Realizing and adiabatically preparing bosonic integer and fractional quantum Hall states in optical lattices

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Realizing and Adiabatically Preparing Bosonic Integer and Fractional Quantum Hall states in Optical Lattices

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We study the ground states of 2D lattice bosons in an artificial gauge field. Using state of the art DMRG simulations we obtain the zero temperature phase diagram for hardcore bosons at densities n_b with flux n_ϕ per unit cell, which determines a filling $\nu = n_b/n_\phi$. We find the bosonic Jain sequence ($\nu = p/(p + 1)$) states, in particular a bosonic integer quantum Hall phase (BIQH) at $\nu = 2$, are pretty robust in the hard-core boson limit, In addition to identifying Hamiltonians whose ground states realize these phases, we discuss their preparation beginning from independent chains, and ramping up interchain couplings. Using time dependent DMRG simulations, these are shown to reliably produce states close to the ground state for experimentally relevant system sizes. Our proposal only utilizes existing experimental capabilities.

The two-dimensional Bose-Hubbard model is one of the simplest many body systems that exhibits nontrivial physics. Initially proposed as a model for the superconductor insulator transition in solid state system[1,2], it was later realized most cleanly in optical lattices of ultracold atoms[3,4]. It has been widely studied by varying the ratio of hopping to interaction strength t/U, and the filling n_b of bosons per site. A third natural parameter is the magnetic flux n_ϕ, tuning of which has been demonstrated recently in ultra-cold atomic systems in periodically driven optical lattices[5]. The phase diagram as a function of magnetic flux through the unit cell is less understood. This is the bosonic analog of the Harper-Hofstadter problem of free electrons in a tight binding model with magnetic flux[6]. However to realize interesting phases, the bosonic problem is necessarily interacting (also see related study of fermions[7–11]).

At finite flux density, quantum Hall phases (QH)[12–14] of bosons might appear if the filling factor $\nu = n_b/n_\phi$ is appropriate. One interesting state corresponds to $\nu = 2$, it is called bosonic integer quantum Hall state (BIQH)[15,16]. It belongs to the newly discovered symmetry protected topological (SPT) phase[17–19], different from all other fractional quantum Hall states that are intrinsically topologically ordered. This BIQH state was theoretically found before, e.g. two-component bosons or higher Chern number flat bands model[20–27]. The BIQH indeed can be constructed using the well-known composite fermion approach[28]. Specifically one can first attach one flux quantum to the bosons, converting them into composite fermions, and letting them form a $\nu_{CF} = p$ integer quantum Hall state. This construction gives the so called Jain sequence states at filling factor $\nu = p/(p+1)$, taking $p = -2$ gives the BIQH state[29].

For a given filling factor ν, there could be different competing phases (e.g. different QHs, ordered states). Which phase is the groundstate is an energetic problem that usually differs case by case, but it is very useful if one can learn some general knowledge about the appearance of Jain's composite fermion states. For fermions, it is found that the Jain sequence state systematically appears in the lowest Landau level. However, bosons behave quite differently: the half of the Jain sequence (the hole part with $p < 0$) is missing in the continuum limit with the lowest Landau level of bosons ($U \ll t\phi$) (e.g. see a review[30] for numerics and Ref.[31,32] for analytical results). Therefore it is interesting to pass to the lattice, on which one can achieve the infinite interaction limit $U/t \to \infty$ that may not be continuously connected with the continuum limit. Indeed, early work motivating the search for lattice effects reported a candidate BIQH at low densities ($n = 1/7, 1/9$)[29]. Also, previous ED calculations on small system size found several Jain's composite fermion states with $p = 1, 2$[29,33–35]. In this paper, we systematically show that the full Jain sequence (at least up to $p = \pm 5$), in particular the BIQH state, appears in the $U \to \infty$ limit.

