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Measuring space-group symmetry fractionalization in Z_2 spin liquids

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The interplay of symmetry and topological order leads to a variety of distinct phases of matter, the Symmetry Enriched Topological (SET) phases. Here we discuss physical observables that distinguish different SETs in the context of Z_2 quantum spin liquids with $SU(2)$ spin rotation invariance. We focus on the cylinder geometry, and show that ground state quantum numbers for different topological sectors are robust invariants which can be used to identify the SET phase. More generally these invariants are related to 1D symmetry protected topological phases when viewing the cylinder geometry as a 1D spin chain. In particular we show that the Kagome spin liquid SET can be determined by measurements on one ground state, by wrapping the Kagome in a few different ways on the cylinder. In addition to guiding numerical studies, this approach provides a transparent way to connect bosonic and fermionic mean field theories of spin liquids. When fusing quasiparticles, it correctly predicts nontrivial phase factors for combining their space group quantum numbers.

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I. INTRODUCTION

In contrast to conventional phases that are distinguished by Landau order parameters, topologically ordered states with emergent anyonic excitations remain distinct even in the absence of symmetry. However, the presence of symmetries,

which is natural in most physical contexts, leads to further distinctions, the so called symmetry enriched topological phases (SETs). Well known manifestations include fractional charge of anyons in the fractional quantum Hall states (and in fractional Chern insulators) and spin-charge separation in quantum spin liquids. Recently, rapid progress in the theoretical understanding of SETs is being made[1–11]. This is partly driven by conceptual advances in the related theory of strongly interacting Symmetry Protected Topological phases (SPTs), where despite the absence of anyon excitations in the bulk, nontrivial boundary excitations emerge [7, 12–18]. What would be particularly welcome at this stage, is a physically well motivated example of an SET. To make progress in this direction we will need to understand how different SET phases can be distinguished.

Further motivation for studying measurable characteristics of SETs is the recent progress in the search for quantum spin liquids in frustrated magnets, both in experiments and in numerics. A number of $S=1/2$ quantum magnets in 2D and 3D frustrated lattice have been identified, which appear to evade magnetic order [19, 20]. These include the $S=1/2$ Kagome material, Herbertsmithite, which shows no sign of ordering down to temperatures that are a thousand times smaller than the exchange constant[21]. Evidence for an energy gap in these materials, [22] a requirement for topological order, is a matter of ongoing study, and is presumably complicated by extrinsic effects due to impurities (although other explanations have also been suggested[23–25]). Meanwhile, early density matrix renormalization group (DMRG) simulations of the nearest neighbor Kagome Heisenberg antiferromagnet (KHA) suggested (i) a gapped ground state that respects all symmetries [26, 27] and (ii) a topological entanglement[27, 28] entropy of $\log 2$. In reference [29] it was argued that the ground state must possess Z_2 (toric code) topological order to be compatible with (i) and (ii). More recent DMRG studies, however, have found evidence that the KHA model is gapless.[30] Regardless, even if the Kagome antiferromagnet is gapped, identifying the precise phase of matter realized, i.e. the SET order, remains an important open question. While a complete solution would necessitate extensive numerical input, and is beyond the scope of this paper, here we will relate SET phases to physical properties that are readily measurable in numerical simulations.

A prerequisite is a measurement of the topological order itself. Entanglement entropy [31, 32] provides one signature, although it does not uniquely specify the topological order. A complete characterization is obtained, from either the entanglement spectrum [33] (in certain cases), or the S and T matrices [34–40] some of which are well suited to numerical calculations [41–44]. Related techniques can be used to diagnose 2D SPT phases [45].

Here we discuss measurable properties that distinguish SETs. For the reasons above we focus on the $S=1/2$ KHA, assuming Z_2 (toric code) topological order, which has a pair of emergent $S=1/2$ spinons (one bosonic and one fermionic) and a vortex (vison). The different SETs differ in their realization of space group symmetries and their interplay with time reversal.

We consider systems on both finite and infinite cylinder geometries, which are well suited to DMRG calculations. We show that 1) the many-body quantum numbers of a finite-cylinder ground state under space group symmetries such as reflection, translation etc. provide a powerful diagnostic of the underlying SET and 2) when viewing an infinitely long cylinder as a 1D spin chain, the 1D SPT order of the spin chain for various geometries and topological sectors completely determines the SET order, at least within the space of mean-field parton ansatz.

A reflection quantum number is a probe of quantum entanglement: if a state is odd under reflection, then the two halves of the system are entangled. But how can quantum numbers of a global symmetry, which are always integral, be related to symmetry fractionalization? As a simple example - consider creating a pair of identical anyons from the vacuum. Say that the pair is related to one another by a symmetry (such as a reflection or rotation). If the excitations carry charge, a unit charge for this state actually implies half charge for each excitation since they are constrained by symmetry. This is one of the key ideas that we will exploit. Its implementation is more involved when the second symmetry is not charge, but also a space group (or time reversal) symmetry. Nevertheless such arguments establish the relative quantum numbers between different topological sectors.

To relate our results to an established classification scheme, we use a specific model of SETs obtained by a parton decomposition of the spin operator into bosonic (Schwinger bosons) or fermionic (Abrikosov fermions) partons. Symmetry fractionalization is encoded in the Projective Symmetry Group (PSG) [1, 46, 47], which determines how the partons transform under symmetry. A parton mean field theory combined with Gutzwiller projection leads to a spin wave-function whose quantum numbers in each topological sector are completely determined, which reflects the underlying SET. In particular we show that the Kagome spin liquid SET can in principle be uniquely determined by measurements on *one* ground state in a few different finite-cylinder geometries. This information is numerically superior to the relative quantum numbers between topological sectors, since DMRG numerics on KHA do not obtain all topological sectors in the finite sized systems studied.

A different perspective on our approach is to view it as a ‘dimensional reduction’ in which we view a 2D SET phase on a cylinder as a 1D SPT. The nature of the 1D SPT depends on the topological sector being studied, the geometry of the cylinder, and the SET. In addition to its utility as a diagnostic in numerics, our approach provides a theoretical tool to study connections between different representations of the same SET. The 1D SPT invariants for the degenerate ground states (which are labeled by distinct quasiparticles of the topological order) are shown to follow the same multiplicative law as the fusion rules for the Abelian quasiparticles. This allows us to determine from the PSGs of two anyons types in the Z_2 topological order the PSG of the third anyon type, which is found to obtain nontrivial phases in certain cases. Some of these nontrivial phases associated with mirror reflection and time reversal symmetries were missed in previous studies[4]

and serve to correctly relate bosonic and fermionic mean field states on the Kagome lattice [5]. In particular this leads us to equate two popular states, the $Q_1 = Q_2$ Schwinger boson state [48] and the $Z_2[0, \pi]_\beta$ fermionic mean field state [5, 49, 50].

In addition, we show how dimensional reduction can be used to completely identify the four topological sectors of a cylinder; this is highly useful for DMRG studies, and does not require simultaneous knowledge of all four ground states. In particular, we have found a 1D SPT invariant that distinguishes between the bosonic and fermionic spinon.

Earlier work employed a similar dimensional reduction approach in the case of internal symmetries with projective representations[51], and here we find it to be much more generally applicable in the presence of space group symmetries. Related work specializing to the case of just translation symmetry has recently appeared [52]. The connection between 1D SPT invariants and global quantum numbers was previously noted [13, 53], and other works have utilized global many-body quantum numbers to identify topological phases [54, 55].

II. REVIEW OF Z_2 SPIN-LIQUIDS

According to the arguments of Hastings, Oshikawa, Lieb, Shultz and Mattis, [56–58] a quantum magnet with half-integer spin per unit cell is either gapless, breaks a symmetry, or is a gapped ‘spin liquid’ with emergent anyonic excitations. In the latter case, the simplest possibility consistent with time-reversal is the Z_2 ‘toric code’-type spin-liquid. [29] This phase has emergent $S = 1/2$ excitations, the ‘spinons,’ even though a truly local excitation (the magnons) must carry $S = 1$. These emergent spinons are anyonic excitations with non-trivial braiding and statistics. The Z_2 spin-liquid has four anyon types: the local excitations (‘1’); the bosonic spinon (‘ b ’), which carries $S = 1/2$; the vison (‘ v ’), which behaves like a π -flux for the spinon; and the fermionic spinon (‘ f ’), formed from the composite of $f = bv$, and which also carries $S = 1/2$. Each particle is its own anti-particle, $v^2 = f^2 = b^2 = 1$ (hence ‘ Z_2 ’). The braiding and statistics of the Z_2 spin-liquid are equivalent to Z_2 gauge theory (the ‘toric code’). In the language of Z_2 gauge theory, b is the electric charge e ; v is the magnetic flux m ; and f is the dyon $f = em$ composed of flux and charge.

The simplest SET aspect of the Z_2 spin-liquid is its behavior under $SO(3)$ spin rotations: 1 and v carry integer spin, while the spinons b, f carry half-integer spin. The half-integer spin carried by the spinons is ‘fractionalized’ because any local excitation of the constituent $S = 1/2$ spins transforms as $S = 1$. We can look for additional SET distinctions based on the transformation properties of anyons under space-group symmetries, which is the subject of this work.

A. Minimally entangled states

The Z_2 spin liquid has a 4-fold topological ground state degeneracy on the torus or cylinder. Throughout this work we rely on a special basis for the ground-state manifold called the ‘minimally entangled states’ (MES).[35] To construct the MES basis, let x run along the infinite length of the cylinder, and let \mathcal{F}_x^a denote the adiabatic process in which a pair of anyons a/\bar{a} are created from the vacuum and dragged in opposite directions $\pm\hat{x}$ out to infinity. The process \mathcal{F}_x^a returns the system to the ground state, so is a unitary operation in the ground-state manifold. We say that \mathcal{F}_x^a ‘threads anyonic flux a ’ through the cylinder. Since the model is Abelian, $\mathcal{F}_x^a \mathcal{F}_x^b \propto \mathcal{F}_x^{a \cdot b}$, where $a \cdot b$ denotes the fusion of Abelian anyons.

The MES basis is the unique basis in which \mathcal{F}_x^a is realized as a *permutation* of the basis states for all $a \in \{1, b, v, f\}$. \mathcal{F}_x^a acts as a permutation in the MES basis because each MES has definite topological flux threading the cylinder (its ‘topological sector’).

In previous discussions of the MES it is often assumed that each of the four MES can be uniquely identified with an anyon type $1, b, v, f$, so that each MES can be labeled with an anyon type $|a\rangle$. Then threading topological flux is realized as $\mathcal{F}_x^b |a\rangle = |b \cdot a\rangle$. This is the case for Z_2 spin-liquids on *even* circumference cylinders, but for an $S = 1/2$ model on an odd circumference cylinder there is a subtlety that arises because the MES double the unit-cell along the x -direction. The unit cell doubles because $S = 1/2$ models behave as if there is a topological flux piercing each unit cell, so (as for a magnetic field) the net topological flux through the cylinder changes along the direction x . [29] Strictly speaking it is more precise to view the MES as a torsor for the fusion group, but this subtlety does not affect the measurements proposed in this work, so for notational simplicity we will label the MES by anyon types.

