrm{i}|K'\rangle)/\sqrt{2}$ are recombined valley modes, for which we define a valley sign factor

$$(-)^Q=\begin{cases}+1 & \text{if } Q=K_+,\\ -1 & \text{if } Q=K_-.\end{cases} \tag{44}$$

Then the chiral symmetry $\mathbb{Z}_2^{\mathcal{S}}$ is implemented as

$$\mathcal{S}:\mathrm{c}_{Q\sigma}\to\mathcal{K}(-)^Q\mathrm{i}\gamma^0\mathrm{c}_{\bar{Q}\sigma}. \tag{45}$$

The translation and the chiral symmetry together is sufficient to rule out all the fermion bilinear masses. This can be understood from an anomaly argument. The idea is to construct an "anomalous" antiunitary symmetry $\mathbb{Z}_2^{\mathcal{T}}$ from the translation and chiral symmetries. Suppose the translation symmetry can be enlarged from the $\mathbb{Z}_3$ valley rotation to the U(1) rotation (as an emergent symmetry in the field theory), by treating the conserved valley momentum as the conserved "charge", then it can be used to generate a $\pi/2$ valley rotation $R_{\pi/2}:\mathrm{c}_{Q\sigma}\to(-)^Q\mathrm{c}_{\bar{Q}\sigma}$, which defines another antiunitary transformation $\mathcal{T}=-R_{\pi/2}\mathcal{S}$ from the chiral symmetry transformation $\mathcal{S}$,

$$\mathcal{T}:\mathrm{c}_{Q\sigma}\to\mathcal{K}\mathrm{i}\gamma^0\mathrm{c}_{Q\sigma}. \tag{46}$$

This $\mathbb{Z}_2^{\mathcal{T}}$ is not an on-site symmetry, so the field theory Eq. (41) could behave anomalously under $\mathbb{Z}_2^{\mathcal{T}}$. It turns out that all Majorana fermions $\mathrm{c}_{Q\sigma}$ transform in the same way under $\mathbb{Z}_2^{\mathcal{T}}$ with $\mathcal{T}^2=-1$. This situation is analogous to the (2+1)D Majorna fermions on the surface of the (3+1)D class DIII topological superconductor (TSC) (e.g. the ^{3}He B phase).[54–56] Without interaction, the class DIII TSC is $\mathbb{Z}$ classified in (3+1)D, so the corresponding (2+1)D $\mathbb{Z}_2^{\mathcal{T}}$-symmetric Majorana fermions (as TSC surface states) are anomalous in the non-interacting limit and can not be gapped out by fermion bilinear masses without breaking the symmetry.

However, the $\mathbb{Z}\to\mathbb{Z}_{16}$ interaction reduced classification of the class DIII TSC[37,57–61] implies that sixteen $\mathbb{Z}_2^{\mathcal{T}}$-symmetric Majorana fermions are actually anomaly free in the presence of interaction. So there must be a way to gap out the sixteen Majorana fermions altogether by interaction without breaking the translation and chiral symmetry. Several field theory scenarios of how the sixteen Majorana fermions can be trivially gapped were proposed in an insightful work by Witten,[42] which generally require two separate transitions. In our lattice model Eq. (33), the fermions are gapped by the interlayer interaction J though a single SMG transition, resulting in the featureless Mott state directly. The SMG has been observed to be a continuous quantum phase transition in various different models[14,25–27,30]. We will focus on our model Eq. (33) throughout this work. Another model of SMG with a different symmetry was discussed in Ref. 32, which shares many common features.

B. Field Theory of SMG

Now let us put all pieces of evidence together. The SMG is the tricritical point in the phase diagram Fig. 4 where two BTT critical lines fuse into the semimetal critical line. Each BTT theory (the $N_f=4$ QED theory) contains four gauged Dirac fermions. Fusing two of them together would result in eight gauged Dirac fermions, or sixteen gauged Majorana fermions, which are fermionic partons that transform under the SO(4) symmetry only. On the other hand, the semimetal contains sixteen physical Majorana fermions, which transform under the SO(4) $\times$ SO(3)$_\ell$ symmetry. This suggests that the fermionic parton should originate from the physical fermion by gauging the SO(3)$_\ell$ symmetry. Or more precisely, we could consider fractionalizing the physical fermion into the SO(4)-charged fermionic parton and the SO(3)$_\ell$-charged bosonic parton, such that the gauge structure emerges naturally.

To this end, we propose that the SMG could be described by the following quantum chromodynamics (QCD) field theory (at $r=0$),

$$\begin{aligned}\mathcal{L}_{\mathrm{SMG}}&=\mathcal{L}_\psi+\mathcal{L}_\phi,\\ \mathcal{L}_\psi&=\frac{1}{2}\sum_{Q,\sigma}\bar{\psi}_{Q\sigma}\gamma\cdot(\partial-\mathrm{i}a^a\tau^a-\mathrm{i}A^a_\sigma\mu^a)\psi_{Q\sigma},\\ \mathcal{L}_\phi&=\frac{1}{2}\big((\partial-\mathrm{i}a^a\tau^a-\mathrm{i}A^a_\ell\mu^a)\phi\big)^2+\frac{r}{2}\phi^2+\frac{u}{4}\phi^4,\end{aligned} \tag{47}$$

which contains four SU(2) fundamental fermions $\psi_{Q\sigma}$ (labeled by the valley $Q=K_\pm$ and the spin $\sigma=\uparrow,\downarrow$ indices) and one SU(2) fundamental boson ϕ, both coupled to the internal SU(2) gauge field $a=a^a\tau^a$. The external SU(2) gauge fields $A_\uparrow=A^a_\uparrow\mu^a$, $A_\downarrow=A^a_\downarrow\mu^a$ and $A_\ell=A^a_\ell\mu^a$ are introduced to keep track of the SU(2)$_\uparrow$ $\times$ SU(2)$_\downarrow$ $\times$ SU(2)$_\ell$ symmetries of the physical fermion. The gauge/symmetry charges τ^a and μ^a follow the same definition as in Eq. (42). More specifically, ϕ is a four-component (gauge and particle-hole) real boson field, and $\psi_{Q\sigma}$ is an eight-component (Lorentz, gauge and particle-hole) Majorana fermion field (for each fixed valley Q and spin σ). The matter fields are written in the real/Majorana basis (indicated by their upright font), in order to fully expose their symmetry properties.

The theory $\mathcal{L}_\psi$ for the fermionic parton is very similar to that of the physical fermion in the semimetal phase as $\mathcal{L}_{\mathrm{SM}}$ in Eq. (41) and the only difference is that the symmetry field A_ℓ in $\mathcal{L}_{\mathrm{SM}}$ is replaced by the gauge field a. The SU(2)$_\ell$ symmetry is now carried by the bosonic parton ϕ which also couples to the SU(2) gauge field a. The bosonic mass term $r\phi^2\equiv r\phi^\intercal\phi$ drives the theory between the semimetal and featureless Mott phase across the SMG critical point. The $u\phi^4\equiv u(\phi^\intercal\phi)^2$ term in $\mathcal{L}_\phi$ simply remind us that the bosons are interacting. In general, all symmetry-allowed and gauge-invariant interactions will be present in the Lagrangian. We will not spell them all out explicitly.

The fractionalization can be formulated on the lattice scale. To simplify the presentation, we switch to the basis of complex fermions and complex bosons. Let us introduce four complex fermionic partons $\psi_{i\sigma\tau}$ and two complex bosonic partons $\phi_{i\tau}$ on each site i, where $\sigma=\uparrow,\downarrow$ and $\tau=1,2$ are the spin and the layer indices. They are related to the real matter fields in the field theory Eq. (47), by $\uppsi\equiv(\operatorname{Re}\psi,\operatorname{Im}\psi)^\intercal$ and $\upphi\equiv(\operatorname{Re}\phi,\operatorname{Im}\phi)^\intercal$. We rearrange the complex parton operators ψ and ϕ into the matrix form similar to Eq. (34),

$$\begin{aligned}\Psi_{i\sigma}&=\begin{bmatrix}1&\\&(-)^\sigma\end{bmatrix}\begin{bmatrix}\psi_{i\sigma1}&-\psi_{i\sigma2}^\dagger\\\psi_{i\sigma2}&\psi_{i\sigma1}^\dagger\end{bmatrix}\begin{bmatrix}1&\\&(-)^i\end{bmatrix},\\\Phi_i&=\begin{bmatrix}\phi_{i1}&-\phi_{i2}^\dagger\\\phi_{i2}&\phi_{i1}^\dagger\end{bmatrix}.\end{aligned}\tag{48}$$

Then the fractionalization of the physical fermion $C_{i\sigma}$ can be expressed as[62]

$$C_{i\sigma}=\Phi_i^\dagger\Psi_{i\sigma}.\tag{49}$$

Under the $\mathrm{SU}(2)_\uparrow\times\mathrm{SU}(2)_\downarrow\times\mathrm{SU}(2)_\ell$ symmetry and SU(2) gauge, the parton operators transform as

$$\Phi_i\to G_i\Phi_iV^\dagger,\quad\Psi_{i\sigma}\to G_i\Psi_{i\sigma}U_\sigma^\dagger,\tag{50}$$

for $U_\sigma\in\mathrm{SU}(2)_\sigma$, $V\in\mathrm{SU}(2)_\ell$ and $G_i\in\mathrm{SU}(2)_{\mathrm{gauge}}$. Thus the physical fermion $C_{i\sigma}$ in Eq. (49) is gauge neutral and transforms under the symmetries in the same way as defined in Eq. (37). The fractionalization scheme is depicted in Fig. 5. The corresponding real fields c, $\upphi$ and $\uppsi$ will inherit the similar fractionalization relation $\mathrm{c}_{Q\sigma}\sim\upphi\times\uppsi_{Q\sigma}$ from Eq. (49).

