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Diffusive real-time dynamics of a particle with Berry curvatures

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We study theoretically the influence of Berry phase on the real-time dynamics of the single particle focusing on the diffusive dynamics, i.e., the time-dependence of the distribution function. Our model can be applied to the real-time dynamics of intraband relaxation and diffusion of optically excited excitons, trions or particle-hole pair. We found that the dynamics at the early stage is deeply influenced by the Berry curvatures in real-space (B), momentum-space (Ω), and also the crossed space between these two (C). For example, it is found that Ω induces the rotation of the wave packet and causes the time-dependence of the mean square displacement of the particle to be linear in time t at the initial stage; it is qualitatively different from the t^3 dependence in the absence of the Berry curvatures. It is also found that Ω and C modifies the characteristic time scale of the thermal equilibration of momentum distribution. Moreover, the dynamics under various combinations of B, Ω and C shows singular behaviors such as the critical slowing down or speeding up of the momentum equilibration and the reversals of the direction of rotations. The relevance of our model for time-resolved experiments in transition metal dichalcogenides is also discussed.

I. INTRODUCTION

The role of Berry phase[1] in wave mechanics has been attracting intensive attention. The effects from both the geometry characterized by the Berry curvature, which can be understood as a modification of commutation relations between phase space coordinates[2,3], and its global aspects captured by the topological indices are the focus of recent studies. The former includes the anomalous Hall effect[4], spin Hall effect[5,6], and magnon Hall effect[7], while the topological insulators and topological superconductors are the examples of the latter[8,9]. Berry phase has been discussed for the ground states and the linear responses near the thermal equilibrium[4–7], the general cyclic evolution of a quantum state[10], and the periodically driven systems[11–14].

On the other hand, the role of Berry phase in the real-time dynamics far from the equilibrium has been less studied. Especially the diffusion processes[15] are fundamental for propagation of particles, chemical reactions, and even biological phenomena[16]. Especially, the real-time dynamics becomes a tractable issue experimentally due to the technological developments, e.g., ultra-fast time-resolved spectroscopies in cold atom systems[17,18] and in solids[19,20]. Although there have been some proposals and experiments in cold atom systems[21–27] and photonic lattice systems[28] for measuring the Berry curvatures in momentum space, the diffusive dynamics has not been explored.

In this work, we study the role of Berry phase in diffusion processes[16,29]. We consider the Berry curvatures in real-space (B), momentum-space (Ω), and also the crossed space between these two (C). These three curvatures play distinct roles in the real-time dynamics of diffusion starting from the initial condition of fixed position and momentum. Therefore, the results offer yet another method to disentangle the Berry curvatures in terms of time-resolved experiments. Also it is found that the interference between them results in rich phenomena including the singular behaviors as shown below.

II. MODEL AND RESULTS

A. Semiclassical stochastic equation

The semiclassical equation for the wave packet localized both in position and momentum space is, if we include the friction and fluctuation caused by a heat bath[15,29,30] (see Appendix C for derivation),

$$\dot{r}_i = \frac{\partial \epsilon(\boldsymbol{r},\boldsymbol{p})}{\partial p_i} - \left((\hat{\Omega}_{pp})_{ij}\dot{p}_j + (\hat{\Omega}_{pr})_{ij}\dot{r}_j\right), \tag{1}$$

$$\dot{p}_i = -\frac{\partial \epsilon(\boldsymbol{r},\boldsymbol{p})}{\partial r_i} + \left((\hat{\Omega}_{rp})_{ij}\dot{p}_j + (\hat{\Omega}_{rr})_{ij}\dot{r}_j\right) - m\gamma\dot{r}_i + \sqrt{2m\gamma k_B T}\,\xi_i(t), \tag{2}$$

where m is the mass of the particle, γ is the friction constant, k_B is the Boltzmann constant, T is the temperature of the system, $i,j = 1,\ldots,d$ and d is the spatial dimension of the system. $(\hat{\Omega}_{XX})_{\alpha\beta}$ ($\boldsymbol{X} = (\boldsymbol{r},\boldsymbol{p})$, and $\alpha,\beta = 1,\ldots,2d$ are the coordinates of phase space.) is the Berry curvature, and $\epsilon(\boldsymbol{r},\boldsymbol{p})$ is the energy of the particle. $\xi_i(t)$ is the Gaussian fluctuation force and satisfies $\langle \xi_i(t)\xi_j(t')\rangle = \delta(t-t')\delta_{ij}$ and $\langle \xi_i(t)\rangle = 0$, where the bracket denotes the ensemble average.

From now on, we will assume that the spatial dimension $d = 2$, $\epsilon(\boldsymbol{p}) = \boldsymbol{p}^2/(2m)$, $\hat{\Omega}_{pp} = (\Omega/\hbar)i\sigma_y$,

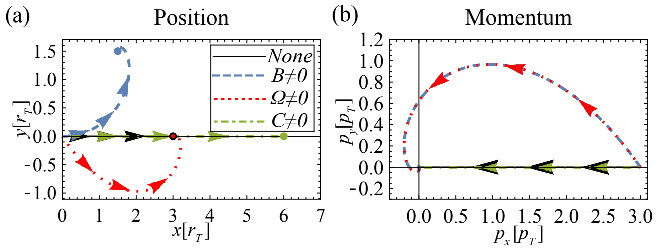

FIG. 1. The plots for the time evolutions of the averages of the position (a) and momentum (b), measured in units of $p_T = \sqrt{2mk_BT}$ and $r_T = p_T/(m\gamma)$. The initial condition is $P(\boldsymbol{X}, t = 0) = \prod_\alpha \delta(X_\alpha - X^0_\alpha)$ and $(r_{x0}, r_{y0}, p_{x0}, p_{y0}) = (0, 0, 3p_T, 0)$. The "None", "$B \neq 0$", "$\Omega \neq 0$" and "$C \neq 0$" lines are the behaviors at dimensionless parameters $(qB/(m\gamma), m\gamma\Omega/\hbar, C) = (0, 0, 0)$, $(-1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, -1)$, respectively, as is shown in the inset of (a). The final position of the particle is denoted by the dots in (a). Note that the endpoints of "None" and "$\Omega \neq 0$" line in (a), $B \neq 0$ and $\Omega \neq 0$ line in (b), and "None" and $C \neq 0$ line in (b), coincide. The momentum relaxes to 0 by friction for all the cases. In the case of $\Omega \neq 0$, the directions of $\langle p_i \rangle$ and $\frac{d}{dt}\langle r_i \rangle$ do not coincide; to see this in the figure, we note that, although the initial momentum is purely x direction, the initial $\dot{\vec{r}}$ contains y component, because of the finite anomalous velocity.

$\hat{\Omega}_{rr} = qBi\sigma_y$ and $\hat{\Omega}_{rp} = CI_2$, where q is the charge of the particle and I_2 and σ_y are 2×2 unit matrix and y component of Pauli matrices, respectively. We set $\hbar = 1$ henceforth. Here we assumed Ω, qB and C to be constant. We defer the discussion for the applicability of our model to real experiments to the end of the paper. Here, B and Ω are the real space magnetic field perpendicular to our two dimensional system and Berry curvature in momentum space, respectively. As for C, in the presence of elastic deformation field $u_i(\boldsymbol{r})$, $\hat{\Omega}_{rp}$ can be calculated as[31]

$$(\hat{\Omega}_{rp})_{ij} = \frac{\partial u_j}{\partial r_i}\left(1 - \frac{m_0}{m}\right) =: w_{ji}\left(1 - \frac{m_0}{m}\right), \quad (3)$$

where m_0 is an unrenormalized bare mass of the particle. Here we restrict our attention to the symmetric part of w_{ij} ($w^s_{ij} = \frac{1}{2}(w_{ij} + w_{ji})$). If we consider the case where the system is under the uniform, isotropic and weak pressure, according to Hooke's law[32], $w^s_{ij} \propto \delta_{ij}$. Moreover, for the system with $m \ll m_0$, small amount of deformation leads to large $C \sim 1$.

As we mentioned in the introduction, varying Berry curvatures amount to modifying the commutation relation. We will see, at particular parameter range, i.e., $C = 1$ and $qB\Omega = 1$, the dynamics becomes singular. It can be attributed to the singularity of the commutation relation of the dynamics. For example, C=1 indicates that the r and p commute each other and both can be determined simultaneously, i.e., the uncertain principle does not apply in this case.

