Quantum disentangled liquid in the half-filled Hubbard model

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Quantum-disentangled liquid in the half-filled Hubbard model

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We investigate the existence of quantum disentangled liquid (QDL) states in the half-filled Hubbard model on bipartite lattices. In the one dimensional case we employ a combination of integrability and strong coupling expansion methods to argue that there are indeed finite energy-density eigenstates that exhibit QDL behaviour in the sense of J. Stat. Mech. P10010 (2014). The states exhibiting the QDL property are atypical in the sense that while their entropy density is non-zero, it is smaller than that of thermal states at the same energy density. We argue that for $U \gg t$ these latter thermal states exhibit a weaker form of the QDL property, which carries over to the higher dimensional case.

I. INTRODUCTION

The last decade has witnessed significant advances in our understanding of relaxation in isolated many-particle quantum systems. Many of these were driven by advances in ultra-cold atom experiments, which have made it possible to study the real-time dynamics of almost isolated systems in exquisite detail[1–11]. At present there are three established paradigms for the relaxational behaviour in such systems. According to the eigenstate thermalization hypothesis[12–15] "generic" systems relax towards thermal equilibrium distributions. In this case the only information about the initial state that is retained at late times is its energy density. Quantum integrable models cannot thermalize in this way as they exhibit additional conservation laws, but they relax to generalized Gibbs ensembles instead[16–21]. Finally, disordered many-body localized systems fail to thermalize as well[22–27]. This is related to the existence of conservation laws[27–29], although in contrast to integrable models no fine-tuning of the Hamiltonian is required.

Recently it has been proposed that certain isolated quantum systems may exhibit behaviour that is not captured by these three known paradigms[30]. The resulting state of matter has been termed a "quantum disentangled liquid" (QDL). A characteristic feature of such systems is that they comprise both "heavy" and "light" degrees of freedom, which will be made more precise below. The basic premise of the QDL concept is that while the heavy degrees of freedom are fully thermalized, the light degrees of freedom, which are enslaved to the heavy degrees of freedom, are not independently thermalized. A convenient diagnostic for such a state of matter is the bipartite entanglement entropy (EE) after a projective measurement of the heavy particles. Specifically, we define a QDL to be a state in which the entanglement entropy reduces from a volume-law to an area-law upon a projective measurement of the heavy particles. The possibility of realizing a QDL in the one dimensional Hubbard model was subsequently investigated by exact diagonalization of small systems in Ref. 31. Given the limitations on accessible system sizes it is difficult to draw definite conclusions from these results. Motivated by these studies we explore the possibility of realizing a QDL in the half-filled Hubbard model on bipartite lattices by analytic means. We employ methods of integrability as well as strong-coupling expansion techniques[32–35] to analyze properties of finite energy-density eigenstates of the Hubbard Hamiltonian

$$\begin{aligned} H &= -t\sum_{\langle j,k\rangle}\sum_{\sigma=\uparrow,\downarrow} c^\dagger_{j,\sigma}c_{k,\sigma} + c^\dagger_{k,\sigma}c_{j,\sigma} \\ &+ U\sum_j \big(n_{j,\uparrow}-\frac{1}{2}\big)\big(n_{j,\downarrow}-\frac{1}{2}\big). \end{aligned} \tag{1}$$

Here $n_{j,\sigma} = c^\dagger_{j,\sigma}c_{j,\sigma}$ and $\langle j,k\rangle$ denote nearest neighbour links that connect only sites from different sub-lattices A and B. At half-filling (one electron per site) (1) is invariant under the η-pairing SU(2) symmetry with generators[36]

$$\begin{aligned} \eta^z &= \frac{1}{2}\sum_j \left(n_{j,\uparrow}+n_{j,\downarrow}-1\right) \ , \\ \eta^\dagger &= \sum_{j\in A} c^\dagger_{j,\uparrow}c^\dagger_{j,\downarrow} - \sum_{k\in B} c^\dagger_{k,\uparrow}c^\dagger_{k,\downarrow} \equiv (\eta)^\dagger \, . \end{aligned} \tag{2}$$

In the strong interaction limit of the Hubbard model ($U \gg t$), the bandwidths of the collective spin and charge degrees of freedom are proportional to t^2/U and t respectively. In this regime the spin (charge) degrees of freedom therefore correspond to "heavy" ("light") particles.

The outline of our paper is as follows. In section II we employ methods of integrability to show that it is possible to construct certain ("Heisenberg sector") macrostates at finite energy densities for which the charge degrees of freedom do not contribute to the volume term in the bipartite EE. This strongly suggests that these states have the QDL property. In section III we then turn to strong-coupling expansion methods in order to examine the QDL diagnostic proposed in Ref. 30. In the one dimensional case this analysis shows that in the framework of a t/U-expansion a projective measurement of the spin

degrees of freedom in a Heisenberg sector state indeed leaves the system is a state that is only area-law entangled. Our strong-coupling analysis further suggests the existence of a weaker version of the QDL property, where a projective measurement of the heavy degrees of freedom leaves the system in a state characterized by a volume-law entanglement entropy, but with a pre-factor that is exponentially small in U/t. Section IV contains a summary and discussion of our results. Various technical aspects of our calculations are presented in three appendices.

II. INTEGRABILITY

The one dimensional Hubbard model is exactly solvable by the Bethe ansatz method[37]. This provides a description of energy eigenstates as well as the corresponding macro states in the thermodynamic limit. We now use this framework to identify a class of macro states that exhibits a property that is characteristic of a QDL. In the framework of the string hypothesis general macro states in the one dimensional Hubbard model are described by sets of particle and hole densities $\{\rho^p(k), \rho^h(k), \sigma_n^p(\Lambda), \sigma_n^h(\Lambda), {\sigma'_n}^p(\Lambda), {\sigma'_n}^h(\Lambda) | n \in \mathbb{N}\}$. These are analogs of the well-known particle and hole densities used to characterize macro states in the ideal Fermi gas. The main complication in integrable models is that there are in general (infinitely) many different species of excitations, which interact with one another. In the case at hand the $\rho^{p/h}$ describe pure charge excitations, the $\sigma_n^{p/h}$ correspond to elementary spin excitations as well as their bound states and ${\sigma'_n}^{p/h}$ represent bound states between spin and charge degrees of freedom. As a consequence of the interacting nature of the Hubbard chain the root densities are related in a non-trivial manner: they are subject to the thermodynamic limit of the Bethe Ansatz equations[37]

$$
\begin{aligned}
\rho^p(k) + \rho^h(k) &= \frac{1}{2\pi} + \cos k \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\Lambda \; a_n(\Lambda - \sin k) \left[{\sigma'_n}^p(\Lambda) + \sigma_n^p(\Lambda) \right] \; , \\
\sigma_n^h(\Lambda) &= -\sum_{m=1}^{\infty} \int_{-\infty}^{\infty} d\Lambda' A_{nm}(\Lambda - \Lambda') \; \sigma_m^p(\Lambda') + \int_{-\pi}^{\pi} dk \; a_n(\sin k - \Lambda) \; \rho^p(k) \; , \\
{\sigma'_n}^h(\Lambda) &= \frac{1}{\pi} \mathrm{Re} \frac{1}{\sqrt{1 - (\Lambda - inu)^2}} - \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} d\Lambda' A_{nm}(\Lambda - \Lambda') \; {\sigma'_m}^p(\Lambda') - \int_{-\pi}^{\pi} dk \; a_n(\sin k - \Lambda) \; \rho^p(k) \; . \qquad (3)
\end{aligned}
$$