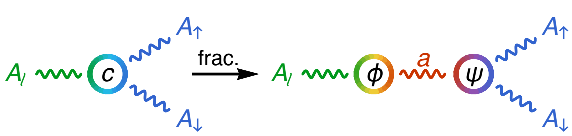

FIG. 5: The physical fermion c carries three SU(2) symmetry charges. Two of them, the $\mathrm{SU}(2)_\uparrow\times\mathrm{SU}(2)_\downarrow$ charges (in blue), are carried by the fermionic parton ψ; and the remaining $\mathrm{SU}(2)_\ell$ charge (in green) is carried by the bosonic parton ϕ. Both partons carry the SU(2) gauge charge (in red). The real fields c, $\upphi$ and $\uppsi$ will inherit the charge assignments.

For the convenience of later discussion, let us also define the O(4) and O(3) vectors for the fermionic parton $\uppsi_{Q\sigma}$, in analogy to that of the physical fermion $\mathrm{c}_{Q\sigma}$ following Eq. (38). In terms of the matrix-form operator in Eq. (48),

$$\begin{aligned}\boldsymbol{N}_i&=(-)^i\operatorname{Tr}\left(\Psi_{i\downarrow}^\dagger\Psi_{i\uparrow}(\sigma^0,\mathrm{i}\sigma^1,\mathrm{i}\sigma^2,\mathrm{i}\sigma^3)\right),\\\tilde{\boldsymbol{M}}_i&=(-)^i\sum_\sigma(-)^\sigma\tfrac{1}{2}\operatorname{Tr}(\Psi_{i\sigma}^\dagger\boldsymbol{\tau}\Psi_{i\sigma}).\end{aligned}\tag{51}$$

The parton O(3) vector $\tilde{\boldsymbol{M}}_i$ is different from $\boldsymbol{M}_i$ of the physical fermion defined in Eq. (38), since $\tilde{\boldsymbol{M}}_i$ rotates under the SU(2) gauge transformation while $\boldsymbol{M}_i$ is gauge neutral and rotates under the $\mathrm{SU}(2)_\ell$ symmetry transformation. This difference is emphasized by the tilde in the notation of $\tilde{\boldsymbol{M}}_i$. On the other hand, the O(4) vector $\boldsymbol{N}_i$ still represents the physical bosonic order parameter as in Eq. (38), which is also consistent with the fractionalization scheme in Eq. (5) in the BTT theory. The symmetry and gauge charges are summarize in Tab. IV.

TABLE IV: The $\mathrm{SU}(2)_\uparrow\times\mathrm{SU}(2)_\downarrow\times\mathrm{SU}(2)_\ell$ symmetry and the SU(2) gauge charges $(s_\uparrow,s_\downarrow,s_\ell;s_{\mathrm{gauge}})$ of various operators.

field	(symmetry; gauge) charge
physical fermion c_i	$(\frac{1}{2},0,\frac{1}{2};0)\oplus(0,\frac{1}{2},\frac{1}{2};0)$
fermionic parton $\uppsi_i$	$(\frac{1}{2},0,0;\frac{1}{2})\oplus(0,\frac{1}{2},0;\frac{1}{2})$
bosonic parton $\upphi_i$	$(0,0,\frac{1}{2};\frac{1}{2})$
ψ-parton O(4) vector $\boldsymbol{N}_i$	$(\frac{1}{2},\frac{1}{2},0;0)$
ψ-parton O(3) vector $\tilde{\boldsymbol{M}}_i$	$(0,0,0;1)$

Besides the $\mathrm{SO}(4)\times\mathrm{SO}(3)_\ell$ continuous symmetry, let us also briefly discuss the translation and the chiral symmetry of the partons. From the on-site fractionalization scheme in Eq. (49), it is obvious that the fermionic and the bosonic partons will translate together in the same way as the physical fermion. What is non-trivial is the chiral symmetry $\mathbb{Z}_2^{\mathcal{S}}$ (and its derived symmetry $\mathbb{Z}_2^{\mathcal{T}}$), whose action on the partons is subject to SU(2) gauge transformations. Such symmetry-gauge combined transformations form the projective symmetry group (PSG)[43], which characterizes the symmetry fractionalization pattern of the partons. As analyzed in Ref. 42, the PSG of the fermionic parton must be non-trivial in the SMG theory. In our context for example, one non-trivial choice of the PSG can be

$$\mathcal{T}:\uppsi_{Q\sigma}\to\mathcal{K}\mathrm{i}\gamma^0\mathrm{i}\uptau^2\uppsi_{Q\sigma},\upphi\to\mathcal{K}\mathrm{i}\uptau^2\upphi,a\to a.\tag{52}$$

In contrast to the $\mathcal{T}^2=-1$ for the physical fermion $\mathrm{c}_{Q\sigma}$ in Eq. (46), the $\mathcal{T}^2$ signature is fractionalized to $\mathcal{T}^2=+1$ on the fermionic parton $\uppsi_{Q\sigma}$ and $\mathcal{T}^2=-1$ on the bosonic parton $\upphi$. This completely changes the anomaly classification for the fermionic parton, as its symmetry class is shifted from DIII to BDI. Given that the (3+1)D class BDI TSC has a trivial classification, the Majorana fermions in parton QCD theory $\mathcal{L}_\uppsi$ (see Eq. (47)) is free from the $\mathbb{Z}_2^{\mathcal{T}}$ anomaly even on the fermion bilinear level. This implies that a bilinear mass term for the fermionic parton is now allowed by the $\mathbb{Z}_2^{\mathcal{T}}$ PSG. This points out a plausible route to get rid of the fermionic partons at low energy,[42] which would eventually lead to the featureless Mott phase, as to be elaborated in Sec. III C.

C. From Semimetal to Featureless Mott Insulator

The bosonic parton mass r in Eq. (47) is a relevant and symmetric perturbation at the SMG critical point. The semimetal phase can be accessed from the SMG critical point by condensing the bosonic parton. When $r < 0$, the bosonic parton condenses, i.e. $\langle \phi \rangle^2 \neq 0$. Using the SU(2) gauge freedom (which has three real gauge parameters), one can always gauge the condensation direction to $\langle \phi \rangle = \phi_0 (1,0,0,0)^\intercal$, where $\phi_0 \in \mathbb{R}$ is the condensation amplitude. Or equivalently, $\langle \phi \rangle$ can be written in the matrix form as

$$\langle \Phi \rangle = \phi_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{53}$$

As $\Phi \to G \Phi V^\dagger$ for $V \in \mathrm{SU}(2)_\ell$ and $G \in \mathrm{SU}(2)_{\text{gauge}}$, to keep $\langle \Phi \rangle$ invariant, we must have $G = V$, implying that the SU(2) gauge field a and the external $\mathrm{SU}(2)_\ell$ symmetry field A_ℓ are locked together. Therefore the $\mathrm{SU}(2)_\ell$ symmetry remains unbroken. Its symmetry charge is transferred to the fermionic parton $\psi_{Q\sigma}$, which is now also equivalent to the physical fermion as $c_{Q\sigma} \sim \langle \phi \rangle \times \psi_{Q\sigma}$. So we have recovered the effective field theory of the semimetal phase in Eq. (41). The translation and the chiral symmetry that protects the gaplessness of the semimetal also remains unbroken.

The featureless Mott phase corresponds to the $r > 0$ phase of Eq. (47) where the bosonic parton is gapped out. At low-energy (below the bosonic parton gap), we are left with the QCD theory of the fermionic parton coupled to the SU(2) gauge field, described by $\mathcal{L}_\psi$ in Eq. (47). The fate of the QCD theory is not completely clear yet. Our conjecture is that an SU(2) gauge triplet bilinear mass is spontaneously generated for the fermionic parton. We will supply this conjecture with more evidence by making the connection to the BTT theory later. If we accept such a spontaneous mass generation of the QCD theory, it will gap out all the fermionic partons and also break the SU(2) gauge structure down to U(1). The compact U(1) gauge field will then confine itself, since all matter fields have been gapped out at this stage. Therefore we end up with a featureless ground state with no gapless excitations, describing the featureless Mott phase.

