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Machine learning spatial geometry from entanglement features
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I. INTRODUCTION

Holographic duality\textsuperscript{1–3} is a duality proposed between a quantum field theory (the boundary theory) and a gravitational theory (the bulk theory) in one higher dimension. In 2006, S. Ryu and T. Takayanagi proposed the Ryu-Takayanagi (RT) formula\textsuperscript{4}, which relates the entanglement entropy of a boundary region to the area of the minimal surface in the bulk that is homologous to the same region. The RT formula and its generalizations\textsuperscript{5–8} point out that entanglement plays a fundamental role in holographic duality. One perspective to understand the entanglement-geometry correspondence is to consider a tensor network representation of a quantum many-body state\textsuperscript{9,10}, and view the network geometry as a representation of the dual spatial geometry\textsuperscript{11,12}. Many different schemes of tensor network approaches have been investigated\textsuperscript{9,10,13–28}. Tensor network states with various entanglement properties similar to holographic theories have been constructed\textsuperscript{29–36}. In particular, the random tensor network (RTN) states\textsuperscript{31} are shown to satisfy the Ryu-Takayanagi formula\textsuperscript{4} and the quantum error correction properties\textsuperscript{37} in the large bond dimension limit. The RTN states on all possible graphs form an overcomplete basis of the boundary Hilbert space\textsuperscript{32}, so that a generic many-body state of the boundary can be mapped to a superposition of RTN’s with different geometry. For states with a semi-classical bulk dual, one expects the superposition to be strongly peaked around a “classical geometry”, which provides a best approximation to entanglement entropy of different regions in the given state. In other words, finding the best RTN description of a given many-body state can be considered as a variational problem similar to a familiar variational wavefunction approach, except that the criteria of the optimization is not minimizing energy but reproducing entanglement features of the state, such as entanglement entropy and Renyi entropies of various subsystems. For deeper understanding of holographic duality, such as understanding how boundary dynamics are mapped to bulk gravitational dynamics, it is essential to develop a systematic approach of finding the optimal network geometry for generic many-body states.

In this paper, we propose that the RTN optimization problem can be mapped to a deep learning problem\textsuperscript{38–40}, because the paradigm of neural network based deep learning is precisely about how to adjust the network connectivity (geometry) to achieve a certain optimization goal. More specifically, we propose a learning approach, called the entanglement feature learning (EFL), which learns the entanglement features in the quantum many-body state and encodes the entanglement structures in the neural network connectivity. Interestingly, the deep learning approach provides not only a technical tool to optimize the RTN, but also a profound connection between tensor networks and neural networks in terms of their geometric interpretations. Base on this interpretation, the holographic dual spatial geometry of a quantum many-body state could emerge as the neural network geometry from machine learning the entanglement features. In other words, special geometry is just an efficient way to encode entanglement features. The corresponding tensor network can be viewed as a disentangling circuit that gradually resolves the entanglement features at different layers, which is the common idea underlying other tensor network holography approaches.\textsuperscript{36} For simplicity we will consider the second Renyi entropy $S^{(2)}(A)$ of all subregions $A$ as entanglement features of a

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Conceptual connections between holographic duality and deep learning.}
\end{figure}
state. Using the second Renyi entropy of some regions as the training data, the goal of the neural network is to give a best prediction to the second Renyi entropy of other regions. As the learning is done, the geometric structure of the neural network can be interpreted as the emergent holographic bulk geometry. This draws a direct connection between holographic duality and the deep learning, as illustrated in Fig. 1. This connection was also made in a recent work Ref. 41, based on the similarity in their relations to the renormalization group\textsuperscript{11–15,42–46}. The relation between neuron networks and tensor networks have also been discussed recently in Refs. 47–52.

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In this work, we point out that for “machine learning holography”, what should be learned are the entanglement features of the quantum many-body states. We also develop a concrete EFL algorithm that can be implemented in numerics. A terminology dictionary of EFL is summarized in Tab. I. Our EFL approach is based on the planar RTN already has sufficient expression power strict to the RTN on planar graphs. It turns out that each RTN can be mapped to a DBM with the same architecture in order to make EFL a practical (rather than theoretical) algorithm. To this end, we will re-structure the EFL in 1D free fermion systems and show how the holographic bulk geometry can arise from learning the entanglement features.

II. RANDOM TENSOR NETWORKS

A. Definition of RTN States

We will briefly review the definition of random tensor network (RTN) state following the projected entangled pair state (PEPS) approach.\textsuperscript{31,32} A RTN state is specified by an edge-weighted graph $G = (\mathcal{V}; \mathcal{E}; I)$ comprising the vertex set $\mathcal{V}$ and the edge set $\mathcal{E}$ along with a weighting function $I : \mathcal{E} \rightarrow \mathbb{R}_+$, such that each edge $e \in \mathcal{E}$ is associated with a real and positive edge weight $I_e$. On each vertex $v \in \mathcal{V}$, we define a local Hilbert space $\mathcal{H}_v = \bigotimes_{\mu \in \partial e} \mathcal{H}_e^{\mu}$, where $\partial e$ denotes the set of edges adjacent to the vertex $v$. $\mathcal{H}_e^{\mu}$ is subspace on the vertex $v$ to be connected to the incident edge $e$ (as the small blue circle in Fig. 2). Let $|\psi_e^{\mu}\rangle$ (labeled by $\mu_e^{\mu} = 1, 2, \ldots$) be a complete set of basis states of the Hilbert space $\mathcal{H}_e^{\mu}$.

![FIG. 2: The structure of a RTN state.](image)

We then define a random state $|\psi_v\rangle \in \mathcal{H}_v$ on each vertex $v$ (as the big red circle in Fig. 2).

$$|\psi_v\rangle = \sum_{[\mu_v]} T[\mu_v] \bigotimes_{e \in \partial v} |\psi_e^{\mu_e}\rangle. \quad (1)$$

The coefficient tensor $T$ is a random tensor, whose tensor elements are independently drawn from normal distributions following $P(T) \propto e^{-\frac{1}{2} \sum_{[\mu_v]} |T[\mu_v]|^2}$. On each edge, we define an entangled pair state $|I_e\rangle$ (as the blue link in Fig. 2) in the Hilbert space $\bigotimes_{\mu_e} \mathcal{H}_e^{\mu}$ (where $\partial e$ denotes the set of two vertices at the end of the edge $e$),

$$|I_e\rangle = \sum_{[\mu_e]} \lambda[\mu_e] \bigotimes_{v \in \partial e} |\psi_e^{\mu_e}\rangle. \quad (2)$$

The entanglement of $|I_e\rangle$ across the edge is characterized by the edge mutual information $I_e$. Each edge could have a different $I_e$ in general. If we treat the coefficient $\lambda[\mu_e] = \lambda_{\mu_1 \mu_2}$ as a matrix, the $n$th Renyi mutual information can...
be expressed as
\[ I_e^{(n)} = \frac{2}{1 - n} \ln \text{Tr}(\lambda^e)^n. \] (3)

In the following, we will focus on the case of Renyi index \( n = 2 \) and take \( I_e = I_e^{(2)} \) unless otherwise specified. It is free to choose \( \lambda \) on each edge, as long as the edge mutual information \( I_e \) matches the edge weight \( I_v \) of the graph \( G \). There is also a set of special edges (the thick edges in Fig. 2) on the boundary of the network. They are the external edges (physical legs) that connects to the physical degrees of freedom. On these edges, we assume that the entangled pair states are maximally entangled, hence the edge mutual information is \( 2 \ln D_\partial \) with \( D_\partial \) being the bond dimension of the external leg.