III. SPACE-GROUP QUANTUM NUMBERS: ROBUST SET INVARIANTS

In this section we consider the space-group quantum numbers of a finite length cylinder: even though the structure of the two edges is non-universal, we argue that the space-group quantum numbers are. In particular, consider creating a pair of well-separated excitations from the vacuum which are related by a mirror plane, and separating them out to the edges of the cylinder. If the reflection quantum number flips sign after this process, the excitations must be anyons which are connected by an invisible string which is *odd* under reflection. Since the pair of anyons together transforms as $R = -1$, it as if $R = \sqrt{-1}$ acting on each anyon individually, so we say R is ‘fractionalized.’

While global quantum numbers are only well defined for finite systems, at a later point we will show that they leave their imprint on the bulk entanglement spectrum in a way which can be measured on an infinitely long cylinder as well.

A. Reflection quantum numbers

Consider a large but finite cylinder with a reflection symmetry \hat{R}_x that exchanges its two edges. We argue that the global \hat{R}_x quantum number of any symmetric state $|\Lambda, a\rangle$ with no excitations in the interior of the cylinder depends only on a) the SET order of the bulk phase of matter b) the details of the geometry, such as its dimensions, which we denote by ‘ Λ ’, and c) the topological sector a of the cylinder. Throughout this paper we will denote these global quantum numbers by Q ,

$$\hat{R}_x |\Lambda, a\rangle = Q_{R_x}(\Lambda, a) |\Lambda, a\rangle. \quad (1)$$

To elaborate on c), note that an *infinitely* long cylinder has the same topological ground state degeneracy as the torus. While the edges may reduce the ground state degeneracy, we only require that there are no excitations in the *bulk* of the cylinder, so are left with the same bulk degeneracy as the torus. We label these topological sectors of the finite cylinder by a , and as discussed earlier we assume a indexes a special ‘minimally entangled’ basis which has definite topological flux a threading the cylinder. This work focuses mainly on Z_2 topological order because the states $|\Lambda, a\rangle$ will *break* R_x if a is not its own anti-particle.

The global quantum number Q_{R_x} is insensitive to any details of the edge state or bulk Hamiltonian. To show insensitivity to the edge, note that perturbing the edges amounts to acting with unitaries U_L, U_R localized at the edges of the cylinder. When U_L and U_R are spatially well separated, using R_x symmetry we can require that $U_R = \hat{R}_x U_L \hat{R}_x^{-1}$. Since the perturbation $U_L \hat{R}_x U_L \hat{R}_x^{-1}$ commutes with \hat{R}_x , the quantum number is unchanged. Q is insensitive to the bulk Hamiltonian because $Q = \pm 1$, so is quantized and can only change during a bulk phase transition.

However, $Q_{R_x}(\Lambda, a)$ can depend on the topological sector a , since changing the topological sector from $a \rightarrow b \cdot a$ requires separating an anyon pair b/\bar{b} out to the edges using \mathcal{F}_x^b , which is a string-like operation. We will find that the dependence on Λ cancels if look at the relative quantum number between topological sectors, [59]

$$Q_{R_x}^{(b)} \equiv \frac{Q_{R_x}(\Lambda, ba)}{Q_{R_x}(\Lambda, a)}. \quad (2)$$

These ratios have a particularly simple relationship to the SET order: if $Q_{R_x}^{(b)} = -1$, it implies that a pair of anyons related by R_x each carry half the $R_x = -1$ quantum number, which we consider to be ‘fractional.’ In contrast, our earlier argument implies that a pair of truly local excitations must always have $Q_{R_x} = 1$.

Now suppose there is an additional reflection symmetry R_y which does *not* exchange the two edges. In the absence of the edge-exchanging R_x symmetry, the R_y quantum numbers are not robust, because nothing then prevents the perturbation U_L from being odd under R_y while U_R is even. But if both R_x and R_y are present, we can instead measure the combination $I = R_x R_y$; since I exchanges the edges, according to our earlier reasoning Q_I is also a protected invariant.

Finally, if lattice doesn’t have a C_4 symmetry (as for the Kagome model) there are distinct ways to compactify the geometry into a cylinder: for one choice, R_x exchanges the edges, while for the second, R_y does. We can then measure quantum numbers Q_{R_x} in the first cylinder, Q_{R_y} under the second cylinder, and Q_I in either. This give three independent quantum numbers for each anyon type, $Q_{R_x/R_y/I}^{(b/f/v)}$, which we will find almost fully characterizes Z_2 SETs (at least within the PSG framework). The remaining information relates to the commutation relations of time-reversal \mathcal{T} and the reflections R , which will lead to protected edge degeneracies we discuss in Sec.V C.

B. Translation quantum numbers

In a magnetic field the translations T_x, T_y form a ‘magnetic algebra’ $T_x T_y T_x^{-1} T_y^{-1} = e^{i\Phi}$ which is a *projective* representation of the translation group. Even in the absence of a physical magnetic field, in a topologically ordered phase the anyons may experience an effective magnetic field. The magnetic field experienced by anyon a is encoded in the projective relation $(T_x T_y T_x^{-1} T_y^{-1})^{(a)} = \eta_{xy}^{(a)}$.

For an Abelian theory the above projective relations must obey the fusion rule $\eta_{xy}^{(a)} \eta_{xy}^{(b)} = \eta_{xy}^{(ab)}$, since $\eta_{xy}^{(a)}$ is the Berry phase acquired when a circles a unit cell. For a Z_2 spin-liquid in a model with an odd number of spin-1/2 per unit cell, such as on square, triangular and Kagome lattices, we always have the relation $\eta_{xy}^{(v)} = -1$. This can be argued in the language of Z_2 gauge theory, where all $S = 1/2$ objects are the source of Z_2 electric flux (for example the spinons b, f , which map on to the electrically charged e, f particles in the gauge theory). This includes the microscopic $S = 1/2$ in each unit cell, which implies that the system behaves as if there is electric flux piercing each unit cell. Consequently under e - m duality, the flux m (the vison) experiences a background flux of π per unit cell, so $\eta_{xy}^{(v)} = -1$. Since $\eta_{xy}^{(1)} = 1, \eta_{xy}^{(v)} = -1$, and $\eta_{xy}^{(b)} \eta_{xy}^{(v)} = \eta_{xy}^{(f)}$, there is a single sign left undetermined, which is the most basic SET distinction between Z_2 spin-liquids.

To probe η_{xy} , consider a cylinder of length L_x and circumference L_y in topological sector a , and measure the momentum quantum number $Q_{T_y}(\Lambda, a) = \hat{T}_y |\Lambda, a\rangle$. Note that when L_y is odd the MES double the unit cell in the x direction, so we must restrict to L_y even. T_y symmetry alone does not protect Q_{T_y} , since the edge excitations can carry an arbitrary momentum, but the combination of T_x and T_y allows us to define a robust ‘momentum per unit length.’ Recall that in the Landau gauge, the momentum T_y of a particle in a magnetic field is proportional to its position x . Since $|\Lambda, a\rangle$ has an a particle localized near the edge, as we grow L_x its momentum grows linearly with x , ie, it is a momentum per unit length of cylinder. To define the momentum per unit length operationally, we need to grow the length of the cylinder L_x while keeping the topological flux and edge state the same, as otherwise Q_{T_y} could contain a spurious contribution coming from the changing edge state. Concretely, we require the

reduced density matrix for the edge be kept constant as L_x grows. The momentum per unit length $\eta_{xy}^{(a)}$ is then

$$Q_{T_y}(\Lambda, a) = q_{T_y}(\partial\Lambda, a) \left(\eta_{xy}^{(a)} \right)^{L_x}. \quad (3)$$

$q_{T_y}(\partial\Lambda, a)$ depends on the edge, while the bulk contribution reveals the SET invariant $\eta_{xy}^{(a)}$.

The momentum per unit length is *trivial* to measure in any tensor network ansatz. For MPS, it is the invariant $\eta_{xy}^{(a)} = r_{T_y}$ defined in Eq. (48)) when viewing $g = T_y$ as an ‘onsite’ symmetry in the 1D representation of the cylinder. r_{T_y} is a byproduct of the algorithm used to calculate the momentum-resolved entanglement spectrum of infinite-DMRG studies, so presumably has already been computed in existing studies.

IV. SYMMETRY ENRICHED ORDER: DETECTING THE PROJECTIVE SYMMETRY GROUP

The preceding discussion is independent of any classification of space-group SETs, since we have argued on general grounds that these quantum numbers are robust SET invariants. Nevertheless, we would like to identify these invariants within a general classification scheme. Recently it has been proposed that distinct Z_2 spin liquids that preserves symmetry group G_s (including both onsite and spatial symmetries) is classified by 2nd group cohomology $\mathcal{H}^2(G_s, G_f)$ [4, 8, 10] where $G_f = Z_2 \times Z_2$ is the fusion group of Abelian anyons in a Z_2 spin liquid. While this abstract classification provides limited insights on how to detect the SET invariants, the parton construction provides a rich zoo of concrete Z_2 spin liquid states whose symmetry properties can be analyzed using Wen’s ‘projective symmetry group’ (PSG)[1]. The PSG provides at least a partial classification of space-group SETs. In this section, we show how to compute the quantum numbers $Q_U(\Lambda, a)$ within the parton construction, thereby identifying them with invariants of the PSG.

A. The Parton Construction and the Projective Symmetry Group

The resonating valence bond (RVB) picture proposed by Phil Anderson provided the first intuition for a spin liquid ground state as a quantum superposition of different dimer configurations covering a lattice of spin-1/2 particles. Each dimer (denoted by $\bullet - \bullet$) is a singlet pair formed by two spin-1/2 particles:

$$|\bullet - \bullet\rangle = |\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2.$$

The state is a ‘liquid’ because the quantum superposition of dimer patterns restores the translational symmetry. One type of elementary excitation in these systems is created by breaking a dimer into a pair of particles carrying spin-1/2 each, which were coined *spinons*.

The *parton* construction is a systematic formalism for writing down ansatz RVB wavefunctions in which each spinon is

realized either as a fermionic parton f_σ or bosonic parton b_σ , where $\sigma = \uparrow, \downarrow$. In the fermionic description, each dimer is realized as an s-wave Cooper pair of partons; breaking a Cooper pair generates a pair of spinons. The microscopic spins \vec{S}_r are related to the partons through the bilinears

$$\vec{S}_r = \frac{1}{2} \sum_{\alpha, \beta = \uparrow, \downarrow} f_{r, \alpha}^\dagger \vec{\sigma}_{\alpha, \beta} f_{r, \beta} = \frac{1}{2} \sum_{\alpha, \beta = \uparrow, \downarrow} b_{r, \alpha}^\dagger \vec{\sigma}_{\alpha, \beta} b_{r, \beta}. \quad (4)$$

where $\vec{\sigma}$ are the three Pauli matrices. In order for this mapping to generate a sensible $S = 1/2$ wavefunction, the partons cannot be free particles: they obey the ‘single-occupancy’ constraint of one parton per lattice site:

$$\sum_{\sigma = \uparrow, \downarrow} f_{r, \sigma}^\dagger f_{r, \sigma} = \sum_{\sigma = \uparrow, \downarrow} b_{r, \sigma}^\dagger b_{r, \sigma} = 1, \quad \forall \text{ lattice site } r. \quad (5)$$

This constraint can be implemented by a gauge field which couples to the partons.