To be more precise, let us write down the effective field theory for the featureless Mott phase. The SU(2) triplet mass corresponds to the O(3) vector $\tilde{\boldsymbol{M}}$ of the fermionic parton defined in Eq. (51). It is only a matter of gauge choice to align $\langle \tilde{\boldsymbol{M}} \rangle$ along the $\langle \tilde{\boldsymbol{M}} \rangle \propto (0,0,1)$ direction. With this gauge choice, the featureless Mott phase can be describe by

$$\begin{aligned} \mathcal{L}_{\text{Mott}} &= \mathcal{L}_\psi + m_{\text{Mott}} \tilde{M}^3 \\ &= \frac{1}{2} \sum_{Q,\sigma} \bar{\psi}_{Q\sigma} \gamma \cdot (\partial - \mathrm{i} a^3 \tau^3 - \mathrm{i} A^a_\sigma \mu^a) \psi_{Q\sigma} \\ &\quad + \frac{1}{2} \sum_{Q,\sigma} m_{\text{Mott}} (-)^{Q+\sigma} \bar{\psi}_{Q\sigma} \mathrm{i} \tau^3 \psi_{\bar{Q}\sigma}. \end{aligned} \tag{54}$$

If such a mass term $\boldsymbol{M}$ were introduced to the physical fermion, it would break the $\mathrm{SO}(3)_\ell = \mathrm{SU}(2)_\ell / \mathbb{Z}_2$ symmetry and the chiral symmetry $\mathbb{Z}_2^{\mathcal{S}}$ (or equivalently $\mathbb{Z}_2^{\mathcal{T}}$), because $\boldsymbol{M}$ transforms as a vector of the $\mathrm{SO}(3)_\ell$ and is also odd under $\mathbb{Z}_2^{\mathcal{T}}$ (i.e. $\mathcal{T} : \boldsymbol{M} \to -\boldsymbol{M}$). However for the fermionic parton, the vector $\tilde{\boldsymbol{M}}$ preserves all these symmetries. First of all, the $\mathrm{SO}(3)_\ell$ is not broken because the $\mathrm{SO}(3)_\ell$ symmetry charge has been carried away by the bosonic parton, which is now in a gapped and disordered state. The parton mass $\tilde{\boldsymbol{M}}$ only rotates under the SU(2) gauge transformation (as an SU(2) triplet). Then, because the fermionic parton mass $\tilde{\boldsymbol{M}}$ is not gauge neural and can be flipped by the gauge transformation, this leaves us rooms to restore the "broken" symmetry by the PSG. One can see that the PSG transformation of $\mathbb{Z}_2^{\mathcal{T}}$ in Eq. (52) is indeed chosen to keep $\tilde{M}^3$ invariant. So the fermionic parton mass $m_{\text{Mott}} \tilde{M}^3$ does not break any symmetry or introduce any gauge anomaly.

With the SU(2) gauge triplet mass m_{Mott}, all the fermions are gapped out. The SU(2) gauge field $a = a^a \tau^a$ is reduced to a compact U(1) gauge field a^3 by the Higgs mechanism. The expectation is that the compact U(1) gauge field will get confined by the non-perturbative monopole effect. However, there is the concern that the U(1) gauge flux might carry some symmetry charges or projective representations, such that the monopole operator would be forbidden by the symmetry and the confinement could not occur. Here we show that this is not the case.

First, we check the SO(4) symmetry. Integrating out the gapped fermion $\psi_{Q\sigma}$ in Eq. (54), no Chern-Simons term is generated between the gauge field a^3 and the symmetry probe fields A^a_σ, so the U(1) gauge flux $\mathrm{d}a^3$ does not carry any SO(4) symmetry charge.

Next, we check the translation symmetry, by studying how the U(1) gauge flux transforms under translation. To this purpose, we calculate the on-site gauge charge $\langle \psi_i^\intercal \tau^3 \psi_i \rangle$ and $\langle \phi_i^\intercal \tau^3 \phi_i \rangle$ in the matter field sector. It turns out that $\langle \psi_i^\intercal \tau^3 \psi_i \rangle = \langle \phi_i^\intercal \tau^3 \phi_i \rangle = 0$, i.e. the matter field background is gauge neutral. So the U(1) gauge flux $\mathrm{d}a^3$ does not see any background "magnetic field" as it moves around on the lattice, meaning that the translation symmetry is not fractionalized on the U(1) gauge flux ($T_1 T_2 T_1^{-1} T_2^{-1} = +1$). One may wonder why the SU(2) triplet mean-field $\langle \tilde{\boldsymbol{M}} \rangle$ does not lead to any gauge charge polarization in the fermionic sector. This is because $\langle \tilde{\boldsymbol{M}} \rangle$ polarizes the gauge charge oppositely in different spin $\sigma = \uparrow, \downarrow$ sectors, as seen from Eq. (54), so there is no net gauge charge polarization on each site. However, this also implies that the U(1) gauge flux $\mathrm{d}a^3$ does carry the quantum number of a spin-dependent translation, where $\uparrow$ and $\downarrow$ spins translate in opposite directions (see Appendix A for derivation). But this spin-dependent translation has been explicitly broken by the interaction H_{int} in the model Hamiltonian, so it imposes no symmetry constraint on the monopole operator.

Finally, we check the chiral symmetry. The U(1) gauge flux is reversed $\mathrm{d}a^3 \to -\mathrm{d}a^3$ under the $\mathbb{Z}_2^{\mathcal{S}}$ PSG, such that the monopole operator $\mathcal{M}_{a^3}$ that creates the gauge flux will be conjugated as $\mathcal{S} : \mathcal{M}_{a^3} \to \mathcal{M}_{a^3}^\dagger$. We also verified numerically on the lattice that there is no sign/phase change of the monopole operator $\mathcal{M}_{a^3}$ associated to this conjugation. Therefore the monopole terms like $\mathcal{M}_{a^3} + \mathcal{M}^\dagger_{a^3}$ are allowed by symmetries in the Lagrangian. Such terms will drive the gauge theory to the confined phase and gap out the U(1) photon from the low-energy sector. In the end, all excitations in the theory are gapped out, and we are left with a featureless Mott insulator.

D. Accessing BSPT Phases

In order to further check the consistency of the SMG theory, we can perturb the SMG critical point by the Kane-Mele spin-obit coupling λ, which breaks the chiral symmetry $\mathbb{Z}_2^S$ explicitly. As shown in the phase diagram Fig. 4, λ is a relevant perturbation, which drives the system to the BSPT phases. Within the framework of the SMG theory proposed in Eq. (47), the spin-orbit coupling λ in the lattice model Eq. (33) should correspond to a QSH mass m_{QSH} for the fermionic partons:

$$\mathcal{L}_{\text{BSPT}} = \mathcal{L}_{\text{SMG}} + \frac{1}{2} m_{\text{QSH}} \sum_{Q,\sigma} (-)^\sigma \bar{\psi}_{Q\sigma} \psi_{Q\sigma}, \tag{55}$$

because the m_{QSH} mass term is a relevant perturbation like λ that also preserves the $\text{SO}(4) \times \text{SO}(3)_\ell$ and the translation symmetry and breaks the chiral symmetry.

The topological response in the BSPT phase is easily found from the fermionic parton sector,

$$\begin{split}\mathcal{L}_{\text{BSPT}} = \frac{1}{2} \sum_{Q,\sigma} \bar{\psi}_{Q\sigma} \big(\gamma \cdot (\partial - \mathrm{i} a^a \tau^a - \mathrm{i} A^a_\sigma \mu^a) \\ + m_{\text{QSH}} (-)^\sigma \big) \psi_{Q\sigma}.\end{split} \tag{56}$$

In the presence of the m_{QSH} mass, the fermionic parton $\psi_{Q\sigma}$ is fully gapped. Integrating them out, we obtain the Chern-Simon term for the $\text{SU}(2)_\sigma$ symmetry probe fields $A_\sigma = A^a_\sigma \mu^a$[71] as proposed in Eq. (7),

$$\mathcal{L}_A = \frac{\mathrm{i}\nu}{4\pi} (\mathsf{CS}[A_\uparrow] - \mathsf{CS}[A_\downarrow]). \tag{57}$$

where the topological index is given by $\nu = \operatorname{sgn} m_{\text{QSH}}$. No Chern-Simons term is generated for the gauge field a or between a and A_σ. $\mathcal{L}_A$ describes the response theory of the $\text{SO}(4) = (\text{SU}(2)_\uparrow \times \text{SU}(2)_\downarrow)/\mathbb{Z}_2$ symmetric BSPT state, with $\text{SU}(2)_\uparrow$ and $\text{SU}(2)_\downarrow$ currents running oppositely on the boundary.