Even if a quantum Hall state is the groundstate of a simple Harper-Hofstadter model, it remains challenging for cold-atom experiments to realize. Cooling into a nontrivial ground state poses special challenges particularly in the context of driven systems such as the Floquet engineered optical lattice systems[5,36–44]. We need a cooling scheme to overcome this issue. One way of cooling, called adiabatic preparation[45–49], begins with a trivial state with low entropy, which is then slowly ramped to the desired final state. Such adiabatic preparation schemes in general require a continuous phase transition between the initial state and the final state. For a quantum Hall state, an adiabatic preparation scheme is even more difficult, since usually an exotic phase transition e.g. deconfined phase transition) will be involved[48]. Finding an appropriate adiabatic preparation scheme for optical lattice quantum Hall states is the second question on which we will make progress, and in particular our scheme appears to work for most quantum Hall phases, at least for the system sizes relevant for experiments.

We will first present our DMRG simulation[50–52] which numerically finds robust Jain sequence states $p/(p + 1)$ (e.g. $p = 1, \pm 2, \cdots, \pm 5$) on the lattice with a relatively high particle density. In particular, the BIQH state (at $p = -2$) is found robust with a short correlation length
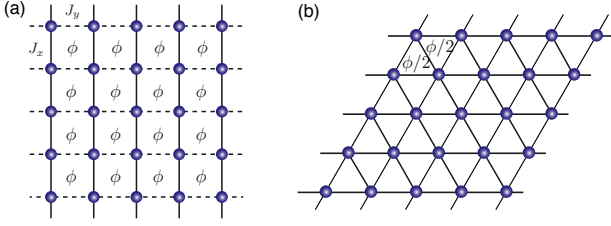
FIG. 1. Harper-Hofstadter model on (a) square lattice with flux $\phi = 2\pi n_\phi$ on each square plaquette, (b) triangular lattice with flux $n_\phi/2$ on each triangle plaquette.

and quantized Hall conductance. A related state was observed in the low density limit in Ref.[29]. Next we use time-dependent DMRG simulations[53–56] as well as exact diagonalization to discuss the adiabatic preparation scheme for quantum Hall phases, focusing on the BIQH state. The basic idea is beginning with the independent chain limit of 1D Luttinger liquids, and ramping up interchain couplings that also introduce the flux. To benchmark the effectiveness of our preparation scheme, we utilize the wave-function overlap between the state generated by the time ramp and the true ground state as an indicator. We also discuss a physical diagnosis using two-point correlation function to detect the gapless edge state of quantum Hall phases.

Model and Phases.—We consider the Bose-Hubbard model (Harper-Hofstadter model) on a square (triangular) lattice,

$$H = -J\sum_{\langle ij\rangle} e^{iA_{ij}} a_i^\dagger a_j + U\sum_i n_i(n_i - 1) \qquad (1)$$

The first term is the nearest neighbor hopping subject to a background flux A_{ij}, with $\sum A = \phi = 2\pi n_\phi$ on each square plaquette (or $n_\phi/2$ on each triangle). The second term is the on-site Hubbard interactions, and we mainly consider the limit $U \to \infty$ that gives the hard-core boson constraint $n = 0, 1$. Numerically we confirm the phases also survive under finite U.

One may expect quantum Hall phases for certain filling factor $\nu = n_b/n_\phi$, where n_b is the boson density per site. The simplest possibility is the Jain sequence with $\nu = p/(p + 1) = n_b/n_\phi$ from the composite fermion approach[28]. First, one can attach one flux quantum to the boson, yielding the composite fermion. The composite fermions still have density n_b and see an effective flux $n_\phi - n_b = n_b/p$, then naturally they will form an integer quantum Hall state with $\nu_{\rm CF} = p$. Naively, the continuum limit, which can be formulated as lowest Landau level with contact Haldane's pseudo-potential $V\delta(r - r')$, is the most ideal platform for quantum Hall phases. In that limit, however, several states particularly the BIQH state ($p = -2$) were not found in the extensive study (e.g. see a review[30]).