Here $u = U/4t$ and

$$
\begin{aligned}
a_n(x) &= \frac{1}{2\pi} \frac{2nu}{(nu)^2 + x^2} \; , \\
A_{nm}(x) &= \delta(x) + (1 - \delta_{m,n}) a_{|n-m|}(x) + 2a_{|n-m|+2}(x) \\
&+ \cdots + 2a_{|n+m|-2}(x) + a_{n+m}(x). \qquad (4)
\end{aligned}
$$

The energy and thermodynamic entropy per site are then given by

$$
\begin{aligned}
e &= u + \int_{-\pi}^{\pi} dk \left[-2\cos k - \mu - 2u \right] \rho^p(k) \\
&+ 4 \sum_{n=1}^{\infty} \int d\Lambda \, {\sigma'_n}^p(\Lambda) \, \mathrm{Re} \sqrt{1 - (\Lambda + inu)^2}, \\
s &= \int_{-\pi}^{\pi} dk \; \mathcal{S} \left[\rho^p(k), \rho^h(k) \right] \\
&+ \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\Lambda \; \mathcal{S} \left[{\sigma'_n}^p(\Lambda), {\sigma'_n}^h(\Lambda) \right] \\
&+ \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\Lambda \; \mathcal{S} \left[{\sigma_n}^p(\Lambda), {\sigma_n}^h(\Lambda) \right] , \qquad (5)
\end{aligned}
$$

where we have defined

$$
\begin{aligned}
\mathcal{S}[f, g] &= \left[f(x) + g(x) \right] \ln \left(f(x) + g(x) \right) \\
&- f(x) \ln \left(f(x) \right) - g(x) \ln \left(g(x) \right) \; . \qquad (6)
\end{aligned}
$$

The ground state of the half-filled Hubbard model in zero magnetic field is obtained by choosing

$$
\rho^h(k) = 0 = \sigma_1^h(\Lambda) \; , \qquad {\sigma'}^p_n(\Lambda) = 0 = \sigma^p_{n \geq 2}(\Lambda) \; . \qquad (7)
$$

A. The "Heisenberg sector"

Motivated by Ref. 31 we now consider a particular class of macro states, which we call *Heisenberg sector states*. The principle underlying their construction is to "freeze" the charge degrees of freedom in its ground state configuration while imposing a finite energy density in the spin sector. This is achieved by requiring

$$
{\sigma'}^p_n(\Lambda) = 0 = \sigma^p_n(\Lambda) \; , \; n = 1, 2, \ldots \qquad (8)
$$

Condition (8) corresponds to the absence of bound states between spin and charge degrees of freedom (i.e. no k-Λ

strings) and a completely filled "Fermi sea" of elementary charge degrees of freedom. Importantly the thermodynamic entropy per site of Heisenberg sector states depends only on the spin sector

$$s = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\Lambda \; \mathcal{S}\left[{\sigma_n}^p(\Lambda), {\sigma_n}^h(\Lambda)\right]. \tag{9}$$

As shown in Appendix A the total number of Heisenberg sector states $N_{\rm HS}$ fulfils

$$\lim_{L\to\infty} \frac{\ln\left(N_{\rm HS}\right)}{L} = \ln(2). \tag{10}$$

The result (10) suggests that for large, finite L there are approximately 2^L Heisenberg sector states.

B. Entanglement entropy of Heisenberg sector states

We now make use of the relation between the volume term in the EE and the thermodynamic entropy density for eigenstates of short-ranged Hamiltonians. Consider a finite energy density eigenstate $|\Psi\rangle$ and a large subsystem A of size $|A|$. Then the volume term in the EE of the state $|\Psi\rangle$ is given by

$$S_{\rm vN,A} = s|A| + o(|A|) \; . \tag{11}$$

where s is the thermodynamic entropy density of the macro state associated with $|\Psi\rangle$.

Combining (9) with (11), we conclude that *for Heisenberg sector states the volume term in the entanglement entropy is entirely due to the spin degrees of freedom, and the charge degrees of freedom do not contribute.* This is very much in line with what one would expect for a QDL state.

C. Thermal states in the large-U limit

It is very instructive to contrast the entanglement properties of Heisenberg sector states to those of typical states. The maximum entropy states at a given energy density are thermal and can be constructed by the Thermodynamic Bethe Ansatz (TBA). This provides a system of coupled non-linear integral equations for the ratios of the hole and particle densities[37,39]

$$\zeta(k) = \frac{\rho^h(k)}{\rho^p(k)} \; , \; \eta_n(\Lambda) = \frac{\sigma_n^h(\Lambda)}{\sigma_n^p(\Lambda)} \; , \; \eta_n'(\Lambda) = \frac{{\sigma_n'}^h(\Lambda)}{{\sigma_n'}^p(\Lambda)} \; . \tag{12}$$

For the sake of completeness we present the TBA equations in Appendix B. While in general the TBA equations can only be solved numerically, in the limit of strong interactions analytic results can be obtained[40–42]. For simplicity we focus on the "spin-disordered regime"

$$\frac{4t^2}{U} \ll T \ll U \; . \tag{13}$$

This regime corresponds to temperatures that are small compared to the Mott gap, but large compared to the exchange energy. Here one has[42]

$$\rho^h(k) = \mathcal{O}\big(e^{-u/T}\big) \; , \quad {\sigma'}_n^{p,h}(\Lambda) = \mathcal{O}\big(e^{-u/T}\big) \; , \tag{14}$$

where we have set $t = 1$ as our energy scale. Substituting this into the general expression (5) for the thermodynamic entropy density we obtain

$$s = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\Lambda \; \mathcal{S}\left[{\sigma_n}^p(\Lambda), {\sigma_n}^h(\Lambda)\right] + \mathcal{O}\Big(\frac{u}{T} e^{-u/T}\Big). \tag{15}$$