What about the physics of the bosonic parton sector then? As the QSH mass m_{QSH} gaps out the fermionic parton and generates the response theory $\mathcal{L}_A$, the bosonic parton and the SU(2) gauge field are left untouched. They are described by the following theory

$$\mathcal{L}_\phi = \frac{1}{2} \big((\partial - \mathrm{i} a^a \tau^a - \mathrm{i} A^a_\ell \mu^a)\phi\big)^2 + \frac{1}{2} r\phi^2 + \frac{1}{4} u \phi^4, \tag{58}$$

which is decoupled from the response theory $\mathcal{L}_A$. Despite the freedom to tune the parameter r, the theory $\mathcal{L}_\phi$ has only one single phase (independent of r). When $r < 0$, the bosonic parton condenses, which Higgs out the SU(2) gauge field and attaches the $\text{SU}(2)_\ell$ symmetry charge to the fermionic parton, such that the physical fermion is restored (and remains gapped). When $r > 0$, the bosonic parton is gapped and the fluctuating SU(2) gauge field will get confined, which binds the partons into physics fermions in a gapped spectrum. In any case, all excitations are gapped and response theory $\mathcal{L}_A$ is the same as that of the BSPT state. So there should be no physical transition across the $r = 0$ line (the dashed line in Fig. 4) inside the BSPT phase. Both $\nu = \pm 1$ BSPT phases are accessible from the SMG critical point simply by turning on the spin-orbit coupling for the physical fermions.

E. Bridging SMG and BTT

Now we are in the position to connect the SMG and BTT field theories. Within the framework of the SMG theory, the competition between the featureless Mott phase and the BSPT phase is just a matter of competing mass terms m_{Mott} and m_{QSH} (on the $r > 0$ side where the bosonic partons are gapped). Introducing both mass terms to the fermionic parton (by merging Eq. (54) and Eq. (56)), the BTT theory can be derived as follows

$$\begin{split}\mathcal{L}_{\text{BTT}} = \frac{1}{2} \sum_{Q,\sigma} \bar{\psi}_{Q\sigma} \gamma \cdot (\partial - \mathrm{i} a^3 \tau^3 - \mathrm{i} A^a_\sigma \mu^a) \psi_{Q\sigma} \\ + \frac{1}{2} \sum_{Q,\sigma} m_{\text{Mott}} (-)^{Q+\sigma} \bar{\psi}_{Q\sigma} \mathrm{i} \tau^3 \psi_{\bar{Q}\sigma} \\ + \frac{1}{2} \sum_{Q,\sigma} m_{\text{QSH}} (-)^\sigma \bar{\psi}_{Q\sigma} \psi_{Q\sigma}.\end{split} \tag{59}$$

The fermionic parton field $\psi_{Q\sigma}$ is still the same as in the SMG theory. But the SU(2) gauge structure is now broken down to U(1) in the presence of the gauge triplet mass m_{Mott}. The remaining U(1) gauge field a^3 is compact. We may choose to fix $m_{\text{Mott}} > 0$ using the SU(2) gauge freedom in the SMG theory.

Because the two masses m_{Mott} and m_{QSH} commute, they compete with each other to gap out the fermionic parton in different manners. The BTT happens when the two masses reaches a balance $|m_{\text{Mott}}| = |m_{\text{QSH}}|$, where half of the fermionic partons in the theory will be gapped and the other half remain gapless. The number of gapless fermions corresponds to eight Majorana or four Dirac, matching the fermion falvor in the $N_f = 4$ QED theory for the BTT. Obviously, the driving parameter of BTT is the difference between m_{Mott} and m_{QSH}, denoted as

$$m = m_{\text{QSH}} - m_{\text{Mott}}. \tag{60}$$

Herein we assume $m_{\text{QSH}} > 0$ by focusing on the BTT to the $\nu = +1$ BSPT phase. In the vicinity of the BTT, we have $|m| \ll m_{\text{QSH}}$ and m_{Mott}, so there is a separation of the fermion mass scale. The four massive Dirac fermions can be grouped into $\mathrm{SU}(2)_\uparrow$ and $\mathrm{SU}(2)_\downarrow$ fundamentals with opposite masses $\pm(m_{\mathrm{QSH}} + m_{\mathrm{Mott}})$, providing the following "level-1/2" background response

$$\mathcal{L}_{\mathrm{bg}}[A] = \frac{\mathrm{i}}{8\pi}(\mathsf{CS}[A_\uparrow] - \mathsf{CS}[A_\downarrow]). \tag{61}$$

The remaining four Dirac fermions are close to critical, which can be describe by (see Appendix A for derivation)

$$\sum_\sigma \bar{\psi}_\sigma\big(\gamma\cdot(\partial - \mathrm{i}a - \mathrm{i}A^a_\sigma \mu^a) + m(-)^\sigma\big)\psi_\sigma, \tag{62}$$

where we have switched back to the complex fermion basis (indicated by the italic font ψ_σ) and replace the compact U(1) gauge field a^3 by a. Compared with the SMG theory, the absence of the valley index Q in ψ_σ reflects the fact that half of the fermions are effectively removed from low-energy. In each spin $\sigma =\uparrow,\downarrow$ sector, the ψ_σ fermion transforms as the fundamental representation of $\mathrm{SU}(2)_\sigma$, so the $\mathrm{SO}(4) = \mathrm{SU}(2)_\uparrow \times \mathrm{SU}(2)_\downarrow/\mathbb{Z}_2$ symmetry can be implemented. Putting together Eq. (61) and Eq. (62) and including the SO(4) symmetry allowed interactions, we arrive at the compact $N_f = 4$ QED theory in Eq. (6) that was proposed to describe the SO(4) symmetric BTT.

The connection to the BTT theory provides a piece of supportive evidence for the existence of the SU(2) gauge triplet mass m_{Mott} in the Mott phase, which is one important assumption in our explanation of the SMG. Another assumption we made is that m_{Mot} must be spontaneously generated once the bosonic parton ϕ is gapped, which we resorted to the instability of the $N_f = 4$ SU(2) QCD theory. If this assumption is challenged, i.e. the bosonic parton gap $\sqrt{r}$ and the fermionic parton gap m_{Mott} do not open at the same point, an intermediate phase will set in between the semimetal and the featureless Mott phase. In Sec. III F, we will discuss one such scenario of the intermediate phase.

F. $\mathrm{SO}(3)_\ell$ Symmetry Breaking Phase

In the previous discussion of the semimetal to featureless Mott transition, we focus on the scenario of a direct transition via the SMG critical point. However, another possibility is that an intermediate $\mathrm{SO}(3)_\ell$ SSB phase may set in, splitting the SMG transition into two separate transitions. Starting from the semimetal phase, the system can first develop a long-range order of the O(3) vector $\langle \boldsymbol{M}\rangle \neq 0$, gapping out the physical fermions from the low-energy sector. Then the order is destroyed upon the increasing interaction strength, which restores the $\mathrm{SO}(3)_\ell$ symmetry in the featureless Mott phase.

These two scenarios can be distinguished from the different behaviors of the excitation gaps as we tune the interaction. We will focus on the following excitation gaps, defined via the correlation functions

$$\begin{aligned}\langle c_i^\dagger(\tau)c_j(0)\rangle &\sim e^{-\Delta_c\tau},\\ \langle \boldsymbol{N}_i(\tau)\cdot\boldsymbol{N}_j(0)\rangle &\sim e^{-\Delta_N\tau},\\ \langle \boldsymbol{M}_i(\tau)\cdot\boldsymbol{M}_j(0)\rangle &\sim e^{-\Delta_M\tau},\end{aligned} \tag{63}$$

where Δ_c the single-particle gap, Δ_N is the O(4) vector gap and Δ_M is the O(3) vector gap. In the first scenario Fig. 6(a), all excitation gaps open up at the single SMG critical point. In the second scenario Fig. 6(b), Δ_c and Δ_N first opens at a Gross-Neveu[63] critical point J_{c1}, where the $\mathrm{SO}(3)_\ell$ symmetry is spontaneously broken. Since the $\boldsymbol{N}$-vector boson excitation involves two fermion excitations, so a gap $\Delta_N \simeq 2\Delta_c$ is expected on the mean-field level. Gapless Goldstone bosons of $\boldsymbol{M}$ appear in the low-energy spectrum, so Δ_M remains zero in the $\mathrm{SO}(3)_\ell$ SSB phase. With stronger interaction, the Δ_M gap eventually opens at the O(3) Wilson-Fisher[64] critical point J_{c2}, where the Goldstone modes of $\boldsymbol{M}$ are gapped.