In this paper, we focus mainly on the limit with $U \to \infty$. Unexpectedly we numerically find that the Jain sequence states ($p = 1, \pm 2, \cdots, \pm 5, \cdots$) systematically

TABLE I. A brief summary of Jain's sequence on the square lattice with small $p = 1, \pm 2$ obtained in our DMRG simulations. n_b is the density per site. n_ϕ is the flux per square plaquette. The simulations are mainly carried on an infinite cylinder with circumference $L = 6, \cdots, 12$.

$\sigma^{xy} = \frac{p}{p+1}$	n_ϕ	n_b
$p = 1$	1/4	1/8
$\sigma^{xy} = 1/2$	1/5	1/10
	1/6	1/12
Laughlin State	…	…
$p = 2$	1/4	1/6
$\sigma^{xy} = 2/3$	1/5	2/15
	1/6	1/9
Halperin's (221) State	…	…
$p = -2$	1/6	1/3
$\sigma^{xy} = 2$	1/8	1/4
Bosonic Integer	1/10	1/5
Quantum Hall	…	…

appear in this limit. We also note that even if $n_\phi \ll 1$, the system we consider here is still different from the continuum limit. It is because the infinite on-site interaction U will be much larger than the Landau level spacing ($\sim n_\phi J$), making the simple Landau level physics invalid. Theoretically the flux attachment requires the boson to be a hard-core object, hence the infinite U may energetically help the flux attachment to happen. This may be an intrinsic mechanism for our numerical observation.

Several methods were applied to study this problem before[29,33–35,57–61], here we will use the infinite DMRG simulation[52] to tackle it. We numerically observe Jain sequence states of bosons at filling factor $\nu = p/(p + 1)$ for $p = 1, \pm 2, \cdots, \pm 5$. Generally the instability of the Jain's states grows with p. A consequence is that, to realize a larger p one needs a more dilute density (meaning a smaller n_ϕ and n_b). On the other hand, we also find that the Jain sequence states are more stable on the triangular lattice[62]. Here and the following we mainly focus on small $p = 1, \pm 2$ on the square lattice as summarized in Table I. The results of larger p are summarized in the supplementary materials[62]. We study infinite cylinder with circumference $L_c = 6, \cdots 12$ and different sizes give consistent results. For a smaller flux density n_ϕ than we show in the Tables, we expect the same quantum Hall state still exists. A particularly interesting state corresponds to $p = -2$, that is the BIQH state at $\nu = 2$. Unlike fractional QH, BIQH doesn't possess topological order, instead it is a SPT (protected by the $U(1)$ charge conservation). So here our results provide a very simple setting for experimentally realizing the putative interacting SPT phase in spatial dimension higher than $d = 1$.

We numerically diagnose those quantum Hall phases through their quantized Hall conductance (many-body
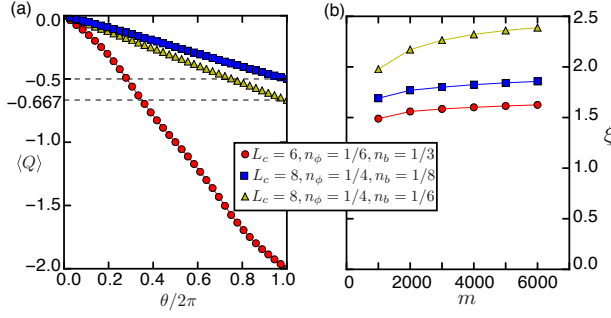
FIG. 2. Numerical diagnosis of quantum Hall states $\nu = 2, 1/2, 2/3$, L_c is the circumference. (a) Quantized Hall conductance measured from flux insertion on an infinite cylinder. Charge transfer as a function of flux: $\langle Q \rangle = -\sigma_{xy} \frac{\theta}{2\pi}$. (b) The correlation length ξ of quantum Hall state versus bond dimension m in DMRG simulations showing convergence. The truncation error of DMRG simulation is around $10^{-8} \sim 10^{-10}$. The correlation length is small compared with the circumference, indicating that our DMRG simulation is reliably producing 2D physics.

