

This is the accepted manuscript made available via CHORUS. The article has been published as:

Anomaly manifestation of Lieb-Schultz-Mattis theorem and topological phases

Gil Young Cho, Chang-Tse Hsieh, and Shinsei Ryu

Phys. Rev. B **96**, 195105 — Published 2 November 2017

DOI: [10.1103/PhysRevB.96.195105](https://doi.org/10.1103/PhysRevB.96.195105)

Anomaly Manifestation of Lieb-Schultz-Mattis Theorem and Topological Phases

Gil Young Cho,^{1,2,*} Chang-Tse Hsieh,^{3,*} and Shinsei Ryu⁴

¹*School of Physics, Korea Institute for Advanced Study, Seoul 02455, Korea*

²*Department of Physics, Korea Advanced Institute of Science and Technology, Daejeon 305-701, Korea*

³*Department of Physics, University of Illinois, 1110 W. Green St., Urbana, Illinois 61801-3080, USA*

⁴*James Franck Institute and Kadanoff Center for Theoretical Physics, University of Chicago, Illinois 60637, USA*

(Dated: October 17, 2017)

The Lieb-Schultz-Mattis (LSM) theorem dictates that emergent low-energy states from a lattice model cannot be a trivial symmetric insulator if the filling per unit cell is not integral and if the lattice translation symmetry and particle number conservation are strictly imposed. In this paper, we compare the one-dimensional gapless states enforced by the LSM theorem and the boundaries of one-higher dimensional strong symmetry-protected topological (SPT) phases from the perspective of quantum anomalies. We first note that, they can be both described by the same low-energy effective field theory with the same effective symmetry realizations on low-energy modes, wherein non-on-site lattice translation symmetry is encoded as if it is an internal symmetry. In spite of the identical form of the low-energy effective field theories, we show that the quantum anomalies of the theories play different roles in the two systems. In particular, we find that the chiral anomaly is equivalent to the LSM theorem, whereas there is another anomaly, which is not related to the LSM theorem but is intrinsic to the SPT states. As an application, we extend the conventional LSM theorem to multiple-charge multiple-species problems and construct several exotic symmetric insulators. We also find that the (3+1)d chiral anomaly provides only the perturbative stability of the gapless-ness local in the parameter space.

CONTENTS

I. Introduction	1	C. Vanishing \mathbb{Z}_N anomaly in Perturbatively-gappable Electronic Systems	15
A. LSMOH and no-go theorem; On-site v.s. non-on-site symmetries	2	D. Dirac Semimetals and Chiral Anomaly	16
B. LSMOH v.s. SPT anomalies	3	1. Dirac semimetal: disorted spinel	16
C. Effective symmetry and perturbative stability	3	2. Dirac semimetal: Na_3Bi and Cd_3As_2	16
D. Summary	4	References	17
II. 2d SPTs and 1d lattice models	4		
A. (2+1)d QSHE	4		
B. (1+1)d lattice spinless fermions	5		
C. Effective symmetry	6		
D. Other examples	6		
III. Anomaly and LSMOH theorem	6		
A. Chiral anomaly and the LSMOH theorem	7		
1. Chiral Anomaly: Field Theory	7		
2. Chiral Anomaly: Pumping Argument	8		
B. Extension of the LSMOH theorem: a stability analysis	9		
IV. \mathbb{Z}_N anomaly and Local Stability	11		
V. (3+1)d Chiral Anomaly and Weyl Semimetals	12		
VI. Conclusions and Outlooks	13		
Acknowledgments	14		
A. Derivation of the two anomaly indices in (14)	14		
B. Chiral anomaly from bosonization	15		

I. INTRODUCTION

Predicting possible macroscopic behaviors of many-body systems from a given kinematical input data, such as spatial dimensions, the presence of a certain set of symmetries, etc., is a central question in many-body physics. More precisely, predicting spectral properties (e.g., presence/absence of a spectral gap above ground states) and the nature of ground states (e.g., long/short-range entangled, trivial, etc.) would be of great interest.

In this regard, we will discuss the following three classes of problems (systems) in this paper:

- (i) *The LSMOH theorem:* The Lieb-Schultz-Mattis theorem and its generalization by Oshikawa and Hastings¹⁻⁴ dictates that when the lattice translation symmetry and $U(1)$ charge (electric charge, spin, etc.) conservation are preserved, the system must be gapless or its ground state must be long-range entangled if the particle number (or spin quantum number) per unit cell is fractional (non-integral). In one spatial dimension, this in particular means that the system has to be gapless.
- (ii) *SPT boundaries:* The boundaries of a symmetry-protected topological (SPT) phase⁵⁻⁷ cannot be

gapped trivially, i.e., they must be either gapless or exhibit topological order, so far as the symmetries protecting the bulk SPT phase are enforced. For (1+1)-dimensional boundaries of (2+1)-dimensional SPT phases, this in particular means that they have to be gapless.

- (iii) *Fermi “surfaces”*: There are a class of lattice fermion systems in which the single-particle spectrum supports zeros in the Brillouin zone, i.e., Fermi (nodal) points/lines/surfaces, etc., in the presence of a certain set of symmetries. For notational simplicity, we will call such zeros of the single particle spectrum Fermi “surfaces”, although one should bear in mind that such zeros can form a hypersurface of various dimensions.