Given the random state \( |\psi_v\rangle \) on each vertex \( v \) and the entangled pair state \( |I_e\rangle \) on each edge \( e \), the RTN state can be constructed by projecting the entangled pair states to random vertex states via the following partial inner product
\[ |G\rangle = \bigotimes_{v \in V} \bigotimes_{e \in E} (|\psi_v\rangle |I_e\rangle). \] (4)

The remaining subspaces (as solid circles in Fig. 2) on the dangling ends of the external edges are not touched by the projection. They form the physical Hilbert space \( H_{\text{phys}} = \bigotimes_{v \in V_\partial} H_{\text{phys}} \) in which the RTN state \( |G\rangle \) is supported. Here \( V_\partial \) denotes the set of boundary vertices, i.e. the subset of \( V \) whose vertices are connected to the external edges. It is worth mentioning that \( |G\rangle \) should better be treated as an ensemble of RTN states, instead of a single specific state, due to the randomness in \( |\psi_v\rangle \). All states in the ensemble are labeled by the same edge-weighted graph \( G \) and share the similar entanglement feature.

\section{B. Entanglement Features of RTN States}

The entanglement feature of a quantum many-body state refers to the full set of entanglement entropies over all entanglement subregions. In general, one could include all orders of Renyi entropies in the definition, but we will only focus on the 2nd Renyi entropies in the following and leave the generic discussion to the last section.

Given an ensemble of RTN states \( |G\rangle \) and a subregion \( A \subseteq V_\partial \), the ensemble-typical 2nd Renyi entropy \( S_G(A) \) over the subregion \( A \) is defined via
\[ e^{-S_G(A)} = \frac{E \cdot \text{Tr}_A(\text{Tr}_\bar{A} |G\rangle \langle G|)^2}{(\text{Tr} |G\rangle \langle G|)^2}, \] (5)
where \( E \) takes the RTN ensemble expectation value (i.e. averaging over the random states \( |\psi_v\rangle \) on all vertices), and \( \bar{A} = V_\partial \setminus A \) denotes the complement region of \( A \). We have explicitly introduced the denominator \( \text{Tr} |G\rangle \langle G| \) to ensure the normalization of the RTN density matrix. An important result of Ref. 31 is to show that the entanglement entropy \( S_G(A) \) can be expressed in term of the free energies of a classical Ising model on the same graph \( G \) in the large bond dimension limit. A more general treatment away from that limit is provided in a related work Ref. 68, but in this work we will only consider the large bond dimension limit.

To specify the Ising model, we first introduce a set of Ising spins \( \sigma_v = \pm 1 \) for all \( v \in V \) and an additional set of Ising variables \( \tau_v = \pm 1 \) on the boundary \( v \in V_\partial \) only. The model is described by the energy functional
\[ E_\partial[\sigma, \tau] = -\sum_{v \in \partial} J_e \prod_{e \in \partial v} \sigma_v - h \sum_{v \in V_\partial} \tau_v \sigma_v. \] (6)
The Ising coupling \( J_e \equiv I_e/4 \) is set by the edge mutual information \( I_e \) of the RTN state. The external field \( h \equiv \frac{1}{2} \ln D_\partial \) is set by the local Hilbert space dimension \( D_\partial \) of the physical degrees of freedom (which is also the bond dimension of the external leg). Only \( \sigma_v \) spins are dynamical, and \( \tau_v \) are just Ising variables that specifies the directions of the external pinning field \( h \tau_v \) on the boundary. The configuration of \( \tau_v \) is determined by the choice of the entanglement region \( A \)
\[ \tau_v(A) = \begin{cases} -1 & v \in A, \\ +1 & v \notin A. \end{cases} \] (7)

Tracing out the dynamical spins \( \sigma_v \), the free energy \( F[\tau] \) of the boundary spins \( \tau_v \) can be defined via
\[ e^{-F_\partial[\tau]} = \sum_{[\sigma]} e^{-E_\partial[\sigma, \tau]} \] (8)
In the large bond dimension limit \( (I_e \gg 1) \), it was shown\(^{31} \) that the typical 2nd Renyi entropy of the RTN state \( |G\rangle \) is given by the free energy difference
\[ S_G(A) = F_\partial[\tau(A)] - F_\partial[\tau(\emptyset)], \] (9)
where \( \tau(A) \) denotes the boundary pinning field configuration specified in Eq. (7) and \( \tau(\emptyset) \) denotes the configuration of \( \tau_v = +1 \) for all \( v \in V_\partial \). The derivation of Eq. (9) is reviewed in Appendix A. The physical intuition of Eq. (9) comes from the interpretation\(^4 \) of the entanglement entropy as the area of the minimal surface that separates the region \( A \) from \( \bar{A} \) in the holographic bulk. Correspondingly, the free energy difference \( F[\tau(A)] - F[\tau(\emptyset)] \) measures the energy cost of the domain wall that separates the part \( A \) from \( \bar{A} \) in the tensor network (see Fig. 3), which matches the holographic interpretation of the entanglement entropy in the large bond dimension limit. Technically, the advantage of RTN over other types of tensor networks also lies in the fact that the 2nd Renyi entropy of the RTN state can be efficiently estimated from the free energy of the corresponding Ising model as in Eq. (9). For a generic tensor network, calculating its entanglement entropy requires to diagonalize the reduced density matrix, which could be much more difficult than solving the Ising model in many cases.
network connections and obtain the optimal holographic deep learning technique to optimize the random tensor neural network based deep learning. We will apply the complete graph, and the latter is a typical problem of the graph can be disconnected by setting its weight edge weights $I_e$ (the graph with all-to-all connections) by adjusting the interpreted as the dual bulk geometry. In principle, all quantum many-body state, how to determine the optimal entropy. However, in this work, we would like to consider the entanglement region $A$ defined in Eq. (7), which is just another way to specify the entanglement region $A$. Usually it is not practical to collect entanglement entropies for all possible subregions $A \subseteq V_0$, so only a subset of the entanglement feature will be used in the EFL (how to sample the subset will be explained in details later). Once the entanglement feature is collected, we will make no further reference to the original quantum state $|\Psi\rangle$.

We wish to fit the entanglement feature of the given state $|\Psi\rangle$ by the RTN state $|G\rangle$. We would like to emphasize that we are not intended to find the tensor network representation of the state $|\Psi\rangle$, which could be a much harder task. We just want to find the optimal tensor network geometry such that the entanglement features between $|\Psi\rangle$ and $|G\rangle$ match as much as possible. In fact, as the tensors are random in the RTN, the RTN state $|G\rangle$ would be very different from (most likely orthogonal to) the given state $|\Psi\rangle$. To learn the tensor network geometry from the entanglement feature, there are two possible learning approaches: supervised learning or unsupervised learning.

In supervised learning, each training sample is a pair $(\tau(A), S_\Psi(A))$ consisting of the Ising configuration $\tau(A)$ as the input object and the entanglement entropy $S_\Psi(A)$ as the desired output value. The supervised learning will seek for a fitting function $S_G(A)$ based on the RTN model with minimal prediction error. The supervised EFL is essentially a regression problem. We can choose to minimize the mean square error loss function, which is commonly used for regression problems

$$L(G) = \operatorname{avg}_{A \subseteq V_0} (S_G(A) - S_\Psi(A))^2.$$  

The variational parameters will be the edge weights $I_e$ that parameterize the graph $G$ (and the RTN model).

In unsupervised learning, the training samples $\tau(A)$ are “unlabeled”, but they appear with an empirical probability distribution

$$P_\Psi[\tau(A)] \propto e^{-S_\Psi(A)}.$$  

Such training set can be prepared by Monte Carlo sampling the entanglement region $A$ following the Boltzmann.