B. Fokker-Planck equation and real-time dynamics of diffusion

From the Langevin equations (1) and (2), we can derive the Fokker-Planck equation, which describes the time evolution of the probability distribution function $P(\boldsymbol{X}, t)$ (Details of the derivation are in Refs. 15, 29, and 30) and Appendix B:

$$\frac{\partial P(\boldsymbol{X}, t)}{\partial t} = (\hat{G})_{\alpha\beta}\nabla_\alpha[(\nabla_\beta\epsilon)P] + \frac{k_BT}{2}(\hat{G} + \hat{G}^T)_{\alpha\beta}\nabla_\alpha\nabla_\beta P, \quad (4)$$

where the matrix $\hat{G}$ is the inverse of

$$\hat{G}^{-1} = \begin{pmatrix} m\gamma I_2 - qBi\sigma_y & (1 - C)I_2 \\ -(1 + C)I_2 & -\Omega i\sigma_y \end{pmatrix}. \quad (5)$$

Here we assumed that the matrix $\hat{G}^{-1}$ is regular:

$$\det \hat{G}^{-1} = \left[(1 - C)^2 - qB\Omega\right]^2 + (m\gamma\Omega)^2 \neq 0. \quad (6)$$

We will discuss what happens if G^{-1} is singular later.

Now we study the time-evolution of the distribution function $P(\boldsymbol{X}, t)$. Because of the assumption of quadratic dispersion of $\epsilon(\boldsymbol{p})$ and constant Berry curvatures, we can exactly solve Eq. (4) with the initial condition of fixed position and momentum[16]: $P(\boldsymbol{X}, t = 0) = \prod_\alpha \delta(X_\alpha - X^0_\alpha)$, where $\boldsymbol{X}_0 = (\boldsymbol{r}_0, \boldsymbol{p}_0)$ denotes the initial coordinate and momentum. Since the solution is the Gaussian distribution, it is enough to calculate the first and second moments for specifying the probability distribution.

The time evolution of the first moment is shown in Fig. 1 in the case of only one of B, Ω and C is nonzero. We define two time scales which characterize the dynamics, $1/\gamma_1$ and $1/\gamma_2$:

$$\gamma_1 = \frac{(1 - C)^2\gamma}{[(1 - C)^2 - qB\Omega]^2 + m^2\gamma^2\Omega^2}, \quad (7)$$

$$\gamma_2 = \frac{-qB(1 - C)^2 + (q^2B^2 + m^2\gamma^2)\Omega}{m\{[(1 - C)^2 - qB\Omega]^2 + m^2\gamma^2\Omega^2\}}, \quad (8)$$

where $1/\gamma_1$ is the relaxation time toward the final position and momentum, and γ_2 represents the frequency of the characteristic rotational motion. We can see the characteristic rotational motion when $\gamma_2 \neq 0$, i.e., $B \neq 0$ or $\Omega \neq 0$ in Fig. 1.

As for the second moment, we define the correlation function $\langle\langle X_\alpha(t)X_\beta(t)\rangle\rangle = \langle(X_\alpha(t) - \langle X_\alpha(t)\rangle)(X_\beta(t) - \langle X_\beta(t)\rangle)\rangle$. Then the long time behavior of $\langle\langle r_ir_j\rangle\rangle$ is, as $t \to \infty$,

$$\langle\langle r_i(t)r_j(t)\rangle\rangle = \left(\frac{2m\gamma k_BT}{q^2B^2 + m^2\gamma^2}t + \frac{mk_BT(1 - C)^2}{(q^2B^2 + m^2\gamma^2)^2}(q^2B^2 - 3m^2\gamma^2) + \mathcal{O}(e^{-\gamma_1 t})\right)\delta_{ij}. \quad (9)$$

On the other hand, the short time behavior of $\langle\langle r_i r_i \rangle\rangle$ (no summation) is, as $t \to 0$,

$$\langle\langle r_i(t) r_i(t) \rangle\rangle = R_1 t - R_2 t^2 + R_3 t^3 + \mathcal{O}(t^4), \qquad (10)$$

where

$$R_1 = \frac{2m\gamma\Omega^2 k_B T}{\det\hat{G}^{-1}}, \qquad (11)$$

$$R_2 = \frac{2(1-C)^2 m\gamma^2\Omega^2 k_B T}{(\det\hat{G}^{-1})^2}, \qquad (12)$$

$$R_3 = 2(1-C)^2\gamma k_B T \times \frac{[(1-C)^2 - qB\Omega]^3 + m^2\gamma^2\Omega^2[3(1-C)^2 - qB\Omega]}{3m(\det\hat{G}^{-1})^3}. \qquad (13)$$

The correlations of the momenta are,

$$\langle\langle p_i(t) p_j(t) \rangle\rangle = m k_B T \left(1 - e^{-2\gamma_1 t}\right)\delta_{ij}. \qquad (14)$$

This quantity eventually relaxes to mk_BT with the relaxation time $1/(2\gamma_1)$, since the probability distribution relaxes to the thermal equilibrium, see Appendix B.

Finally, the cross-correlations between the position and momentum are,

$$\langle\langle r_i(t) p_j(t) \rangle\rangle = \frac{m k_B T}{q^2B^2 + m^2\gamma^2}(f_1(t)\delta_{ij} - f_2(t)(i\hat{\sigma}_y)_{ij}), \qquad (15)$$

where $f_1(t) = m\gamma(1-C) + e^{-2\gamma_1 t} m\gamma(1-C)[1 - 2e^{\gamma_1 t}\cos(\gamma_2 t)]$ and $f_2(t) = qB(1-C) - e^{-2\gamma_1 t}[qB - 2m\gamma(1-C)e^{\gamma_1 t}\sin(\gamma_2 t)]$. The antisymmetric correlation of r_i and p_j, i.e., the second term in the right hand side of Eq. (15), represents the orbital angular momentum.

Among the Berry curvatures B, Ω and C, the long time behavior of the diffusive dynamics, i.e., $t >> 1/\gamma_1$, is characterized mainly by B and C: If $B \neq 0$, the rotational motion from the Lorentz force (Fig. 2(d)) leads to the slow diffusion, i.e., the small diffusion coefficient, at long time (Fig. 2(b))[33–35]; the value of $\langle\langle r_i r_i \rangle\rangle$ (no summation) is affected when $C \neq 0$, see Eq. (9) and Fig. 2(b). At long time, we do not see any effect of Ω, see Fig. 2. The reason is that, after the relaxation of momentum distribution ($t > 1/\gamma_1$), the force on the particle is balanced and $\dot{p}_i = 0$, so the anomalous velocity term vanishes at the equilibrium of the momentum distribution. However, the effect of Ω does appear in the short time dynamics at $t < 1/\gamma_1$.

C. Effect of each Berry curvature on the short time dynamics

Now we study the effect of individual Berry curvature B, Ω, and C on the short time dynamics by putting only one of them nonzero. The interference between them will be discussed later.