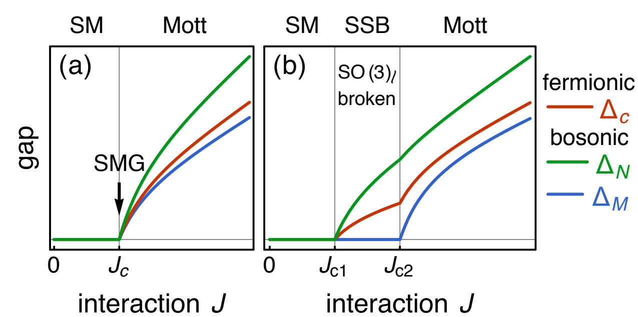

FIG. 6: Illustration of excitation gaps through the semimetal to featureless Mott transition via (a) an SMG point, (b) an intermediate $\mathrm{SO}(3)_\ell$ symmetry breaking phase. J_c is an SMG critical point, J_{c1} is a Gross-Neveu critical point, and J_{c2} is an O(3) Wilson-Fisher critical point. In the strong interaction $J\to\infty$ limit, we expect $\Delta_c \sim 9J/8$, $\Delta_N \sim 3J/2$ and $\Delta_M \sim J$ to match the on-site interaction spectrum listed in Tab. III.

In the parton field theory, the SMG critical point corresponds to the case when the gap opening of the bosonic parton ϕ and the spontaneous mass generation of the fermionic parton ψ happen at the same point. If the fermion mass generation happens before the gapping of bosons, our theoretical framework will allow an intermediate $\mathrm{SO}(3)_\ell$ SSB phase. In the parton language, the Gross-Neveu transition J_{c1} corresponds to the mass generation of fermionic partons in the presence of the condensed bosonic partons. However the parton description is not necessary in this case because the condensed bosonic parton will Higgs out the gauge field and make the fermionic parton equivalent to the physical fermion, then the Gross-Neveu transition is just the conventional mass generation for the physical fermion.

The O(3) Wilson-Fisher transition J_{c2} turns out to be a more interesting one, which is a transition of the bosonic parton on the background of gapped fermionic partons. It is accessible from the SMG theory Eq. (47) as

$$\mathcal{L}_{\mathrm{WF}} = \mathcal{L}_{\mathrm{SMG}} + m_{\mathrm{Mott}} \tilde{M}^3, \tag{64}$$

assuming the fermionic parton has developed an SU(2) gauge triplet mass along the direction $\tilde{\boldsymbol{M}} \propto (0,0,1)$. The mass m_{Mott} gaps out the fermionic parton and Higgs down the SU(2) gauge group to U(1), so the remaining low-energy theory only contains bosonic partons coupled to the U(1) gauge field

$$\mathcal{L}_{\mathrm{WF}} = \frac{1}{2}\big((\partial - \mathrm{i}a^3\tau^3 - \mathrm{i}A_\ell^a \mu^a)\phi\big)^2 + \frac{r}{2}\phi^2 + \frac{u}{4}\phi^4, \tag{65}$$

which is exactly the CP^1 field theory description of the O(3) Wilson-Fisher critical point. The SSB-Mott transition is driven by the bosonic mass term r. Since the physical fermion excitation involves the excitations of both the fermionic and the bosonic parton, so the gap opening of the bosonic parton will create a kink in the single-particle gap Δ_c across the transition J_{c2}, as illustrated in Fig. 6(b).

IV. SUMMARY

In this work we have given an alternative description of the bosonic topological transition (BTT), a transition between the bosonic symmetry protected topological (BSPT) phase and the featureless Mott phase, and demonstrated that how the BTT is connected to another exotic quantum phase transition which we call the symmetry mass generation (SMG). Previously the BTT is described by the $N_f = 2$ non-compact QED with an emergent O(4) symmetry in the infrared, while in our work we show that the $N_f = 4$ compact QED is instable against a SU(4) to SO(4) breaking deformation, and flows to a stable fixed point with $\mathrm{O}(4) = \mathrm{SO}(4) \rtimes \mathbb{Z}_2$ symmetry, which we identify with the infrared limit of the $N_f = 2$ non-compact QED.

The SMG is a direct transition between the Dirac semimetal and the featureless Mott insulator, where the Dirac fermions are gapped by their interaction without breaking any symmetry. Conventionally, within Landau's paradigm, two transitions, a Gross-Neveu followed by a Wilson-Fisher, are expected between the semimetal and the featureless Mott phases. As the two transitions merge into a single one, exotic quantum criticality emerges beyond Landau's paradigm. We propose that the SMG is a new type of deconfined quantum critical point (DQCP) where the physical fermion is fractionalized into bosonic and fermionic partons.

In particular, motivated by recent numerics, we studied a SO(4) × SO(3) symmetric (2+1)D fermion model that exhibits the SMG transition. At the critical point, the SO(3) symmetry quantum number is carried by the bosonic parton and the SO(4) symmetry quantum number is carried by the fermionic parton. They both couple to an emergent SU(2) gauge field as fundamental representations. We propose that the SMG critical point can be described by the SU(2) QCD-Higgs theory, where both the bosonic and fermionic partons become critical. Several key ingredients of the SMG theory are summarized as follows. First, the theory must contain a Higgs field (e.g. the bosonic parton), whose condensation can break the gauge structure completely and bring the theory back to the semimetal phase. Second, the $\mathbb{Z}_2^{\mathcal{T}}$ symmetry acting on the fermionic parton must be followed by a non-trivial gauge transformation, such that a parton bilinear mass is allowed by the PSG to gap out the fermionic partons in the featureless Mott phase. The parton bilinear mass (e.g. the SU(2) triplet mass) must not be gauge neutral, in order for the gauge transformation to come into play. Finally, after the parton bilinear mass condensation, the remaining unbroken gauge group (if not trivial) should be confined by itself, such that a direct continuous SMG transition becomes possible. These features are shared among the SMG theories with other symmetries as discussed in Ref. 32,42.

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Appendix A: Majorana Basis

1. Lattice Model Basis

In this section, we derive the low-energy effective theory from the lattice model. Let us start from the Haldane model of *spinless* fermion on the honeycomb lattice:

$$H = -t\sum_{\langle ij\rangle} c_i^\dagger c_j + \lambda \sum_{\langle\langle ij\rangle\rangle} \mathrm{i}\nu_{ij} c_i^\dagger c_j + \text{h.c.}. \tag{A1}$$

Switching to the momentum space and introducing $c_{\boldsymbol{k}} = (c_{\boldsymbol{k}A}, c_{\boldsymbol{k}B})^\intercal$, we have

$$\begin{aligned} H &= \sum_{\boldsymbol{k}} c_{\boldsymbol{k}}^\dagger \begin{bmatrix} g(\boldsymbol{k}) & f^*(\boldsymbol{k}) \\ f(\boldsymbol{k}) & -g(\boldsymbol{k}) \end{bmatrix} c_{\boldsymbol{k}}, \\ f(\boldsymbol{k}) &= -t\Big(e^{\mathrm{i}k_y} + 2e^{-\mathrm{i}\frac{k_y}{2}} \cos\frac{\sqrt{3}k_x}{2}\Big), \\ g(\boldsymbol{k}) &= -4\lambda \sin\frac{\sqrt{3}k_x}{2}\Big(\cos\frac{\sqrt{3}k_x}{2} - \cos\frac{3k_y}{2}\Big). \end{aligned} \tag{A2}$$

Expand $f(\boldsymbol{k})$ and $g(\boldsymbol{k})$ around the momentum points $K, K' = (\pm\frac{4\pi}{3\sqrt{3}}, 0)$, we get

$$\begin{aligned} f(K+\boldsymbol{k}) &= v_F(k_x - \mathrm{i}k_y), \quad & g(K) &= m_{\mathrm{IQH}}, \\ f(K'+\boldsymbol{k}) &= v_F(-k_x - \mathrm{i}k_y), \quad & g(K') &= -m_{\mathrm{IQH}}, \end{aligned} \tag{A3}$$

where $v_F = 3t/2$ is the Fermi velocity and $m_{\mathrm{IQH}} = 3\sqrt{3}\lambda$ is the integer quantum Hall mass. In the following, we will set $v_F = 1$ as the energy unit. In the complex fermion basis (valley $\otimes$ sublattice)

$$c = \left[\begin{smallmatrix} K \\ K' \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right], \tag{A4}$$

the low-energy effective Hamiltonian (density) reads

$$\mathcal{H} = c^\dagger(-\mathrm{i}\partial_x\sigma^{31} + \mathrm{i}\partial_y\sigma^{02} + m_{\mathrm{IQH}}\sigma^{33})c. \tag{A5}$$

Throughout this appendix, we will follow the convention to denote the tensor product of Pauli matrices as

$$\sigma^{abc\cdots} \equiv \sigma^a \otimes \sigma^b \otimes \sigma^c \otimes \cdots, \tag{A6}$$

where $a, b, c, \cdots = 0, 1, 2, 3$ are called the *Pauli indices*.