The set of entanglement entropies $\{S_G(A) | A \subseteq V_0\}$ constitutes the entanglement feature of the RTN state, which only depends on the graph $G$ and its edge weights $I_e$. The RTN state thus provides us a model to encode the entanglement region $A$, i.e., the dual bulk geometry. This is the essential idea behind the tensor network holography. In many previous approaches, a bulk geometry is first given and a tensor network is tiled on the background geometry. The resulting tensor network state then produces the entanglement feature on the holographic boundary that is dual to the holographic bulk geometry. For example, Fig. 3 demonstrates how different network structures lead to different scaling behaviors of the single-interval entanglement entropy. However, in this work, we would like to consider the inverse problem: given the entanglement feature of a quantum many-body state, how to determine the optimal holographic geometry? We will show that this problem can be mapped to a machine learning problem, which we called the entanglement feature learning (EFL).

III. ENTANGLEMENT FEATURE LEARNING

A. General Algorithm

The goal of EFL is to develop an RTN ensemble that best matches the entanglement feature of the given many-body state $|\Psi\rangle$. The graph geometry of the RTN is then interpreted as the dual bulk geometry. In principle, all graph geometries can be realized on a complete graph (the graph with all-to-all connections) by adjusting the edge weights $I_e$. For example, an edge in the complete graph can be disconnected by setting its weight $I_e = 0$ to zero. Therefore optimizing the graph geometry is equivalent to optimizing the set of edge weights on the complete graph, and the latter is a typical problem of the neural network based deep learning. We will apply the deep learning technique to optimize the random tensor network connections and obtain the optimal holographic geometry of the given quantum many-body state.

Given a quantum many-body state $|\Psi\rangle$ (to learn), we first extract its entanglement feature by collecting the 2nd Renyi entanglement entropies $S_\Psi(A)$ over different entanglement subregions $A$

$$S_\Psi(A) = -\ln \operatorname{Tr}_A(\rho_A |\Psi\rangle\langle\Psi| A).$$  

Admittedly, calculating the entanglement entropy of a generic many-body state is difficult. However, let us assume that these data can be in principle collected, for example by experimental measurements\textsuperscript{69-72}. Then they can be used to construct the training set:

$$\{ (\tau(A), S_\Psi(A)) | A \subseteq V_0 \},$$  

where $\tau(A)$ is the boundary pinning field configuration defined in Eq. (7), which is just another way to specify the entanglement region $A$. Usually it is not practical to collect entanglement entropies for all possible subregions $A \subseteq V_0$, so only a subset of the entanglement feature will be used in the EFL (how to sample the subset will be explained in details later). Once the entanglement feature is collected, we will make no further reference to the original quantum state $|\Psi\rangle$.

In supervised learning, each training sample is a pair $(\tau(A), S_\Psi(A))$ consisting of the Ising configuration $\tau(A)$ as the input object and the entanglement entropy $S_\Psi(A)$ as the desired output value. The supervised learning will seek for a fitting function $S_G(A)$ based on the RTN model with minimal prediction error. The supervised EFL is essentially a regression problem. We can choose to minimize the mean square error loss function, which is commonly used for regression problems

$$L(G) = \operatorname{avg}_{A \subseteq V_0} (S_G(A) - S_\Psi(A))^2.$$  

The variational parameters will be the edge weights $I_e$ that parameterize the graph $G$ (and the RTN model).

In unsupervised learning, the training samples $\tau(A)$ are “unlabeled”, but they appear with an empirical probability distribution

$$P_\Psi[\tau(A)] \propto e^{-S_\Psi(A)}.$$  

Such training set can be prepared by Monte Carlo sampling the entanglement region $A$ following the Boltzmann.
weight $e^{-S_{\Psi}(A)}$ on the given state $|\Psi\rangle$. The goal of the unsupervised learning is to train a generative model that could reproduce the samples $\tau(A)$ with the probability distribution close to the empirical distribution as much as possible. If the goal is achieved, it is believed that the generative model has captured the hidden features in the training data. For our purpose, we take the RTN model as the generative model, which generates the sample $\tau(A)$ with the model probability

$$P_\Psi[\tau(A)] \propto e^{-S_{\Psi}(A)} \propto e^{-F_\psi[\tau(A)]},$$

or more precisely,

$$P_\Psi[\tau] = \frac{1}{Z_\Psi} \sum_{\sigma} e^{-E_\Psi[\sigma,\tau]},$$

$$Z_\Psi = \sum_{\tau} e^{-F_\psi[\tau]} = \sum_{\sigma,\tau} e^{-E_\Psi[\sigma,\tau]},$$

where the energy model $E_\Psi[\sigma, \tau]$ is given by Eq. (6). If we treat the bulk spins $\sigma$ as hidden units and the boundary spins $\tau$ as visible units, the model is precisely mapped to the Boltzmann machine in machine learning. The goal is to approximate the empirical distribution $P_\Psi[\tau]$ by the distribution $P_\Psi[\tau]$ produced by the Boltzmann machine. To measure how similar the two distributions are, the Kullback-Leibler divergence is typically used as the objective function

$$\mathcal{L}(G) = \sum_{\tau} P_\Psi[\tau] \ln \frac{P_\Psi[\tau]}{P_\Psi[\tau]},$$

which is minimized if $P_\Psi[\tau] \rightarrow P_\Psi[\tau]$. Because the empirical distribution $P_\Psi[\tau]$ was constructed in Eq. (13) to encode the entanglement feature of $|\Psi\rangle$, if the Boltzmann machine manages to reproduce this distribution after training, the entanglement feature should have been learnt and encoded in the neural network connectivity, which gives us a representation of the emergent holographic bulk geometry.

For both supervised and unsupervised learning, the training procedure is to minimize objective function $\mathcal{L}(G)$, which is formally a functional of the edge-weighted graph $G$. As mentioned before, we can always embed the graph $G$ in a large enough complete graph and take the edge weights $I_e$ (or equivalently the Ising couplings $J_e = I_e/4$) as the variational parameters. Hence, we can use a gradient descent algorithm over $\mathcal{L}(G)$ to find its minimum according to the following update rule

$$I_e \leftarrow I_e - r_t \frac{\partial \mathcal{L}(G)}{\partial I_e},$$

where $r_t$ denotes the learning rate. The whole EFL algorithm is summarized as the computation graph in Fig. 4. In the training process, the neural network learns the entanglement feature of the input quantum state $|\Psi\rangle$. As the training converges, we open up the neural network and extract the network connectivity from the parameters $I_e$, which parameterize the optimal edge mutual information of the RTN as well as the optimal graph geometry in the holographic bulk.

In this work, we will adopt the supervised learning approach and leave the unsupervised learning approach for future investigation. There are still two technical obstacles that we have to overcome to make the EFL really a practical (rather than theoretical) algorithm for tensor network holography. In the remainder of this section, we will analyze the obstacles and provide solutions to them.