— *Real-space magnetic field B*

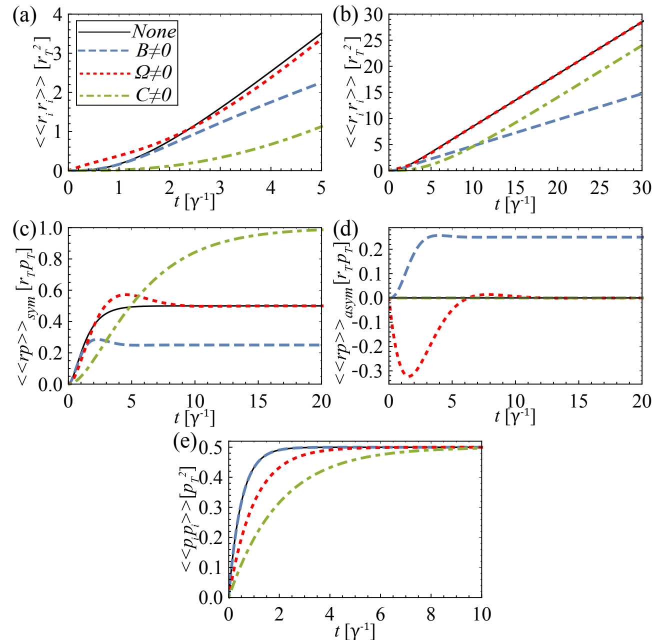

FIG. 2. The plots for the time evolutions of, (a,b) $\langle\langle r_i r_i \rangle\rangle$, (c) $\langle\langle rp \rangle\rangle_{sym} = \langle\langle r_i p_i \rangle\rangle$, (d) $\langle\langle rp \rangle\rangle_{asym} = \frac{1}{2}(\langle\langle r_x p_y \rangle\rangle - \langle\langle p_x r_y \rangle\rangle)$ and (e) $\langle\langle p_i p_i \rangle\rangle$ (no summation of repeated indices here), where $p_T = \sqrt{2mk_BT}$ and $r_T = p_T/(m\gamma)$. Note the differences of the time scale of each panel. (a) is the zoom up of (b). The initial condition is $P(\boldsymbol{X}, t=0) = \prod_\alpha \delta(X_\alpha - X_\alpha^0)$, so $\langle\langle X_\alpha X_\beta \rangle\rangle = 0$ for $\forall\alpha, \beta$ at $t = 0$. The "None", "$B \neq 0$", "$\Omega \neq 0$" and "$C \neq 0$" lines are the behaviors at dimensionless parameters $(qB/(m\gamma), m\gamma\Omega/\hbar, C) = (0,0,0)$, $(-1,0,0)$, $(0,1,0)$ and $(0,0,-1)$, respectively, as is shown in the inset of (a). (a) From Eq. (10), the short time behavior of $\langle\langle r_i r_i \rangle\rangle$ is $\mathcal{O}(t)$ for $\Omega \neq 0$ and $\mathcal{O}(t^3)$ for $\Omega = 0$. (b) We can see the difference of slope and value of $\langle\langle r_i r_i \rangle\rangle$ at long time, see Eq. (9). (c,d) The fact that $\frac{1}{2}(\langle\langle r_x p_y \rangle\rangle - \langle\langle p_x r_y \rangle\rangle)$ does not vanish indicates the finite angular momentum at long time for $B \neq 0$. (e) The characteristic relaxation time of $\langle\langle p_i p_i \rangle\rangle$ is different in three cases, see Eq. (14). Note that the "None" and "$B \neq 0$" line coincide in (e). These results do not depend on the initial values of X_α, i.e., X_α^0.

The rotational motion caused by the Lorentz force affects the diffusive dynamics. By the rotational motion (Fig. 2(d)), the diffusion is suppressed, although R_3 in Eq. (10) is not affected by B (Fig. 2(a)). The relaxation of the momentum distribution is not affected by B (Fig. 2(e)), from Eqs. (7) and (14).

— *Momentum space Berry curvature* Ω

The anomalous velocity term, combined with the friction and fluctuation terms in Eq. (2), result in the modification of the diffusive dynamics at short time. Namely, the spread in real space $\langle\langle r_i r_i \rangle\rangle$ becomes fast; it is linear in t in stark contrast to the usual t^3 behavior without Ω, see Eqs. (10), the definitions of R_1, R_2, R_3 and Fig. 2(a). We note that our model is not a Smoluchowski equation, which describes the long time scale dynamics and gives t linear behavior in the absence of Berry curvatures. The coefficient is $R_1 = (2k_BTx^2)/[m\gamma(1+x^2)]$, where

$x = m\gamma\Omega$. The finite angular momentum at short time can be seen in Fig. 2(d); as we noted above, this behavior is independent of the initial momentum p_0, and can be understood as the internal rotational motion of the wave packet of the probability distribution in real space. From Eq. (7), the characteristic relaxation time $1/\gamma$ is modified as $1/\gamma_1 = (1+x^2)/\gamma$, and the relaxation of the momentum distribution toward the equilibrium becomes slower, see Fig. 2(e).

— *Berry curvature in crossed space C*

The dynamics does not contain the rotational motion, since $\hat{\Omega}_{rp}$ is the diagonal matrix and the system is symmetric in the left-handed and right-handed direction. The effect of C appears in the modification of the relaxation time $1/\gamma_1 = (1-C)^2/\gamma$ from Eq. (7) (Fig. 2(c,f)) and the diffusion at short time (Fig. 2(a)) and at long time (Fig. 2(b)). In particular, for $0 < C < 1$ ($C < 0$), $1/\gamma_1$ is reduced (enhanced) and the relaxation become faster (slower). When $C = 1$, from Eq. (6) the matrix $\hat{G}^{-1}$ is singular and γ_1 diverges. We will discuss this singular case below.

D. Interference between Berry curvatures

Now we consider the effects due to the coexistence of different Berry curvatures. In particular, it often happens that both qB (C) and Ω are finite[2], e.g., when the external magnetic field (the elastic deformation) is applied to the system with the band structure of finite Ω, so we discuss these cases.

— *The interference between Ω and B*

Because of the term $1 - qB\Omega$ in the denominator, the presence of both B and Ω leads to the enhancement of $\gamma_1 = \gamma/[(1-qB\Omega)^2 + (m\gamma\Omega)^2]$, which is the reciprocal of the characteristic time scales of the relaxation. This is in sharp contrast to the case where only Ω is finite and the effect is only the reduction of γ_1. In particular, if we regard γ_1 and γ_2 as functions of qB, γ_1 obeys the Lorentzian distribution with a peak of height $1/(m^2\gamma\Omega^2)$ at $qB = 1/\Omega$ with a half width at half maximum $m\gamma$. When $qB\Omega = 1$ and $\gamma = 0$, it is known that the degrees of freedom of the system is reduced, and we get the constrained system[36,37]. Here, γ and $\xi_i(t)$ remove the singularity of the $\det \hat{G}^{-1}$ in Eq. (6), as was pointed out in Ref. 15. However, the anomalous behavior appears in the diffusive dynamics: The minimum of γ_2 at $B = 1/(q\Omega) - m\gamma/(qB)$ ($\Omega > 0$) dips below zero for $\gamma < 1/(2m\Omega)$, and the characteristic rotational motion for short time changes the sign of the angular momentum twice as we sweep B from $-\infty$ to $+\infty$. Since the ratio of peak values of γ_1 and γ_2 is $|\gamma_{2,\mathrm{peak}}/\gamma_{1,\mathrm{peak}}| = |m\gamma\Omega - 1/2|$, if $m\gamma\Omega \ll 1$, it is possible to detect the rotational motion before the average of the momentum and position relaxes to the equilibrium.

— *The interference between Ω and C*

In the presence of both Ω and C, $\gamma_1 = [(1-C)^2\gamma]/[(1-C)^2 + (m\gamma\Omega)^2]$ and $\gamma_2 = [\gamma(m\gamma\Omega)]/[(1-C)^2 + (m\gamma\Omega)^2]$. From these two quantities, we can see the resonant behavior as we vary $1-C$, and this behavior crucially depends on whether $\Omega = 0$ or not, as shown below.

When $C = 1$ and $\Omega = 0$, the dynamics of p_i and r_i completely decouples, and we get the constraint $p_i = 0$. In this case, the system is governed by the dynamics of r_i only, and we get the Langevin equation for the Brownian particle. In fact, as $C \to 1$, $\gamma_1 = \gamma/(1-C)^2 \to \infty$ and the system becomes overdamped for all the time scale.

When $C = 1$ and $\Omega \neq 0$, the singularity of $\hat{G}^{-1}$ is removed, see Eq. (6). However, the dynamics of p_i and r_i is still decoupled. As $C \to 1$, we get $\gamma_1 = [(1-C)^2\gamma]/[(1-C)^4 + (m\gamma\Omega)^2] \to 0$, and the system becomes underdamped for all the time scale, and the effect of the friction and fluctuation on p_i vanishes. In this case, we get the singular rotational motion: The solution of Eq. (2) is $(p_x, p_y) = p_0(\cos[t/(m\Omega)+\phi], \sin[t/(m\Omega)+\phi])$ ($(p_{x0}, p_{y0}) = p_0(\cos\phi, \sin\phi)$), so the dynamics of p_i is purely rotational motion with the frequency $1/(m\Omega)$ ($= \gamma_2$), which is singular at $\Omega = 0$. The dynamics of r_i is the same as $\Omega = 0$ case discussed above. Here we see modification of the commutation relation by the Berry curvatures decouples the dynamics of p_i and r_i.