In practice, we use the parton construction to create ansatz wave-functions. If $|\text{MF}\rangle$ is a ‘mean-field ansatz’ state for the partons which need not obey the single occupancy constraint (for example, a BCS superconductor of fermionic partons f_σ), we enforce the constraint via Gutzwiller projection to obtain an $S = 1/2$ wavefunction:

$$\langle \uparrow_1 \downarrow_2 \cdots \sigma_i \cdots | \Psi \rangle = \langle 0 | f_{r_1, \uparrow} f_{r_2, \downarrow} \cdots f_{r_i, \sigma_i} \cdots | \text{MF} \rangle, \quad (6)$$

and similarly in the bosonic construction. Note that in the fermionic case, we must choose and fix an ordering of sites r_1, r_2, \dots in order to maintain the correct relative sign between different spin configurations. For the purposes of calculation $|\text{MF}\rangle$ is usually taken to be a free wavefunction, such as a mean-field BCS superconductor or pair-superfluid for the fermionic / bosonic constructions considered here. Gutzwiller projecting the creation of a *single* parton f^\dagger/b^\dagger results in a highly non-trivial excitation: an $S = 1/2$ anyonic excitation, the spinon.

A crucial question in the parton construction is how the symmetries $\{U \in \text{SG}\}$ of the $S = 1/2$ wavefunction (such as global $\text{SO}(3)$ spin rotations and space-group symmetries) are realized in the partons and their mean-field ansatz $|\text{MF}\rangle$. The simplest possibility is that the partons form a linear representation of the symmetry group SG. But is this the only option? The answer is no, because according to Eq. (4) applying a gauge transformation $b_{r, \sigma} \rightarrow e^{i\phi_r} b_{r, \sigma}$ leaves the physical spin operators unchanged, and accordingly the Gutzwiller projection in Eq. (6) is a many-to-one mapping insensitive to $U(1)$ gauge transformations. Consequently under a series of symmetry operations $\{U_i\} \in \text{SG}$ which yield the identity operation e

$$U_1 U_2 \cdots U_n = e \quad (7)$$

a single-parton operator $b_{r, \sigma}$ (or f) may acquire a nontrivial phase factor $e^{i\phi} \neq 1$ instead of remaining invariant. In this case the partons transform *projectively*, rather than linearly, under the symmetry group SG. The symmetry operations $\{U \in \text{SG}\}$ are accompanied by certain gauge transformations $\{G_U | U \in \text{SG}\}$ on partons, forming a ‘projective symmetry group’ $\text{PSG} \equiv \{G_U U | U \in \text{SG}\}$ which is a

central extension of the symmetry group SG. [1] The center of such an extension is called the ‘‘invariant gauge group’’ $IGG \equiv PSG/SG$. The IGG are those gauge transformations which leave the mean-field ansatz $|\text{MF}\rangle$ invariant.

For the Z_2 spin liquids which are the focus of this work, the invariant gauge group is $IGG = Z_2$. This means under a series of symmetry operations $\{U_i\}$ in (7), each parton acquires a Z_2 phase factor of ± 1 . This is consistent with the mean-field ansatz being a singlet BCS superconductor of spinons. For example, for symmetry a symmetry $UU = e$ we have the relation

$$(\hat{G}_U \hat{U})^2 f_{\mathbf{r}_i, \sigma_i} (\hat{G}_U \hat{U})^{-2} = \eta_U^f f_{\mathbf{r}_i, \sigma_i}, \quad \eta_U^f = \pm 1. \quad (8)$$

These ± 1 -valued η_U^a for anyons $a \in \{1, b, v, f\}$ correspond to elements in 2nd group cohomology $\mathcal{H}^2(\text{SG}, Z_2 \times Z_2)$, which are proposed to fully characterize the symmetry fractionalization of a 2+1-D SET phase [4, 8, 10]. The primary goal of this work is to show how to detect these ± 1 -valued SET invariants associated with symmetry group SG.

B. Minimally entangled states within the parton construction

We first must review how to generate the MES within the parton construction.

There is a two-fold degeneracy associated with threading a vison v for both finite and infinite cylinders. In the parton ansatz this arises because the boundary conditions of a cylinder of circumference L_y can be either periodic (P) or anti-periodic (AP). At the level of the parton Hamiltonian, this is accomplished by assigning an additional sign of -1 to all matrix elements which cross a line at some fixed $y = y_0$; the choice of y_0 is a gauge choice. When acting with T_y or R_y , the location of the twist y_0 must be restored by an additional contribution to the gauge transformations G_{T_y/R_y} , leading to new PSG relations on a cylinder:

$$(\hat{G}_{T_y} \hat{T}_y)^{L_y} = (-1)^{\text{bc}} \quad (9)$$

$$(\hat{G}_{T_y} \hat{T}_y)^{L_y/2} (\hat{G}_{R_y} \hat{R}_y) (\hat{G}_{T_y} \hat{T}_y)^{L_y/2} (\hat{G}_{R_y} \hat{R}_y)^{-1} = (-1)^{\text{bc}} \quad (10)$$

where $\text{bc} = 0/1$ for P/AP . The second equation should be interpreted as relative to the PSG in the plane geometry. Hence in all that follows the PSG implicitly depends on the boundary condition (bc) of the mean-field ansatz $|\text{MF}\rangle$.

The two-fold degeneracy associated with threading a spinon is a bit more subtle, as in the finite case the degeneracy is split by the edges. If we make a bipartition of the cylinder at some x_0 , the parton parity $(-1)^{N_{b/f}}$ in the left half of the mean-field ansatz $|\text{MF}\rangle$ fluctuates across the cut (note the parton number itself is not conserved). After Gutzwiller projection, Eq. (6), the parton parity to the left is fixed by the number of sites to the left. But in the infinite case there is an ambiguity, since the number of sites is infinite. This means that when Gutzwiller projecting we can freely choose either the sector with even (E) or odd (O) parton parity to the left of

the cut at x_0 , which generates an additional 2-fold degeneracy on the infinite cylinder.

The choice of P/AP boundary conditions combined with E/O parton parity generates the 4-fold degeneracy of the infinite cylinder. These sectors are identified with the anyon types in a manner that depends on the parton construction:

$$\begin{aligned} 1, v, b, f &\leftrightarrow (\text{P}, \text{E}), (\text{AP}, \text{E}), (\text{P}, \text{O}), (\text{AP}, \text{O}) \quad (\text{bosonic}) \quad (11) \\ 1, v, b, f &\leftrightarrow (\text{AP}, \text{E}), (\text{P}, \text{E}), (\text{P}, \text{O}), (\text{AP}, \text{O}) \quad (\text{fermionic}) \quad (12) \end{aligned}$$

Note that in the no-spinon sector the role of P/AP is flipped between the two constructions.

For an even circumference cylinder the E/O parity is the same for all cuts x_0 , since an even number of sites intervene between cuts. But for an odd circumference cylinder, the E/O assignment alternates with x_0 . This alternation doubles the physical unit cell.

C. Computation of global quantum numbers from the PSG

Since the PSG determines how the partons transform under symmetry operations, we can compute the (crystal) symmetry quantum numbers of any projected wavefunction (Eq. (6)) constructed from a parton mean-field ansatz. If U is a space-group operation which permutes the sites according to $i \rightarrow U(i)$, the wavefunction transforms as

$$\langle \{\sigma_i\} | U \Psi \rangle = \langle \{\sigma_{U^{-1}(i)}\} | \Psi \rangle = \langle 0 | \prod_i f_{\mathbf{r}_i, \sigma_{U^{-1}(i)}} |\text{MF}\rangle, \quad (13)$$

and similarly for the bosonic ansatz. We can split the resulting quantum number into two contributions. First, there is a part $Q_U(\Lambda, \text{bc})$ coming from the Gutzwiller projection,

$$\langle 0 | \prod_i f_{\mathbf{r}_i, \sigma_{U^{-1}(i)}} = Q_U(\Lambda, \text{bc}) \langle 0 | \prod_i f_{\mathbf{r}_i, \sigma_i} \hat{G}_U \hat{U}, \quad (14)$$

which depends only on the geometry Λ and the PSG (which is modified by bc); we will show how to compute this in the subsequent section. The second contribution comes from the quantum number of $|\text{MF}\rangle$. Inserting Eq. (14) into Eq. (13) we find that the total quantum number factorizes as

$$\hat{G}_U \hat{U} |\text{MF}\rangle \equiv Q_U(\text{MF}) |\text{MF}\rangle \quad (15)$$

$$\hat{U} |\Psi\rangle = Q_U(\Lambda, \text{bc}) Q_U(\text{MF}) |\Psi\rangle \quad (16)$$

D. Ratios of edge-exchanging quantum numbers in different topological sectors

In the first scenario, we suppose we have access to all topological sectors on the same geometry. We find that for edge-exchanging symmetries, the ratio between the quantum number before and after threading anyonic flux a reveals the PSG of anyon a .

1. Spinon insertion

We fix the geometry Λ and compute the relative reflection quantum number Q_R between two states which differ by the insertion of a pair of spinons at the edges. To generate the appropriate pair of mean-field ansatz $|\text{MF}\rangle$ and $|c \cdot \text{MF}\rangle$ which differ by spinon flux $c = b/f$ (depending on the construction), let c_L^\dagger create an arbitrary bosonic / fermionic parton near the left edge. To ensure that $c \cdot \text{MF}$ is symmetric under R we must create a corresponding spinon on the right using $c_R^\dagger \equiv (\hat{G}_R \hat{R}) c_L^\dagger (\hat{G}_R \hat{R})^{-1}$, so

$$|c \cdot \text{MF}\rangle \equiv c_L^\dagger (\hat{G}_R \hat{R}) c_L^\dagger (\hat{G}_R \hat{R})^{-1} |\text{MF}\rangle. \quad (17)$$

$Q_U(\Lambda, bc)$ is unchanged, and it is straightforward to verify

$$\hat{G}_R \hat{R} |b/f \cdot \text{MF}\rangle = (-1)^F (\hat{G}_R \hat{R})^2 Q_R(\text{MF}) |b/f \cdot \text{MF}\rangle \quad (18)$$

where the sign $(-1)^F$ occurs for fermionic partons as we must exchange the creation operators and we use $(\hat{G}_R \hat{R})^2$ to denote the parton PSG associated with $\hat{R}^2 = e$:

$$\begin{aligned} (\hat{G}_R \hat{R})^2 c_i (\hat{G}_R \hat{R})^{-2} &= \eta_R^c c_i, \quad \forall c_i, \\ (\hat{G}_R \hat{R})^2 &\equiv \eta_R^c. \end{aligned} \quad (19)$$

Consequently for any geometry the change in the quantum number Q_R after inserting a bosonic / fermionic spinon is

$$Q_R^{(b/f)} \equiv \frac{Q_R(\Lambda, b/f \cdot \text{MF})}{Q_R(\Lambda, \text{MF})} = (-1)^F \eta_R^{b/f}. \quad (20)$$

Hence the spinon PSG can be recovered by measuring the relative quantum number between different topological sectors.