### B. Deterministic Learning on Planar Graph

The gradient descent method is not practical for training generic Boltzmann machines with unrestricted connections (i.e. on a complete graph). One major reason lies in the lack of efficient inference method: the complexity to evaluate the free energy $F_G[\tau]$ (or the marginal distribution $P_G[\tau]$) of the Boltzmann machine grows exponentially with the number of hidden units. Adding restrictions to the network structure allows for more efficient training algorithms, such as the restricted Boltzmann machine (RBM). By stacking RBM layers, one obtains a deep architecture known as the deep Boltzmann machine (DBM), which better fits the purpose of EFL to produce the geometry deep in the holographic bulk. However, the original proposal to estimate the learning gradient for the DBM is based on the Monte Carlo method. It is found that the deep layers typically receive very weak gradient signals, which can be easily overwhelmed by the thermal fluctuations introduced by the Monte Carlo process. The net effect is that the thermal noise will drive the edge weights $I_e$ in the deep layers to follow a random walk until the activation saturates.
Therefore the deep layers cannot be trained by stochastic learning algorithms. Instead, we need deterministic learning\textsuperscript{61} algorithms. The idea is to avoid the Monte Carlo sampling and evaluate the Ising model free energy analytically. Several approximate methods have been developed, including the belief propagation\textsuperscript{78–80} and the high-temperature series expansion.\textsuperscript{81}

Instead of approximate approaches, if we restrict the network geometry to planar graphs, there are exact learning methods\textsuperscript{82–84} exploiting the exact solvability of planar graph Ising models\textsuperscript{85–87} (by mapping them to free Majorana fermion problems on related but different graphs, see Appendix B for details). Naively, it seems too restricted to study planar graphs, which are very special among all graphs. However, the RTN on a planar graph can already model a variety of entanglement features on the holographic boundary, as demonstrated in Fig. 3. For example, the volume-law entanglement can be described by a planar network geometry with flat or positive curvature, because in that case the minimal surface is pushed to the boundary. Therefore the planar graphs can describe a large family of states of interest,\textsuperscript{104} including, for example, area-law\textsuperscript{62–64} ground states of local Hamiltonians and volume-law\textsuperscript{65–67} excited eigenstates satisfying the eigenstate thermalization hypothesis\textsuperscript{88–90}.

Details of the Ising-Majorana fermion mapping is reviewed in Appendix B. As a short summary, the free energy $F_\phi[\tau]$ can be calculated from the Pfaffian of the lattice adjacency matrix $A[J, h\tau]$ (with edges weighted by $J$ and $h\tau$) on which the dual fermions live:

$$F_\phi[\tau] = \sum_{v \in V} J_v + h \sum_{v \in V} \tau_v - \ln pf A[J, h\tau].$$

The computational complexity is of the cubic order of the graph size. The gradient can also be calculated efficiently from $\frac{d}{dA} \ln pf A = \frac{1}{2} \text{Tr} A^{-1} dA$. Because there is no thermal fluctuation in the gradient signal, the edge weights in deep layers can be trained towards their optimal values deterministically. On the other hand, considering the DBM with planar graph architecture is also physically plausible for the purpose of the tensor network holography, because the planar graph is naturally a discretized description of the 2D spatial part of the (2+1)D holographic geometry (as the holographic dual to the (1+1)D quantum many-body state).

### C. Architecture and Regularization

Besides the deterministic learning, another technical challenge of EFL is the redundancy in the graphical representation of the entanglement feature. For example, consider an Ising model with three spins as shown in Fig. 5(a), described by

$$E[\sigma, \tau] = -J_1 \sigma_1 \sigma_2 - J_2 (\sigma_1 + \sigma_2) \sigma_3 - h (\tau_1 \sigma_1 + \tau_2 \sigma_2),$$

which is parameterized by two Ising couplings $J_1$ and $J_2$. But the free energy $F[\tau] = -\ln \sum_\sigma e^{-E[\sigma, \tau]}$ only depends on an effective coupling (obtained by first tracing out the $\sigma_3$ spin)

$$J_{\text{eff}} = J_1 + \frac{1}{2} \ln \cosh 2J_2.$$  \hspace{1cm}(20)

So there is a trade-off between $J_1$ and $J_2$: as long as $J_{\text{eff}}$ remains unchanged, adjusting $J_1$ and $J_2$ in the opposite way will not change the free energy $F[\tau]$, and thus will not affect the objective function. As illustrated in Fig. 5(b), there will be a flat channel along which all different edge weights are degenerated in the objective function.

![FIG. 5: Redundancy in a three-vertex graph. (a) The network structure. (b) The objective function in the $J_1$-$J_2$ plane. A flat channel indicates the redundant direction.](image)

This phenomenon can be viewed as a discrete analog of the diffeomorphism redundancy of the gravity theory. It also poses a problem to the EFL, because each time the training will end up with a different edge weight configuration along the flat direction, which makes it hard to compare the network geometries between two training.

Before coming up with a systematic classification of these redundancies, we have to introduce a “gauge fixing” by hand. This is done by imposing more restrictions on the architecture.

In the following we will consider two particular architectures: the cylindrical and the hyperbolic network as shown in Fig. 6. In particular, the hyperbolic network in Fig. 6(b) can be viewed as a variation of the convolutional deep Boltzmann machine architecture. Both networks have layered structure. Within each layer, the horizontal (intra-layer) bonds and the zig-zag (inter-layer) bonds can trade off each other (approximately), similar to the situation in Fig. 5. To fix this redundancy, we lock the inter-layer coupling to the intra-layer coupling on the UV side (i.e. the side closer to the boundary), see Fig. 6. If the training data is translation invariant along $x$-direction, we will also set the coupling uniform within each layer to respect the translation symmetry.

The learning signal originates from the training data and is passed down layer-by-layer from the boundary into the bulk. Suppose at the beginning, all couplings are initialized to zero. When the training data is presented to the machine, the first layer learns the local spin correlation in the training samples and develops the coupling $J_1$ to match the correlation. Due to the interlayer couplings, the spin correlation in the first layer will induce
fermion flavor number). The central charge $c$ and the

the residual spin correlation in the second layer. The
residual correlation is then presented to the second layer
to train the coupling $J_2$ and so on. So the deeper layer
should be designed to resolve the residual correlations
that can not be resolved in the previous layers. Bearing
this physical picture in mind, we propose the following feasible domain

$$J_1 \geq J_2 \geq J_3 \geq \cdots \geq 0,$$

(21)

where $J_z$ is the coupling strength in the $z$th layer. In
the algorithm implementation, the condition Eq. (21) is
checked at each training step. If the condition is viol-
ated, the parameters $J_z$ will be pulled back to the nearest
boundary point of the feasible domain. In the machine
learning terminology, Eq. (21) can be considered as a reg-
ularization that coordinates the training among different
layers and effectively prevents overfitting in the first sev-
eral layers.

IV. NUMERICAL RESULTS

A. Training Set Preparation

Computing the entanglement entropies for a generic quantum many-body state is difficult. As a proof of con-
cept, we choose the free fermion system to demonstrate the idea of EFL. Consider $N$ copies of the (1+1)D Ma-
ajorana fermion chain, described by the Hamiltonian

$$H = \sum_{a=1}^{N} \sum_i \left(1 + m(-1)^j\right) \chi_{i,a} \chi_{i+1,a},$$

(22)

where $\chi_{i,a}$ is the Majorana fermion operator of the fla-
vor $a$ on the site $i$, satisfying $\{\chi_{i,a}, \chi_{j,b}\} = \delta_{ij}\delta_{ab}$. The
Majorana coupling $1 + m(-1)^i$ has a staggered pattern
along the chain, such that each unit cell contains two
sites, $m \in (0,1)$ and $m \in (-1,0)$ correspond to two dif-
ferent gapped topological phases of the fermions, which are separated by the quantum phase transition at $m = 0$.
The critical point $m = 0$, the fermion become gapless and
the system is described by a (1+1)D conformal field
theory (CFT) with central charge $c = N/2$ (where $N$ is the
fermion flavor number). The central charge $c$ and the

fermion mass $m$ are two parameters that control the en-
tanglement feature of the Majorana chain. We will tune
these two parameters to study their effects on the holo-
graphic geometry.