III. DISCUSSION

The results given above offer enough information to determine Berry curvatures from the measurements of real-time diffusive dynamics. The relaxation of the momentum distribution is affected in the presence of "magnetic field" in momentum space just like the diffusion coefficient is modified in the presence of magnetic field in real space, and the behavior we saw is expected to occur universally also in more complex models.

As for the coexistence of both Ω and B, a promising candidate is the surface state of magnetic topological insulator[7,8]. Due to the exchange gap induced at the surface state leads to the Berry curvature Ω and quantized anomalous Hall effect[38]. Recently, it is found that the skyrmions are produced during the magnetization process of this system[39], which produces the real-space Berry curvature B due to the scalar spin chirality[40]. In this situation, by tuning the exchange gap and the size of the skyrmion, the product $qB\Omega$ can be of the order of unity. Note that the real-space Berry curvature produced by the Skyrmion crystal are modulated spatially, but its effect on the electrons with small wavenumber is identical to that of the uniform B[41].

Even more direct relevance to our model is the dynamics of optically excited excitons and trions at K and K' point in transition metal dichalcogenides[42]. In this material, when the circularly polarized light is injected, one can selectively create the bound exciton at only K or K' point depending on the polarization. The exchange coupling leads to strong mixing between K and K' excitons, and the Hamiltonian for the center of mass momentum of excitons $\vec{k} = k(\cos\phi, \sin\phi)$ is $H_{\mathrm{D}} =$

$vk(\cos(2\phi)\sigma_x + \sin(2\phi)\sigma_y)$, where σ_i is K and K' valley pseudo-spin and $v \sim 0.79\,\mathrm{eV\AA}$ represents the mixing from the exchange coupling[43]. If we apply magnetic field B, by valley Zeeman effect[44–50], the gap $H_{\mathrm{gap}} = \Delta\sigma_z$, where $\Delta \sim 2.3\,\mathrm{meV}$ with $B \sim 10\,\mathrm{T}$, is induced between K and K' excitons. And if the temperature is low enough to satisfy $k_T := \sqrt{2k_BT\Delta}/v \leq \Delta/v$, i.e., $T \leq 13\,\mathrm{K}$, Berry curvature can be regarded as constant $\Omega \sim 1.2 \times 10^5\,\mathrm{\AA}^2$ and at the same time the dispersion of the upper band can be approximated as quadratic. Also, the authors of Ref. 43 suggested that binding another doped electron at K or K' point to form a trion leads to a Dirac type dispersion with a mass term, coming from exchange coupling between exciton and electron, $H_{\mathrm{gap}} = \Delta\sigma_z s_z$, where σ_i and s_i represent valley degrees of freedom of constituting exciton and electron, respectively. The estimated value is $\Delta \sim 3\,\mathrm{meV}$, so if $T \leq 17\,\mathrm{K}$, our model with $\Omega \sim 6.9 \times 10^4\,\mathrm{\AA}^2$ is applicable for the same reason as above. Moreover, since trion is a charged particle, by applying magnetic field $B = (\hbar/q)/\Omega \sim -970\,\mathrm{mT}$, we expect the singular behavior of γ_1 and γ_2 as we discussed above. Here, B is so small that we can neglect the effect of Zeeman energy. In both cases, for laser spot of $0.5\,\mu\mathrm{m}$, the uncertainty in momentum space is $\Delta k \sim 2 \times 10^{-4}\,\mathrm{\AA}^{-1}$, and well within Δ/v. The time- and space-resolved spectra of light emission can detect the diffusive dynamics of these particles. The time-scale of the relaxation $\gamma_{1,2}^{-1}$ is typically pico second for electronic systems, which is now within the range of experimental access.

Besides above two, another candidate is the cold atom systems. Recently, the topological band structure, i.e., Haldane model, is realized in optical lattice[51]. It is also realized that the local defect is introduced as the initial condition and trace the time-evolution of the system after it[52,53]. In the case of cold atoms in optical lattice, the random force and dissipation is rather weak, and one needs to design the coupling of the atoms to the heat bath such as the electromagnetic field. However, the time scale in this case is much longer, i.e., typically $\sim$ 10msec[53], and the observation of the dynamics of a single particle is expected to be easier than the electronic systems.

Finally, we point out the difference between our work and the work in the previous literature[28,54]. In Ref. 28, the method of measuring the momentum space Berry curvature in the lossy photonic lattice systems was discussed. Although the idea of measuring the Berry curvature through the optical excitation and the resultant real-space distribution has some resemblance to our proposal, there are important differences: They discussed the effect of momentum space Berry curvature on a steady state property (especially $\langle x \rangle$) of a lossy system with a continuous pumping at zero temperature, while we discussed the effect of phase space Berry curvatures on the diffusive transient dynamics (including the first and second moment in phase space) after the irradiation of light at finite temperature. In Ref. 54, the diffusive dynamics of an electron in a Landau level was discussed. Although their treatment is fully quantum mechanical and ours is semiclassical, our model is more general when restricted to the semiclassical regime: Since projecting onto a Landau level corresponds to neglecting the kinetic term, their model in the semiclassical, high temperature regime corresponds to the special case of our model with $qB \neq 0$ and $m \to 0$ with $m\gamma$ fixed in Eq. (2).

In summary, we find Berry curvatures modify the relaxation time of the probability distribution in momentum space and the diffusion coefficient. In particular, the short time behavior contains useful information and hence the time-resolved experiments will provide useful information on Berry curvatures.

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Appendix A: Semiclassical equation in the presence of Berry curvatures

The physical meaning of each term in the semiclassical equation, Eqs. (1) and (2) is the followings. $\epsilon(\boldsymbol{r}, \boldsymbol{p})$ is the energy of the particle and reflects the potential energy and the dispersion relation of the band. To understand the origin of the terms containing the Berry curvature in the equation, it is important to note that Berry connection is defined as the inner product of the adjacent wave functions in some parameter space; here, the parameter space is a phase space spanned by the position and momentum of a particle. Since the particle is represented by the wave packet composed of the neighboring wave functions, the dynamics of the particle is affected by Berry connections. The Berry connections appear in the Lagrangian of the system, derived by the time dependent variational principle[55]. Except the last two terms in the right hand side of Eq. (2), Eqs. (1) and (2) are derived from the effective Lagrangian of the system,

$$L = p_i\dot{r}^i + A_i(\boldsymbol{r}, \boldsymbol{p})\dot{r}^i + a_i(\boldsymbol{r}, \boldsymbol{p})\dot{p}^i - \epsilon(\boldsymbol{r}, \boldsymbol{p}), \qquad \text{(A1)}$$

where $A_i(\boldsymbol{r}, \boldsymbol{p})$ and $a_i(\boldsymbol{r}, \boldsymbol{p})$ are Berry connections of the wave function in real space and momentum space, respectively. To see the role of each term in the Lagrangian, we rewrite Eqs. (1) and (2) as,

$$\hat{G}^{-1}\begin{pmatrix}\dot{\boldsymbol{r}}\\ \dot{\boldsymbol{p}}\end{pmatrix}=-\begin{pmatrix}\nabla_{\boldsymbol{r}}\\ \nabla_{\boldsymbol{p}}\end{pmatrix}\epsilon(\boldsymbol{r},\boldsymbol{p})+\sqrt{2m\gamma k_BT}\begin{pmatrix}\boldsymbol{\xi}(t)\\ 0\end{pmatrix}, \tag{A2}$$

where

$$\hat{G}^{-1}=\begin{pmatrix}m\gamma\hat{I}_d & 0\\ 0 & 0\end{pmatrix}+\left[\begin{pmatrix}0 & \hat{I}_d\\ -\hat{I}_d & 0\end{pmatrix}-\begin{pmatrix}\hat{\Omega}_{rr} & \hat{\Omega}_{rp}\\ \hat{\Omega}_{pr} & \hat{\Omega}_{pp}\end{pmatrix}\right]; \tag{A3}$$

$\hat{I}_d$ is a $d\times d$ unit matrix; the $d\times d$ matrices $\hat{\Omega}_{rr},\hat{\Omega}_{rp},\hat{\Omega}_{pr}$ and $\hat{\Omega}_{pp}$ represent the Berry curvatures and are defined as the field strengths in phase space:

$$(\hat{\Omega}_{rr})_{ij}=\partial_{r_i}A_j-\partial_{r_j}A_i,\ (\hat{\Omega}_{rp})_{ij}=\partial_{r_i}a_j-\partial_{p_j}A_i, \tag{A4}$$

$$(\hat{\Omega}_{pr})_{ij}=\partial_{p_i}A_j-\partial_{r_j}a_i,\ (\hat{\Omega}_{pp})_{ij}=\partial_{p_i}a_j-\partial_{p_j}a_i. \tag{A5}$$

From the term in the square bracket in Eq. (A3), we can see that the first three terms in Eq. (A1) represent the symplectic structure of the system. In particular, the first term in the parenthesis on the right hand side of Eq. (1) is known as a source of Hall effect, and is called the anomalous velocity term[4].