We distinguish systems in Class (iii) from those in the other classes by their *perturbative* stability. I.e., the gapless nature in Class (iii) is only *perturbatively* or *locally* stable; The impossibility of trivial gapped states dictated by the LSMOH theorem and at SPT boundaries is *non-perturbative* in the sense that it relies only on the kinematical input data (e.g., the filling fraction, symmetries), but not on the interaction strength. On the other hand, the local stability in Class (iii) excludes, in the parameter space, symmetric trivial insulators only in the vicinity of a given gapless low-energy theory. In other words, a trivial insulator may exist if the system is perturbed far away from the low-energy theory. This may become particularly important when the Fermi “volume” inside the surface is zero, e.g., nodal points and lines. One way to understand the perturbative stability of Fermi surfaces is to note that since by the fermion-doubling theorem, these Fermi surfaces always appear in pair. Hence unless enough symmetry conditions are imposed, by adding strong enough perturbations, these systems are ultimately gappable by “pair annihilating” these Fermi surfaces. Nevertheless, some of Fermi surfaces are expected to be stable locally or perturbatively. With further symmetry constraints, it would be possible to turn systems in Class (iii) into systems in Class (i), which are stable beyond the perturbative level. For example, for the case of Fermi surfaces with a finite Fermi volume in Class (i), imposing translation and charge conservation symmetries turns the system into the class (i).

Among (i-iii), the “mechanism” behind the obstruction for trivially gapping out the SPT boundaries [class (ii)] is understood in terms of quantum anomalies.^{8–15} In particular, when the symmetries protecting SPT phases are strictly on-site, i.e., for such “strong” SPT phases, the relevant anomalies are ’t Hooft anomalies. Here, a ’t Hooft anomaly is an obstruction to gauge on-site global symmetries of the theory.

More precisely, for bosonic systems whose Hilbert space \mathcal{H} is factorized into local Hilbert spaces, $\mathcal{H} = \prod_x \mathcal{H}_x$ where \mathcal{H}_x is the Hilbert spaces for a given “lattice site” x , a unitary symmetry g is said to be on-site if g factorizes similarly as $g = \prod_x g_x$. (This property of g is

also called splittable.) For fermionic systems, there is no natural factorization of the fermion Fock space due to the Fermi statistics. Nevertheless, we will assume that the similar notion of on-site symmetries exists, when a unitary operator g transforms fermion creation/annihilation operators purely locally.

In the typical setting of SPT phases, we start from bulk phases where symmetry actions are purely on-site. At non-trivial SPT boundaries, however, symmetries cannot be made purely on-site. This is in fact another way to state that SPT boundaries suffer from (or enjoy) quantum anomalies (’t Hooft anomalies). Boundaries of topologically distinct SPT phases are characterized by different ’t Hooft anomalies. In fact, the topological invariants characterizing bulk SPT phases are in one-to-one correspondence with ’t Hooft anomalies of SPT boundaries – the fact known as the bulk-boundary correspondence. In other words, the boundaries of SPT phases cannot exist on their own (i.e., cannot be put on a proper lattice), if we require the relevant symmetries be strictly on-site – the boundary theories of an SPT phase cannot be decoupled or “disentangled” from its bulk because of the anomalies. The impossibility of realizing boundaries of SPT phases as an isolated local system is usually called as the no-go theorem.

The purpose of this paper is, by taking simple examples, to give a detailed comparison between the LSMOH theorem and SPT boundaries. In particular, given that the impossibility of trivially gapping SPT boundaries is due to quantum anomalies (’t Hooft anomalies), we will make an attempt to interpret the LSMOH theorem in terms of quantum anomalies. For precursors of the current work, discussing the relationship between SPT phases and the LSMOH theorem, see, for example, Refs. 16–20. We will also touch upon the origin of the perturbative stabilities of Fermi surfaces using quantum anomalies. It should be noted that the stability of Fermi surfaces has been so far discussed mainly at the level of the single particle physics. Our discussion using quantum anomalies should shed light on the stability of Fermi surfaces in the presence of interactions.

In the rest of the Introduction, we will list and briefly describe some of the key issues in discussing the similarities and distinctions among the three classes of problems (i-iii). See Sec. IA, IB and IC below. They also serve as a short summary for Sec. II, III, and IV in the main text.

A. LSMOH and no-go theorem; On-site v.s. non-on-site symmetries

To explore a possible connection between the LSMOH theorem and quantum anomalies (and SPT boundaries), we will first note that the low-energy physics of lattice models dictated to be gapless by the LSMOH theorem and SPT boundaries can be described by an identical continuum field theory. For example, in Sec. II, we will dis-

cuss a (1+1)d lattice fermion model at fractional filling. For the low-energy effective field theory of this model, we will find that there is a (2+1)d SPT phase (a version of the quantum spin Hall effect) whose boundary is described by the same continuum field theory. Here, the relevant symmetry in the (1+1)d lattice fermion model is the $U(1)$ particle number conservation and lattice translation symmetry, whereas on the SPT side the relevant symmetry is the $U(1)$ particle number conservation and $U(1)$ or \mathbb{Z} internal (spin) rotation symmetry.