The relative quantum numbers can be computed for any space-group symmetry which exchanges the edges; by using different cylinder compactifications we can measure the R_x, R_y , and $I = R_x R_y$ quantum numbers. There may be distinct π -rotations depending on whether the rotation is site or bond / plaquette centered; on a square lattice, for example, $I' = T_x R_x T_y R_y$, reveals an independent PSG relation. In TABLE I and II we tabulate the relative quantum numbers for the square and Kagome lattices.

2. Vison insertion

We again fix Λ , and compute the relative reflection quantum number Q_R between two states which differ by the insertion of a vison. As discussed, threading a vison switches between P/AP boundary conditions, so $Q_U(\Lambda, bc)$ may change due to the PSG relations of Eq. (9). The PSG relations associated with $U^2 = e$ are only modified if U takes an odd number of sites across the twist boundary condition modified by the vison. For both the square and Kagome lattice, for the geometries in which U exchanges the edges of the cylinder we

a	b	f	v	$Q^{(f)} = Q^{(v)} Q^{(b)}$
$Q_{R_x}^{(a)}$	$\eta_{R_x}^b$	$-\eta_{R_x}^f$	1	$\eta_{R_x}^b = -\eta_{R_x}^f$
$Q_{R_y}^{(a)}$	$\eta_{R_y}^b$	$-\eta_{R_y}^f$	1	$\eta_{R_y}^b = -\eta_{R_y}^f$
$Q_{I_{p/b}}^{(a)}$	$\eta_{R_x}^b \eta_{R_y}^b \eta_{R_x, R_y}^b$	$-\eta_{R_x}^f \eta_{R_y}^f \eta_{R_x, R_y}^f$	1	$\eta_{R_x, R_y}^b = -\eta_{R_x, R_y}^f$
$Q_{I_s}^{(a)}$	$\eta_{x_y}^b \eta_{R_x}^b \eta_{R_y}^b \eta_{R_x, R_y}^b$	$-\eta_{x_y}^f \eta_{R_x}^f \eta_{R_y}^f \eta_{R_x, R_y}^f$	-1	$\eta_{x_y}^b = -\eta_{x_y}^f$

TABLE I. Relative quantum numbers $Q_U^{(a)}$ between topological sectors a of a **square** lattice spin-liquid. $I_{p/b}$ is a plaquette or bond centered π -rotation, while I_s is a site-centered π -rotation.

have

$$Q_{R_y}(\Lambda, \text{AP}) = Q_{R_y}(\Lambda, \text{P}) \quad (21)$$

$$Q_{R_x}(\Lambda, \text{AP}) = Q_{R_x}(\Lambda, \text{P}) \quad (22)$$

$$Q_{I_{b/p}}(\Lambda, \text{AP}) = Q_{I_{b/p}}(\Lambda, \text{P}) \quad (23)$$

$$Q_{I_s}(\Lambda, \text{AP}) = -Q_{I_s}(\Lambda, \text{P}) \quad (24)$$

where $I_{b/p}$ is a bond or plaquette centered π -rotation, while I_s is site-centered. The extra -1 sign for I_s comes from the vison string sweeping over an odd number of lattice sites (including the inversion center) upon I_s operation.

Next we argue that $Q_U(\text{MF})$ is unchanged when threading a vison. To start, suppose $|\text{MF}\rangle$ is in the same phase as the ground state of a BCS superconductor / pair super-fluid. In the subsequent section, we show that $Q_U(\text{MF}) = 1$ regardless of boundary condition, so is unchanged by vison insertion. Now suppose we modify the ground state $|\text{MF}\rangle$ with arbitrary U -symmetric edge perturbations of fixed parton parity. The resulting $Q_U(\text{MF})$ can depend only on the parton parity of the edge perturbation, but not on the boundary condition, because the vison-modified PSGs of Eq. (9) will necessarily act on U -related partons and the signs will cancel. This shows $Q_U(\text{MF})$ is unchanged by vison insertion, and we conclude that

$$Q_{R_y}^{(v)} = 1, \quad Q_{R_x}^{(v)} = 1, \quad Q_{I_{b/p}}^{(v)} = 1, \quad Q_{I_s}^{(v)} = -1. \quad (25)$$

A possible loophole in our argument is that we assumed that for the ground state, $|\text{MF}\rangle$ could be taken to be in the same phase as a BCS / pair superfluid, but our result agrees with earlier discussions.[5]

3. Fusion and unification: $Q^{(f)} = Q^{(v)} Q^{(b)}$

Our derivation of the vison quantum numbers shows that the relative reflection quantum numbers obey fusion, $Q_U^{(f)} = Q_U^{(v)} Q_U^{(b)}$, because vison insertion changes the quantum number by $Q^{(v)}$ regardless of the parton parity at the edge. As tabulated in Table I and Table II, we can use this fusion rule to relate the bosonic and fermionic PSGs, unifying the two approaches[5].

a	b	f	v	$Q^{(f)} = Q^{(v)}Q^{(b)}$
$Q_{R_x}^{(a)}$	$(-1)^{p_2+p_3}$	$-\eta_\sigma$	1	$(-1)^{p_2+p_3} = -\eta_\sigma$
$Q_{R_y}^{(a)}$	$(-1)^{p_2}$	$-\eta_\sigma\eta_\sigma c_6$	1	$(-1)^{p_3} = \eta_\sigma c_6$
$Q_{I_h}^{(a)}$	$(-1)^{p_1+p_3}$	$-\eta_{C_6}$	1	$(-1)^{p_1} = -\eta_{C_6}\eta_\sigma c_6$
$Q_{I_s}^{(a)}$	$(-1)^{p_3}$	$-\eta_{12}\eta_{C_6}$	-1	$(-1)^{p_1} = -\eta_{12}$

TABLE II. Relative quantum numbers $Q_U^{(a)}$ between topological sectors a in a **Kagome** lattice spin-liquid. $I_{h/s}$ is a hexagon / site centered π -rotation. The bosonic PSGs are expressed through the invariants (p_1, p_2, p_3) of Ref. [46], while the fermionic PSGs are expressed through the invariants η of Ref. [49]. The fermionic invariants satisfy $\eta_{12}\eta_{C_6}\eta_\sigma c_6 = 1$ tautologically.

E. The quantum numbers of the ground state

A given geometry Λ generically has a lowest energy state which is SO(3) symmetric on both edges, and two-fold degeneracy for vison insertion. We can compute the quantum numbers for such an SO(3) symmetric state by assuming $|\text{MF}\rangle$ is in the same universality class as a BCS superconductor / pair superfluid in the fermionic / bosonic constructions. $|\text{MF}\rangle$ is then invariant under any symmetry: $Q_U(\text{MF}) = 1$. This is because in both constructions we have

$$|\text{MF}\rangle = \exp \left[\sum_{\mathbf{i}, \mathbf{j}} g(\mathbf{i} - \mathbf{j}) c_{\mathbf{i}, \uparrow}^\dagger c_{\mathbf{j}, \downarrow}^\dagger \right] |0\rangle, \quad (26)$$

$$g(\mathbf{i} - \mathbf{j}) = -(-1)^F g(\mathbf{j} - \mathbf{i})$$

where $c = f/b$ and $(-1)^F = -1/1$ for Abrikosov-fermions and Schwinger-bosons respectively. Because $|\text{MF}\rangle$ always contains the parton Fock vacuum $|0\rangle$ as a component in the Taylor expansion of the exponential, and $|0\rangle$ is invariant under any symmetry operation, $|\text{MF}\rangle$ must also be invariant. Therefore the quantum number of the ground state depends only on the geometry Λ and the PSG:

$$Q_U(\Lambda, \text{bc}, \text{MF}) = Q_U(\Lambda, \text{bc}) \quad (27)$$

where bc will depend on the two-fold degeneracy associated vison insertion and $Q_U(\Lambda, \text{bc})$ is defined in Eq. 14. This result is particularly useful numerically, since by modifying Λ we can probe the PSG using only a *single* topological sector.

In the following we specifically illustrate how to obtain the eigenvalues $Q_U(\Lambda, \text{bc})$ of two crystal symmetry operators, inversion I and mirror reflection R , for a projected wavefunction (6) on a finite-size lattice from parton PSGs.

1. Eigenvalue of plaquette-centered inversion I_p

First we consider an inversion symmetry I_p whose inversion center lies on a plaquette. For a finite-size lattice with plaquette-centered inversion I_p , the number of lattice sites N_s must be even. All lattice sites must be exchanged in pairs under inversion operation, since no lattice site remains invariant under I_p operation. More specifically, spinon $f_{\mathbf{r}_i, \sigma_i}$ must appear altogether with its inversion counterpart $f_{\hat{I}_p \mathbf{r}_i, \sigma_i} = \hat{G}_{I_p} \hat{I}_p f_{\mathbf{r}_i, \sigma_i} (\hat{G}_{I_p} \hat{I}_p)^{-1}$ in the many-spinon operator $\prod_i f_{\mathbf{r}_i, \sigma_i}$.

Note that in projected wavefunction (6) there is always a particular ordering for the real-space positions $\{\mathbf{r}_i\}$ of the many-spinon operator $\prod_i f_{\mathbf{r}_i, \sigma_i}$. Here we simply choose a ordering in which a pair of spinons related by inversion show up together i.e.

$$\prod_i f_{\mathbf{r}_i, \sigma_i} \equiv \prod_i' f_{\mathbf{r}_i, \sigma_i} \cdot \hat{G}_{I_p} \hat{I}_p f_{\mathbf{r}_i, \sigma_i} (\hat{G}_{I_p} \hat{I}_p)^{-1}. \quad (28)$$

where $'$ denotes the product over half lattice sites that are unrelated by inversion. Clearly under inversion operation \hat{I}_p the above many-spinon operator transform as

$$\hat{G}_{I_p} \hat{I}_p \prod_i f_{\mathbf{r}_i, \sigma_i} (\hat{G}_{I_p} \hat{I}_p)^{-1} = \prod_i' \hat{G}_{I_p} \hat{I}_p f_{\mathbf{r}_i, \sigma_i} (\hat{G}_{I_p} \hat{I}_p)^{-1} \cdot (\hat{G}_{I_p} \hat{I}_p)^2 f_{\mathbf{r}_i, \sigma_i} (\hat{G}_{I_p} \hat{I}_p)^{-2}.$$

By definition of PSGs we have

$$(\hat{G}_{I_p} \hat{I}_p)^2 f_{\mathbf{r}_i, \sigma_i} (\hat{G}_{I_p} \hat{I}_p)^{-2} = \eta_{I_p}^f f_{\mathbf{r}_i, \sigma_i}, \quad \eta_{I_p}^f = \pm 1. \quad (29)$$

since $I_p^2 = e$ yields the identity operation. We used η^f to denote the PSGs for fermionic spinons (Abrikosov fermions) and η^b for bosonic spinons (Schwinger bosons). Notice that for Abrikosov-fermion representation, exchange of two spinons $f_{\mathbf{r}_i}$ and $f_{\hat{I}_p \mathbf{r}_i}$ gives rise to an extra -1 sign due to Fermi statistics. As a result we obtain

$$\hat{G}_{I_p} \hat{I}_p \prod_i f_{\mathbf{r}_i, \sigma_i} (\hat{G}_{I_p} \hat{I}_p)^{-1} = (-\eta_{I_p}^f)^{N_s/2} \prod_i f_{\mathbf{r}_i, \sigma_i} \quad (30)$$

Hence the eigenvalue of plaquette-centered inversion I_p for projected wavefunction (6) on a N_s -site lattice ($N_s = \text{even}$) is

$$Q_{I_p}(N_s) = (-\eta_{I_p}^f)^{N_s/2} \quad (31)$$

for Abrikosov-fermion representation. On the other hand, in the Schwinger-boson representation, exchange of two spinons won't yield a -1 sign and we have

$$Q_{I_p}(N_s) = (\eta_{I_p}^b)^{N_s/2} \quad (32)$$

2. Eigenvalue of site-centered inversion I_s

Now let's take one more step to consider an inversion symmetry I_s whose inversion center lies on one or more lattice sites. Let's assume inversion centers contain N_I sites and N_s is the total number of lattice sites. For those $(N_s - N_I)$ sites other than the inversion centers, their contribution to the I_s eigenvalue follows exactly the same form as (31) and (32), except that we need to replace N_s by $N_s - N_I$.