Appendix B: Langevin equation in the presence of Berry curvatures

The Langevin equation of the particle with the energy $\epsilon(\boldsymbol{r},\boldsymbol{p})$ in the presence of Berry curvatures is[15,29,30],

$$(\hat{G}^{-1})_{\alpha\beta}\dot{X}_\beta=-\nabla_\alpha\epsilon(\boldsymbol{X})+N_{\alpha\beta}\xi_\beta(t)\Leftrightarrow\dot{X}_\alpha=-G_{\alpha\beta}\nabla_\beta\epsilon(\boldsymbol{X})+(\hat{G}\hat{N})_{\alpha\beta}\xi_\beta(t), \tag{B1}$$

where $\boldsymbol{X}=(\boldsymbol{r},\boldsymbol{p})$ and

$$\hat{G}^{-1}=\hat{Q}+\left[\begin{pmatrix}0 & \hat{I}_d\\ -\hat{I}_d & 0\end{pmatrix}-\begin{pmatrix}\hat{\Omega}_{rr} & \hat{\Omega}_{rp}\\ \hat{\Omega}_{pr} & \hat{\Omega}_{pp}\end{pmatrix}\right]. \tag{B2}$$

Here, $\hat{Q}$ is some $2d\times 2d$ symmetric matrix which represents the effect of friction and $\xi_\alpha(t)$ is the Gaussian fluctuation force:

$$\langle\xi_\alpha(t)\xi_\beta(t')\rangle=\delta(t-t')\delta_{\alpha\beta},\quad \langle\xi_\alpha(t)\rangle=0. \tag{B3}$$

The subscript $\alpha,\beta=1,\ldots,2d$ represent the coordinates of phase space, $\hat{I}_d$ is a $d\times d$ unit matrix. This stochastic differential equation does not necessarily describe the dynamics of a particle coupled with a thermal bath; we need to impose the condition which ensures the relaxation of the system toward the equilibrium (Eq. (B21)). This condition can be derived from the Fokker-Planck equation, which is equivalent to the Langevin equation equipped with the interpretation of the noise term. From now on, we assume that $\hat{G}\hat{N}$ does not depend on $\boldsymbol{X}$ to avoid the subtlety of the interpretation of the noise term. In general, given some stochastic differential equation,

$$\dot{x}_i(t)=g_i(\boldsymbol{x}(t))+h_{ij}\xi_j(t), \tag{B4}$$

we can derive the time evolution of the probability distribution $P(\boldsymbol{r},t)=\langle\prod_i\delta(r_i-x_i(t))\rangle$. First,

$$x_i(t+\epsilon)=x_i(t)+\int_t^{t+\epsilon}dt_1\,g_i(\boldsymbol{x}(t_1))+h_{ij}\int_t^{t+\epsilon}dt_1\,\xi_j(t_1). \tag{B5}$$

To evaluate this up to $\mathcal{O}(\epsilon)$, we note that for some arbitrary function $L(\boldsymbol{x}(t_1))$,

$$\begin{aligned}L(\boldsymbol{x}(t_1))&=L(\boldsymbol{x}(t)+\boldsymbol{x}(t_1)-\boldsymbol{x}(t))\\ &=L(\boldsymbol{x}(t))+(x_k(t_1)-x_k(t))\nabla_kL(\boldsymbol{x}(t))+\ldots\\ &=L(\boldsymbol{x}(t))+\int_t^{t_1}dt_2\,\dot{x}_k(t_2)\nabla_kL(\boldsymbol{x}(t))+\ldots\\ &=L(\boldsymbol{x}(t))+\int_t^{t_1}dt_2\,\left[g_k(\boldsymbol{x}(t_2))+h_{kl}\xi_l(t_2)\right]\nabla_kL(\boldsymbol{x}(t))+\ldots.\end{aligned} \tag{B6}$$

To evaluate the order of the second term, we note that

$$\begin{aligned}&\langle\int_t^{t+\epsilon}dt_1\,\xi_i(t_1)\int_t^{t+\epsilon}dt_1\,\xi_j(t_1)\rangle\\ &=\int_t^{t+\epsilon}dt_1\int_t^{t+\epsilon}dt_2\langle\xi_i(t_1)\xi_j(t_2)\rangle=\delta_{ij}\epsilon.\end{aligned} \tag{B7}$$

So,

$$\int_t^{t+\epsilon}dt_1\,\xi_i(t_1)=\mathcal{O}(\epsilon^{\frac{1}{2}}). \tag{B8}$$

Then the right hand side of Eq. (B5) can be evaluated as

$$x_i(t) + \epsilon g_i(\boldsymbol{x}(t)) + \mathcal{O}(\epsilon^{\frac{3}{2}}) + h_{ij}\int_t^{t+\epsilon} dt_1\, \xi_j(t_1). \qquad \text{(B9)}$$

From Eq. (B9), we can calculate the first and second moments,

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\langle x_i(t+\epsilon) - x_i(t)\rangle = g_i(\boldsymbol{x}(t)) =: a_i(\boldsymbol{x}(t)), \qquad \text{(B10)}$$

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\langle (x_i(t+\epsilon) - x_i(t))(x_j(t+\epsilon) - x_j(t))\rangle = h_{ik}h_{jl}\delta_{kl} =: a_{ij}(\boldsymbol{x}(t)). \qquad \text{(B11)}$$

And higher order moments are $\mathcal{O}(\epsilon^{\frac{3}{2}})$. From these moments, with the Chapman-Kolmogorov equation for this Markov process and its Kramers-Moyal expansion[56],

$$\begin{aligned}
P(\boldsymbol{x}, t+\epsilon) &= \int d\boldsymbol{x}' P(\boldsymbol{x}, t+\epsilon|\boldsymbol{x}', t)P(\boldsymbol{x}', t) && \text{(B12)}\\
&= \int d\boldsymbol{x}' P((\boldsymbol{x}-\boldsymbol{x}') + \boldsymbol{x}', t+\epsilon|\boldsymbol{x}-\boldsymbol{x}', t)P(\boldsymbol{x}-\boldsymbol{x}', t)\\
&=: \int d\boldsymbol{x}' F(\boldsymbol{x}-\boldsymbol{x}', \boldsymbol{x}'; t+\epsilon, t)\\
&= \sum_{i_1,\ldots,i_D}\frac{(-1)^{i_1+\cdots+i_D}}{i_1!\ldots i_D!}\left(\frac{\partial}{\partial x_1}\right)^{i_1}\cdots\left(\frac{\partial}{\partial x_D}\right)^{i_D}\int d\boldsymbol{x}' x_1'^{i_1}\ldots x_D'^{i_D} F(\boldsymbol{x}, \boldsymbol{x}'; t+\epsilon, t) && \text{(B13)}\\
&= P(\boldsymbol{x}, t) - \left(\frac{\partial}{\partial x_i}\right)(\epsilon a_i(\boldsymbol{x})P(\boldsymbol{x}, t)) + \frac{1}{2}\left(\frac{\partial^2}{\partial x_i\partial x_j}\right)(\epsilon a_{ij}(\boldsymbol{x})P(\boldsymbol{x}, t)) + \mathcal{O}(\epsilon^{\frac{3}{2}}), && \text{(B14)}
\end{aligned}$$

where D is the dimension of the system. So, if we take $\epsilon \to 0$, we obtain the Fokker-Planck equation:

$$\frac{\partial P(\boldsymbol{x}, t)}{\partial t} = -\left(\frac{\partial}{\partial x_i}\right)(a_i(\boldsymbol{x})P) + \frac{1}{2}\left(\frac{\partial^2}{\partial x_i x_j}\right)(a_{ij}(\boldsymbol{x})P). \qquad \text{(B15)}$$