Since the LSMOH theorem concerns isolated lattice systems without referring to any higher dimensional bulk systems, this may look seemingly against the no-go theorem. The trick of evading the no-go theorem is that, while symmetries in the bulk SPT phases are realized *on-site*, symmetries entering into the corresponding LSMOH theorem are *non-on-site*. In the typical setting of SPT phases, we start from bulk phases where symmetry actions are purely on-site. On the other hand, in the context of the LSMOH theorem, it typically involves non-on-site symmetries. E.g., lattice translation symmetries. This is the reason why, even if the LSMOH theorem may be related to some sort of quantum anomalies, relevant systems can still be put on a lattice without having a higher dimensional bulk. Evading the no-go “theorem” is also possible in higher dimensions. For example, the “duality” between the composite Fermi liquid in the half-filled Landau level and the (2+1)d boundary of (3+1)d topological insulators has been discussed extensively recently.^{21–26} See Sec. II.

We will also note in Sec. II that the lattice translation symmetry within the low-energy field theory can be encoded as an *effective* symmetry. In the rational filling $\nu = p/q$ for mutually-prime p and q , the translation symmetry (up to some gauge choice and changes in band structures) can be further reduced as \mathbb{Z}_q , i.e., there may be symmetry-reduction $G = \mathbb{Z} \rightarrow G_{eff} = \mathbb{Z}_q$. Hence, we may consider the effective translation symmetry $G_{eff} = \mathbb{Z}_q$ as an on-site internal symmetry in the low-energy limit.

B. LSMOH v.s. SPT anomalies

Having confirmed that the low-energy theories for the fractionally filled 1d lattice fermion model and for the SPT boundary are identical, we will discuss quantum anomalies within the low-energy theory in Sec. III. The identification/computation of quantum anomalies can be done within the low-energy theories since anomalies are preserved along the RG flow – the ’t Hooft anomaly matching.²⁷ If the low-energy effective theory has a ’t Hooft anomaly, one would then expect that the theory at any energy scale and at any interaction strength cannot be deformed to a symmetrical trivial insulator.

That an SPT boundary and a lattice model for which we apply the LSMOH theorem can be described by the same low-energy effective theory would imply that the

both systems have the same anomalies. However, we will show that there are some subtleties – instead of the full ’t Hooft anomaly, we need to consider the chiral anomaly for the LSMOH theorem.

In Sec. III we will illustrate this by considering the (1+1)-dimensional lattice fermion model at filling ν , in the presence of the lattice translation symmetry and global $U(1)$ charge conservation symmetry. Here, by the full ’t Hooft anomaly, we mean the ’t Hooft anomaly of the whole global symmetries (= an effective on-site version of lattice translation \times charge $U(1)$). On the other hand, the chiral anomaly involves the two symmetries and partially gauging the symmetries, e.g., only one of the two symmetries. It effectively “measures” the conflict of the two symmetries, or violation of one global symmetry when the other symmetry is gauged. This chiral anomaly implies that both the symmetries cannot be gauged consistently and thus the obstruction to a symmetric trivial insulator. In some sense, the chiral anomaly can be thought of as a part of (a subset of) the full ’t Hooft anomaly.

As the chiral anomaly is the subset of the full ’t Hooft anomaly the chiral anomaly gives rise to a “cruder” classification of SPT phases when it comes to the interacting classification. We will show that the no-go condition for a symmetric insulator from the LSMOH theorem is identical to the non-trivial chiral anomaly. The chiral anomaly hence provides a non-perturbative stability of the gaplessness.

C. Effective symmetry and perturbative stability

In contrast to the chiral anomaly, we will discuss the other part of the full ’t Hooft anomaly (the system with vanishing chiral anomaly) in Sec. IV. For the theory emergent from the (1+1)d lattice system, we will argue that the other part of the full ’t Hooft anomaly implies the perturbative stability of Fermi surfaces. This perturbative stability detected by the anomaly is the one-dimensional analogue of the classification of (some) nodal fermions emerging from accidental band crossings in higher dimensions, e.g., classification of the nodal fermions in (3+1)d systems.^{28–33} This is particularly important when the filling is rational $\nu = p/q$ (for mutually prime p and q). For the filling, when the band structure is fine-tuned, the low-energy translation symmetry can be effectively reduced to \mathbb{Z}_q , a subset of full translation symmetry \mathbb{Z} . Then the anomaly signals that the system must be gapless only when the translation symmetry is strictly \mathbb{Z}_q , but not bigger than this. In other words, when the full translation symmetry \mathbb{Z} is considered, the theory with this anomaly only can be gapped out symmetrically. However, to have a symmetric gap to the spectrum, we need non-perturbative processes, e.g., to introduce extra “trivial” degrees of freedom, non-quadratic interaction terms or to change the band structures. [Here, the opposite of the non-perturbative stability, i.e., the pertur-

batively gappability, is equivalent to the condition that we can gap out the spectrum within the quadratic terms (“mass terms”) without any further modification of the given low-energy theory. This is slightly different from the “perturbative” stability in the renormalization group theory sense, i.e., the absence of relevant directions of the theory in the parameter space. (Note that when the strong forward scattering is present, some multi-fermion terms may become relevant in Luttinger liquids.)]