Finally, using the relation between thermodynamic and EE (11) we conclude that for thermal states in the spin-disordered regime the contribution of the charge degrees of freedom contribute to the volume term is

$$\begin{aligned} S_{\rm vN,A} &= \big(s_{\rm spin} + s_{\rm charge}\big)|A| + o\big(|A|\big) \; , \\ s_{\rm spin} &= \mathcal{O}(1) \; , \quad s_{\rm charge} = \mathcal{O}\Big(\frac{u}{T} e^{-u/T}\Big) \; . \end{aligned} \tag{16}$$

Here $s_{\rm charge}$ includes the contributions from pure charge degrees of freedom as well as bound states of spin and charge. Importantly, unlike Heisenberg sector states, typical states have a contribution from the charge degrees of freedom to the volume term. However, this contribution is exponentially small in u/T and therefore only visible for extremely large subsystems. While we have focussed on the spin-disordered regime, the behaviour (16) extends to thermal states for all $0 < T \ll U$.

III. t/U-EXPANSION

The analysis presented above provides a strong indication that the Heisenberg sector states realize the QDL concept. However, the exact solution does not presently allow one to examine the QDL diagnostic proposed in Ref. 30, which requires the calculation of the EE after a partial measurement. We therefore now turn to a complementary approach, namely a strong-coupling expansion in powers of t/U. This is most conveniently implemented by following Ref. 33. At a given site j there are four possible states $|0\rangle_j$, $|\uparrow\rangle_j = c^\dagger_{j,\uparrow}|0\rangle_j$, $|\downarrow\rangle_j = c^\dagger_{j,\downarrow}|0\rangle_j$ and $|2\rangle_j = c^\dagger_{j,\uparrow}c^\dagger_{j,\downarrow}|0\rangle_j$. Defining Hubbard operators by $X^{ab}_j = |a\rangle_{jj}\langle b|$, the Hamiltonian (1) can be expressed in the form

$$H = UD + t\left(T_0 + T_1 + T_{-1}\right), \tag{17}$$

where $D = \frac{1}{4}\sum_j X^{22}_j + X^{00}_j - X^{\uparrow\uparrow}_j - X^{\downarrow\downarrow}_j$ and $T_a = \sum_j T_{a,j}$ are correlated hopping terms that change the number of doubly occupied sites by a

$$\begin{aligned} T_{0,j} &= -\sum_\sigma \big(X^{2\sigma}_j X^{\sigma 2}_{j+1} + X^{\sigma 0}_j X^{0\sigma}_{j+1} + \text{h.c.}\big), \\ T_{1,j} &= T^\dagger_{-1,j} = -\sum_\sigma \sigma \left[X^{2\bar\sigma}_j X^{0\sigma}_{j+1} + X^{0\bar\sigma}_j X^{2\sigma}_{j+1}\right] \; . \end{aligned}$$

The t/U-expansion is conveniently cast in the form of a unitary transformation[33]

$$H' = e^{iS} H e^{-iS} = H + [iS, H] + \frac{1}{2}[iS, [iS, H]] + \ldots, \quad (18)$$

where the generator iS is chosen as a power series in t/U $iS = \sum_{n=1}^{\infty} \left(\frac{t}{U}\right)^n iS^{[n]}$. The operators $S^{[1]}, \ldots, S^{[k]}$ can be chosen such that the first $k+1$ terms in the t/U-expansion of H' will not change the number of doubly occupied sites. It follows from Ref. 33 that the unitary transformation can be written as

$$e^{-iS} = \sum_{k \geq 0} \sum_{[m]} \left(\frac{t}{U}\right)^k \alpha^{(k)}[m] T^{(k)}[m], \quad (19)$$

$$T^{(k)}[m] = T_{m_1} T_{m_2} \ldots T_{m_k}, \quad m_j \in \{-1, 0, 1\}, \quad (20)$$

where $\alpha^{(k)}[m]$ are suitably chosen coefficients.

A. Heisenberg sector states in the t/U expansion

We now need to identify the Heisenberg sector states in the framework of the t/U-expansion. We propose that they are characterized by their property that, in the framework of the t/U-expansion, they are connected by our unitary transformation to states without any double occupancies

$$|\psi_H\rangle\Big|_{t/U} = |\psi\rangle = e^{-iS} \sum_{\alpha_j = \uparrow, \downarrow} f_{\alpha_1 \ldots \alpha_L} \Big[\prod_{j=1}^{L} X^{\alpha_j 0} \Big] |0\rangle. \quad (21)$$

Our identification is based on the fact that in the Bethe ansatz solution exact eigenstates are labelled by sets of (half-odd) integer quantum numbers, which makes it possible to follow a particular state when changing the interaction strength U. When sending U/t to infinity for a large but fixed system size one finds that in this limit the lowest energy eigenstates belong to the Heisenberg sector, which in turn leads to the identification (21). In the following we will use the eigenstate $|\psi_H\rangle$ and its t/U-expansion $|\psi\rangle$ interchangeably and leave a discussion of contributions not captured by the t/U-expansion for our concluding remarks. The ground state of the half-filled Hubbard model is a particular example of a Heisenberg state. Heisenberg sector states have the important property that they are singlets under the η-pairing SU(2) algebra. This follows from the easily established fact that

$$[\eta, T_m] = 0 = [\eta^\dagger, T_m] = [\eta^z, T_m], \quad m = 0, \pm 1. \quad (22)$$

The commutation relations (22) imply that $T^{(k)}[m]$ commute with the η-pairing operators, and this in turn implies by virtue of (19) that

$$[\eta, e^{-iS}] = 0 = [\eta^\dagger, e^{-iS}] = [\eta^z, e^{-iS}]. \quad (23)$$

This, together with the fact that all states with only singly occupied sites are annihilated by both η and $\eta^\dagger$, establishes that all Heisenberg states are η-pairing singlets

$$\eta|\psi\rangle = \eta^\dagger|\psi\rangle = \eta^z|\psi\rangle = 0. \quad (24)$$