The entanglement entropy of a free fermion state $|\Psi\rangle$
can be efficiently calculated from the fermion correlation function.$^{92,93}$ Let $C_A = \langle\Psi|\chi\chi^T|\Psi\rangle_A$ be the fermion cor-
relation restricted to the entanglement subregion $A$, the
2nd Renyi entanglement entropy is then given by

$$S_\Psi(A) = -\frac{1}{2} \text{Tr} \ln \left( C_A^2 + (1 - C_A)^2 \right).$$

(23)

We can then collect the entropy $S_\Psi(A)$ over arbitrary
region $A$. The entanglement cut is always placed between
the unit cells (i.e. the region $A$ always contains complete
unit cells). Therefore the local Hilbert space dimension
in each unit cell is $D_0 = 2^N = 2^{2c}$. Correspondingly,
the external pinning field in the Ising model is set by $h = \frac{\pi}{2} \ln D_0 = c \ln 2$.

In the following, we will perform the EFL on the
ground state of the Majorana fermion chain. The lattice
is fixed to the size of 64 sites (i.e. 32 unit cells) with the
periodic boundary condition. The entanglement features
are collected from Eq. (23) and then served to the ma-
chine as the training data. For the 32-unit-cell fermion
chain, there are altogether $2^{32}$ possible choices of the en-
tanglement region $A$ (as each unit-cell can choose to be
included in the region $A$ or not). Obviously, it is both unfeasible and unnecessary to collect $S_\Psi(A)$ for all these
$2^{32}$ regions. We will only collect a subset of them. There
are several options to choose the sampling ensemble of
entanglement regions.

Option (1) is to sample all of them with equal prob-
ability. With this sampling scheme, most of the entan-
glement regions will contains multiple small and discon-
nected intervals. Consequently, this sampling is not effi-
cient at conveying large-scale entanglement features for
large single intervals (which represent the correlations be-
tween far-separated entanglement cuts in the Ising model
language).

Option (2) is to sample only single interval regions. As
the interval length varies, these regions cover different scales of entanglement features, but the multi-par-ite en-
tanglement features are missing. We will use this single-
interval data for some testing cases to see if the machine
has the generalization ability to predict multi-interval en-
tropies form single-interval data.

Option (3), the most comprehensive one, is to weight the entanglement region $A$ by the number of intervals
$n_A$ in $A$, such that the probability distribution $p(A) \sim e^{-n_A/\bar{n}}$ is controlled by the average interval number $\bar{n}$.
We may tentatively take $\bar{n} = 2$, which provides a nice
balance between the single-interval and the multi-interval
entanglement features. We call this the interval-weighted
sampling scheme for the entanglement regions. As we
have checked in our numerics, the choice of $\bar{n}$ does not
affect training result much (which may be an indication of
the internal consistency in entanglement features col-
lected at different interval numbers).

B. Choosing the Central Charge

We first fix the fermion mass to \( m = 0 \) and run the EFL on the hyperbolic network architecture. The visible layer has 32 units, matching the 32 unit cells of the fermion chain. Each deeper layer halves the number of units, so the number of units per layer vanishes after five layers, and the network can not go deeper. A uniform weight \( I_z \) (or equivalently the Ising coupling \( J_e = I_z / 4 \)) is assigned to all links in the same layer, where \( z = 1, 2, \cdots, 5 \) labels the layer depth.

We adopt the supervised learning approach described in Eq. (12). The EFL algorithm is implemented\(^{105} \) on the TensorFlow\(^{94} \) system using Adam\(^{95} \) optimizer. We use the interval-weighted scheme to sample the entanglement regions and prepare the training data for this study. As shown in Fig. 7(a), the (relative) loss \( \mathcal{L} \) decreases with the training steps and converges to \( \sim 10^{-3} \) eventually. Although the learning algorithm is deterministic, noise is still introduced by the randomly batched training data, leading to the fluctuations of \( \mathcal{L} \). Nevertheless, the noise in the training data will not wash out the gradient signals in deep layers, thus the deep network is still trained efficiently.

![Figure 7: Typical training curves of (a) the objective function \( \mathcal{L} \) and (b) the edge weight \( I_z \) in each layer.](image)

Driven by the training data, the edge weight \( I_z \) develops one layer after another as shown in Fig. 7(b). Apart from the first layer weight \( I_1 \), the rest of the weights all converge to the same value controlled by the regularization Eq. (21). If the regularization condition is lifted, we observe that the machine has the tendency to develop unphysical weights to overfit the data.

We take the final values of the weights \( I_z \) and plot them in Fig. 8(a). As we tune the central charge \( c \) of the fermion chain, the behavior of \( I_z \) undergoes a transition around \( c = 2 \). When the central charge is smaller than that (e.g., \( c = 1/2, 1 \)), the deep layers will not be trained. This corresponds to an order-disorder transition of the Boltzmann machine. Smaller central charge means weaker entanglement and smaller edge mutual information in the RTN. Since the edge mutual information \( I \) maps to the Ising coupling \( J = I / 4 \), decreasing the coupling \( J \) could drive the system into the paramagnetic phase. Then the original assumption on the large edge mutual information fails and the physical picture of representing the entanglement entropy by the domain wall energy in the holographic bulk no longer holds. To estimate the critical coupling \( J_c \) on the hyperbolic network, we first pin the boundary spins to the same direction and then measure the magnetization of the spin at the deepest layer to see if the magnetization can propagate through the system all the way from the boundary to the deepest layer in the bulk. As shown in Fig. 8(b), we found an activation behavior in the magnetization curve, which roughly divides the coupling \( J \) into paramagnetic-like or ferromagnetic-like regimes. Although the transition is smeared out in the finite-sized system, we can still give an estimate of the critical \( J_c \approx 0.15 \) (or \( I_c \approx 0.6 \)) from the extrapolation of the activation slope. Fig. 8(a) indeed shows that as \( I_z \) drops below the level of \( I_c \), the training signal disappears and the deep layer weights cease to develop.

![Figure 8: (a) Final values of the edge weights \( I_z \) on different central charges. The dashed line marks the level of the critical weight \( I_c \approx 0.6 \). (b) The IR spin magnetization under UV pinning vs the Ising coupling \( J \). The critical \( J_c \approx 0.15 \) is estimated from extrapolation the activation slope.](image)

In the AdS/CFT duality, the central charge \( c \) of a holographic CFT\(_2\) is universally given by \( c = 3\ell^2 / 2G_N \),\(^{4,96} \) where \( \ell \) is the AdS radius and \( G_N \) is the Newton constant in three dimensional gravity. Our approach of fixing the tensor network architecture and training the edge weights corresponds to fixing the AdS radius. Then changing the central charge \( c \) effectively changes the gravitational constant \( G_N \). Large \( c \) corresponds to small \( G_N \) and hence weakly coupled classical gravity. The classical holographic geometry can be represented by the classical network geometry that can be trained by the EFL. As the central charge \( c \) gets small, the gravity crosses over from classical to quantum and the EFL ceases to produce a sensible result. Therefore, in the following, we will fix the central charge at \( c = 4 \) on the classical side.