The different origins of the low-energy symmetries in the two systems are at the heart of the different roles of the SPT anomaly in the two systems. Though the translation symmetry of the lattice model at the low-energy limit may look identical to an on-site symmetry of some SPT phase, the translation symmetry is intrinsically non-on-site. Hence, it can never be gauged in the precise manner, and, the SPT anomaly, an obstruction of gauging global symmetries, may not have much implication on the “non-perturbative” nature of the theory emergent from the lattice. (In this context, it may be interesting to ask: When it is possible to gap the system trivially (i.e., anomaly-free), is there any way one can adiabatically deform the system to make translation symmetry on-site?)

In Sec. V, we will also consider the (3+1)d chiral anomalies and relate the anomaly to the “perturbative” stability. In contrast to the (1+1)d chiral anomaly detecting the no-go conditions for the LSMOH theorem, we show that (3+1)d chiral anomaly only detects the stability of the gaplessness only near the given low-energy theory in the parameter space.

D. Summary

The above considerations can be summarized from the view point of continuum field theories as follows. Let us consider a $(d+1)$ -dimensional continuum field theory \mathcal{F} . To be concrete, suppose \mathcal{F} is a theory of *relativistic* fermion. This theory may or may not arise as a low-energy effective theory of a given lattice model of the same spacetime dimension. Let there be a global symmetry G respected by \mathcal{F} . Let there be a ’t Hooft anomaly for G . The ’t Hooft anomaly has a one-to-one correspondence with $\Omega_{\text{Spin}/\text{Spin}^c}^{d+2, \text{tors}}(BG)$, the Pontryagin dual of the torsion subgroup of the equivariant spin/spin^c bordism groups with the symmetry group G .³⁴ Taking this ’t Hooft anomaly “naively”, one would conclude that the theory must be realized as a boundary theory of a $(d+2)$ -dimensional bulk theory.

Let us now assume that we actually know that \mathcal{F} is a low-energy effective theory of a $(d+1)$ -dimensional lattice model. Then, at least one of the following must be true: (a) G is not on-site for the $(d+1)$ -dimensional lattice model. (b) G is not the true symmetry of the problem; It is a symmetry only emergent in the low-energy physics. These two possibilities correspond to the LSMOH theorem and Fermi surfaces. In addition, it should be also

noted that, in particular for the case of the LSMOH theorem, there is no reason to consider relativistic fermions to start with. In other words, we need to care about the high-energy scale origins of the low-energy symmetries and interpret the meaning of the anomalies carefully.

The rest of the paper is organized as follows.

- In Section II, we first review briefly the edge of the 2d quantum spin Hall effect (QSHE) and show that the exactly same low-energy theory can arise from the 1d lattice model with spinless fermion at fractional filling.

- In Section III and IV, We discuss the implications of the anomalies. With the anomaly-based understandings of the LSMOH theorem, in section III, we will extend the conventional LSMOH theorem, concerning singly-charged particles in general, to the multiple particle species with the different charge assignments, e.g., a mixed system of charge-2 spinless boson at half-filling with charge-1 spinless electrons at integral filling. We construct several novel symmetric insulators, which cannot be adiabatically deformed into a Slater-type insulator.

- In Section V, we consider the (3+1)d chiral anomalies^{35–37} in Weyl and Dirac semimetals^{28–33} and relate the anomaly to the “local” stability. In contrast to the (1+1)d chiral anomaly detecting the no-go conditions for the LSMOH theorem, we show that (3+1)d chiral anomaly only detects the stability of the gaplessness only near the low-energy theory in the parameter space. We apply these results to the (3+1)d Weyl and Dirac semimetals.

- We finish by providing conclusions and outlooks in Section VI.

- *Note Added:* After the completion of the work, we became aware of the work by Jian, Bi, and Xu,³⁸ in which the similar consideration is made.

II. 2D SPTS AND 1D LATTICE MODELS

In this section, we consider 1d lattice models, which are enforced to be critical by the LSMOH theorem, and compare their low-energy theory to the edge theories of 2d SPTs. We will find that the lattice model gives rise to exactly the same effective low-energy theory as the edge of the SPTs. We will discuss the quantum anomalies relevant for the low-energy theories.

A. (2+1)d QSHE

We start by revisiting the simplest (2+1)d fermionic SPT phase, the QSHE, protected by unitary on-site $U(1)_Q \times U(1)_{S_z}$ symmetry. The 2d bulk of this SPT phase can be constructed on the honeycomb lattice by following Kane and Mele³⁹

$$H = -t \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} \Psi_{\mathbf{r}}^\dagger \Psi_{\mathbf{r}'} + i\lambda \sum_{\langle\langle \mathbf{r}, \mathbf{r}' \rangle\rangle} \Psi_{\mathbf{r}}^\dagger \sigma_z \hat{z} \cdot (\hat{d}_{\mathbf{r}\mathbf{r}'}^1 \times \hat{d}_{\mathbf{r}\mathbf{r}'}^2) \Psi_{\mathbf{r}'},$$

where \mathbf{r} labels the lattice site and $\Psi_{\mathbf{r}} = (c_{\mathbf{r}\uparrow}, c_{\mathbf{r}\downarrow})^T$ a spinful fermion; $\hat{d}_{\mathbf{r}\mathbf{r}'}^{1,2}$ is a vector connecting the next nearest-neighbor sites \mathbf{r} and \mathbf{r}' on the honeycomb lattice. The lattice Hamiltonian clearly respects the symmetry $U(1)_Q \times U(1)_{S_z}$ at the ultraviolet (UV) scale

$$\begin{aligned} U(1)_Q : \quad \Psi_{\mathbf{r}} &\rightarrow e^{i\phi} \Psi_{\mathbf{r}}, \\ U(1)_{S_z} : \quad \Psi_{\mathbf{r}} &\rightarrow e^{i\sigma_z \theta/2} \Psi_{\mathbf{r}}. \end{aligned} \quad (1)$$

The ground state is simply the combinations of the completely filled Chern band with Chern number $\nu = 1$ for spin- \uparrow electrons and the completely filled Chern band with $\nu = -1$ for spin- \downarrow electrons.