Using the results of Refs 43,44 we may construct a basis for the space of η-pairing singlets. Let us select a set $A = \{a_1, \ldots, a_{2q}\}$ of $2q$ lattice sites and denote the complementary set by $\bar{A}$. We take all $\bar{A}$ sites to be singly occupied by spin-σ_j fermions, while we form singlet η-pairing dimers $|S(a,b)\rangle = |2\rangle_a|0\rangle_b + (-1)^{a+b-1}|0\rangle_a|2\rangle_b$ on the A sites. This gives an overcomplete set of states

$$|\boldsymbol{k}; \boldsymbol{\sigma}\rangle = \prod_{i=1}^{q} |S(k_{2i-1}, k_{2i})\rangle \prod_{j \in \bar{A}} |\sigma_j\rangle_j, \quad (25)$$

where $\boldsymbol{k} = k_1, \ldots, k_{2q}$ is a permutation of $\{a_1, \ldots, a_{2q}\}$. It is easily verified that the states (25) are η-pairing singlets. A linearly independent set of states (25) is formed by imposing a "non-crossing" constraint on the permitted values of $\boldsymbol{k}$ as follows. We first impose an ordering of sites $\{a_1, \ldots, a_{2q}\}$, e.g. $a_1 < a_2 < \cdots < a_{2q}$. In one dimension this is simply the natural order of the sites. We then connect the q pairs of sites $\{(k_{2i-1}, k_{2i})\}$ by lines, *cf.* Fig. 1. If no lines cross, the state corresponding to the pairing $\boldsymbol{k}$ is permitted. A basis $\mathfrak{B}$ of all η-pairing singlet states in the half-filled Hubbard model is obtained by taking into account all "sectors" $0 \leq 2q \leq L$ and all distinct sets $\{a_1, \ldots, a_{2q}\}$ of lattice sites in a given sector. All eigenstates in the Heisenberg sector can then be

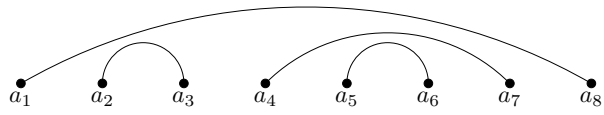

FIG. 1: The pairing $\boldsymbol{k} = \{a_1, a_8, a_2, a_3, a_4, a_7, a_5, a_6\}$ does not involve crossed lines and fulfils the constraint.

expressed in the form

$$|\psi\rangle = \sum_{|\boldsymbol{k};\boldsymbol{\sigma}\rangle \in \mathfrak{B}} A_{\boldsymbol{k};\boldsymbol{\sigma}} \, |\boldsymbol{k}; \boldsymbol{\sigma}\rangle \, . \quad (26)$$

B. One spatial dimension

The t/U-expansion allows us to determine the dependence of the amplitudes $A_{\boldsymbol{k};\boldsymbol{\sigma}}$ on U. It is useful to define the "total bond length" by

$$\mathcal{D}\Big(\sum_{|\boldsymbol{k};\boldsymbol{\sigma}\rangle \in \mathfrak{B}} A_{\boldsymbol{k};\boldsymbol{\sigma}} \, |\boldsymbol{k}; \boldsymbol{\sigma}\rangle \Big) = \max_{\substack{|\boldsymbol{k};\boldsymbol{\sigma}\rangle \in \mathfrak{B} \\ A_{\boldsymbol{k};\boldsymbol{\sigma}} \neq 0}} \mathcal{D}\left(|\boldsymbol{k}; \boldsymbol{\sigma}\rangle\right), \quad (27)$$

where we take $\mathcal{D}\big(|\boldsymbol{k}; \boldsymbol{\sigma}\rangle\big) = \sum_{i=1}^{q} (\|k_{2i-1} - k_{2i}\| + 1)$. It is a straightforward matter to show (see the Supplementary Material) that $\mathcal{D}\big(T_n|\psi\rangle\big) \leq \mathcal{D}\big(|\psi\rangle\big) + n + 1$ where $n = 0, \pm 1$ and $|\psi\rangle$ is any Heisenberg sector state (26).

This in turn implies that $\mathcal{D}\left[T^{(k)}[m]|\psi_0\rangle\right] \leq k+q$, where $q=\sum_{i=1}^{k} m_i$ and $|\psi_0\rangle$ is any state with only singly occupied sites. Applying this to the expressions (21), (19) for Heisenberg states, we conclude that the expansion coefficients in (26) fulfil

$$A_{\boldsymbol{k};\boldsymbol{\sigma}} = \mathcal{O}\Big(\big(t/U\big)^{\sum_{i=1}^{q}\|k_{2i}-k_{2i-1}\|}\Big). \tag{28}$$

These results cannot be straightforwardly generalized to $D>1$ because our definition of a total bond length hinges on $|\boldsymbol{k};\boldsymbol{\sigma}\rangle$ forming a basis of states, which imposes constraints on the allowed values of $\boldsymbol{k}$. In a typical Heisenberg sector state we have a finite density of doubly occupied sites and the coefficients (28) are of an extremely high order in t/U. In order to proceed we will assume that the t/U-expansion for the wave-function (26) has a finite radius of convergence. We know this to be the case for certain quantities such as the ground state energy[37].

C. Quantum disentangled diagnostic

According to Refs 30,31 a QDL can be diagnosed by preparing the system in a finite energy-density eigenstate with volume-law bipartite EE, and then to carry out a projective measurement of the z-component of spin on each site of the lattice. If the resulting state is characterized by an area-law EE, the original state realizes a QDL.

We now address this proposal in the framework of the t/U-expansion. As our initial state we choose a finite energy density eigenstate (21) in the Heisenberg sector. These generically have volume-law entanglement entropies as can be seen from the fact that the corresponding macro-states have finite thermodynamic entropy densities. As the 1D Hubbard model is integrable there also exist finite energy density eigenstates with area-law EE, but these are the exception rather than the rule. Let us assume that the outcome of our projective spin measurement is that we obtain spin zero at all sites in the set $A=\{a_1,a_2,\dots a_{2q}\}$ and spin $\sigma_j=\pm 1/2$ everywhere else. Then the state of the system after the projective measurement can be written as

$$\begin{aligned}|\psi_{\rm proj}\rangle \;&=\; \frac{1}{\sqrt{\mathcal{N}}}\prod_{j_1=1}^{a_1-1} X_{j_1}^{\sigma_{j_1}\sigma_{j_1}}\left(X_{a_1}^{00}+X_{a_1}^{22}\right)\\ &\times\prod_{j_2=a_1+1}^{a_2-1} X_{j_2}^{\sigma_{j_2}\sigma_{j_2}}\left(X_{a_2}^{00}+X_{a_2}^{22}\right)\cdots|\psi\rangle,\end{aligned} \tag{29}$$

where $\mathcal{N}$ is a normalisation factor. Using our results (26) for the structure of Heisenberg states in the t/U expansion, we can rewrite this in the form

$$|\psi_{\rm proj}\rangle = \sum_{Q\in S_{2q}}{}' W(Q)\prod_{i=1}^{q}|S(k_{Q_{2i-1}},k_{Q_{2i}})\rangle|\boldsymbol{\sigma}\rangle. \tag{30}$$