What about the contribution from the N_I inversion centers? First of all, if there is an odd number of inversion centers ($N_I = \text{odd}$), the inversion eigenvalue is not a gauge invariant quantity since the symmetry operations on a single spinon can always be followed by an arbitrary gauge transformation. In fact if $N_I = \text{odd}$, the many-body ground states must have a two-fold degeneracy due to Kramers theorem. In the case when $N_I = \text{even}$, if the inversion centers are not related by any other symmetry, again they can acquire extra gauge transformations independently under symmetry operations, and again the inversion eigenvalue is not a topological invariant.

If an even number of inversion centers are related by symmetry, on the other hand, one can compute their contribution to I_s eigenvalue from parton PSGs in a universal manner. Without loss of generality, let's consider a pair of inversion centers related by certain crystal symmetry \hat{P} (e.g. it could be a mirror reflection or a translation on a finite cylinder), this spinon pair operator transforms under inversion \hat{I}_s as

$$\begin{aligned} & \hat{G}_{I_s} \hat{I}_s [f_{\mathbf{r},\sigma} \cdot \hat{G}_P \hat{P} f_{\mathbf{r},\sigma} (\hat{G}_P \hat{P})^{-1}] (\hat{G}_{I_s} \hat{I}_s)^{-1} \\ &= \hat{G}_{I_s} \hat{I}_s f_{\mathbf{r},\sigma} (\hat{G}_{I_s} \hat{I}_s)^{-1} \cdot \hat{G}_{I_s} \hat{I}_s \hat{G}_P \hat{P} f_{\mathbf{r},\sigma} (\hat{G}_P \hat{P})^{-1} (\hat{G}_{I_s} \hat{I}_s)^{-1} \\ &= \eta_{I_s, P}^f \hat{G}_{I_s} \hat{I}_s f_{\mathbf{r},\sigma} (\hat{G}_{I_s} \hat{I}_s)^{-1} \\ &\quad \cdot \hat{G}_P \hat{P} \hat{G}_{I_s} \hat{I}_s f_{\mathbf{r},\sigma} (\hat{G}_{I_s} \hat{I}_s)^{-1} (\hat{G}_P \hat{P})^{-1} \\ &= \eta_{I_s, P}^f \eta_{I_s}^f [f_{\mathbf{r},\sigma} \cdot \hat{G}_P \hat{P} f_{\mathbf{r},\sigma} (\hat{G}_P \hat{P})^{-1}]. \end{aligned} \quad (33)$$

where we defined spinon PSGs

$$\begin{aligned} & \hat{G}_{I_s} \hat{I}_s \hat{G}_P \hat{P} f_{\mathbf{r},\sigma} (\hat{G}_P \hat{P})^{-1} (\hat{G}_{I_s} \hat{I}_s)^{-1} = \quad (34) \\ & \eta_{I_s, P}^f \hat{G}_P \hat{P} \hat{G}_{I_s} \hat{I}_s f_{\mathbf{r},\sigma} (\hat{G}_{I_s} \hat{I}_s)^{-1} (\hat{G}_P \hat{P})^{-1}, \\ & (\hat{G}_{I_s} \hat{I}_s)^2 f_{\mathbf{r},\sigma} (\hat{G}_{I_s} \hat{I}_s)^{-2} = \eta_{I_s}^f f_{\mathbf{r},\sigma}. \end{aligned} \quad (35)$$

Consequently, for a N_s -site lattice with N_I inversion centers ($N_s, N_I = \text{even}$) which are pairwise related by crystal symmetry \hat{P} , the inversion eigenvalue of projected wavefunction (6) is given in terms of spinon PSGs by

$$\begin{aligned} Q_{I_s}(N_s, N_I) &= (-\eta_{I_s}^f)^{(N_s - N_I)/2} (\eta_{I_s, P}^f \eta_{I_s, P}^f)^{N_I/2} \\ &= (-\eta_{I_s}^f)^{N_s/2} (-\eta_{I_s, P}^f)^{N_I/2}. \end{aligned} \quad (36)$$

for Abrikosov fermions and

$$Q_{I_s}(N_s, N_I) = (\eta_{I_s}^b)^{N_s/2} (\eta_{I_s, P}^b)^{N_I/2}. \quad (37)$$

for Schwinger bosons.

A crucial point is that the PSG $\eta_{I_s, P}$ can depend on the boundary condition of the cylinder (for example, if $P = T_y^{L/2}$).

3. Eigenvalue of mirror reflection operator R

The eigenvalues of mirror reflection operator R can be computed completely in parallel to the case of inversion symmetry as discussed previously. Again let's assume N_R lattice sites lie on the mirror reflection axis on a N_s -site lattice. As argued earlier, only when N_R is even and these N_R sites are related to each other by other crystal symmetries, will the R eigenvalue be a topological invariant that is fully determined by parton PSGs. Let's assume these N_R sites are exchanged in pairs by crystal symmetry \hat{P} . Similar to the case of inversion symmetry I_s we can compute the reflection eigenvalue of projected wavefunction (6) as

$$Q_R(N_s, N_R) = (-\eta_{R}^f)^{N_s/2} (-\eta_{R, P}^f)^{N_R/2}. \quad (38)$$

for Abrikosov fermions and

$$Q_R(N_s, N_R) = (\eta_{R}^b)^{N_s/2} (\eta_{R, P}^b)^{N_R/2}. \quad (39)$$

for Schwinger bosons. The parton PSGs are defined as

$$\begin{aligned} & \hat{G}_R \hat{R} \hat{G}_P \hat{P} f_{\mathbf{r},\sigma} (\hat{G}_P \hat{P})^{-1} (\hat{G}_R \hat{R})^{-1} = \quad (40) \\ & \eta_{R, P}^f \hat{G}_P \hat{P} \hat{G}_R \hat{R} f_{\mathbf{r},\sigma} (\hat{G}_R \hat{R})^{-1} (\hat{G}_P \hat{P})^{-1}, \\ & (\hat{G}_R \hat{R})^2 f_{\mathbf{r},\sigma} (\hat{G}_R \hat{R})^{-2} = \eta_{R}^f f_{\mathbf{r},\sigma}. \end{aligned} \quad (41)$$

for Abrikosov fermions and similarly for Schwinger bosons.

4. Unifying bosonic and fermionic PSGs

If an Abrikosov-fermion state and a Schwinger-boson state describes the same Z_2 spin liquid state, their symmetry quantum numbers on any finite lattice must be the same for arbitrary crystal symmetries. Therefore from the eigenvalues of inversion and reflection symmetries summarized previously, we can achieve a unification of Abrikosov-fermion and Schwinger-boson representation: i.e. their PSGs must satisfy the following correspondence:

$$-\eta_{I_s}^f = \eta_{I_s}^b; \quad (42)$$

$$-\eta_{I_s}^f = \eta_{I_s}^b; \quad (43)$$

$$-\eta_{I_s, P}^f = \eta_{I_s, P}^b, \quad (44)$$

\forall crystal symmetry P satisfying $PI_s = I_s P$;

$$-\eta_{R}^f = \eta_{R}^b; \quad (45)$$

$$-\eta_{R, P}^f = \eta_{R, P}^b, \quad (46)$$

\forall crystal symmetry P satisfying $PR = RP$.

These relations are in agreement with our conclusions based on the relative quantum numbers.

As pointed out in Ref.[4], the SET invariants $\{\eta_U^a\}$ must obey the fusion rule of anyons, i.e. $\eta_U^{b \cdot v} \equiv \eta_U^f = \omega_U^t \eta_U^b \eta_U^v$, where $\omega_U^t = \pm 1$ is the twisting factor associated with symmetry operation $U \in \text{SG}$ in a Z_2 spin liquid. As shown above, here we find a nontrivial twisting factor $\omega_R^t = -1$ for the square of any mirror reflection $R^2 = e$, in contrast to results in Ref.[4]. Similarly nontrivial twisting factor also shows up for $I^2 = RPR^{-1}P^{-1} = IPI^{-1}P^{-1} = e$. These twisting factors will be elaborated in the language of 1d SPT invariants in the next section.

In the next section we'll establish the correspondence between Schwinger-boson and Abrikosov-fermion representations for those PSGs concerning time reversal symmetry. This is achieved by relating the 2D parton PSGs to 1D SPT invariants by considering projected wavefunctions on a thin but long cylinder.

V. DIMENSIONAL REDUCTION AND ENTANGLEMENT SIGNATURES

A 2D model defined on a cylinder can be viewed as a 1D system by grouping one ring of the cylinder into a single super-site. This point of view is useful because the interplay of symmetry, topology and entanglement has been completely understood in 1D through the recent classification of 1D SPT

phases. [12–15] In this section we explain how 2D SET order manifests itself as 1D SPT order under this dimensional reduction. In particular, we find the \mathbb{Z}_2 PSG relations have a one-to-one correspondence with the U(1) projective representations that classify 1D SPT phases. While generally U(1) projective representations are a coarse-grained version of \mathbb{Z}_2 projective representations, space-group symmetries actually have an *anti*-unitary character under the dimensional reduction, and for this special case the correspondence becomes one-to-one.

A. A review of 1D SPT phases.

While the classification of 1D SPTs can be discussed in terms of Schmidt decomposition, the most compact treatment uses the formalism of matrix product states. We refer to previous works for a more detailed review. [12–15]

1. Matrix product states

Let $|j_n\rangle$ span the local Hilbert spaces of a spin chain with sites at n . A MPS $|\Psi\rangle$ is characterized by a sequence of rank-3 tensors $\{\Gamma_{\alpha_n\alpha_{n+1}}^{j_n}\}$ and rank-1 vectors $\{s_{\alpha_n}\}$ through the ansatz

$$\langle\{j_n\}|\Psi\rangle = \sum_{\alpha_n=1}^{\chi_n} \prod_n s_{\alpha_n} \Gamma_{\alpha_n\alpha_{n+1}}^{j_n}. \quad (47)$$

The indices α_n which are summed over are called the auxiliary indices, with dimension χ_n . We have assumed the MPS is in the ‘canonical form,’ which means that each s_{α_n} is the set of Schmidt weights for a bipartition of the system between sites $n-1, n$.