If we calculate the moments from Eq. (B1), we get

$$a_\alpha(\boldsymbol{X}) = -G_{\alpha\beta}\nabla_\beta\epsilon, \qquad \text{(B16)}$$

$$a_{\alpha\beta}(\boldsymbol{X}) = (\hat{G}\hat{N})_{\alpha\gamma}(\hat{G}\hat{N})_{\beta\delta}\delta_{\gamma\delta} = (\hat{G}\hat{N}\hat{N}^T\hat{G}^T)_{\alpha\beta}. \qquad \text{(B17)}$$

The system will eventually relax to the thermal equilibrium if the fluctuations and frictions are caused by a heat bath. From this physical assumption, we impose the condition that, the equilibrium distribution,

$$P_{\rm eq} = \exp\left(-\frac{\epsilon(\boldsymbol{X})}{k_B T}\right), \qquad \text{(B18)}$$

where k_B is the Boltzmann constant and T is the temperature, is the stationary solution of Eq. (B15). We note that ϵ in Eq. (B18) is the same as the one in Eq. (B1), since the effect of Berry curvatures are the modification of the symplectic structure of the system and the energy of the system is not modified. As we assumed that Berry curvature terms are constant in phase space, the modification of the density of states[2,57] is constant and can be ignored. From this condition,

$$0 = \left(\frac{\partial}{\partial X_\alpha}\right)(G_{\alpha\beta}\nabla_\beta\epsilon P_{\rm eq}) + \frac{1}{2}\left(\frac{\partial^2}{\partial X_\alpha X_\beta}\right)((\hat{G}\hat{N}\hat{N}^T\hat{G})_{\alpha\beta}P_{\rm eq}) \qquad \text{(B19)}$$

$$\begin{aligned}
\Leftrightarrow 0 = (\nabla_\alpha\nabla_\beta\epsilon)&\left(G_{\alpha\beta} - \frac{1}{2k_BT}(\hat{G}\hat{N}\hat{N}^T\hat{G}^T)_{\alpha\beta}\right)\\
&+ (\nabla_\alpha\epsilon)(\nabla_\beta\epsilon)\left(-\frac{1}{k_BT}G_{\alpha\beta} + \frac{1}{2(k_BT)^2}(\hat{G}\hat{N}\hat{N}^T\hat{G}^T)_{\alpha\beta}\right). && \text{(B20)}
\end{aligned}$$

As a result, we obtain the condition

$$\frac{1}{2}\left(\hat{G}^{-1} + (\hat{G}^{-1})^T\right) = \frac{1}{2k_BT}\hat{N}\hat{N}^T. \qquad \text{(B21)}$$

This condition relates friction terms to fluctuation terms, and is called the fluctuation-dissipation relationship.

Up to now, as far as the condition Eq. (B21) is satisfied, we can choose arbitrary form for $\hat{Q}$ and $\hat{N}$. Here we consider the microscopic derivation of the Langevin equation (B1) by coupling the system with a bath to decide the form of $\hat{Q}$ and $\hat{N}$ in that situation.

Appendix C: Derivation of Eq. (B1) from Feynman and Vernon's influential functional

To derive the form of friction and fluctuation terms in Eq. (B1), we consider the Caldeira-Leggett model[58,59] in the presence of Berry curvatures. The argument here closely follows the one in Ref. 59. We set $\hbar = 1$ and $k_B = 1$ in this section. The action of the system is,

$$S_{\rm sys} = \int_C d\tau \left(p_i \dot{r}_i + A_i(\boldsymbol{r},\boldsymbol{p})\dot{r}_i + a_i(\boldsymbol{r},\boldsymbol{p})\dot{p}_i - \frac{p_i^2}{2m} \right), \tag{C1}$$

where C is the closed time contour and $C = C_+ \cup C_- = \{t_i + i0, t_f + i0\} \cup \{t_f - i0, t_i - i0\}$. Here we consider two dimensional system and the form of Berry curvatures are

$$\hat{\Omega}_{pp} = \begin{pmatrix} 0 & \Omega \\ -\Omega & 0 \end{pmatrix}, \quad \hat{\Omega}_{rr} = \begin{pmatrix} 0 & qB \\ -qB & 0 \end{pmatrix},$$
$$\hat{\Omega}_{rp} = \begin{pmatrix} C & A \\ -A & C \end{pmatrix}, \quad \hat{\Omega}_{pr} = \begin{pmatrix} -C & A \\ -A & -C \end{pmatrix}. \tag{C2}$$

Then,

$$a_i = \frac{\Omega}{2}\epsilon_{ji} p_j + A\epsilon_{ji} r_j, \quad A_i = \frac{qB}{2}\epsilon_{ji} r_j - C p_i, \tag{C3}$$

where ϵ_{ji} is the antisymmetric tensor. We define $r_i(t + i0) =: r_i^+(t)$, $r_i(t - i0) =: r_i^-(t)$, and $r_i^{\rm cl(q)}(t) =: \frac{1}{2}(r_i^+(t) \pm r_i^-(t))$. We use the same definition also for all the fields in the Keldysh space. Then we get

$$S_{\rm sys} = 2\int_{t_i}^{t_f} d\tau \Big(p_i^{\rm q}\dot{r}_i^{\rm cl} + p_i^{\rm cl}\dot{r}_i^{\rm q} + qB\epsilon_{ji} r_j^{\rm q}\dot{r}_i^{\rm cl} + \Omega\epsilon_{ji} p_j^{\rm q}\dot{p}_i^{\rm cl}$$
$$+ A\epsilon_{ji} r_j^{\rm q}\dot{p}_i^{\rm cl} + A\epsilon_{ji} r_j^{\rm cl}\dot{p}_i^{\rm q} - C p_i^{\rm q}\dot{r}_i^{\rm cl} - C p_i^{\rm cl}\dot{r}_i^{\rm q} - \frac{p_i^{\rm cl} p_i^{\rm q}}{m} \Big). \tag{C4}$$

Here, we couple the system with a bath which is a collection of oscillators labeled by s[59]:

$$S_{\rm bath} = \frac{1}{2}\sum_{s,i} \int_{-\infty}^{+\infty} dt\, \vec{\phi}_{s,i}^T(t) \hat{D}_s^{-1}(t) \vec{\phi}_{s,i}(t), \tag{C5}$$

$$S_{\rm int} = \sum_{s,i} g_s \int_{-\infty}^{+\infty} dt\, \left(r_i^+ \phi_{s,i}^+ - r_i^- \phi_{s,i}^- \right) = \sum_{s,i} 2g_s \int_{-\infty}^{+\infty} dt\, \vec{r}_i^T(t) \hat{\sigma}_x \vec{\phi}_{s,i}(t), \tag{C6}$$

where $\hat{\sigma}_x$ is the x component of the Pauli matrix in Keldysh space; the vector represents

$$\vec{r}_i^T = (r_i^{\rm cl}, r_i^{\rm q}), \quad \vec{\phi}_{s,i}^T = (\phi_{s,i}^{\rm cl}, \phi_{s,i}^{\rm q}); \tag{C7}$$

$\hat{D}_s^{-1}$ is a 2×2 matrix in Keldysh space:

$$\hat{D}_s^{-1}(t) = \begin{pmatrix} 0 & [D_s^{-1}]^{\rm A}(t) \\ [D_s^{-1}]^{\rm R}(t) & [D_s^{-1}]^{\rm K}(t) \end{pmatrix},$$
$$\hat{D}_s(t) = \begin{pmatrix} D_s^{\rm K}(t) & D_s^{\rm R}(t) \\ D_s^{\rm A}(t) & 0 \end{pmatrix}. \tag{C8}$$

And from the dispersion relationship of the harmonic oscillator and the fluctuation-dissipation relationship for the heat bath, in the Fourier transformed basis,

$$D_s^{\rm R(A)}(\epsilon) = \frac{1}{2}\frac{1}{(\epsilon \pm i0)^2 - \omega_s^2},$$
$$D_s^{\rm K}(\epsilon) = \coth\frac{\epsilon}{2T}\left[D_s^{\rm R}(\epsilon) - D_s^{\rm A}(\epsilon)\right]$$
$$\cong \frac{2T}{\epsilon}\left[D_s^{\rm R}(\epsilon) - D_s^{\rm A}(\epsilon)\right], \tag{C9}$$

where in the last equation, we assume the temperature is high compared to the characteristic frequency of the oscillator (semiclassical approximation); ω_s is the frequency of the oscillator s.