Under translation $T_{1,2} : \boldsymbol{r} \to \boldsymbol{r} + \boldsymbol{a}_{1,2}$, with $\boldsymbol{a}_{1,2} = (\sqrt{3}/2, \mp 3/2)$ according to Fig. 3. So the fermion transforms as

$$T_{1,2} : c \to e^{\mathrm{i}\frac{2\pi}{3}\sigma^{30}}c. \tag{A7}$$

The translation is implemented as a three-fold rotation in the valley subspace. For the chiral symmetry $\mathcal{S} : c_i \to \mathcal{K}(-)^i c_i^\dagger$, so $c_Q \to \mathcal{K}\sigma^3 c_Q^\dagger$. Therefore, we conclude

$$\mathcal{S} : c \to \mathcal{K}\sigma^{03}c^\dagger. \tag{A8}$$

In the Majorana fermion basis (valley $\otimes$ sublattice $\otimes$ particle-hole)

$$\mathrm{c} = \left[\begin{smallmatrix} K \\ K' \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} \mathrm{Re}\, c \\ \mathrm{Im}\, c \end{smallmatrix}\right], \tag{A9}$$

the Hamiltonian in Eq. (A5) becomes

$$\mathcal{H} = \tfrac{1}{2}\mathrm{c}^\intercal(-\mathrm{i}\partial_x\sigma^{310} + \mathrm{i}\partial_y\sigma^{022} + m_{\mathrm{IQH}}\sigma^{332})\mathrm{c}. \tag{A10}$$

We follow the convention of using the italic letter c for complex fermion and the upright letter c for Majorana (real) fermion. The symmetry actions can be written in the Majorana basis as

$$\begin{aligned} T_{1,2} &: \mathrm{c} \to e^{\mathrm{i}\frac{2\pi}{3}\sigma^{302}}\mathrm{c}, \\ \mathcal{S} &: \mathrm{c} \to \mathcal{K}\sigma^{030}\mathrm{c}. \end{aligned} \tag{A11}$$

We introduce the spin and the layer degrees of freedom by extending the Majorana basis to

$$\mathrm{c} = \begin{bmatrix} K \\ K' \end{bmatrix} \otimes \begin{bmatrix} A \\ B \end{bmatrix} \otimes \begin{bmatrix} \uparrow \\ \downarrow \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 2 \end{bmatrix} \otimes \begin{bmatrix} \mathrm{Re}\, c \\ \mathrm{Im}\, c \end{bmatrix}. \tag{A12}$$

The Hamiltonian in Eq. (A10) is also extended to

$$\mathcal{H} = \tfrac{1}{2}\mathrm{c}^\intercal(-\mathrm{i}\partial_x\sigma^{31000} + \mathrm{i}\partial_y\sigma^{02002} + m_{\mathrm{IQH}}\sigma^{33002})\mathrm{c}. \tag{A13}$$

The discrete symmetries is extended from Eq. (A11) to

$$\begin{aligned} T_{1,2} &: \mathrm{c} \to e^{\mathrm{i}\frac{2\pi}{3}\sigma^{30002}}\mathrm{c}, \\ \mathcal{S} &: \mathrm{c} \to \mathcal{K}\sigma^{03000}\mathrm{c}. \end{aligned} \tag{A14}$$

The O(4) vector $\boldsymbol{N}$ and the O(3) vector $\boldsymbol{M}$ can be written down following Eq. (38),

$$\begin{aligned} \boldsymbol{N} &= \tfrac{1}{2}\mathrm{c}^\intercal(\sigma^{03132}, -\sigma^{10213}, -\sigma^{10211}, \sigma^{03230})\mathrm{c}, \\ \boldsymbol{M} &= \tfrac{1}{2}\mathrm{c}^\intercal(\sigma^{03012}, \sigma^{03020}, \sigma^{03332})\mathrm{c}. \end{aligned} \tag{A15}$$

As we can see, the rotation among these order parameters are interwound with the valley and the sublattice degrees of freedom in the lattice model basis.

2. SMG Field Theory Basis

To separate the action subspace of the discrete and the continuous symmetries explicitly, we introduce the SMG field theory basis by the following basis transformation from the lattice model basis,

$$\mathrm{c} \to \mathsf{C}^{\sigma^{00300}}_{\sigma^{00030}}\mathsf{C}^{\sigma^{03000}}_{\mathrm{i}\sigma^{00002}}\mathsf{C}^{\sigma^{01000}}_{\mathrm{i}\sigma^{30002}}\mathsf{C}^{\sigma^{10000}}_{\mathrm{i}\sigma^{00002}}\mathrm{c}, \tag{A16}$$

where $\mathsf{C}^X_Y \equiv \frac{1}{2}(1 + X + Y - XY)$ represents a controlled gate. Under the transformation, the Hamiltonian in Eq. (A13) becomes

$$\mathcal{H} = \tfrac{1}{2}\mathrm{c}^\intercal(-\mathrm{i}\partial_x\sigma^{03000} + \mathrm{i}\partial_y\sigma^{01000} + m_{\mathrm{IQH}}\sigma^{02000})\mathrm{c}. \tag{A17}$$

The symmetry actions in Eq. (A14) become

$$\begin{aligned} T_{1,2} &: \mathrm{c} \to e^{\mathrm{i}\frac{2\pi}{3}\sigma^{20000}}\mathrm{c}, \\ \mathcal{S} &: \mathrm{c} \to \mathcal{K}\sigma^{22000}\mathrm{c}, \end{aligned} \tag{A18}$$

which leads to Eqs. (43, 45) in the complex fermion basis. Note that the C_6 transform can be equivalently written as $C_6 : \mathrm{c} \to \mathrm{i}\sigma^{30002}e^{\mathrm{i}\frac{\pi}{6}\sigma^{02000}}\mathrm{c}$. The generator σ^{20000} of the translation indicates that the valley basis has been transformed to $|K_\pm\rangle = (|K\rangle \pm \mathrm{i}|K'\rangle)/\sqrt{2}$. The fermion bilinear order parameters in Eq. (A15) become

$$\begin{aligned} \boldsymbol{N} &= \tfrac{1}{2}\mathrm{c}^\intercal(\sigma^{22102}, \sigma^{22123}, \sigma^{22121}, \sigma^{22200})\mathrm{c}, \\ \boldsymbol{M} &= \tfrac{1}{2}\mathrm{c}^\intercal(\sigma^{22312}, \sigma^{22320}, \sigma^{22332})\mathrm{c}. \end{aligned} \tag{A19}$$

The continuous symmetry $\mathrm{SO}(4) \times \mathrm{SO}(3)_\ell$ acts only in the spin $\otimes$ layer $\otimes$ particle-hole subspace. The representation of their generators in the Majorana basis can be derived from the commutator of the vectors $\boldsymbol{N}$ and $\boldsymbol{M}$ in the same representation, which are concluded in Tab. V From Tab. V we can see the SO(4) generators splits to those of the $\mathrm{SU}(2)_\uparrow \times \mathrm{SU}(2)_\downarrow$ group. In each spin sector, the $\mathrm{SU}(2)_\sigma$ group acts only in the layer $\otimes$ particle-hole subspace, whose generators are represented as

$$\boldsymbol{\mu} = \sigma^{000} \otimes (\sigma^{23}, \sigma^{21}, \sigma^{02}). \tag{A20}$$

TABLE V: Representations of the $\mathrm{SO}(4)\times\mathrm{SO}(3)_\ell$ generators in the spin $\otimes$ layer $\otimes$ particle-hole subspace. The generator at row-a column-b is given by $\Gamma_{ab}\sim \mathrm{i}[\Gamma_a,\Gamma_b]$ (where the sign and pre-factors are omitted).

SO(4) generators

	σ^{102}	σ^{123}	σ^{121}	σ^{200}
σ^{102}		σ^{021}	σ^{023}	σ^{302}
σ^{123}			σ^{002}	σ^{323}
σ^{121}				σ^{321}
σ^{200}				

$\mathrm{SO}(3)_\ell$ generators

	σ^{312}	σ^{320}	σ^{332}
σ^{312}		σ^{032}	σ^{020}
σ^{320}			σ^{012}
σ^{332}			

The $\mathrm{SO}(3)_\ell \simeq \mathrm{SU}(2)_\ell$ group also acts only in the layer $\otimes$ particle-hole subspace, whose generators are represented as

$$\boldsymbol{\tau} = \sigma^{000}\otimes(\sigma^{12},\sigma^{20},\sigma^{32}). \tag{A21}$$

So the symmetry charge are indeed given by Eq. (42). For the consistency in the context, here we have filled in the omitted identity operator σ^{000} in the valley $\otimes$ sublattice $\otimes$ spin subspace.