The MPS ansatz includes both finite and infinite spin chains. In the finite case with L sites, $\chi_1 = \chi_{L+1} = 1$. In the infinite case with a unit cell of length L , we can always choose the tensors to share this unit cell: $\Gamma_{\alpha_{n+L}\alpha_{n+1+L}}^{j_{n+L}} = \Gamma_{\alpha_n\alpha_{n+1}}^{j_n}$ and likewise for s and χ .

2. Onsite symmetries

If a spin chain is invariant under an onsite symmetry $g \in G$ (e.g. a spin rotation), it is natural to ask how the symmetry is encoded in the tensors Γ . The representation of the onsite symmetry decomposes into its action on each site, $\hat{g} = \otimes_n \hat{g}_n$. For notational simplicity, we will drop the site index n . An MPS is symmetric under g if and only if the Γ and s transform as [60]

$$\sum_k g_{jk} \Gamma_{\alpha\beta}^k = r_g \sum_{\alpha',\beta'} U_{g;\alpha\alpha'} \Gamma_{\alpha'\beta'}^j U_{g;\beta'\beta}^\dagger \quad (48)$$

$$U_{g;\alpha\beta} s_\beta = s_\alpha U_{g;\alpha\beta} \quad (49)$$

where the U_g are unitary matrices and $r_g \in \text{U}(1)$.

The phases r_g form a U(1) representation of the group, and encode the g -charge per unit length. But for the unitaries U_h there is another possibility: U_g may be a *projective* representation, meaning that the requirements of a group representation are satisfied only up to U(1) phases. This subtlety arises because the phase of U_g isn’t fixed by the transformation law of Eq. 48. Arbitrarily fixing the phase of each U_g , the U(1) phases ω are encoded in the relations

$$U_g U_h = \omega(g, h) U_{gh}, \quad \omega(g, h) \in \text{U}(1). \quad (50)$$

The phases ω are called cocycles, or the factor set. If we consider the phase ambiguity $U_g \rightarrow \theta(g) U_g$, $\theta(g) \in \text{U}(1)$ to be a ‘gauge transformation,’ we see that ω is not gauge invariant. The classification of gauge-inequivalent ω is given by the 2nd group cohomology $[\omega] \in \mathcal{H}^2[G, \text{U}(1)]$, resulting in a classification of 1D phases symmetric under G . Note that for symmetry group $G = \mathbb{Z}_N$, there are no projective representations, and hence no 1D SPT phases.

An important physical signature of an SPT phase under onsite G is degenerate edge states. It can be shown there are edge states which transform under G with the same projective representation $[\omega]$ as the U_g ; if $[\omega]$ is non-trivial, the projective representation must be multi-dimensional, implying a degeneracy.

3. Time-reversal symmetry

The transformation laws are modified for time-reversal $\mathcal{T} = \otimes_n \hat{u}_\mathcal{T} K$ because of the complex conjugation K . Similar to before, the MPS can be taken to transform as

$$\sum_k u_{\mathcal{T};jk} \bar{\Gamma}_{\alpha\beta}^k = \sum_{\alpha',\beta'} U_{\mathcal{T};\alpha\alpha'} \Gamma_{\alpha'\beta'}^j U_{\mathcal{T};\beta'\beta}^\dagger \quad (51)$$

$$U_{\mathcal{T};\alpha\beta} s_\beta = s_\alpha U_{\mathcal{T};\alpha\beta} \quad (52)$$

(note we can remove $r_\mathcal{T}$ by redefining the U(1) phase of the state). But for an anti-unitary symmetry like time-reversal \mathcal{T} , the projective relations are modified to

$$U_\mathcal{T} U_h^* = \omega(\mathcal{T}, h) U_{\mathcal{T}h}, \quad (53)$$

$$U_h U_\mathcal{T} = \omega(h, \mathcal{T}) U_{h\mathcal{T}} \quad \omega \in \text{U}(1) \quad (54)$$

In contrast to an onsite $G = \mathbb{Z}_2$, for the anti-unitary time-reversal ($G = \mathbb{Z}_2^T$) we can form the gauge-invariant relation

$$\gamma_\mathcal{T} \equiv U_\mathcal{T} U_\mathcal{T}^* = \pm 1. \quad (55)$$

This $\gamma_\mathcal{T}$ gives a \mathbb{Z}_2 SPT classification.

There is also an interplay between the other symmetries and \mathcal{T} . If G contains $H \times \mathbb{Z}_2^T \subset G$ as a subgroup, where H is onsite, for $h \in H$ there is a projective relation

$$U_\mathcal{T} U_h^* U_\mathcal{T}^{-1} = \frac{\omega(\mathcal{T}, h)}{\omega(h, \mathcal{T})} U_h \quad (56)$$

For our purposes it will be sufficient to understand $\mathcal{H}^2[\mathbb{Z}_2 \times \mathbb{Z}_2^T, \text{U}(1)] = (\mathbb{Z}_2)^2$; since $h\mathcal{T}$ is itself anti-unitary \mathbb{Z}_2 symmetry we can form another gauge-invariant relation

$$U_{h\mathcal{T}} U_{h\mathcal{T}}^* = \gamma_{h\mathcal{T}} = \pm 1, \quad (\text{when } h^2 = 1), \quad (57)$$

an additional \mathbb{Z}_2 invariant. Alternatively, we have the relation

$$(U_h^2)(U_{\mathcal{T}}U_h^*U_{\mathcal{T}}^{-1}U_h^{-1}) = \gamma_h\mathcal{T}\gamma_{\mathcal{T}} \quad (58)$$

4. Reflection symmetry

Finally, consider a reflection R which spatially inverts the 1D chain, $R|j_n\rangle = u_{R;j_n,k_{-n}}|k_{-n}\rangle$. The unitary matrix u_R encodes any internal rotation in the definition of the reflection. The transformation law is [13]

$$\sum_k u_{R;jk}(\Gamma^T)_{\alpha\beta}^k = r_R \sum_{\alpha',\beta'} U_{R;\alpha\alpha'}\Gamma_{\alpha'\beta'}^j U_{R;\beta'\beta}^\dagger \quad (59)$$

$$U_{R;\alpha\beta}S_\beta = s_\alpha U_{R;\alpha\beta} \quad (60)$$

where $r_R = \pm 1$ provides the first \mathbb{Z}_2 invariant, the ‘parity per unit length.’ Somewhat surprisingly, combining this transformation law with those of an onsite h we find the projective relations are the same as those of a *anti*-unitary symmetry:

$$U_R U_h^* = \omega(R, h) U_R h, \quad (61)$$

$$U_h U_R = \omega(h, R) U_h R \quad \omega \in \mathbf{U}(1). \quad (62)$$

The origin of this similarity is that transposition T and complex conjugation behave analogously when acting on the unitary U_h . This point is important, as it implies inversion has the same \mathbb{Z}_2 invariants as time-reversal \mathcal{T} :

$$U_R U_R^* = \gamma_R = \pm 1 \quad (63)$$

$$U_{hR} U_{hR}^* = \gamma_{hR} = \pm 1 \quad (\text{when } h^2 = 1) \quad (64)$$

$$(U_h^2)(U_R U_h^* U_R^{-1} U_h^{-1}) = \gamma_{hR} \gamma_R = \pm 1 \quad (65)$$

B. Identification of 1D SPT order and the space-group PSGs

Earlier we argued that space-group quantum numbers are topological invariants in the presence of reflection symmetries, and calculated these quantum numbers using the PSG. We now show that under the dimensional reduction these quantum numbers can be calculated from the 1D SPT invariants $r_R, \gamma_R, \gamma_{hR}$. This clarifies the origin of their stability, since 1D SPT phases are robust in the presence of symmetries, and provides a dictionary between the 2D SET and 1D SPT order.

To equate the 2D SET order with the 1D SPT invariants, we must compute the quantum numbers of a finite chain given the *infinite* chain 1D SPT invariants. For notational simplicity, we will assume the dimensional reduction results in a one-site 1D unit cell. The notation is more complex for the Kagome model, since under the dimensional reduction it is more convenient to use a unit cell of two, so we delay the discussion of this case. Since the system is a 2D cylinder, in addition to a reflection R which exchanges the edges of the chain, there may be an orthogonal reflection R' which does not exchange the edges, but instead behaves as an ‘onsite’ symmetry under the dimensional reduction. We combine these two reflections

to form a 180 degree inversion $I = RR'$, which again exchanges the edges. Consequently there are \mathbb{Z}_2 1D SPT invariants $\gamma_R, \gamma_I = \gamma_{RR'}$.

For any R/I symmetric state on a finite chain with L sites (L -odd implies a site-centered inversion) it can be proven that, independent of any details of the edge, the quantum numbers of the chain are

$$Q_R(L) = \gamma_R (r_R)^L \quad (66)$$

$$Q_I(L) = \gamma_I (r_I)^L \quad (67)$$

where γ, r are the 1D SPT invariants of the bulk phase.

To relate the 1D SPT invariants to the PSG, fix the transverse geometry Λ_\perp of the cylinder (such as the circumference), and find the four ground states of the infinite cylinder. Each topological sector a has 1D SPT invariants $\gamma_U(\Lambda_\perp, a), r_U(\Lambda_\perp, a)$.

Following our earlier discussion, we first compute the ratio of quantum numbers after threading a spinon b/f :

$$Q_U^{(b/f)} = \frac{Q_U(\Lambda, b/f \cdot a)}{Q_U(\Lambda, a)} = \frac{\gamma_U(\Lambda_\perp, b/f \cdot a)}{\gamma_U(\Lambda_\perp, a)} \left(\frac{r_U(\Lambda_\perp, b/f \cdot a)}{r_U(\Lambda_\perp, a)} \right)_L \quad (68)$$

Since the result is *independent* of the geometry, we have the following:

$$Q_U^{(b/f)} = \frac{\gamma_U(\Lambda_\perp, b/f \cdot a)}{\gamma_U(\Lambda_\perp, a)} \quad (69)$$

$$r_U(\Lambda_\perp, b/f \cdot a) = r_U(\Lambda_\perp, a) \quad (70)$$

We find that **the ratio of 1D SPT invariants γ_U between the topological sectors of an infinite cylinder reveals the PSG relation $Q_U^{(b/f)}$** . Furthermore, the parity per unit length is unchanged by threading a spinon.