C. Single-Interval Entanglement Entropy

For the critical fermion chain \( m = 0 \), it is known that the single-interval Renyi entanglement entropy (i.e. the entanglement region \( A \) is a single continuous interval)
follows the logarithmic law $S(L_A) \sim \ln L_A$ in the thermodynamic limit. To see how well the RTN can reproduce this logarithmic entropy scaling after training, for this study we serve the machine with only the single-interval 2nd Renyi entanglement entropies taken from a critical fermion chain of 32 unit cells (calculated from Eq. (23) using the lattice model). After the training, we ask the machine to reproduce the entanglement entropies over the trained intervals and compare the predictions with the actual values. The result is shown in Fig. 9.

\[ S(L_A) \sim \ln L_A \]

In Fig. 9(a), we see the actual entropy values compared to the predicted values for the hyperbolic architecture. The hyperbolic network is able to capture the logarithmic scaling, but there are some deviations at smaller interval lengths. In Fig. 9(b), we see the same comparisons for the cylindrical architecture. The cylindrical network also follows the logarithmic entropy scaling, but the deviation is more significant at smaller interval lengths.

D. Multi-Interval Entanglement Entropy

To test the prediction power of the RTN model, we train a hyperbolic network using single-interval entanglement entropies and ask if the network can predict multi-interval entanglement entropies. Let us use different colors to label the entanglement entropies over different numbers of intervals, and plot the predicted entropy against the actual entropy in Fig. 10. In the training phase, only the single-interval data is presented to the machine. After the training, the machine was able to predict multi-interval entanglement entropies, which was not in the training set. If the prediction is perfect, then all the points should fall along the diagonal line in Fig. 10. We can see the points do line up nicely, especially when the number of intervals is small. The overall prediction accuracy is $\sim 95\%$.

This demonstrates the prediction power of the RTN model. However, this may not be very surprising. Since the multi-interval entanglement entropy is related to the single-interval ones

\[ S(A \cup B) = S(A) + S(B) - I(A, B). \]  

If the mutual information $I(A, B)$ is small, the multi-interval entropy is dominated by the additive part $S(A \cup B) \approx S(A) + S(B)$, which is relatively easy to capture. So we will turn to the sub-additive part (i.e. the mutual information) in the following.

E. Mutual Information

We found that for adjacent intervals, the RTN model can still fit the mutual information well, as shown in Fig. 11(a). There is a geometric interpretation of this type of mutual information in the holographic bulk. According to the Ryu-Takayanagi formula, the entanglement entropy $S(A)$ of the interval $A$ is proportional to the area of the minimal surface $\gamma_A$, which, in the AdS$_3$ space, is also the geodesic line connecting the two bound-
ary points of the interval $A$. Therefore the mutual information of adjacent intervals $A$ and $B$ corresponds to

$$I(A, B) = S(A) + S(B) - S(AB) = \frac{1}{4G_N}(|\gamma_A| + |\gamma_B| - |\gamma_{AB}|). \quad (25)$$

$\gamma_A$, $\gamma_B$ and $\gamma_{AB}$ form the three sides of a triangle in the holographic bulk. The mutual information measures how much is the sum of the two sides greater than the third side. This indicates that the machine gets a grasp of the holographic geometry in its neural network, so it can provide a good prediction of the mutual information that has classical geometric interpretations.

![Diagram](image)

**FIG. 11:** Fitting the mutual information of (a) two adjacent equal-length intervals and (b) two separated equal-length intervals with the separation region of the same length as the interval length.

In contrast, for separated intervals, the predicted mutual information is obviously less than the actual value by quite a large amount, as shown in Fig. 11(b). This is actually not a problem of our algorithm, but has a deep physical origin. States with semi-classical dual which satisfies RT formula are necessarily strongly correlated and contain a lot of multi-partite entanglement. For example, it is known that holographic states have large and negative tripartite information, in contrast from the free fermion theory. RTN is designed to describe holographic states, which have much smaller mutual information between separated intervals compared to that in the free fermions. The free fermion conformal field theory has many low-dimension operators, which corresponds to light matter fields in the dual gravity theory. In our approach, these matter field fluctuations are not taken into account, which partially explains the the deficit of mutual information in Fig. 11(b). Also, our approach only captures the optical classical geometry and does not include the quantum fluctuation of geometries around the classical saddle points. How to go beyond the planar graph EFL and include the fluctuation effect of both matter fields and geometries is an interesting topic for future research.

**F. Emergent Holographic Geometry**

Finally, we turn on the fermion mass $m$. The fermion correlation length $\xi$ becomes finite and is given by

$$\xi^{-1} = \frac{1}{2} \ln \frac{1 + |m|}{1 - |m|}. \quad (26)$$

In the holographic bulk, the fermion mass caps off the IR region at the scale $z_{1R} \sim \ln \xi$. Because the entanglements are resolved in the UV layers of the RTN at this scale, the network ceases to grow deeper and the holographic space ends. As the mass $m$ is turned on, the edge weight will start to fade away from the deepest layer, as shown in Fig. 12(a). With increasing mass, the fade-off scale $z_{1R}$ moves from IR (large $z$) toward UV (small $z$), see Fig. 12(a).

![Diagram](image)

**FIG. 12:** (a) The edge weight $I_z$ in each layer, trained from the fermion model with different mass $m$. (b) Color plot of the edge weight $I_z$ as a function of mass $m$ and the layer depth $z$. The dashed line is the curve of $\log_2 \xi$ with $\xi$ taken from Eq. (26).

We scan over a range of mass $m \in [0, 0.5]$. At each $m$, we train the machine and obtain the edge weight $I_z$. The result is shown in Fig. 12(b). There is a clear boundary where the holographic space terminates. This boundary matches the theoretical expectation $z_{1R} = \log_2 \xi$ nicely (we take $\log_2$ here because of each deeper layer halves the number of unit in the hyperbolic network architecture). This demonstrates how the AdS$_3$ spacial geometry emerges as we gradually decrease the mass $m$ and drive the boundary system toward the CFT$_2$.

**V. DISCUSSIONS AND SUMMARY**

In this work, we have restricted the entanglement feature to the 2nd Renyi entropies. It is actually conceptually more natural to include all orders of Renyi entropies over all regions in the entanglement feature. Ref. 31 shows that the $n$th Renyi entropy of the RTN state can be mapped to the free energy difference of an $S_n$ model in the large bond dimension limit. In the $S_n$ model, each vertex $v \in V$ hosts a permutation group element $\sigma_v \in S_n$, coupled together via the energy functional

$$E_G[\sigma, \tau] = -\sum_{e \in E} \chi_e \left( \prod_{v \in \partial e} \sigma_v \right) - \sum_{v \in V_0} \chi_0 (\tau_v^{-1} \sigma_v). \quad (27)$$
where $\chi_e(g)$ and $\chi_\sigma(g)$ are class functions that only depend on the cycle type $l^a_\sigma$ of the permutation $g$ (i.e. $l^a_\sigma$ is the length of the $\alpha$th cycle in $g$). More specifically, we have

$$
\chi_e(g) = \frac{1}{2} \sum_{\alpha} (l^a_g - 1) I_e(l^a_\sigma),
$$

$$
\chi_\sigma(g) = \sum_{\alpha} (l^a_g - 1) \ln D_\sigma.
$$

(28)

The $\chi_e$ parameter is factorized by the edge mutual information $I^E_l$ for Renyi index $l = 2, \cdots, n$. The $\chi_\sigma$ term describes the boundary pinning field that pins the boundary configuration to another set of permutation group elements $\tau_v \in S_n$. By defining $e^{-F_G[\tau]} = \sum_{|\sigma|} e^{-E_G[\sigma,\tau]}$, we can consider $S_{\tau}[\tau] = F_G[\tau] - F_G[\tau = 1]$. If we allow $\tau_v$ to take all group elements in $S_n$ (not limited to the cyclic permutations), the function $S_{\tau}[\tau]$ actually includes the RTN entanglement entropies over all regions for all Renyi indices up to $n$. In principle, using the $S_n$ model, entanglement entropies of different Renyi indices (up to $n$) can all be put together as the training data for the EFL, and the edge mutual information of all Renyi indices (up to $n$) will be trained simultaneously. However, the efficient training method for Boltzmann machines on $S_n$ models is still lacking, so the above idea is still not practical yet.