In the SMG field theory basis, there are only six mass terms of the form $\mathsf{c}^\intercal\mathsf{M}\mathsf{c}$ that commute with all the $\mathrm{SO}(4)\times\mathrm{SO}(3)_\ell$ generators:

$$\begin{split}\mathsf{M} = \sigma^{02000},\sigma^{12000},\sigma^{32000},\\ \sigma^{02300},\sigma^{12300},\sigma^{32300}.\end{split} \tag{A22}$$

In the first line, the first one (σ^{02000}) is the IQH mass. The following two (σ^{12000} and σ^{32000}) are the Kekulé masses (dimerization according to the Kekulé pattern) which breaks the translation symmetry, as they do not commute with the translation generator σ^{20000} given in Eq. (A18). The second line is the spin-dependent version of the first line. For example, the first one σ^{02300} corresponds to the QSH mass. Therefore if we consider translation invariant masses, we are left with the IQH and QSH masses only. It is easy to see that these two masses are further ruled out by the reflection σ_h, the time-reversal $\mathcal{T}$ and the chiral $\mathcal{S}$ symmetries, given their definitions in Eq. (A18).

The fermionic parton can be written in the same field theory basis as the physical fermion by replacing $\mathsf{c}\to\psi$ in all equations. In the featureless Mott phase, we expect a $\boldsymbol{M}$ mass (for example M^3) will be generated for the fermionic partons. From Eq. (A18), we can see M^3 changes sign under $\mathcal{S}$ transformation. So the chiral symmetry should act projectively (i.e. should be followed by the gauge transformation $\psi\to\mathrm{i}\sigma^{00020}\psi$ to revert the sign change of the mean-field mass M^3).

$$\begin{split}T_{1,2}:\psi\to e^{\mathrm{i}\frac{2\pi}{3}\sigma^{20000}}\psi,\\ \mathcal{S}:\psi\to\mathcal{K}\mathrm{i}\sigma^{22020}\psi.\end{split} \tag{A23}$$

For the fermionic partons, the $\mathrm{SU}(2)_\ell$ symmetry is promoted to the SU(2) gauge group. As the fermionic partons are gapped out by the mass $M\psi^\intercal\sigma^{22332}\psi$, the SU(2) gauge group will be Higgs down to its U(1) subgroup. The U(1) gauge flux $\mathrm{d}a^3\tau^3$ can acquire a quantum number due to the Hall-like response. The quantum number is determined by evaluating the following product

$$\mathsf{q}=\gamma^0 M^3\tau^3=\sigma^{02000}\sigma^{22332}\sigma^{00032}=\sigma^{20300}. \tag{A24}$$

The transformation $\psi\to e^{\mathrm{i}\theta\mathsf{q}}\psi$ generated by this quantum number correspond to a spin-dependent translation (recall that the translation generator is σ^{20000} in Eq. (A23) while $\sigma^{00300}=(-)^\sigma$ is the sign of the spin and q is just the product of them). Since the spin-dependent translation is not a symmetry at the UV scale, the U(1) gauge field a^3 will remain compact.

3. BTT Field Theory Basis

Near the bosonic topological transition, the masses M and m_{QSH} competes in the fermionic parton sector,

$$\begin{split}\mathcal{H}=\tfrac{1}{2}\psi^\intercal(-\mathrm{i}\partial_x\sigma^{03000}+\mathrm{i}\partial_y\sigma^{01000}\\ +M\sigma^{22332}+m_{\mathrm{QSH}}\sigma^{02300})\psi.\end{split} \tag{A25}$$

At the transition $|M|=|m_{\mathrm{QSH}}|$, half of the fermions ψ will be gapped out. To focus on the remaining gapless fermions, we wish to explicitly separate gapped and gapless subspaces. This can be done by further applying the following transformation to the field theory basis,

$$\psi\to\mathsf{C}^{\sigma^{10030}}_{-\mathrm{i}\sigma^{00002}}\mathsf{C}^{\sigma^{00030}}_{\sigma^{00003}}\psi, \tag{A26}$$

under which the Hamiltonian becomes

$$\begin{split}\mathcal{H}=\tfrac{1}{2}\psi^\intercal(-\mathrm{i}\partial_x\sigma^{03000}+\mathrm{i}\partial_y\sigma^{01000}\\ +M\sigma^{32300}+m_{\mathrm{QSH}}\sigma^{02300})\psi.\end{split} \tag{A27}$$

In this basis, the masses M and m_{QSH} only differed by the factor of σ^{30000}, then the gapped and gapless subspace are clearly separated by

$$\sigma^{30000}\psi=\begin{cases}+\psi & \text{gapped},\\ -\psi & \text{gapless}.\end{cases} \tag{A28}$$

The PSG transformations in Eq. (A23) becomes

$$\begin{split}T_{1,2}:\psi\to e^{\mathrm{i}\frac{2\pi}{3}\sigma^{30002}}\psi,\\ \mathcal{S}:\psi\to\mathcal{K}\mathrm{i}\sigma^{22023}\psi.\end{split} \tag{A29}$$

The translation generator is transformed back to σ^{30002}, implying that the valley subspace is represented in the $|K\rangle$, $|K'\rangle$ basis again (nominally). So Eq. (A28) simply suggest that the fermions ψ_K from the K-valley are gapped and those from the K'-valley $\psi_{K'}$ remain gapless. Nevertheless, the statement should not be taken too seriously, since the basis transformation has mixed the valley and gauge charge together. In the new basis, the $\boldsymbol{N}$ and $\boldsymbol{M}$ vectors are represented as

$$\begin{split}\boldsymbol{N}[\psi]=\tfrac{1}{2}\psi^\intercal(\sigma^{32130},\sigma^{32110},\sigma^{32122},\sigma^{32202})\psi,\\ \boldsymbol{M}[\psi]=\tfrac{1}{2}\psi^\intercal(\sigma^{22321},\sigma^{22323},\sigma^{32300})\psi,\end{split} \tag{A30}$$

and the $\mathrm{SU}(2)_\sigma$ and $\mathrm{SU}(2)_\ell$ generators are represented as

$$\begin{aligned}\boldsymbol{\mu} &= \sigma^{000}\otimes(\sigma^{12},\sigma^{20},\sigma^{32}),\\ \boldsymbol{\tau} &= (\sigma^{10023},\sigma^{10021},\sigma^{00002}).\end{aligned} \tag{A31}$$

Now we project to the low-energy subspace of $\sigma^{30000}\psi = -\psi$, this amounts to removing the first Pauli index in all Pauli operators. Off-diagonal operators $\sigma^{1\cdots},\sigma^{2\cdots}$ will not survive the projections (i.e. they can not be represented in the low-energy subspace). The Hamiltonian Eq. (A27) is reduced to

$$\mathcal{H} = \tfrac{1}{2}\psi^\intercal(-\mathrm{i}\partial_x\sigma^{3000} + \mathrm{i}\partial_y\sigma^{1000} + m'_{\mathrm{QSH}}\sigma^{2300})\psi, \tag{A32}$$

where $m'_{\mathrm{QSH}} = m_{\mathrm{QSH}} - M$ is the reduced QSH mass. This low-energy subspace basis is the BTT field theory basis.

From Eq. (A29), we can see that the chiral symmetry $\mathcal{S}$ does not survive the the projection, because the QSH mass m_{QSH} has explicitly broken the symmetry. The $\mathrm{SO}(4)\simeq\mathrm{SU}(2)_\uparrow\times\mathrm{SU}(2)_\downarrow$ symmetry is preserved. The O(4) vector is projected to

$$\boldsymbol{N} = -\tfrac{1}{2}\psi^\intercal(\sigma^{2130},\sigma^{2110},\sigma^{2122},\sigma^{2202})\psi. \tag{A33}$$

The $\mathrm{SU}(2)_\sigma$ generators becomes $\boldsymbol{\mu} = \sigma^{00}\otimes(\sigma^{12},\sigma^{20},\sigma^{32})$. The $\mathrm{SO}(3)_\ell\simeq\mathrm{SU}(2)_\ell$ symmetry is now gauged for the fermionic partons. The mass M explicitly Higgs down the gauge group from SU(2) to U(1) and the remaining U(1) gauge charge is $\tau^3 = \sigma^{00002}$. So in the BTT field theory basis, the U(1) gauge transformation is simply the phase rotation $\psi\to e^{\mathrm{i}\theta}\psi$ of the complex fermion ψ.