We then compute the ratio of quantum numbers after threading a vison v :

$$Q_U^{(v)} = \frac{Q_U(\Lambda, v \cdot a)}{Q_U(\Lambda, a)} = \frac{\gamma_U(\Lambda_\perp, v \cdot a)}{\gamma_U(\Lambda_\perp, a)} \left(\frac{r_U(\Lambda_\perp, v \cdot a)}{r_U(\Lambda_\perp, a)} \right)_L \quad (71)$$

When U is not site-centered, our earlier discussion found that $Q_U^{(v)} = 1$, so by comparison we expect

$$\gamma_U(\Lambda_\perp, v \cdot a) = \gamma_U(\Lambda_\perp, a) \quad (72)$$

$$r_U(\Lambda_\perp, v \cdot a) = r_U(\Lambda_\perp, a), \quad U \in \{R, I_{h/p}\}. \quad (73)$$

On the other hand, the site-centered inversion I_s requires odd L , and from $Q_{I_s}^{(v)} = -1$ we find

$$\frac{r_{I_b}(\Lambda_\perp, v \cdot a)}{r_{I_b}(\Lambda_\perp, a)} = -1 \quad (74)$$

1. Translating between the \mathbb{Z}_2 PSG and the 1D SPT $U(1)$ PSG

The bosonic \mathbb{Z}_2 PSG is encapsulated by the projective relations

$$X \cdot Y = \omega^b(X, Y)(XY) \quad (75)$$

where $\omega^b \in \mathbb{Z}_2$. The 1D SPT relations are similar, but $\omega \in U(1)$, which is in general a much coarser classification as there are more phase ambiguities. Yet we have shown that the 1D SPT relations recover the 2D PSG. So how do the 2D PSG relations ‘descend’ to the 1D SPT relations? The key point is that if the symmetry R exchanges the edges of the cylinder, we can transcribe the 2D PSG relations into 1D SPT relations if we remember that reflection / inversion becomes anti-unitary in the 1D SPT realization:

$$R^2 = \eta_R^b \Rightarrow U_R U_R^* = \gamma_R \quad (76)$$

Due to the anti-unitary nature, the $U(1)$ phase ambiguity does not affect the robustness of γ_R . In general, **the 2D PSG relations descend to 1D SPT relations if we treat edge-exchanging symmetries as anti-unitary**. This explains the equality of the quantum numbers found in Eq. (69).

C. Identification of 1D SPT order and the $RTR^{-1}T^{-1}$ PSGs

In the fermionic parton construction there is an additional fermion PSG associated with the interplay of time-reversal T and a reflection R :

$$\eta_{RT}^f = R^{-1}T^{-1}RT \quad (77)$$

Viewing R as on-site, under the dimensional reduction we have a similar 1D relation

$$U_R U_T = \frac{\omega(R, T)}{\omega(T, R)} U_T U_R^* \quad (78)$$

But $\omega(R, T)$ is *not* $U(1)$ gauge invariant. Instead, we may consider the 1D SPT invariant defined by Eq. (57). Comparing the 2D and 1D SPT PSG relations,

$$(R^2)(R^{-1}T^{-1}RT) = \eta_R^f \eta_{RT}^f \quad \text{2D PSG} \quad (79)$$

$$(U_R)^2 (U_R^{-1} U_T U_R^* U_T^{-1}) = \gamma_{RT} \gamma_T \quad \text{1D SPT} \quad (80)$$

we obtain the following 2D PSG to 1D SPT reduction:

$$\eta_R^f \eta_{RT}^f \rightarrow \gamma_{RT} \gamma_T. \quad (81)$$

This identification can be verified by checking for the physical signature of $\eta_R^f \eta_{RT}^f = -1$. The fermionic spinon always has a 2-fold degeneracy associated with it’s Kramer’s degeneracy $\eta_T^f = -1$, but when $\eta_R^f \eta_{RT}^f = -1$ the spinon excitation is at least four-fold degenerate, as this is the minimal dimension of these projective relations. The 1D SPT relation $\gamma_T = -\gamma_{RT}$ implies *precisely* this additional two-fold degeneracy.

In summary, we can determine $\eta_R^f \eta_{RT}^f$ by computing the change in the 1D SPT invariant $\gamma_T \gamma_{RT}$ (with R onsite) when the topological flux changes by f . In Ref. 5 it argued that when $\eta_R^f = -1$, $\eta_{RT}^f = 1$ (where $R = \sigma$ in the notation of the Kagome model), there *may* be gapless edge modes protected by a non-trivial *vison* PSG $R^{-1}T^{-1}RT = -1$ (associated with topological superconductivity in the fermionic mean-field ansatz). Presumably the vison PSG can also be computed from the change in the 1D SPT invariant γ_{RT} ($\gamma_T = 1$ for the vison) when threading a vison.

VI. INTRINSIC TOPOLOGICAL ORDER: DETECTING THE TOPOLOGICAL FLUX

It is important to have a method for measuring the topological flux a of an MES independent of the SET order; in the finite cylinder, we must detect whether a vison v threads the cylinder, and for a torus / infinite cylinder we must distinguish between all of $1, b, v, f$. It has previously been shown that the topological S and T matrices can be calculated from the MES on both the torus and infinite cylinder, which can then be used to label the MES.[35] In practice this is not so simple as all four-sectors must be found on a finite circumference cylinder, which is frustrated by finite size effects which lead to a sizable splitting of the topological degeneracy. However, for a Z_2 spin-liquid we find there is a simpler procedure to uniquely label the MES individually.

Note that an $S = 1/2$ model must have finite topological-flux per unit cell [29], so the topological flux through an entanglement cut depends on the location of the cut. So to simplify the discussion, we restrict to even circumferences $L_y \in 2\mathbb{Z}$, which contain an even number of $S = 1/2$ within each ring of the cylinder, and always consider ‘vertical’ entanglement cuts which lie between 1D super-sites under the 2D to 1D dimensional reduction.

A. $\{1, v\}$ vs $\{b, f\}$

The first distinction detects the fractional $S = 1/2$ spin carried by the spinons b and f . Under the dimensional reduction, an even circumference cylinder is an $SO(3)$ invariant integer-spin chain, which has a \mathbb{Z}_2 1D SPT classification associated with the emergence of two-fold degenerate $S = 1/2$ edge states (protected either by time-reversal or $SO(3)$). The simplest non-trivial example is the $S = 1$ AKLT state. In 2D, when topological flux b or f terminates at the edge of the cylinder it also produces a spinon excitation near the edge carrying $S = 1/2$. If the system is $SO(3)$ symmetric, this emergent edge spin carries a two-fold degeneracy. Hence under the dimension reduction, the b, f sectors are non-trivial 1D SPT states under $SO(3)$, while $1, v$ are trivial.

We conclude that the $1, v$ sectors will have an entanglement spectrum that transforms under integer representations of $SO(3)$, and hence will have singlets in the entanglement spectrum, while the entanglement spectrum of the b, f sectors will transform under half-integral representations of $SO(3)$, leading to a minimum two-fold entanglement degeneracy.

B. 1 vs v

In an $SO(3)$ symmetric spin liquid, threading a vison v through the system is topologically equivalent to threading 2π - flux with respect to S^z spin-rotations. To distinguish between the 1 and v sectors we can detect the change in momentum induced by the flux threading. Viewing a cylinder of length L_x and circumference L_y as a spin chain on a periodic ring of length L_y , the unit cell of the chain contains integral /

half-integral spin when L_x is even / odd. In the half-integral case, threading flux with respect to S^z spin rotations is known to increase the y -momentum by $e^{i\pi} = -1$. [56] In the integral case, the threading flux will not change the momentum.

We conclude that threading a vison through the system will increment the y -momentum by π when L_x is odd, and by 0 if L_x is even. This increment is simply the y -momentum per unit length $\eta_{xy}^{1/v}$ introduced in Sec. III B. Hence for the vacuum, $\eta_{xy}^1 = 1$, while for the vison, $\eta_{xy}^v = -1$. This result is straightforward to check in any parton construction.

We know $\eta_{xy}^b = -\eta_{xy}^f$, since they differ by vison insertion, but *which* of the two carries $\eta_{xy}^a = -1$ will in fact depend on the PSG relation $(T_x T_y)^a = \eta_{xy}^a (T_y T_x)^a$.

C. b vs f

A fermionic anyon has topological spin $\theta_f = -1$, which we can use to distinguish between the b and f spinons. We will show that the fermion's topological spin is encoded in an additional 2x degeneracy in the entanglement spectrum (ES); combined with the 2x spin degeneracy, the fermionic spinon ES is 4x degenerate. Intuitively, the ES of the fermionic sector should have anti-periodic boundary conditions, meaning that the momenta k are quantized as $k \in \frac{2\pi}{L_y}(\mathbb{Z} + \frac{1}{2})$. With either time-reversal or reflection symmetry, the momenta k and $-k$ will be degenerate, so there is a 2x degeneracy. We will show this 2x degeneracy arises from non-trivial 1D SPT order under a combination of \mathbb{Z}_{L_y} rotational symmetry and either reflection or time-reversal.

1. Review of momentum polarization

Momentum polarization is a procedure to detect topological spin using a translation T_y that rotates a cylinder of circumference L_y . [38, 39] To review, each left Schmidt state $|\alpha\rangle$ of the Schmidt decomposition $\{e^{-E_\alpha}, |\alpha\rangle\}$ can be assigned definite momentum e^{ik_α} , meaning that $T_y |\alpha\rangle = e^{ik_\alpha} |\alpha\rangle$. In a convention in which $k_\alpha \in \frac{2\pi}{L_y}\mathbb{Z}$, the momentum polarization is

$$\left(\sum_{\alpha} e^{-E_\alpha + ik_\alpha} \right)^{L_y} = \mathcal{T}_a e^{-(\alpha - i\eta_H)L_y^2} + \mathcal{O}(e^{-L_y/\xi}), \quad (82)$$

$$\mathcal{T}_a = e^{2\pi i(h_a - c/24)}. \quad (83)$$

Here α is non-universal real constant related to the area-law fluctuation of momentum of the cut; η_H is the ‘Hall-viscosity’ (which isn't quantized on a lattice), and $\mathcal{T}_a = e^{2\pi i(h_a - c/24)}$ is the desired entry of the modular \mathcal{T} -matrix.

The momentum polarization is *not* necessarily a 1D SPT invariant, since there are no 1D SPTs associated with an onsite \mathbb{Z}_{L_y} symmetry. As a consequence, generally the momentum polarization only becomes quantized in the $L_y \rightarrow \infty$ limit, and a scaling analysis is required.

2. A 1D SPT invariant for detecting fermionic topological flux

However, in the presence of a mirror reflection $y \leftrightarrow -y$ or time reversal, we can prove that the momentum polarization is a \mathbb{Z}_2 1D SPT invariant that detects whether the topological flux is bosonic or fermionic. Consider a cylinder of even circumference L_y with a mirror reflection $R_y : (x, y) \rightarrow (x, -y)$ that acts as an onsite symmetry in the 1D picture (the result for time reversal is analogous). When acting on the entanglement spectrum, a π -rotation (translation by $L_y/2$) may anti-commute with the inversion R_y :

$$U_{R_y} (U_{T_y})^{L_y/2} U_{R_y}^{-1} = (-1)^F (U_{T_y})^{L_y/2}, \quad F = 0, 1 \quad (84)$$

giving a 1D \mathbb{Z}_2 invariant $F = 0, 1$. To show F is a 1D SPT invariant, note the symmetry group generated by T_y, R_y is $G = \mathbb{Z}_{L_y} \rtimes \mathbb{Z}_2$ (for even L_y). The cohomology classification is

$$\mathcal{H}^2[\mathbb{Z}_{L_y} \rtimes \mathbb{Z}_2, U(1)] = \mathbb{Z}_2. \quad (85)$$

The relation F of Eq. (84) is a gauge invariant, so must label these two possibilities.