Despite the technical difficulties, the philosophy behind EFL is clear. For a quantum many-body state with a given tensor factorization of the Hilbert space (which specify the “real space basis”), one can forget about operator-specific information such as particular correlation functions, and focus on the local unitary invariant information. All local unitary invariant properties of the wavefunction can be considered as entanglement features of the wavefunction, which include the bipartite entanglement properties and also many more multipartite entanglement properties. From the point of view of gravitational dual, it is interesting to make an analog with the black hole no-hair theorem. The non-invariant features are removed and the geometry only encodes the local unitary invariant features, in the same way how the area of the black hole is proportional to its entropy and is independent from details of the initial state. The random average in RTN serves as a technical tool to remove “hairs” of a many-body state, where the operator specific information is erased by the random tensor, leaving only the entanglement features encoded in the network structure. Consequently, RTN can be potentially a useful framework for characterizing other phenomena in which entanglement features play an essential role, such as the many-body localization-thermalization transition, which is essentially a transition about entanglement structures. The EFL provides us an approach to construct the RTN and to optimize its structures, which could be a useful tool for the study of quantum chaotic dynamics and localization/thermalization.

In summary, the goal of the EFL is to construct an optimal RTN state that best fits the entanglement properties of a given quantum many-body state. The problem similar to the task of feature learning, which extracts the features hidden in the training data and encode them into the structure of the neural network. This analogy is made concrete by mapping the RTN to the Boltzmann machine and train the machine with the entanglement entropies over all subregions. As the entanglement feature is learned, the machine develops a neural network, whose network geometry can be interpreted as the emergent holographic geometry of the given quantum many-body state.

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Appendix A: Entanglement Entropy of RTN States

The RTN entanglement entropy defined in Eq. (5) can be equivalently expressed as

$$
e^{-S_{\tau}(A)} = E \frac{\text{Tr}(|G\rangle\langle G|)^\otimes 2 \hat{\tau}(A)}{\text{Tr}(|G\rangle\langle G|)^\otimes 2},
$$

(A1)

where $E$ denotes the average over the RTN ensemble, and $\hat{\tau}(A)$ is the swap operator in the subregion $A$. It can be factorized to each boundary vertex as

$$
\hat{\tau}(A) = \bigotimes_{v \in V_\partial} \hat{\tau}_v,
$$

$$
\hat{\tau}_v = \begin{cases} \otimes v \in A, \\ I \quad v \notin A. \end{cases}
$$

(A2)

The operator $\hat{\tau}_v$ swaps the replicated local Hilbert space $(\mathcal{H}_v^{\text{rep}})^\otimes 2$ on the vertex $v \in A$, otherwise it is an identity operator acting on the vertex $v \notin A$.

To evaluate Eq. (A1), we first introduce the rules for the ensemble average of the random state. Suppose $|\psi\rangle$ is a random state in an $N$-dimensional Hilbert space, because the random state ensemble is SU($N$) symmetric, due to the Schur’s lemma, the ensemble average of $|\psi\rangle\langle\psi|$ must be proportional to the identity matrix in respect of the SU($N$) symmetry. Under appropriate normalization, we can set $E|\psi\rangle\langle\psi| = 1$. Introducing the graphical representation of the random state,

$$
|\psi\rangle = \bigotimes, (|\langle\psi|\rangle = \bigotimes,
$$

(A3)

the formula $E|\psi\rangle\langle\psi| = 1$ can be represented as

$$E = \bigotimes.
$$

(A4)
For duplicated case, the formula is generalized to
\[
\mathbb{E}(\hat{A}^{\otimes 2}) = \mathbb{E}(\hat{A}^{\otimes 2}) = 1 + \mathbb{E} = \sum_{\sigma \in S_2} \Phi_{\sigma},
\]
(A5)
or \(\mathbb{E}(\hat{\psi}^{\otimes 2}) = \sum_{\sigma \in S_2} \hat{\psi},\) as a result of the SU(N) \(\times S_2\) symmetry.

Consider a RTN state on a graph with two vertices, each connected to an external edge (physical leg).

\[
|\mathcal{G}\rangle = A \otimes A \otimes \bar{A}.
\]
(A6)
Assuming the left and the right external edges are ascribed to the entanglement regions \(A\) and \(\bar{A}\) respectively. Using Eq. (A5), one can evaluate
\[
\mathbb{E} \text{Tr}(|\mathcal{G}\rangle \langle |\mathcal{G}|)^{\otimes 2} \hat{\tau}(A) = \mathbb{E} \text{Tr}(\hat{A}^{\otimes 2} \hat{\tau}(A)) = \sum_{|\sigma|} w(\hat{\sigma}_1, \hat{\sigma}_2) w(\hat{\sigma}_1, \hat{\sigma}_2),
\]
where \(\hat{\sigma}_i \in S_2\) arise from the ensemble average of the random states in the bulk. The weight function \(w(\hat{\sigma}_1, \hat{\sigma}_2)\) is actually a function of \(\hat{\sigma}_1^{-1} \hat{\sigma}_2\), which can be expressed in terms of the 2nd Renyi mutual information \(I_e\) of the entangled pair state along the edge
\[
w(\hat{\sigma}_1, \hat{\sigma}_2) = \begin{cases} 1 & \text{if } \hat{\sigma}_1^{-1} \hat{\sigma}_2 = \mathbb{I}, \\ e^{-I_e/2} & \text{if } \hat{\sigma}_1^{-1} \hat{\sigma}_2 = \mathcal{X}. \end{cases}
\]
(A8)

If we represent the \(S_2\) variable \(\hat{\sigma}_i\) by the Ising variable \(\sigma_i = \pm 1\), the weight function has a more compact form \(w(\sigma_1, \sigma_2) \propto e^{-J_e \sigma_1 \sigma_2}\) where \(J_e = I_e/4\). For external edges, \(J_e\) is replaced by \(h = \frac{1}{2} \ln D_0\) where \(D_0\) is the boundary bond dimension. So Eq. (A7) can be map to the partition function of an Ising model with fixed boundary condition \(\tau(A)\),
\[
\mathbb{E} \text{Tr}(|\mathcal{G}\rangle \langle |\mathcal{G}|)^{\otimes 2} \hat{\tau}(A) = e^{-F_0[\tau(A)]},
\]
(A9)
where \(e^{-F_0[\tau]} = \sum_{|\sigma|} e^{-E_0[\sigma, \tau]}\) and
\[
E_0[\sigma, \tau] = -h \tau_1 \sigma_1 - J_1 \sigma_1 \tau_2 - h \tau_2 \sigma_2.
\]
(A10)

It is straightforward to generalize the energy functional to generic graphs, given in Eq. (6). Correspondingly, Eq. (7) is just a rewritten of Eq. (A2) in terms of the Ising variables.

In the large bond dimension limit (large \(I_e\), the ensemble average of the fraction in Eq. (A1) can be approximated by average of the numerator and the denominator separately.
\[
e^{-S_0(A)} \simeq \frac{\mathbb{E} \text{Tr}(|\mathcal{G}\rangle \langle |\mathcal{G}|)^{\otimes 2} \hat{\tau}(A)}{\mathbb{E} \text{Tr}(|\mathcal{G}\rangle \langle |\mathcal{G}|)^{\otimes 2} \hat{\tau}(A)} = e^{-F[\tau(A)] + F[\tau(\emptyset)]},
\]
(A11)
Hence we have arrived at \(S_0(A) \simeq F[\tau(A)] - F[\tau(\emptyset)]\), verifying the result in Eq. (9). Ref. 31 has shown that the approximation of distributing the ensemble average into the fraction is valid in the large \(I_e\) limit by analyzing the fluctuation. A more careful treatment away from the that limit is provided in Ref. 68.