If we trace them out, there remains the terms which represent the interaction between forward and backward contours of the system. These terms are called the influence functional[60]. Since the argument is exactly the same as the model in the absence of Berry curvatures[59], we just show the results. If we assume the Ohmic bath:

$$J(\omega) := \pi \sum_s \frac{g_s^2}{\omega_s}\delta(\omega-\omega_s) = 2m\gamma\omega, \tag{C10}$$

the contribution of the bath to the effective action for the system coordinate is,

$$\begin{aligned} S_{\mathrm{int}} &= \frac{1}{2}\int\int_{-\infty}^{+\infty} dt\,dt' \sum_i \vec{r}_i^{\,T}(t)\left[-\sum_s (2g_s)^2\hat{\sigma}_x\hat{D}_s(t-t')\hat{\sigma}_x\right]\vec{r}_i(t') \\ &=: \frac{1}{2}\int\int_{-\infty}^{+\infty} dt\,dt' \sum_i \vec{r}_i^{\,T}(t)\hat{\mathfrak{D}}^{-1}(t-t')\vec{r}_i(t'). \end{aligned} \tag{C11}$$

Since

$$\begin{aligned} [\mathfrak{D}^{-1}(\epsilon)]^{\mathrm{R(A)}} &= -\frac{1}{2}\sum_s \frac{4g_s^2}{(\epsilon\pm i0)-\omega_s^2} \\ &= \int_0^{+\infty}\frac{d\omega}{2\pi}\frac{4\omega J(\omega)}{\omega^2-(\epsilon\pm i0)^2} = R\pm 2im\gamma\epsilon, \\ [\mathfrak{D}^{-1}(\epsilon)]^{\mathrm{K}} &\cong \left([\mathfrak{D}^{-1}(\epsilon)]^{\mathrm{R}}-[\mathfrak{D}^{-1}(\epsilon)]^{\mathrm{A}}\right)\frac{2T}{\epsilon} = 8im\gamma T, \end{aligned} \tag{C12}$$

where the constant real part of $[\mathfrak{D}^{-1}(\epsilon)]^{\mathrm{R(A)}}$, R renormalizes the potential of the particle, and we will ignore this term. Then, after Fourier transforming back to the time representation,

$$S_{\mathrm{int}} = -2m\gamma\int dt r_i^{\mathrm{q}}\dot{r}_i^{\mathrm{cl}} + 4im\gamma T\int dt (r_i^{\mathrm{q}})^2. \tag{C13}$$

The second term can be rewritten as

$$e^{-4m\gamma T\int dt (r_i^{\mathrm{q}})^2} = \int D\left[\xi_i(t)\right] e^{-\int dt\left[\frac{\xi_i(t)^2}{4m\gamma T}-2i\xi_i(t)r_i^{\mathrm{q}}(t)\right]}. \tag{C14}$$

As a result, after performing r_i^{q} and p_i^{q} integration, we get the expression for the expectation value of the observable $\Omega(\boldsymbol{r}^{\mathrm{cl}},\boldsymbol{p}^{\mathrm{cl}})$

$$\begin{aligned} \langle\Omega(\boldsymbol{r}^{\mathrm{cl}}(t),\boldsymbol{p}^{\mathrm{cl}}(t))\rangle &= \int D\left[\xi_i(t)\right] e^{-\int dt\,\frac{1}{4m\gamma k_BT}\xi_i(t)^2}\int D\left[r_i^{\mathrm{cl}}(t)p_i^{\mathrm{cl}}(t)\right]\Omega(\boldsymbol{r}^{\mathrm{cl}}(t),\boldsymbol{p}^{\mathrm{cl}}(t)) \\ &\times\prod_i \delta(\dot{r}_i^{\mathrm{cl}}-\frac{p_i^{\mathrm{cl}}}{m}+\Omega\epsilon_{ij}\dot{r}_j^{\mathrm{cl}}+A\epsilon_{ij}\dot{r}_j^{\mathrm{cl}}-C\dot{r}_i^{\mathrm{cl}})\delta(\dot{p}_i^{\mathrm{cl}}-qB\epsilon_{ij}\dot{r}_j^{\mathrm{cl}}-A\epsilon_{ij}\dot{p}_j^{\mathrm{cl}}-C\dot{p}_i^{\mathrm{cl}}+m\gamma\dot{r}_i^{\mathrm{cl}}-\xi_i), \end{aligned} \tag{C15}$$

where we set $T\to k_BT$. This expression represents the Langevin equation (B1) with

$$\hat{N} = \begin{pmatrix}\sqrt{2m\gamma k_BT}\hat{I}_2 & 0\\ 0 & 0\end{pmatrix},\quad \hat{Q} = \begin{pmatrix} m\gamma\hat{I}_2 & 0\\ 0 & 0\end{pmatrix}, \tag{C16}$$

where $\boldsymbol{X}=(\boldsymbol{r},\boldsymbol{p})$. Then, Eq. (B1) is nothing but Eqs. (1) and (2)

Here we note that, the friction term on the right hand side of Eq. (2) is $-m\gamma\dot{r}_i$, not $-\gamma p_i$. The reason is that, the friction term $-\gamma p_i$ and the fluctuation term $\sqrt{2m\gamma k_BT}\xi_i(t)$ do not satisfy Eq. (B21). Also, the microscopic derivation above leads to the friction term $-m\gamma\dot{r}_i$ and the fluctuation term $\sqrt{2m\gamma k_BT}\xi_i(t)$, which satisfy equation (B21). Therefore, as far as this microscopic model is valid for the description of the dynamics, Eqs. (1) and (2) must be used.

From now on, we will use Eq. (C16). Then, from Eq. (B21), Eq. (B15) can be rewritten as

$$\begin{aligned} \frac{\partial P(\boldsymbol{X},t)}{\partial t} &= G_{\alpha\beta}\nabla_\alpha((\nabla_\beta\epsilon)P) \\ &+ \frac{k_BT}{2}(\hat{G}+\hat{G}^T)_{\alpha\beta}\nabla_\alpha\nabla_\beta P. \end{aligned} \tag{C17}$$

Appendix D: Exact results

Given any linear multivariate Fokker-Planck equation,

$$\frac{\partial P(\boldsymbol{X},t)}{\partial t} = -A_{\alpha\beta}\frac{\partial}{\partial X_\alpha}(X_\beta P) + \frac{1}{2}B_{\alpha\beta}\frac{\partial^2 P}{\partial X_\alpha \partial X_\beta}, \tag{D1}$$

where $\hat{A}$ and $\hat{B}$ are the constant matrices, we can exactly solve it with the initial condition[16]

$$P(\boldsymbol{X},0) = \prod_{i=1}^{2d}\delta(X_i - X_{i0}). \tag{D2}$$

If we multiply Eq. (D1) with X_γ and integrate over $\boldsymbol{X}$, we get

$$\frac{\partial}{\partial t}\langle X_\gamma\rangle = A_{\gamma\beta}\langle X_\beta\rangle, \tag{D3}$$

then

$$\langle X_\gamma\rangle(t) = (\exp(t\hat{A}))_{\gamma\beta}X_{\beta 0}. \tag{D4}$$

If we multiply Eq. (D1) with $X_\gamma X_\delta$ and integrate over $\boldsymbol{X}$, we get

$$\frac{\partial}{\partial t}\langle X_\gamma X_\delta\rangle = A_{\gamma\alpha}\langle X_\alpha X_\delta\rangle + A_{\delta\beta}\langle X_\gamma X_\beta\rangle + B_{\gamma\delta}. \tag{D5}$$