To relate the SPT invariant F to the anti-periodic boundary conditions of the entanglement spectrum, suppose we redefine the phase of U_{T_y} to ensure the expected relation $U_{R_y}^{-1} U_{T_y} U_{R_y} = (U_{T_y})^{-1}$. With this gauge choice Eq. (84) requires $(U_{T_y})^{L_y} = (-1)^F$. For $F = 1$, the diagonal basis $U_{T_y} = e^{ik_\alpha}$ requires $k_\alpha \in \frac{2\pi}{L_y}(\mathbb{Z} + \frac{1}{2})$.

We note that the most general commutation relation is

$$U_{R_y} U_{T_y} U_{R_y}^{-1} = \gamma_F U_{T_y}^{-1} \quad (86)$$

with $\gamma_F^{L_y/2} = (-1)^F$.

3. Quantization of momentum polarization by the 1D SPT invariant

We now prove that the momentum polarization is quantized to be $(-1)^F$; this confirms the interpretation that F detects the topological spin $\theta_f = -1$ of the fermionic spinon.

Since U_{R_y} commutes with the entanglement spectrum e^{-E_α} , we have

$$\lambda = \text{Tr}(e^{-E_\alpha} U_{T_y}) = \text{Tr}(e^{-E_\alpha} U_{R_y}^{-1} U_{T_y} U_{R_y}) \quad (87)$$

$$= \text{Tr}(e^{-E_\alpha} \gamma_F U_{T_y}^{-1}) = \gamma_F \text{Tr}(e^{-E_\alpha} U_{T_y}^*) = \gamma_F \lambda^* \quad (88)$$

Using this relation, for even L_y the momentum polarization is

$$\lambda^{L_y} = \lambda^{L_y/2} (\gamma_F \lambda^*)^{L_y/2} = \gamma_F^{L_y/2} |\lambda|^{L_y} = (-1)^F |\lambda|^{L_y}. \quad (89)$$

So long as $|\lambda| \neq 0$, the momentum polarization is equivalent to the SPT invariant. The SPT invariant is even more robust since it is well defined even when $|\lambda| = 0$.

VII. DETECTING SET ORDER ON THE KAGOME LATTICE USING CYLINDER-DMRG

We now propose a procedure to determine the Kagome PSGs which is practically adapted to the constraints of cylinder DMRG. The results of this analysis will be reported in a subsequent work.[61]

Following the notation of the earlier analysis, [49] within the fermionic parton construction we must determine the five invariants $\{\eta_\sigma, \eta_{C_6}, \eta_{\sigma C_6}, \eta_{\sigma T}, \eta_{C_6 T}, \eta_{12} = \eta_{C_6} \eta_{\sigma C_6}\}$, combined with a possibility of mirror-symmetry-protected gapless edge states.[5]

A. Finite DMRG

We first consider a technique for finite length cylinders. By finding the SO(3) invariant ground state after adding or removing an extra spin at each edge of the geometry, DMRG studies can reliably obtain two topological sectors that differ by threading a spinon through the bulk. At the circumferences that can currently be well converged (such as YC8 and XC8), the other two topological sectors are not generally observed. Even without determining whether these two sectors are $1/v$, b/f , the ratio of R_x, R_y, I_h quantum numbers before and after adding the extra sites (i.e. a spinon) will determine $Q_{R_x, R_y, I_h}^{(b/f)}$, and referring to Table II, three of the PSG invariants. Since the symmetries must be edge-exchanging, on the YC type cylinders we obtain $Q_{R_x}^{(b/f)}$ and hexagon centered $Q_{I_h}^{(b/f)}$. On the XC type cylinders, we obtain $Q_{R_y}^{(b/f)}$ and hexagon centered $Q_{I_h}^{(b/f)}$, the latter serving as a double check on the YC data. There is a simple algorithm for measuring space-group quantum numbers in finite DMRG.[61]

To determine η_{12} one must first determine whether the topological sector of the spinon is b/f . To distinguish b/f , we check if there is a 4-fold degeneracy in the entanglement spectrum as predicted by Eq. (84), which would imply the sector is f . Knowing the sector b/f , we know the correct boundary condition for the parton ansatz, and following the techniques of Sec. IV E we can predict the site-centered inversion $Q_{I_s}(\Lambda, b/f)$. The details depend on the cylinder used, but the result always reveals η_{12} .

Once we know the topological sector of the sample, we can further check these results by comparing the *absolute* quantum numbers under $R_x, R_y, I_h/s$ with the computation of Sec. IV E; there are many different cases depending on the sample.

B. Infinite DMRG

An infinitely long cylinder can be studied using iDMRG, which has certain numerical advantages due to the absence of edge effects and the reduced computational costs. iDMRG also reliably finds two topological sectors which differ by a spinon. Here we discuss only even-circumference cylinders.

As discussed, the \mathbb{Z}_2 1D SPT invariants for SO(3) / T determine which state has the spinon. The fermionic 1D SPT invariant distinguishes between b/f . The momentum per unit length η_{xy}^a is trivial to compute in iDMRG, so we distinguish between sectors $1/v$ using $1 = \eta_{xy}^1 = -\eta_{xy}^v$. The momentum per unit length of the spinon sector determines either $\eta_{12} = \eta_{xy}^f$ (if the sector is f) or $-\eta_{12} = \eta_{xy}^b$ (if the sector is b).

To measure the reflection PSGs, one can in principle detect the 1D SPT reflection invariants using established methods.[62] This can be a bit unwieldy in 2D DMRG, as the ordering of the DMRG ‘snake’ breaks the reflection symmetries. A simpler procedure is to generate a *finite* cylinder wavefunction by projecting the left / right regions of the infinite cylinder onto reflection related classical product states, leaving behind a finite segment of spins.[61] Regardless of projection used, the resulting state is a reflection symmetric finite cylinder wavefunction. One can then measure the space-group quantum numbers of the resulting finite cylinder wavefunction in order to determine $Q_{R_x, R_y, I_h}^{(b/f)}$.

C. Determining $RT R^{-1} T^{-1}$

As discussed in Sec. V C, the remaining invariants are related to the onsite 1D SPT invariants γ_{RT} for $R = R_x, R_y$, which can be measured on YC and XC type cylinders respectively using known methods for detecting 1D SPTs. [62] The most obvious signature is the 4-fold degeneracy in the ES required to realize the projective relations $U_T U_T^* = \gamma_T = -1$ and $(U_R)^2 (U_R^{-1} U_T U_R^* U_T^{-1}) = \gamma_T \gamma_{RT} = -1$. The fermionic PSGs are related to the relative 1D SPT order $\gamma^{(f)}$ between the 1 and f sectors via

$$\gamma_T^{(f)} \gamma_{R_x T}^{(f)} = \eta_{\sigma T} \eta_\sigma \quad (90)$$

$$\gamma_T^{(f)} \gamma_{R_y T}^{(f)} = \eta_{\sigma T} \eta_{C_6 T} \eta_\sigma \eta_{\sigma C_6} \quad (91)$$

Of course $\gamma_T^{(f)} = -1$, as it is a spinon.

If the DMRG obtains sectors which differ by f , we are done. If DMRG obtains sectors which differ by b , the analysis depends on whether the vison has a non-trivial $RT R^{-1} T^{-1}$ PSG. If the vison PSG is trivial, there are no gapless edge states and the boson will have the same $RT R^{-1} T^{-1}$ PSG as the fermions, so the bosonic relative SPT order $\gamma^{(b)}$ again recovers the fermionic PSG invariants. If the vison PSG is non-trivial, there are gapless edge states, and $\eta_{\sigma T} \eta_\sigma = -1$ for the R_x edge and $\eta_{\sigma T} \eta_{C_6 T} \eta_\sigma \eta_{\sigma C_6} = -1$ for the R_y edge regardless.

VIII. CONCLUSIONS

We argued that the many-body symmetry quantum numbers are a robust global property ideally suited to detecting distinctions between SETs, and calculated these for several SETs described by slave-particle mean field theories with different projective symmetry groups.

More generally, the SETs may be diagnosed from the 1D SPT invariants in the cylinder geometry. These invariants for different topological sectors (which are labeled by the quasiparticles) combine together in a way that reflects the fusion rules. In contrast, combining PSGs for a pair of quasiparticles to predict the PSG for the fusion product needs to be carefully considered in the case of internal symmetries.

The knowledgeable reader may be puzzled by this correspondence between PSGs and 1D SPTs. The latter is determined by projective representations modulo a phase, or technically $\mathcal{H}^2(G, U(1))$, where G is the symmetry group, while the former is a projective representation modulo $Z_2 \times Z_2$ (the fusion group of Abelian anyons), $\mathcal{H}^2(G, Z_2 \times Z_2)$, for toric-code-type Z_2 topological order. That is, we represent group elements by matrices, whose product satisfies the group multiplication, up to either a $U(1)$ phase or just an overall sign (Z_2) of the matrices. This is because the physically observable quantities are made by combining two identical quasiparticles and so we can only change the overall sign of the matrices. For example, if X is a Z_2 symmetry, then ‘half charge’ of a quasiparticle corresponds to the PSG $X^2 = -1$. However, this is not a 1D SPT invariant. How is this discrepancy reconciled?

The key observation is that if inversion, I , or a equally a reflection is present, these act like anti-unitary symmetries when restricted to the Schmidt states on one side of a bipartition. Thus, while $I^2 = -1$ may again seem to be a PSG relation, regarding it as an antiunitary symmetry turns it into a projective representation even with $U(1)$ phase factors and hence a 1D SPT invariant [13, 53]. Similarly, for a global Z_2 symmetry, X , while by itself $X^2 = -1$ does not produce a 1D SPT invariant, when combined with the effective antiunitary

inversion symmetry $\frac{XIX^{-1}I^{-1}}{X^2}$ is a 1D SPT invariant and the denominator is the fractional charge we are interested in. This type of reasoning has been repeatedly used in this work.

Our procedure is expected to be complete for Z_2 spin liquids, but for more complicated topological orders, such as say Z_3 topological order, there is a Z_3 invariant associated with a C_3 rotation symmetry. Taking into account certain subtleties, global C_3 quantum numbers detect this topological invariant, but the simplest cylinder dimensional reduction will not work. Extensions are left to future work.

Detailed comparison between our predictions and DMRG numerics on the Kagome quantum antiferromagnet will require some care. First, isolating the role of finite size effects is important as the ground state quantum numbers can change with system size. Also, the response of the Kagome model to twisting boundary conditions is significant and has been interpreted as a signal of gapless behavior in recent numerical work[30]. For these reasons, a comparison with numerics will be deferred to a separate publication.

IX. ACKNOWLEDGEMENTS

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