Chern number). To measure the Hall conductance, we wrap the system on a cylinder, and measure the charge pumping by threading 2π flux[63,64]. The pumped charge is exactly the Hall conductance σ^{xy}[65]. As clearly shown in Fig. 2 (a), the Hall conductance is precisely $\sigma^{xy} = 1/2, 2/3, 2$ for three quantum Hall states. Also we find that the states have a short correlation length (Fig. 2 (b)), indicating a fully gapped state.

Adiabatic preparation from the 1D phase.—One important challenge for cold atom experiments is to cool into the ground state. We now discuss one preparation scheme for preparing quantum Hall phases using adiabatic preparation starting from decoupled 1D wires. The idea is that we first turn off hopping along one direction (say $J_y = 0$). In this limit, we get decoupled 1D Luttinger liquids with density n_b. Then we slowly turn on the hopping J_y (that also introduces the flux), which eventually yield a 2D bosonic quantum Hall phase at the isotropic limit $J_y = J_x$. The adiabatic preparation schemes by coupling smaller sub-systems have also been used elsewhere[66].

Numerically we find this scheme can achieve the adiabatic preparation for bosonic (both fractional and integer) quantum Hall phases. One piece of numerical evidence is the properties of the groundstate as we ramp the system from 1D wires to a 2D quantum Hall state. First we find the physical quantities (e.g. the energy and entanglement entropy) evolve continuously as we change the parameter (J_y). Second we observe that the wave-function of the groundstate of the system is changing smoothly, namely the wave-function overlap $|\langle \psi(J_y)|\psi(J_y + dJ_y)\rangle| \to 1$ as $dJ_y \to 0$.

To make a more direct contact with experiments, we also simulate the preparation scheme as the non-equilibrium process. It can be generally described by $|\psi_f\rangle = \int_0^T dt\, e^{-itH(t)}|\psi_0\rangle$. $H(t)$ is the time-dependent
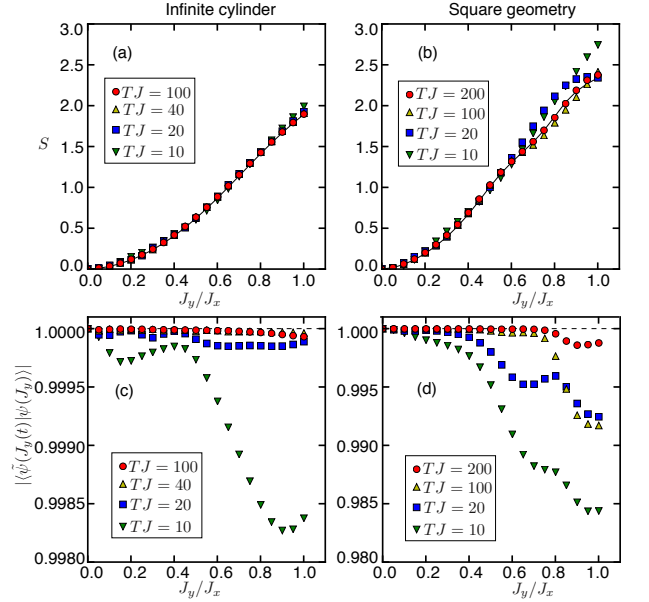

FIG. 3. Non-equilibrium dynamics simulation of the preparation scheme with different ramp time T, and we define $J = J_x$. We show the results of the BIQH state at $n_\phi = 1/6$, $n_b = 1/3$ of a square lattice placed on both the $L_c = 6$ infinite cylinder (a), (c) and the 6×6 square geometry (b), (d). (a), (b) shows the time evolution of entanglement entropy, where the solid line represents the entanglement entropy of the groundstate versus J_y/J_x, the dots represent the entanglement entropy from the time evolution. (c), (d) shows the wave-function overlap per-site between the groundstate $\psi(J_y)$ and the state from time evolution $\tilde{\psi}(J_y(t))$. Intriguingly $TJ = 20$ seems to give a better groundstate than $TJ = 100$, the physical reason is unclear.