When the open boundary condition is imposed, along a spatial direction x , say, gapless edge states emerge. They can be described by the low-energy Hamiltonian

$$H = \int dx \Psi^\dagger(x) (-iv_F \partial_x) \sigma_z \Psi(x), \quad (2)$$

where $\Psi(x) = (\psi_\uparrow(x), \psi_\downarrow(x))^T$ and v_F is the Fermi velocity. The action of the $U(1)_Q \times U(1)_{S_z}$ symmetry on the edge mode is still given by (1), if the fermionic operators there are replaced by their boundary counterparts. It is straightforward to check that there is no gapping term when the symmetry (1) is strictly imposed on the edge. Hence the gaplessness of the edge theory is protected by the symmetry.

To facilitate to establish a connection with filling-enforced gapless states on the 1d lattice, we note that the criticality (as well as the quantum anomaly) of the 1d edge survive even if we lower $U(1)_{S_z}$ down to \mathbb{Z}_{S_z} , i.e., instead of $U(1)_{S_z}$, we can consider the discrete spin rotation

$$\mathbb{Z}_{S_z} : \quad \Psi_{\mathbf{r}} \rightarrow e^{im\theta_F \sigma_z} \Psi_{\mathbf{r}}, \quad m \in \mathbb{Z}, \quad (3)$$

with some $\theta_F \in (0, 2\pi)$. When we fine-tune $\theta_F = \pi/N$, we can further “lower” the symmetry, $\mathbb{Z} \rightarrow \mathbb{Z}_N$. Note also that, at the level of non-interacting fermions, the classification of the 2d SPT with $U(1) \times \mathbb{Z}_N$, for any $N > 1$, is still \mathbb{Z} , since one can easily verify that there is no mass term allowed to the theory (2).

B. (1+1)d lattice spinless fermions

We now consider the model of spinless fermions hopping on a 1d lattice consisting of L lattice sites:

$$H = -t \sum_x (c_x^\dagger c_{x+1} + h.c.) - \mu \sum_x c_x^\dagger c_x. \quad (4)$$

The model is invariant under the charge $U(1)_Q$, and, with periodic boundary condition, lattice translation symmetry \mathbb{Z}_L with $L \gg 1$. In the thermodynamic limit $L \rightarrow \infty$, we have the two symmetries

$$\begin{aligned} U(1)_Q : \quad c_x &\rightarrow e^{i\phi} c_x, \\ \mathbb{Z}_{trans} : \quad c_x &\rightarrow c_{x+1}. \end{aligned} \quad (5)$$

Note that the translation symmetry \mathbb{Z}_{trans} is manifestly non-on-site at this UV scale.

The ground state can be easily found by filling the single particle states below the chemical potential μ ,

$$|GS\rangle \propto \left(\prod_{|k| \leq k_F} c_k^\dagger \right) |\text{vac}\rangle,$$

where $|\text{vac}\rangle$ is the Fock vacuum. The system realizes gapless metal, which can be easily seen from the band structure of (4), if the filling $\nu = \frac{k_F}{\pi} \notin \mathbb{Z}$. Furthermore, the LSMOH theorem² in 1d dictates that if the translation symmetry \mathbb{Z}_{trans} and $U(1)_Q$ are not broken, then the ground state should be always gapless even in the presence of interactions; it is a filling-enforced critical state.

To reveal the connection between this 1d lattice model and the edge of the QSHE, we now proceed to the continuum IR limit of the theory (4)

$$H = \int dx \Psi^\dagger(x) (-iv_F \partial_x) \sigma_z \Psi(x), \quad (6)$$

where $\Psi(x) = (\psi_R(x), \psi_L(x))^T$ is the low-energy fermion field near the Fermi point. Here, the microscopic fermion operator c_x can be expanded in terms of the slowly varying low-energy fields $\psi_{R/L}$ as

$$c_x \approx \psi_R(x) e^{ik_F x} + \psi_L(x) e^{-ik_F x}. \quad (7)$$

Here we take a convention that the k_F is the right-most momentum of the filled state.

The symmetry actions of $U(1)_Q \times \mathbb{Z}_{trans}$ within this low-energy theory (6) can be easily derived,

$$\begin{aligned} U(1)_Q : \quad \Psi(x) &\rightarrow e^{i\phi} \Psi(x), \\ \mathbb{Z}_{trans} : \quad \Psi(x) &\rightarrow e^{ik_F \sigma^z} \Psi(x). \end{aligned} \quad (8)$$

Here, in the continuum (conformal) limit where the UV cutoff, i.e., the lattice constant, is completely ignored, the translation symmetry \mathbb{Z}_{trans} , which is non-on-site at UV scale, acts as if it is a purely on-site internal symmetry on the infrared (IR) field $\Psi(x)$.