Here $|\boldsymbol{\sigma}\rangle=\prod_{j\notin A}|\sigma_j\rangle$ is a product state fixed by the projective measurement, and the sum $\sum'_{Q\in S_{2q}}$ is restricted to the permutations Q such that $a_{Q_1},a_{Q_2},\dots a_{Q_{2q}}$ corresponds to singlet states that satisfy the non-crossing condition. As a result of (28) the amplitudes $W(Q)$ fulfil

$$W(Q)\sim\mathcal{O}\left(\big(t/U\big)^{\sum_{i=1}^{q}\|k_{Q_{2i}}-k_{Q_{2i-1}}\|}\right). \tag{31}$$

In the leading order in the t/U-expansion that is allowed to be non-zero by the above considerations there is only a single term in (30) and we are dealing with a spatially ordered product state of singlet dimers, *cf.* Fig. 2. In the

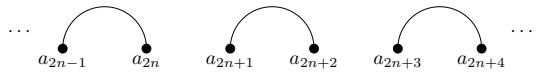

FIG. 2: Structure of the singlet pairings in the leading non-vanishing term of the projected state in the t/U-expansion.

following we will assume for simplicity that the numerical coefficient of this term is indeed different from zero, which will be generically the case. Depending on the choice of the initial Heisenberg state and the outcome of the projective measurement it is possible that this coefficient vanishes. Such cases can be accommodated by relatively straightforward modifications of the following discussion.

D. Entanglement entropy.

At any finite order in the t/U-expansion the projected states (30) are only weakly entangled. Let us consider the generic case, in which the leading non-zero term in the projected state has the form shown in Fig. 2. In this case the leading term in von Neumann entropy of a sub-system A is simply given by $S_{{\rm vN},A}=\ln(2)$ if we cut one of the singlet dimers when we bipartition the system, and zero otherwise. At leading order it is not possible for the bipartition to cut more than one dimer. In order to describe the structure of the leading corrections to this result we take the subsystem A to be the interval $[1,\ell]$, where $a_{2n+1}<\ell<a_{2n+2}$. Here $\{a_j\}$ are the sites in the projected state that are either doubly occupied or empty. We then cast the projected state in the form

$$|\psi_{\rm proj}\rangle = |\psi_L\rangle \Bigg(\Big| \; a_{2n-1}\; a_{2n} \quad a_{2n+1}\, a_{2n+2}\, a_{2n+3}\, a_{2n+4} \Big\rangle + \varepsilon_1 \Big| \; a_{2n-1}\; a_{2n} \quad a_{2n+1}\, a_{2n+2}\, a_{2n+3}\, a_{2n+4} \Big\rangle + \varepsilon_2 \Big| \; a_{2n-1}\; a_{2n}\; a_{2n+1}\, a_{2n+2}\, a_{2n+3}\, a_{2n+4} \Big\rangle \Bigg) |\psi_R\rangle + \ldots , \tag{32}$$

where $|\psi_L\rangle$ ($|\psi_R\rangle$) is the part of the leading term of the projected state that involves the sites $j < a_{2n-1}$ ($j > a_{2n+4}$). The coefficients $\epsilon_{1,2}$ are of order $(t/U)^{2(a_{2n+3}-a_{2n+2})}$ and $(t/U)^{2(a_{2n+1}-a_{2n})}$ respectively. The von Neumann entropy for our subsystem is then

$$S_{{\rm vN},A} = \ln 2 + \frac{9}{16}\varepsilon_1^2\varepsilon_2^2\big(1+2\ln\big(\frac{4}{3}\big)-2\ln(\varepsilon_1\varepsilon_2)\big)+\ldots \tag{33}$$

We note that the correction is positive. The physical picture that emerges is very simple: for small t/U, the projected state is very close to being a product of Bell pairs and the bipartite EE takes the form shown in Fig. 3. The average double occupancy in a Heisenberg state is

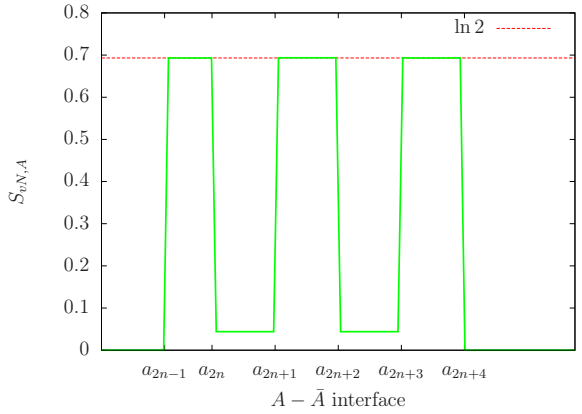

FIG. 3: Form of the bipartite entanglement entropy after projective measurement of a general Heisenberg state. Deviations from $S(x) = \ln 2, 0$ arise from higher order terms, as seen in (32), (33).

$\mathcal{O}(t^2/U^2)$, which implies that the average distance between doubly occupied/unoccupied sites is very large $\sim U^2/t^2$. Hence, if the outcome of our spin measurement is close to its average, the sites $a_1, \ldots, a_{2q}$ are well separated and the leading term (2) is expected to provide an excellent approximation as long as the t/U-expansion converges. We note that the results obtained by the t/U-expansion are fully compatible with those obtained by integrability methods in section II. This, and the fact that the t/U-expansion is known to converge for simple quantities like the ground state energy[37,45], provides support for its applicability.

E. Higher dimensions $D > 1$

A key element of our analysis in the one dimensional case is the notion of a bond length, which fulfils $\mathcal{D}(|\psi_1\rangle) \leq \mathcal{D}(|\psi_1\rangle + |\psi_2\rangle)$ for arbitrary η-pairing singlet states $|\psi_{1,2}\rangle$. Our definition of a bond length utilizes the availability of a convenient basis of such states. It is this aspect which makes the $D = 1$ special. In $D > 1$ we proceed by again first fixing a set $A = \{a_i\}$ of $2q$ unoccupied or doubly occupied sites. We denote by E_{ij} a bond that joins sites a_i and a_j. We then define a dimer configuration $\mathfrak{C}$ as a set of q bonds C_{ij} connecting sites a_i and a_j such that all sites belong to precisely one bond. Each dimer configuration gives rise to η-pairing singlet states of the form

$$|\mathfrak{C};\boldsymbol{\sigma}\rangle = \prod_{C_{ij}\in\mathfrak{C}} |S(a_i,a_j)\rangle \prod_{j\notin A} |\sigma_j\rangle_j \; . \tag{34}$$