If we introduce

$$\langle\langle X_\gamma(t)X_\delta(t)\rangle\rangle = \langle X_\gamma X_\delta\rangle(t) - \langle X_\gamma\rangle(t)\langle X_\delta\rangle(t) =: \Theta_{\gamma\delta}(t), \tag{D6}$$

$$\hat{\Theta}^*(t) := e^{-t\hat{A}}\hat{\Theta}(t)e^{-t\hat{A}^T}, \tag{D7}$$

then $\Theta^*_{\gamma\delta}(0) = 0$ and

$$\frac{\partial}{\partial t}\hat{\Theta}^* = e^{-t\hat{A}}\hat{B}e^{-t\hat{A}^T}. \tag{D8}$$

As a result, we get

$$\begin{aligned}\hat{\Theta}^*(t) &= \int_0^t dt'\, e^{-t'\hat{A}}\hat{B}e^{-t'\hat{A}^T}\\ \Leftrightarrow \hat{\Theta}(t) &= \int_0^t dt'\, e^{(t-t')\hat{A}}\hat{B}e^{(t-t')\hat{A}^T} = \int_0^t dt'\, e^{t'\hat{A}}\hat{B}e^{t'\hat{A}^T}.\end{aligned} \tag{D9}$$

Eqs. (D4) and (D9), are enough to determine the whole dynamics since the process is Gaussian. The solution is,

$$\begin{aligned}P(\boldsymbol{X},t) &= (2\pi)^{-d}(\det\hat{\Theta})^{-\frac{1}{2}}\\ &\times \exp\left[-\frac{1}{2}(\boldsymbol{X}^T - \langle\boldsymbol{X}^T\rangle(t))\hat{\Theta}^{-1}(t)(\boldsymbol{X} - \langle\boldsymbol{X}\rangle(t))\right].\end{aligned} \tag{D10}$$

If we set $\epsilon(\boldsymbol{p}) = \boldsymbol{p}^2/(2m)$, the Fokker-Planck equation with Berry curvatures, Eq. (C17), are linear multivariate and

$$\hat{A} = \frac{1}{m}\begin{pmatrix} 0 & -\hat{G}_{\boldsymbol{rp}} \\ 0 & -\hat{G}_{\boldsymbol{pp}}\end{pmatrix}, \tag{D11}$$

$$B_{\alpha\beta} = k_B T(G_{\alpha\beta} + G_{\beta\alpha}), \tag{D12}$$

where

$$\hat{G} =: \begin{pmatrix} \hat{G}_{\boldsymbol{rr}} & \hat{G}_{\boldsymbol{rp}} \\ \hat{G}_{\boldsymbol{pr}} & \hat{G}_{\boldsymbol{pp}}\end{pmatrix}, \tag{D13}$$

$$\hat{G}_{\boldsymbol{rr}} = M\left[\frac{m\gamma\Omega^2}{D^2+A^2-qB\Omega}\hat{I}_2 - \Omega i\hat{\sigma}_y\right], \tag{D14}$$

$$\hat{G}_{\boldsymbol{rp}} = M\left[\left(-D - \frac{m\gamma\Omega A}{D^2+A^2-qB\Omega}\right)\hat{I}_2 + \left(A - \frac{m\gamma\Omega D}{D^2+A^2-qB\Omega}\right)i\hat{\sigma}_y\right], \tag{D15}$$

$$\hat{G}_{\boldsymbol{pr}} = M\left[\left(D - \frac{m\gamma\Omega A}{D^2+A^2-qB\Omega}\right)\hat{I}_2 + \left(A + \frac{m\gamma\Omega D}{D^2+A^2-qB\Omega}\right)i\hat{\sigma}_y\right], \tag{D16}$$

$$\hat{G}_{\boldsymbol{pp}} = M\left[\frac{m\gamma(A^2+D^2)}{D^2+A^2-qB\Omega}\hat{I}_2 + \left(-qB + \frac{m^2\gamma^2\Omega}{D^2+A^2-qB\Omega}\right)i\hat{\sigma}_y\right], \tag{D17}$$

$D := 1 - C$ and

$$M = \frac{D^2+A^2-qB\Omega}{(D^2+A^2-qB\Omega)^2 + m^2\gamma^2\Omega^2}. \tag{D18}$$

So we just need to calculate Eqs. (D4) and (D9) with matrices Eqs. (D11) and (D12). If we define

$$\gamma_1 = \frac{(D^2+A^2)\gamma}{(D^2+A^2-qB\Omega)^2 + m^2\gamma^2\Omega^2}, \quad \gamma_2 = \frac{-qB(D^2+A^2) + (q^2B^2+m^2\gamma^2)\Omega}{m[(D^2+A^2-qB\Omega)^2 + m^2\gamma^2\Omega^2]}, \tag{D19}$$

and

$$g_1(t) = \frac{1}{q^2B^2+m^2\gamma^2}[AqB + m\gamma D - (AqB+m\gamma D)e^{-\gamma_1 t}\cos(\gamma_2 t) + (m\gamma A - qBD)e^{-\gamma_1 t}\sin(\gamma_2 t)], \tag{D20}$$

$$g_2(t) = \frac{1}{q^2B^2+m^2\gamma^2}[qBD - m\gamma A - (qBD - m\gamma A)e^{-\gamma_1 t}\cos(\gamma_2 t) + (m\gamma D + AqB)e^{-\gamma_1 t}\sin(\gamma_2 t)], \tag{D21}$$

$$f_1(t) = m\gamma D - AqB + e^{-2\gamma_1 t}[m\gamma D + AqB - 2m\gamma e^{\gamma_1 t}(D\cos(\gamma_2 t) + A\sin(\gamma_2 t))], \tag{D22}$$

$$f_2(t) = m\gamma A + qBD - e^{-2\gamma_1 t}[qB - m\gamma A + 2m\gamma e^{\gamma_1 t}(A\cos(\gamma_2 t) - D\sin(\gamma_2 t))], \tag{D23}$$

then Eqs. (D4) and (D9) are,

$$\begin{pmatrix} \langle p_x(t)\rangle \\ \langle p_y(t)\rangle \end{pmatrix} = e^{-\gamma_1 t}\begin{pmatrix} \cos(\gamma_2 t) & -\sin(\gamma_2 t) \\ \sin(\gamma_2 t) & \cos(\gamma_2 t) \end{pmatrix}\begin{pmatrix} p_{x0} \\ p_{y0} \end{pmatrix}, \tag{D24}$$

$$\begin{pmatrix} \langle r_x(t)\rangle \\ \langle r_y(t)\rangle \end{pmatrix} = \begin{pmatrix} g_1(t) & g_2(t) \\ -g_2(t) & g_1(t) \end{pmatrix}\begin{pmatrix} p_{x0} \\ p_{y0} \end{pmatrix} + \begin{pmatrix} r_{x0} \\ r_{y0} \end{pmatrix}, \tag{D25}$$

$$\langle\langle r_i(t) r_j(t)\rangle\rangle = \left[\frac{2m\gamma k_BT}{q^2B^2+m^2\gamma^2}t + \frac{mk_BT(A^2+D^2)}{(q^2B^2+m^2\gamma^2)^2}(B^2-3m^2\gamma^2) + \frac{4m^2\gamma k_BT(A^2+D^2)}{(q^2B^2+m^2\gamma^2)^2}e^{-\gamma_1 t}(m\gamma\cos(\gamma_2 t) + qB\sin(\gamma_2 t)) - \frac{mk_BT(A^2+D^2)}{q^2B^2+m^2\gamma^2}e^{-2\gamma_1 t}\right]\delta_{ij}, \tag{D26}$$

$$\langle\langle r_i(t) p_j(t)\rangle\rangle = \frac{mk_BT}{q^2B^2+m^2\gamma^2}(f_1(t)\delta_{ij} - f_2(t)(i\hat{\sigma}_y)_{ij}), \tag{D27}$$

$$\langle\langle p_i(t) p_j(t)\rangle\rangle = (1-e^{-2\gamma_1 t})\delta_{ij}. \tag{D28}$$

In the main text, we put $A = 0$. We note that,

$$D^2 + A^2 - qB\Omega = 1 - 2C + C^2 + A^2 - qB\Omega = 1 - (\hat{\Omega}_{rp})_{ii} - \epsilon_{\alpha\beta\gamma\delta}(\hat{\Omega}_{XX})_{\alpha\beta}(\hat{\Omega}_{XX})_{\gamma\delta}/8, \tag{D29}$$

where $\epsilon_{\alpha\beta\gamma\delta}$ is the completely antisymmetric tensor, is nothing but the modified density of state of the system[2,57,61].

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