In summary, the continuum IR limit (6) with the symmetry $U(1)_Q \times \mathbb{Z}_{trans}$ (8) is identical to the edge theory (2) of the QSHE with the symmetry $U(1)_Q \times \mathbb{Z}_{S_z}$ (up to the Fermi velocity which is irrelevant for the discussion of quantum anomalies). In the IR limit, the two theories realize the same \mathbb{Z} -symmetry actions (or appropriate subgroup of \mathbb{Z} if the filling is rational fraction – see below) although the symmetry has very different origins at the UV scales. Furthermore, on the lattice scale, the translation symmetry is non-on-site (5), but becomes on-site (8) on the IR scale and looks like the internal symmetry (spin rotation symmetry) of the edge of the 2d QSHE. In other words, the 1d spinless fermion lattice model at fractional filling evades the no-go theorem. This is exactly parallel to the proposed dual description of the half-filled Landau level, which turns out to be identical to the (dual) low-energy theory of the 2d boundary of the 3d topological insulator.

C. Effective symmetry

It is also important to note that when the filling is rational, e.g., $\nu = 1/q$, then the translation symmetry in Eq. (8) can be reduced to an effective \mathbb{Z}_{2q} . If the center of momentum is shifted to $\frac{\pi}{2q}$ (by some fine-tuning of the band structures or by applying the background gauge field), then the translation symmetry is in fact reduced further to \mathbb{Z}_q such that

$$\begin{aligned} U(1)_Q : \quad \Psi(x) &\rightarrow e^{i\phi} \Psi(x), \\ \mathbb{Z}_q : \quad \Psi(x) &\rightarrow e^{i\frac{\pi}{q}(\sigma^z - 1)} \Psi(x). \end{aligned} \quad (9)$$

Thus, the lattice translation symmetry is effectively lowered to \mathbb{Z}_q – this reduced symmetry will be called effective symmetry.

At the free fermion level and within the low-energy theory (6), both the symmetry groups $U(1)_Q \times \mathbb{Z}$ and $U(1)_Q \times \mathbb{Z}_q$ can protect the gaplessness of the theory. At this stage, the lowering of the symmetry as well as treating the translation symmetry as the on-site symmetries are seemingly innocuous. However, when it comes to the multiple copies and interactions, then we will see that these treatments may give rise to subtle effects, i.e., it now matters if the system comes from the lattice or from the edge and if the symmetry in the low-energy limit is effective, as we will see from the discussions of the anomaly.

D. Other examples

Evading the no-go “theorem” is also possible in higher dimensions. For example, the “duality” between the composite Fermi liquid in the half-filled Landau level and the (2+1)d boundary of (3+1)d topological insulators has been discussed extensively recently.^{21–26} In this example, the low-energy theory of the half-filled Landau level is claimed to be the same (in terms of the parity anomaly, field contents, and symmetry actions) as that of the surface of 3d topological insulators. Both of these theories contain a dynamical gauge field and a single Dirac fermion with the anti-unitary symmetry. The anti-unitary symmetry looks on-site in space when acting on the low-energy fermion fields. This seems against the no-go theorem since the Landau level can be constructed from two-dimensional lattices (with the projection to the Landau level). However, the anti-unitary symmetry in the Landau level, which emerges after the projection to the lowest Landau level, is actually non-on-site at lattice scales.

It is also instructive to contrast our work with those which deal with weak SPT phases, e.g., Ref. 19. Ref. 19 considered $(d-1)$ -dimensional lattice models as the surface of d -dimensional weak SPT phases, and then finds the classification of possible topological orders respecting the translational symmetries from the d -dimensional SPT

index, $\mathcal{H}^{d+1}[\mathbb{Z}_{trans}^d \times G, U(1)]$. Here \mathbb{Z}_{trans}^d is the translation symmetry, and G is the on-site symmetry. Through the Kunneth formula, it is found that this index for the weak SPT can be given in terms of those of the lower-dimensional strong SPTs with on-site symmetry G , i.e., $\mathcal{H}^{r+1}[G, U(1)]$ ($r < d$), which is stacked inside the weak SPT. Though this formula helps to understand the index of the weak SPT clearly, this treats the translation symmetry physically different from the on-site symmetry. The anomalous nature of the symmetrically-gapped phases of the $(d-1)$ -dimensional lattice models manifests as the non-trivial indices for this weak SPT phase. On the other hand, we will consider e.g. the $d = 2$ SPT case, or one-dimensional lattice models, which are forced to be critical instead of gapped. The translation symmetry in the lattice models are interpreted as the on-site symmetry in the strong SPT side. This makes the non-on-site translation symmetry and on-site global symmetry, e.g., charge conservation, of the lattice model to be treated on an equal footing in the SPT side. Furthermore, the anomalous nature of the lattice models manifests as the proper generalizations of the chiral anomaly, which are a “more historic” diagnosis of gapless theories than the indices. Currently, the link between the weak SPT index and the anomaly discussed in this paper is not obvious, and thus clarifying the relationships between the cohomological indices \mathcal{H}^{d+1} and chiral anomalies in one-dimensional lattice models and three-dimensional relativistic semimetals will be an interesting future problem.