The states (34) form an overcomplete set. In order to select a set of linearly independent states we use the function $\overline{\mathcal{D}}$ defined by $\overline{\mathcal{D}}\,(|\mathfrak{C};\boldsymbol{\sigma}\rangle) = \sum_{C_{ij}\in\mathfrak{C}} \big(\|a_i - a_j\| + 1\big)$, where $\|a_i - a_j\|$ denotes the Manhattan distance between sites a_i and a_j. Among (34) we select the $\frac{(2q)!}{(q+1)!(q!)}$ states that have the lowest values under the map $\overline{\mathcal{D}}$: it is this criterion that allows the inequalities in 1D to be inherited in the higher-dimensionsal case. Repeating this construction for all values of $2q \leq L$ and all distinct sets A gives a basis $\mathcal{B}$ of η-pairing singlet states. We now define our total bond distance $\mathcal{D}$ in the same way as in (27) with the covering $\mathfrak{C}$ taking the place of the vector $\boldsymbol{k}$. With these definitions in place it is straightforward to show (see the Supplementary Material) that $\mathcal{D}\big(T_n|\psi\rangle\big) \leq \mathcal{D}\big(|\psi\rangle\big)+n+1$ where $n = 0, \pm 1$ and $|\psi\rangle$ is any Heisenberg sector state

$$|\psi\rangle = \sum_{|\mathfrak{C};\boldsymbol{\sigma}\rangle\in\mathfrak{B}} A_{\mathfrak{C};\boldsymbol{\sigma}}\; |\mathfrak{C};\boldsymbol{\sigma}\rangle \; . \tag{35}$$

This in turn implies that $\mathcal{D}\left[T^{(k)}[m]|\psi_0\rangle\right] \leq k+q$, where $q = \sum_{i=1}^{k} m_i$ and $|\psi_0\rangle$ is any state with only singly occupied sites. It follows that the d-dimensional generalisation of (28) is

$$A_{\mathfrak{C};\boldsymbol{\sigma}} = \mathcal{O}\Big(\big(t/U\big)^{\sum_{C_{ij}\in\mathfrak{C}}\|a_i-a_j\|}\Big). \tag{36}$$

The main difference to $D = 1$ is that now there can be several basis states contributing to the leading order term in t/U. To see this we may consider a square lattice and focus on the situation where the sites in A completely fill an $m \times n$ rectangle (with at least one of m, n even). The leading-order term is that generated by $(T_1)^{mn/2}$ produces a superposition of all states in $\mathfrak{B}$ that correspond to nearest-neighbour dimer coverings of this rectangle. Nevertheless, it is apparent that to leading order in the t/U-expansion the structure of the state is such that the entanglement entropy follows an area-law.

IV. CONCLUSIONS

We have shown that there are particular "Heisenberg sector" eigenstates in the one dimensional half-filled Hubbard model that realize the quantum disentangled liquid state of matter in the strong sense proposed in Ref. 30. These states are obtained by freezing the charge degrees of freedom in their ground state configuration in the framework of the Bethe ansatz solution of the Hubbard model. Using methods of integrability we have demonstrated that the charge degrees of freedom ("light particles") do not contribute to the volume term of the bipartite entanglement entropy. Employing strong-coupling expansion techniques we have shown (under the assumption that the expansion converges) that a measurement of the spin degrees of freedom at all sites ("heavy particles") leaves the system in a state that is area-law entangled, which is the defining characteristic of a "strong QDL"[30].

In contrast to Heisenberg sector states, the entanglement entropy for maximal entropy (thermal) states at a given energy density at large values of U/t does have a contribution that involves the charge degrees of freedom, but it is of the form (16), i.e.

$$\begin{aligned} S_{\rm vN,A} &= \big(s_{\rm spin} + s_{\rm charge}\big)|A| + o\big(|A|\big) \ , \\ s_{\rm spin} &= \mathcal{O}(1) \ , \quad s_{\rm charge} = \mathcal{O}\Big(\frac{u}{T} e^{-u/T}\Big) \ , \end{aligned} \tag{37}$$

We expect a similar volume law to occur in the entanglement entropy after a measurement of all spins. This suggests that there is a *weak* variant of a QDL, which is characterized by a volume term in the EE after measurement of the heavy degrees of freedom that is exponentially small in the ratio of masses (i.e. the ratio U/t in the Hubbard model). In this limit the volume term is only observable in enormously (exponentially) large systems and would be practically impossible to detect in numerical simulations. We expect this weak scenario to be realized quite generally in strong coupling limits irrespective of whether the model one is dealing with is integrable or not. In particular, this scenario is compatible with our strong-coupling analysis of the QDL diagnostic in $D = 2$.

An interesting question is how to access Heisenberg sector states in the one dimensional case. In principle this can be achieved by means of a quantum quench[17] from a suitably chosen initial state. At late times the system locally relaxes to a steady state that is described by an appropriate generalized Gibbs ensemble. The parameters of this ensemble are fixed by the initial conditions and can in principle be tuned in such a way that one ends up with a Heisenberg sector state. How to do this in practice is an interesting albeit rather non-trivial question. Fixing the energy density to be small compared to the Mott gap is sufficient to obtain a non-equilibrium steady state that realizes the weak form of a QDL characterized by (16). In order to remove the $s_{\rm charge}$ contribution, the expectation values of higher conservation laws in the initial state have to be chosen appropriately and it would be interesting to investigate this issue further.

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Appendix A: Counting Heisenberg sector states

In order to get a measure of how many Bethe ansatz Heisenberg sector states there are we proceed as follows. We start with the thermodynamic limit of the Bethe ansatz equations (3) for Heisenberg sector states

$$\begin{aligned} \rho^p(k) &= \frac{1}{2\pi} + \cos k \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\Lambda \ a_n(\Lambda - \sin k) \sigma_n^p(\Lambda) \ , \\ \sigma_n^h(\Lambda) &= -\sum_{m=1}^{\infty} \int_{-\infty}^{\infty} d\Lambda' A_{nm}(\Lambda - \Lambda') \ \sigma_m^p(\Lambda') \\ &\quad + \int_{-\pi}^{\pi} dk \ a_n(\sin k - \Lambda) \ \rho^p(k) \ . \end{aligned} \tag{A1}$$