### Appendix B: Planar Graph Ising Model

In this appendix, we will review the systematic approach to calculate the free energy \(F\) of the Ising model on a planar graph \(\mathcal{G} = (\mathcal{V}, \mathcal{E})\), following Ref. 86,87.

\[
Z = e^{-F} = \sum_{|\sigma|} e^{-E[\sigma]}, \quad E[\sigma] = -\sum_{e \in \mathcal{E}} \sum_{\sigma \in \partial e} \sigma_v.
\]
(B1)

First of all, every planar graph can be triangulated by adding virtual edges, across with the Ising coupling \(J_e = 0\) is simple zero. If the boundary spins are also coupled to external Zeeman field \(h_v\), one can consider introducing a fictitious spin at infinity and coupling all the boundary spins to the fictitious spin with the coupling strength set by \(h_v\). This effective doubles the system by its \(Z_2\) symmetry (the Ising spin flip symmetry) counterpart, which only brings a factor 2 to the partition function but does not affect the free energy calculation.

With the tricks of virtual edges and the fictitious spin, we only need to consider the \(Z_2\) symmetric Ising model on the triangulated planar graph.

Every triangulated planar graph has a dual trivalent graph \(\mathcal{G}' = (\mathcal{V}', \mathcal{E}')\). as shown in Fig. 13(a), on which the Ising model is mapped to a loop model. Each Ising domain wall is interpreted as a loop on the dual lattice.

Introduce the \(Z_2\) variable \(l_z\) on the dual edges \(\hat{e}\), such that \(l_z = 1\) corresponds to a loop through the edge \(\hat{e}\) and \(l_z = 0\) corresponds to no loop. The partition function Eq. (B1) can be mapped to
\[
Z = \sum_{|\sigma|} \prod_{e \in \mathcal{E}} e^{-F_0[\sigma_e]} = \sum_{|\sigma|} \prod_{|\hat{e}|} e^{-F_0[\sigma_{\hat{e}}]} \prod_{\hat{e}} \delta_{\hat{e} \in \mathcal{V}} \left( \sum_{l_z \in \mathcal{V}} l_z \right),
\]
(B2)
where the weight is \(w_{\hat{e}} = e^{2l_z}\) (where \(e\) the edge in the original graph that is dual to the edge \(\hat{e}\) in the dual graph) and the factor \(Z_0 = e^{-F_0}\) is given by \(F_0 = \sum_{e \in \mathcal{E}} J_e\). The delta function \(\delta_{\hat{e} \in \mathcal{V}}\) over the \(Z_2\) group imposes the close loop constraint. Unlike conventional loop models, here each segment of the loop (the domain wall) is given a trivial weight 1, while the edge without the loop is given a greater weight \(w_{\hat{e}} = e^{2l_z}\) (for \(J_e \geq 0\)) instead. In this way the domain wall is still relatively suppressed in the partition function. The overall factors generated in this weight rescaling are all absorbed into \(Z_0\).

Further expanding each trivalent site into a triangle, as shown in Fig. 13(b), the loop model can be mapped
to a dimmer model,\textsuperscript{82} where the loop configuration is replaced by the transition graph of dimmer configurations. Let $\Omega$ be the set of all dimmer coverings (perfect matchings) of the extended graph $G'$ in Fig. 13(b), the partition function Eq. (B2) becomes

$$Z = Z_0 \sum_{M \in \Omega} \prod_{e' \in M} w_{e'}$$  \hspace{1cm} (B3)$$

In the dimmer model, each thick edge covered by the dimmer is weighted by $w_{e'} = e^{2J_e}$. The remaining thin edges all share $w_{e'} = 1$.

![Diagram](a) The original graph (in orange) and its dual graph (in green). Each edge $e$ in the original graph is dual to a unique edge $\tilde{e}$ in the dual graph, such that $e$ and $\tilde{e}$ intersect. (b) The extended graph (star lattice) by expanding each site to a three sites in a triangle.

**FIG. 13:** (a) The original graph (in orange) and its dual graph (in green). Each edge $e$ in the original graph is dual to a unique edge $\tilde{e}$ in the dual graph, such that $e$ and $\tilde{e}$ intersect. (b) The extended graph (star lattice) by expanding each site to a three sites in a triangle.

![Diagram](a) Systematic assignment of the Kasteleyn orientation on planar graph. (a) Start from an arbitrary vertex (mark by the red dot) and build a spanning tree. (b) Close the loops respecting the clockwise-odd rule.

**FIG. 14:** Systematic assignment of the Kasteleyn orientation on planar graph. (a) Start from an arbitrary vertex (mark by the red dot) and build a spanning tree. (b) Close the loops respecting the clockwise-odd rule.

The partition function of the dimmer model Eq. (B3) can be formulated as a path integral of free Majorana fermions, with fermion spin structure specified by the Kasteleyn orientation.\textsuperscript{87,103} The insight is that every non-zero term in the Majorana fermion path integral corresponds to a perfect matching on the graph $G'$ (on which the dimmer model is defined). To place the fermion system on the graph $G'$, each edge must be assigned an orientation, such that for every face (except possibly the external face) the number of edges on its perimeter oriented in a clockwise manner is odd, known as the clockwise-odd rule. Any orientation satisfying the clockwise-odd rule is a Kasteleyn orientation, which ensures all dimmer configurations to be mapped to even fermion parity states. The Kasteleyn orientation can be assigned systematically on planar graphs by first choosing an arbitrary vertex in the graph and build a spanning tree from that vertex, then closing the loops respecting the clockwise-odd rule, as demonstrated in Fig. 14.

With the Kasteleyn orientation assigned, we can construct the weighted adjacency matrix $A$ of the graph $G' = (V', E')$, such that $\forall i, j \in V'$: $A_{ij} = 0$ if $\langle ij \rangle$ is not an edge in $E'$, $A_{ij} = w_{ij}$ if the orientation on edge $\langle ij \rangle$ runs from $i$ to $j$, and $A_{ij} = -w_{ij}$ otherwise. The partition function can then be shown to be

$$Z = Z_0 \int \mathcal{D}[\chi] e^{-\frac{1}{2} \chi^T A \chi} = Z_0 \text{pf } A.$$  \hspace{1cm} (B4)$$

So the free energy of the Ising model can be calculated from

$$F = F_0 - \ln \text{pf } A,$$  \hspace{1cm} (B5)$$

where $F_0 = \sum_{e \in E} J_e$ and $A$ is the adjacency matrix of the Kasteleyn oriented extended dual graph $G'$.
There are certainly states that cannot be described by a planar RTN. The simplest example is a state with two long-range EPR pairs between boundary points $x_1, x_3$ and $x_2, x_4$, with the points ordered as $x_1 < x_2 < x_3 < x_4$.

The source code is available at the GitHub repository https://github.com/EverettYou/EFL.

Another idea to realize the hyperbolic geometry on the cylindrical network is to allow the translation symmetry breaking, so that some bonds in the deep layers can be turned off in a pattern similar to the hyperbolic network. In this way, it is possible to realize the logarithmic entropy scaling on a cylindrical network. We will leave this possibility for future research.

More generally, one can consider all local unitary (LU) invariants, which are of the form $\text{tr} \left[ \rho^k \prod_i g_i \right]$ with $g_i$ on each site $i$ an arbitrary component of the $S^k$ permutation group.