III. ANOMALY AND LSMOH THEOREM

Having confirmed that the 1d lattice model and the SPT boundary are described by the same low-energy effective theory with the identical action of the global symmetry $U(1)_Q \times \mathbb{Z}_N$, we now proceed to discuss quantum anomalies. Here, as emphasized in the previous section, one should keep in mind the different origins of the \mathbb{Z}_N symmetry in the LSMOH and SPT contexts. Nevertheless, in the following we will first take the low-energy theory on its own as a relativistic quantum field theory, without asking how it arises. An obstruction to gauge this global symmetry, i.e., ’t Hooft anomaly would be labeled by the elements in the three-dimensional equivariant spin^c cobordism group with \mathbb{Z}_N symmetry⁴⁰

$$\Omega_{\text{Spin}^c}^3(B\mathbb{Z}_N) \cong \mathbb{Z}_{\epsilon_N \cdot N} \times \mathbb{Z}_{N/\epsilon_N}, \quad (10)$$

where $\epsilon_N = 1$ for odd N and $\epsilon_N = 2$ for even N . (For the known results of $\Omega_{\text{Spin}^c}^3(B\mathbb{Z}_N)$, see Ref. 41 and 42.) If the low-energy theory is interpreted as the SPT boundary, $\Omega_{\text{Spin}^c}^3(B\mathbb{Z}_N)$ agrees with the classification of the bulk SPT phases protected by the unitary on-site symmetry $U(1)_Q \times \mathbb{Z}_N$.

More concretely, let us consider the following, slightly more extended version of the (1+1)d continuum theory

with the $U(1) \times \mathbb{Z}_N$ global symmetry:

$$H = \int dx \sum_{a=1}^{N_f} \left[\psi_{L,a}^\dagger i \partial_x \psi_{L,a} - \psi_{R,a}^\dagger i \partial_x \psi_{R,a} \right], \quad (11)$$

with N_f the number of species. Henceforth, the velocity is scaled to 1 for simplicity. We encode the $U(1)_Q$ symmetry into the fermion fields as

$$\begin{aligned} U(1)_Q : \quad \psi_{R,a}(x) &\rightarrow e^{i\delta\phi q_a} \psi_{R,a}(x), \\ \psi_{L,a}(x) &\rightarrow e^{i\delta\phi q_a} \psi_{L,a}(x), \end{aligned} \quad (12)$$

in which the fermion field $\psi_{R/L,a}$ carries an odd integer electric charge q_a . In particular, for the electronic systems, all $q_a = 1$. On the other hand, the \mathbb{Z}_N symmetry acts as

$$\begin{aligned} \mathbb{Z}_N : \quad \psi_{R,a}(x) &\rightarrow e^{2\pi i s_{R,a}/N} \psi_{R,a}(x) \\ \psi_{L,a}(x) &\rightarrow e^{2\pi i s_{L,a}/N} \psi_{L,a}(x), \end{aligned} \quad (13)$$

where $s_{R/L,a}$ are integers such that the whole symmetry group is exactly $U(1) \times \mathbb{Z}_N$ (up to any redefinition of the \mathbb{Z}_N transformation by the $U(1)_Q$ symmetry). Then, the full 't Hooft anomalies of $U(1) \times \mathbb{Z}_N$ in the relativistic field theory (11) can be calculated explicitly, and are characterized by the following two indices (derived in Appendix A)

$$\begin{aligned} \sum_a \nu_a \cdot \frac{s_{R,a} + s_{L,a}}{\epsilon_N} \mod \mathbb{Z}, \\ \sum_a \nu_a q_a \mod \mathbb{Z}. \end{aligned} \quad (14)$$

[Here $\nu_a = (s_{R,a} - s_{L,a})/N$ is equivalent to the lattice filling. See the discussions below.] This matches with $\Omega_{\text{Spin}^c}^3(B\mathbb{Z}_N) \cong \mathbb{Z}_{\epsilon_N \cdot N} \times \mathbb{Z}_{N/\epsilon_N}$. As we will see in details, the second index in (14) is related to the familiar chiral anomaly in (1+1)d.

As the simpler field theory discussed in the previous section, the field theory (11) can be interpreted as either describing a SPT boundary with $U(1)_Q \times \mathbb{Z}_N$ symmetry, or the low-energy effective theory of a 1d LSMOH lattice model. If the theory originates from the 1d lattice, then the filling ν_a of the a -th fermion per unit cell fixes the relative difference between the two Fermi points, i.e., $k_{F,R}^a = 2\pi s_{R,a}/N$ and $k_{F,L}^a = 2\pi s_{L,a}/N$, by $\nu_a = (k_{F,R}^a - k_{F,L}^a)/2\pi$. The specific positions of $k_{F,R/L}^a$ in momentum space depend on the band structure.

Once we specify the microscopic origin of the field theory (11), the two anomaly indices (14) have to be interpreted properly. As mentioned already, if the low-energy theory (11) is interpreted as a SPT boundary, $\Omega_{\text{Spin}^c}^3(B\mathbb{Z}_N)$ agrees with the classification of the bulk SPT phases protected by the unitary on-site symmetry $U(1)_Q \times \mathbb{Z}_N$. In this case, any of non-zero anomalies in the above implies a non-perturbative obstruction for a symmetric trivial gapped state at the boundary.