We then construct the maximum entropy state at a given energy density by extremizing the free energy

$$f = e - \mathcal{T} s - B \left[\int_{-\pi}^{\pi} dk \ \rho^p(k) - \sum_{n=1}^{\infty} 2n \int_{-\infty}^{\infty} d\Lambda \sigma_n^p(\Lambda) \right] , \tag{A2}$$

where e and s are the energy and entropy densities of Heisenberg sector states and are given by (5) and (9) respectively. B plays the role of a magnetic field and will eventually be set to zero. The "temperature" $\mathcal{T}$ plays the role of a Lagrange parameter that allows us to fix the energy density. Extremizing (A2) with respect to the particle and hole densities under the constraints (A1)

fixes the ratios $\eta_n = \frac{\sigma_n^h}{\sigma_n^p}$ for the maximal entropy state at a given value of $\mathcal{T}$ via the TBA-like equations

$$\ln\left(1+\eta_n(\Lambda)\right) = \frac{g_1(\Lambda)}{\mathcal{T}} + \sum_{m=1}^{\infty}\int_{-\infty}^{\infty} d\Lambda'\; A_{nm}(\Lambda-\Lambda')\ln\left[1+\frac{1}{\eta_n(\Lambda')}\right], \quad \text{(A3)}$$

where $g_1(\Lambda) = -4\mathrm{Re}\sqrt{1-(\Lambda-inu)^2} + 4nu + 2nB$. The entropy density of this state is

$$s = \sum_{n=1}^{\infty}\int_{-\infty}^{\infty} d\Lambda\left[\frac{g_1(\Lambda)}{\mathcal{T}}\sigma_n^p(\Lambda) + g_2(\Lambda)\ln\left(1+\eta_n^{-1}(\Lambda)\right)\right], \quad \text{(A4)}$$

where $g_2(\Lambda) = \frac{1}{\pi}\mathrm{Re}\frac{1}{\sqrt{1-(\Lambda+inu)^2}}$. In order to obtain a thermodynamic estimate of the number of Heisenberg sector states we now consider the limit $B, \mathcal{T} \to \infty$ at fixed $B/\mathcal{T}$. In this limit the solution of the system of equations (A3) is[37]

$$\eta_n = \left[\frac{\sinh\left(\frac{(n+1)B}{\mathcal{T}}\right)}{\sinh\left(\frac{B}{\mathcal{T}}\right)}\right]^2 - 1. \quad \text{(A5)}$$

Finally we take $B/\mathcal{T}$ to zero

$$\eta_n = (n+1)^2 - 1. \quad \text{(A6)}$$

Substituting this back into the expression (A3) for the entropy we have

$$\lim_{\mathcal{T}\to\infty} s = \sum_{n=1}^{\infty}\ln\left[\frac{(n+1)^2}{(n+1)^2-1}\right] = \ln(2). \quad \text{(A7)}$$

This tells us that the most likely macro state in the Heisenberg sector has an entropy density of ln(2). All other macro states in this sector are exponentially less likely, and hence we conclude that the total number of Heisenberg sector micro states fulfils (10).

Appendix B: Thermodynamic Bethe Ansatz equations

The Thermodynamic Bethe Ansatz equations for the one dimensional Hubbard model in zero magnetic field are[37,39]

$$\begin{aligned}
\ln\zeta(k) &= \frac{-2\cos k-\mu-2u}{T} + \sum_{n=1}^{\infty}\int_{-\infty}^{\infty} d\Lambda\; a_n(\sin k-\Lambda)\ln\left(1+\frac{1}{\eta_n'(\Lambda)}\right) \\
&- \sum_{n=1}^{\infty}\int_{-\infty}^{\infty} d\Lambda\; a_n(\sin k-\Lambda)\ln\left(1+\frac{1}{\eta_n(\Lambda)}\right), \quad \text{(B1)} \\
\ln\left(1+\eta_n(\Lambda)\right) &= -\int_{-\pi}^{\pi} dk\;\cos(k)\; a_n(\sin k-\Lambda)\ln\left(1+\frac{1}{\zeta(k)}\right) + \sum_{m=1}^{\infty} A_{nm} * \ln\left(1+\frac{1}{\eta_m}\right)\Big|_{\Lambda}, \quad \text{(B2)} \\
\ln\left(1+\eta_n'(\Lambda)\right) &= \frac{4\mathrm{Re}\sqrt{1-(\Lambda-inu)^2}-2n\mu-4nu}{T} - \int_{-\pi}^{\pi} dk\;\cos(k)\; a_n(\sin k-\Lambda)\ln\left(1+\frac{1}{\zeta(k)}\right) \\
&+ \sum_{m=1}^{\infty} A_{nm} * \ln\left(1+\frac{1}{\eta_m'}\right)\Big|_{\Lambda}. \quad \text{(B3)}
\end{aligned}$$

Appendix C: Some inequalities for the total bond length of Heisenberg states

Here we show that

$$\mathcal{D}\left(T_n|\psi\rangle\right) \le \mathcal{D}\left(|\psi\rangle\right) + n + 1, \qquad n = 0, \pm 1. \quad \text{(C1)}$$

We consider the one and higher dimensional cases separately.

1. One Dimension

Any Heisenberg state can be written in the form (26)

$$|\psi\rangle = \sum_{|\boldsymbol{k};\boldsymbol{\sigma}\rangle\in\mathfrak{B}} A_{\boldsymbol{k};\boldsymbol{\sigma}}\;|\boldsymbol{k};\boldsymbol{\sigma}\rangle\;. \quad \text{(C2)}$$

To establish (C1) it therefore suffices to show that

$$\mathcal{D}\left(T_n|\mathbf{k};\boldsymbol{\sigma}\rangle\right) \le \mathcal{D}\left(|\mathbf{k};\boldsymbol{\sigma}\rangle\right) + n + 1\;. \quad \text{(C3)}$$

We next consider the three cases $n = 0, \pm 1$ in turn.

a. Inequality for T_1:

Due to the definition of the bond distance, it is clear that

$$\mathcal{D}\left(\sum_j T_{m,j}|\mathbf{k};\boldsymbol{\sigma}\rangle\right) \leq \sup_j \mathcal{D}\left(T_{m,j}|\mathbf{k};\boldsymbol{\sigma}\rangle\right), \tag{C4}$$

and therefore for $m = \pm 1, 0$, we simply need to find the case which yields the largest value in order to prove the inequality. $T_{1,j}$ creates an η-singlet on two adjacent sites and $\mathcal{D}$ counts the total number of sites spanned by the singlet bonds, applying $T_{1,j}$ increases this by at most that in the case of a system with no doubly occupied sites. We can view the result graphically

$$\begin{aligned}\mathcal{D}\left(T_{1,j}\left|\right.\right.\right. &\leq \mathcal{D}\left(\left|\right.\right.\right), \\ &\leq \mathcal{D}\left(\left|\right.\right.\right) + 2.\end{aligned} \tag{C5}$$

b. Inequality for T_0:

$T_{0,j}$ neither creates nor destroys any η-singlet pairs. Instead, for a given singlet pair, it moves one of the doubly-occupied/empty sites in the pair either left or right one site i.e. it extends the bond distance of the pair by ± 1. The case where the distance *increases* i.e. saturates the inequality, can be represented graphically as

$$\begin{aligned}\mathcal{D}\left(T_{0,2n}\left|\right.\right.\right. &\leq \mathcal{D}\left(\left|\right.\right.\right), \\ &\leq \mathcal{D}\left(\left|\right.\right.\right) + 1.\end{aligned} \tag{C6}$$

c. Inequality for T_{-1}:

It is simple to show that $T_{-1,j}$ "fuses" the ends of singlet pairs together to create a new singlet pair between the unaffected sites. Using this, we can consider the two possible distinct cases graphically. Explicit calculation shows that this intuition holds. The cases are: (i) $a_{2n-1} = a_{2n} - 1$ i.e. they are adjacent

$$\begin{aligned}&\mathcal{D}\left(T_{-1,a_{2n-1}}\left|\right.\right.\right) \\ &\leq \mathcal{D}\left(\left|\right.\right.\right), \\ &\leq \mathcal{D}\left(\left|\right.\right.\right),\end{aligned} \tag{C7}$$

(ii) $a_{2n} = a_{2n+1} - 1$ i.e. two singlets are nested

$$\begin{aligned}&\mathcal{D}\left(T_{-1,a_{2n}}\left|\right.\right.\right) \\ &\leq \mathcal{D}\left(\left|\right.\right.\right), \\ &\leq \mathcal{D}\left(\left|\right.\right.\right).\end{aligned} \tag{C8}$$

2. Higher Dimensions

Any Heisenberg state can be written in the form (35)

$$|\psi\rangle = \sum_{|\mathfrak{C};\boldsymbol{\sigma}\rangle \in \mathfrak{B}} A_{\mathfrak{C};\boldsymbol{\sigma}}\ |\mathfrak{C};\boldsymbol{\sigma}\rangle\ , \tag{C9}$$

so that in order to establish (C1) it suffices to show that

$$\mathcal{D}\left(T_n|\mathfrak{C};\boldsymbol{\sigma}\rangle\right) \leq \mathcal{D}\left(|\mathfrak{C};\boldsymbol{\sigma}\rangle\right) + n + 1\ . \tag{C10}$$

for all basis states $|\mathfrak{C};\boldsymbol{\sigma}\rangle \in \mathfrak{B}$. We consider the three cases $n = 0, \pm 1$ in turn.

a. Inequality for T_0:

We note that by construction we have for all basis states

$$\mathcal{D}(|\mathfrak{C};\boldsymbol{\sigma}\rangle) = \overline{\mathcal{D}}(|\mathfrak{C};\boldsymbol{\sigma}\rangle)\ ,\ |\mathfrak{C};\boldsymbol{\sigma}\rangle \in \mathfrak{B}. \tag{C11}$$

Following through the same steps as in the one dimensional case we see that T_0 can increase the Manhattan distance between any pair in the configuration by at most 1 i.e.

$$\begin{aligned}\mathcal{D}\left(T_0|\mathfrak{C};\boldsymbol{\sigma}\rangle\right) &\leq \overline{\mathcal{D}}\left(T_0|\mathfrak{C};\boldsymbol{\sigma}\rangle\right), \\ &\leq \overline{\mathcal{D}}\left(|\mathfrak{C};\boldsymbol{\sigma}\rangle\right) + 1 = \mathcal{D}(|\mathfrak{C};\boldsymbol{\sigma}\rangle),\end{aligned} \tag{C12}$$

where in the last step we used (C11).

b. Inequality for T_1:

Similarly, acting with T_1 creates a new configuration with an additional edge

$$\begin{aligned}\mathcal{D}\left(T_1|\mathfrak{C};\boldsymbol{\sigma}\rangle\right) &\leq \overline{\mathcal{D}}\left(T_1|\mathfrak{C};\boldsymbol{\sigma}\rangle\right), \\ &\leq \overline{\mathcal{D}}\left(|\mathfrak{C};\boldsymbol{\sigma}\rangle\right) + 2 = \mathcal{D}\left(|\mathfrak{C};\boldsymbol{\sigma}\rangle\right) + 2,\end{aligned} \tag{C13}$$

where in the last step we used (C11).

c. Inequality for T_{-1}:

Finally, acting with T_{-1} always removes a pair of "adjacent" unoccupied/doubly occupied sites. Thinking in terms of configurations, this then either removes a paired bond, or fuses two bonds into one. If we explicitly write the points defining the configuration $\mathfrak{C}$, the first case corresponds to

$$\begin{aligned} T_{-1} : \{\{k_1,k_2\},\{k_3,k_4\},\ldots,\{k_{2q-1},k_{2q}\}\} \\ \to \{\{k_3,k_4\},\ldots,\{k_{2q-1},k_{2q}\}\}\,. \end{aligned} \tag{C14}$$

This implies that

$$\overline{\mathcal{D}}\left(T_{-1}|\mathfrak{C};\boldsymbol{\sigma}\rangle\right) \leq \overline{\mathcal{D}}\left(|\mathfrak{C};\boldsymbol{\sigma}\rangle\right) - \left(\|k_1-k_2\|+1\right). \tag{C15}$$

The second case corresponds to

$$\begin{aligned} T_{-1} : \{\{k_1,k_2\},\{k_3,k_4\},\ldots,\{k_{2q-1},k_{2q}\}\} \\ \to \{\{k_1,k_4\},\ldots,\{k_{2q-1},k_{2q}\}\}\,, \end{aligned} \tag{C16}$$

which implies that

$$\begin{aligned} \overline{\mathcal{D}}\left(T_{-1}|\mathfrak{C};\boldsymbol{\sigma}\rangle\right) \leq \overline{\mathcal{D}}\left(|\mathfrak{C};\boldsymbol{\sigma}\rangle\right) + \left(\|k_1-k_4\|+1\right) \\ - \left(\|k_1-k_2\|+1\right) - \left(\|k_3-k_4\|+1\right). \end{aligned} \tag{C17}$$

By construction k_2 and k_3 are "adjacent", so that

$$\begin{aligned} \left(\|k_1-k_4\|+1\right) - \left(\|k_1-k_2\|+1\right) \\ - \left(\|k_3-k_4\|+1\right) \leq 0. \end{aligned} \tag{C18}$$

Putting everything together we have

$$\mathcal{D}\left(T_{-1}|\mathfrak{C};\boldsymbol{\sigma}\rangle\right) \leq \mathcal{D}\left(|\mathfrak{C};\boldsymbol{\sigma}\rangle\right), \qquad |\mathfrak{C};\boldsymbol{\sigma}\rangle \in \mathcal{B}. \tag{C19}$$

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