Hamiltonian that will be tuned experimentally,

$$H(t) = -J_x \sum_{\langle ij \rangle_x} e^{iA_{ij}} a_i^\dagger a_j - J_y(t) \sum_{\langle ij \rangle_y} e^{iA_{ij}} a_i^\dagger a_j$$
$$+ U \sum_i n_i(n_i - 1) \tag{2}$$

with time-dependent hopping on the y direction, $J_y(t) = J_x t/T$. ψ_0 is the initial state, that is the groundstate of the starting Hamiltonian $H(0)$. Numerically we first discretize the time-evolution operator, $\int_0^T dt\, e^{-itH(t)} \approx \prod_{n=0}^m e^{-i(t_{n+1}-t_n)H(t_{n+1})}$, with $t_n = nT/m$, $m \gg 1$, and the final Hamiltonian $H(t_f = T)$ is the one in Eq. (1). Following the method introduced by Zaletel, et al.[56], we then rewrite the operator $e^{-i(t_{n+1}-t_n)H(t_{n+1})}$ as a matrix product operator, and apply it to the wave-function successively.

Fig. 3 shows the numerical results for the preparation scheme. We carry out simulations for i) the infinite cylinder geometry (y direction is taken to infinite); ii) the finite square geometry that has open boundary condition on both the x and y direction. To quantify how good the adiabatic preparation is, we compare the state from the time-evolution ($\tilde{\psi}(J_y(t))$) with the true groundstate

($\psi(J_y)$) of the static Hamiltonian. Specifically, we compare the entanglement entropy and wave-function overlap between two states. Clearly the adiabatic preparation works well for both schemes, and the quality of the adiabaticity increases as the preparation time becomes longer. In particular the wave-function overlap (per site) can reach 0.9999, which is a strong proof for our adiabatic preparation scheme.

The finite square geometry works worse than the infinite cylinder geometry (e.g. see Fig. 3(c)-(d)). Such behavior is expected since the quantum Hall state on a finite square geometry has gapless edge modes. The gapless modes will inevitably lead to some undesired excitations in an adiabatic preparation scheme. Fortunately, the experimental study as well as our numerical simulations are carried out on a finite system, which has a finite gap $\Delta E \propto 1/L$. Therefore as long as the ramping time is long enough, the adiabatic preparation can be ideally achieved. The preparation scheme can be further optimized by adding a second tuning parameter, the magnetic flux $\phi = 2\pi n_\phi$[47]. Tuning of ϕ has recently been realized using quantum gas microscopes[5]. More details can be found in the supplementary materials[62].

Physical diagnosis of quantum Hall state.— Finally we study a simple correlation function based method to diagnose quantum Hall states in mesoscopic geometries. Although it is presently unclear how to directly measure this quantity in experiments, this will serve as a proxy for other correlation function based approaches to study quantum Hall states. The QH state has a gapped bulk but a gapless edge. To observe this property, one can measure the correlation function $\langle a_0^\dagger(x)a_y(x)\rangle$ along one direction as shown in Fig. 4(a). x represents the position on the $\vec{x}$ direction, and $a_0(x)$ is always placed on the edge. When $x \sim 0$, the two-point correlation function is always measured on the edge, hence will give a power law decaying behavior $\langle a_0^\dagger(x)a_y(x)\rangle \propto 1/y^\alpha$. On the other hand, when x is placed in the middle of the sample ($x \sim L_x/2$), $\langle a_0^\dagger(x)a_y(x)\rangle$ is measuring the correlation function in the bulk yielding an exponentially decay behavior $e^{-y/\xi}$. However, once $a_r(x)$ hits the edge ($r \sim L_y$), $\langle a_0^\dagger(x)a_y(x)\rangle$ will follow a power law decay again. In summary, the two-point correlation functions behave as (we consider $x \le L_x/2$ due to the symmetry),

$$\langle a_0^\dagger(x)a_y(x)\rangle \propto \begin{cases} 1/y^\alpha, & x \sim 0 \\ e^{-y/\xi}, & x \sim L_x/2, y < L_y \\ 1/(y+2x)^\alpha, & x \sim L_x/2, y \sim L_y \end{cases} \quad (3)$$