[1] X.-G. Wen, ArXiv e-prints (2016), 1610.03911.
[2] Z.-C. Gu and X.-G. Wen, Phys. Rev. B **80**, 155131 (2009), 0903.1069.
[3] X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B **83**, 035107 (2011), 1008.3745.
[4] N. Schuch, D. Pérez-García, and I. Cirac, Phys. Rev. B **84**, 165139 (2011), 1010.3732.
[5] F. Pollmann, E. Berg, A. M. Turner, and M. Oshikawa, Phys. Rev. B **85**, 075125 (2012), 0909.4059.
[6] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Phys. Rev. B **87**, 155114 (2013), 1106.4772.
[7] C. Wang and T. Senthil, Physical Review X **5**, 041031 (2015), 1505.05141.
[8] M. A. Metlitski and A. Vishwanath, Phys. Rev. B **93**, 245151 (2016), 1505.05142.
[9] A. Karch and D. Tong, Physical Review X **6**, 031043 (2016), 1606.01893.
[10] N. Seiberg, T. Senthil, C. Wang, and E. Witten, Annals of Physics **374**, 395 (2016), 1606.01989.
[11] C. Wang, A. Nahum, M. A. Metlitski, C. Xu, and T. Senthil, ArXiv e-prints (2017), 1703.02426.
[12] Y.-M. Lu and D.-H. Lee, ArXiv e-prints (2012), 1210.0909.
[13] T. Grover and A. Vishwanath, Phys. Rev. B **87**, 045129 (2013), 1210.0907.
[14] K. Slagle, Y.-Z. You, and C. Xu, Phys. Rev. B **91**, 115121 (2015), 1409.7401.
[15] Y.-Y. He, H.-Q. Wu, Y.-Z. You, C. Xu, Z. Y. Meng, and Z.-Y. Lu, Phys. Rev. B **93**, 115150 (2016), 1508.06389.
[16] Y.-Z. You, Z. Bi, D. Mao, and C. Xu, Phys. Rev. B **93**, 125101 (2016), 1510.04278.
[17] T. Yoshida and N. Kawakami, Phys. Rev. B **94**, 085149 (2016), 1604.00122.
[18] H.-Q. Wu, Y.-Y. He, Y.-Z. You, T. Yoshida, N. Kawakami, C. Xu, Z. Y. Meng, and Z.-Y. Lu, Phys. Rev. B **94**, 165121 (2016), 1606.05822.
[19] L. Tsui, Y.-T. Huang, H.-C. Jiang, and D.-H. Lee, Nuclear Physics B **919**, 470 (2017), 1701.00834.
[20] C. Xu and Y.-Z. You, Phys. Rev. B **92**, 220416 (2015), 1510.06032.
[21] D. F. Mross, J. Alicea, and O. I. Motrunich, Physical Review Letters **117**, 016802 (2016), 1510.08455.
[22] P.-S. Hsin and N. Seiberg, Journal of High Energy Physics **2016**, 95 (2016), ISSN 1029-8479.
[23] A. C. Potter, C. Wang, M. A. Metlitski, and A. Vishwanath, ArXiv e-prints (2016), 1609.08618.
[24] Y. BenTov, Journal of High Energy Physics **7**, 34 (2015), 1412.0154.
[25] S. Catterall, Journal of High Energy Physics **1**, 121 (2016), 1510.04153.
[26] V. Ayyar and S. Chandrasekharan, Phys. Rev. D **93**, 081701 (2016), 1511.09071.
[27] S. Catterall and D. Schaich, ArXiv e-prints (2016), 1609.08541.
[28] V. Ayyar and S. Chandrasekharan, Journal of High Energy Physics **10**, 58 (2016), 1606.06312.
[29] V. Ayyar, ArXiv e-prints (2016), 1611.00280.
[30] Y.-Y. He, H.-Q. Wu, Y.-Z. You, C. Xu, Z. Y. Meng, and Z.-Y. Lu, Phys. Rev. B **94**, 241111 (2016), 1603.08376.
[31] E. Huffman and S. Chandrasekharan, ArXiv e-prints (2017), 1709.03578.
[32] Y.-Z. You, Y.-C. He, C. Xu, and A. Vishwanath, ArXiv e-prints (2017), 1705.09313.
[33] T. Senthil, A. Vishwanath, L. Balents, S. Sachdev, and M. P. A. Fisher, Science **303**, 1490 (2004), cond-mat/0311326.
[34] O. I. Motrunich and A. Vishwanath, Phys. Rev. B **70**, 075104 (2004), cond-mat/0311222.
[35] T. Senthil, L. Balents, S. Sachdev, A. Vishwanath, and M. P. A. Fisher, Phys. Rev. B **70**, 144407 (2004), cond-mat/0312617.
[36] T. Senthil and M. P. A. Fisher, Phys. Rev. B **74**, 064405 (2006), cond-mat/0510459.
[37] Y.-Z. You and C. Xu, Phys. Rev. B **90**, 245120 (2014), 1409.0168.
[38] Z. Bi, R. Zhang, Y.-Z. You, A. Young, L. Balents, C.-X. Liu, and C. Xu, ArXiv e-prints (2016), 1602.03190.
[39] K. Kaveh and I. F. Herbut, Phys. Rev. B **71**, 184519 (2005), cond-mat/0411594.
[40] C. Xu and S. Sachdev, Physical Review Letters **100**, 137201 (2008), 0711.3460.
[41] C. Xu, Phys. Rev. B **78**, 054432 (2008), 0803.0794.
[42] E. Witten, Phys. Rev. B **94**, 195150 (2016), 1605.02391.
[43] X.-G. Wen, Phys. Rev. B **65**, 165113 (2002), cond-mat/0107071.

[44] S. Chatterjee, S. Sachdev, and M. Scheurer, ArXiv e-prints (2017), 1705.06289.
[45] M. Cheng and C. Xu, Phys. Rev. B **94**, 214415 (2016), 1609.02560.
[46] Y. Q. Qin, Y.-Y. He, Y.-Z. You, Z.-Y. Lu, A. Sen, A. W. Sandvik, C. Xu, and Z. Y. Meng, ArXiv e-prints (2017), 1705.10670.
[47] X.-F. Zhang, Y.-C. He, S. Eggert, R. Moessner, and F. Pollmann, ArXiv e-prints (2017), 1706.05414.
[48] Y. You and Y.-Z. You, Phys. Rev. B **93**, 195141 (2016).
[49] Z. Liu, K. Suenaga, P. J. F. Harris, and S. Iijima, Physical Review Letters **102**, 015501 (2009).
[50] Z. Zhou, D. Wang, Z. Y. Meng, Y. Wang, and C. Wu, Phys. Rev. B **93**, 245157 (2016), 1512.03994.
[51] I. Kimchi, S. A. Parameswaran, A. M. Turner, F. Wang, and A. Vishwanath, Proceedings of the National Academy of Science **110**, 16378 (2013), 1207.0498.
[52] S. Jiang and Y. Ran, Phys. Rev. B **92**, 104414 (2015), 1505.03171.
[53] P. Kim, H. Lee, S. Jiang, B. Ware, C.-M. Jian, M. Zaletel, J. H. Han, and Y. Ran, Phys. Rev. B **94**, 064432 (2016), 1509.04358.
[54] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, Phys. Rev. B **78**, 195125 (2008), 0803.2786.
[55] S. Ryu, A. P. Schnyder, A. Furusaki, and A. W. W. Ludwig, New Journal of Physics **12**, 065010 (2010), 0912.2157.
[56] A. W. W. Ludwig, Physica Scripta Volume T **168**, 014001 (2016), 1512.08882.
[57] L. Fidkowski, X. Chen, and A. Vishwanath, Physical Review X **3**, 041016 (2013), 1305.5851.
[58] C. Wang and T. Senthil, Phys. Rev. B **89**, 195124 (2014), 1401.1142.
[59] M. A. Metlitski, L. Fidkowski, X. Chen, and A. Vishwanath, ArXiv e-prints (2014), 1406.3032.
[60] Y. Gu and X.-L. Qi, ArXiv e-prints (2015), 1512.04919.
[61] Q.-R. Wang and Z.-C. Gu, ArXiv e-prints (2017), 1703.10937.
[62] M. Hermele, Phys. Rev. B **76**, 035125 (2007), cond-mat/0701134.
[63] D. J. Gross and A. Neveu, Phys. Rev. D **10**, 3235 (1974).
[64] K. G. Wilson and M. E. Fisher, Phys. Rev. Lett. **28**, 240 (1972).
[65] C.-M. Jian, A. Thomson, A. Rasmussen, Z. Bi, and C. Xu, ArXiv e-prints (2017), 1710.04668.
[66] We will work with Hermitian gauge connections throughout this work.
[67] In Ref. 65, the four flavors of Dirac fermions form a SO(4) vector instead of a spinor. Thus the analysis in this paper will be very different form Ref. 65.
[68] Strictly speaking, the flavor symmetry should be $SU(4)/\mathbb{Z}_4$, but we will suppress the level of exactness in the following.
[69] When performing the SO(7) transformation, one should keep in mind that the vacuum state $|0\rangle_c$ may be rotated as well.
[70] Otherwise, in the original (K, K') basis, the continuous symmetry transformations will also involves rotations in the valley subspace, see Appendix A for more details.
[71] Here the symmetry charges $\mu^a = \sigma^a$ are represented in the complex basis.