Fig. 4(b) shows data of two point correlation functions of the bosonic integer quantum Hall state on a large system, 9×12 cluster (from DMRG). It is consistent with the above scaling behavior, Eq. (3). In particular, when $x \sim L_x/2$, the correlation function shows a non-monotonic behavior, it at first decays fast, but then suddenly increases as a_y hits the edge. Such scaling behavior is also visible on a small system size, e.g. 6×6 cluster in Fig. 4(c). In contrast, a superfluid
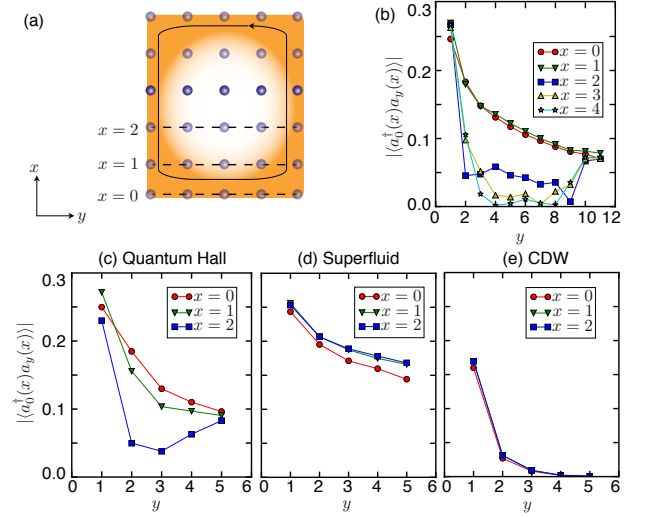

FIG. 4. Diagnosis of quantum Hall state by measuring two point correlation function $|\langle a_0^\dagger(x)a_y(x)\rangle|$. (a) The cartoon picture for a quantum Hall state on a finite $L_x \times L_y$ cluster. For a quantum Hall state, $x = 0, 1$ corresponds to the edge on which the correlation function decays algebraically. $x > 1$ corresponds to the bulk where the correlation function decays exponentially, however if $a_y(x)$ hits the edge ($y \sim L_y$), the correlation function obeys power law. Numerical results: (b) 9×12 cluster, $n_\phi = 1/6$, $n_b = 1/3$, $\nu = 2$ quantum Hall state. (c) 6×6, $n_\phi = 1/6$, $n_b = 1/3$, $\nu = 2$ quantum Hall state. (d) 6×6, $n_\phi = 0$, $n_b = 1/3$, superfluid. (e) 6×6, $n_\phi = 1/6$, $n_b = 1/2$, staggered potential $\Delta = 2$, charge density wave.

(Fig. 4(d)) and a charge-density-wave (Fig. 4(e)) does not show any non-monotonic behavior. The state from our adiabatic preparation protocol also admits such non-monotonic correlations (see supplementary materials[62]) demonstrating that it retains physical characteristics of the ground state.

Conclusion and outlook.—We study quantum Hall phases and their adiabatic preparation scheme in the Harper-Hofstadter model with hardcore bosons. Our theoretical study lends support to Jain's composite fermions picture in a regime where lattice effects play an important role. We note a recent work finds another setting for Jain's composite fermions states of bosons[67]. On the other hand our work indicates a way forward for the experimental study of quantum Hall phases in optical lattices. It is interesting to understand the nature of the phase transition from the 1D wires to 2D quantum Hall phases. Other interesting problem is to come up with experimental measurement protocols such as measuring the Hall conductance[38] or detecting edge state (e.g.[68,69]).

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