On the other hand, in the LSMOH context we show that only the second index, the chiral anomaly, is relevant; The second index is (a slight extension of) the conventional LSMOH theorem. In the usual LSMOH theorem, the charge is taken to be 1, i.e., all $q_a = 1$. Then the absence of the chiral anomaly with $q_a = 1$ for all a is equivalent to the conventional LSM theorem. However, here we allow generic q_a here (we assume the existence of the minimal charge-1 fermion in the spectrum). We will also include the bosons later in the section III A by using the momentum pumping argument.

Given the identification of the chiral anomaly to the LSMOH theorem, the first index must be something intrinsic to the SPT boundaries where \mathbb{Z}_N is truly on-site, and it is unrelated to the LSMOH theorem. We call this the anomaly “ \mathbb{Z}_N anomaly” below to distinguish it from the chiral anomaly. We will find that it does not guarantee the non-perturbative stability for the lattice models but implies only the local stability.

A. Chiral anomaly and the LSMOH theorem

We now give some details for the identification of the second index as the chiral anomaly. We will also see that it is nothing but the LSMOH theorem.

1. Chiral Anomaly: Field Theory

To see the second index is equivalent to the chiral anomaly we “promote” the \mathbb{Z} symmetry (or \mathbb{Z}_N) to the continuous axial $U(1)_A$ and use the standard anomaly equation. For the case of the single-flavor model, the violation of the axial charge conservation in the presence of the electromagnetic (vector) gauge field is quantified by

$$\frac{dQ_5}{dt} = \partial_\mu j_5^\mu = \frac{1}{\pi} \int dx \, q E_x, \quad (15)$$

where the axial charge, the number difference between the left mover and the right mover, corresponds to the momentum up to constant because the momentum is $P = k_F(\psi_R^\dagger \psi_R - \psi_L^\dagger \psi_L) = \nu \pi Q_5$, where Q_5 is the axial charge; E_x is the electric field along the x -direction and q is the electric charge. When we have multiple species of the fermions $\{\Psi_a\}$ with multiple charge q_a at the filling ν_a , the total momentum transfer in the system is the sum of the momentum transfer of each species a , i.e.,

$$\frac{dP_{tot}}{dt} = \sum_a \nu_a \pi \frac{dQ_{5,a}}{dt} = \sum_a \nu_a q_a \int dx \, E_x. \quad (16)$$

Now we imagine to thread a magnetic flux $\int dx \, A_x = 2\pi$ adiabatically to the system. Then during the process, the change in the momentum is

$$\Delta P_{tot} = \sum_a \nu_a q_a \int dx dt \, \partial_t A_x = 2\pi \sum_a \nu_a q_a. \quad (17)$$

Hence, the momentum (axial charge) is not conserved during the process if ΔP_{tot} is non-zero.

Remembering now that the relevant symmetry is \mathbb{Z} (instead of the axial $U(1)_A$), which is derived from lattice translation symmetry, we know that P_{tot} is conserved only modulo 2π . Hence, the anomaly-free condition is given by

$$\sum_a \nu_a q_a = 0 \pmod{\mathbb{Z}}. \quad (18)$$

Otherwise, the translation symmetry and the charge conservation are in conflict. Note that classically $U(1)$ and the translation \mathbb{Z} commute each other and hence we do not expect the change in \mathbb{Z} under the insertion of the flux $U(1)$. This classical expectation is violated and the two symmetries are in conflict at the quantum level.

The above momentum pumping argument for the chiral anomaly can be also computed by using bosonization. We leave the detail of its derivation in Appendix B.

The above result can also be derived in a more general and formal setting. Consider (intrinsically-continuum) Dirac fermions which carry odd electric charges in the presence of the background $U(1)$ gauge field on a closed manifold M . Specifically, we formulate the fermion theory (11) on a generic closed Riemannian two-manifold (M, g) endowed with a spin^c structure, where a well-defined spin^c connection, denoted as A , exists. (Only for this part of the discussion, we temporarily restrict fermions to carry odd charge $q_a \in 2\mathbb{Z} + 1$ (and bosons to carry even charge) to use the spin^c connection.) Here we work in Euclidean signature. [At this stage, one may worry that we are imposing too much structures (e.g., spin structure and continuum manifold with metric g), unrelated to the lattice fermions. However, these are only for the concreteness.] Since there is no (gauge) anomaly for $U(1)_Q$ the partition function $Z_{\{\Psi_a\}}(M; g, A)$ is well-defined on any such two-dimensional spin^c manifold M . However, $Z_{\{\Psi_a\}}(M; g, A)$ might in general not be invariant under $G_{eff} = \mathbb{Z}_q$ (or $G = \mathbb{Z}$), which is a symmetry of the classical action of (11) with background gauge field A . This is a discrete analog of the usual chiral (axial) anomaly of the continuous axial symmetry for Dirac fermions, and one can similarly use Fujikawa's method to compute such a discrete chiral anomaly. The anomaly comes from the nontrivial transformation of the path integral measure $\prod_a \mathcal{D}\Psi_a \mathcal{D}\bar{\Psi}_a$ under G_{eff} .^{43,44}

$$\begin{aligned} & Z_{\{\Psi_a\}}(M; g, A) \\ & \xrightarrow{G_{eff}} \exp\left(2\pi i \sum_a \nu_a \mathcal{I}_{q_a}(M; g, A)\right) Z_{\{\Psi_a\}}(M; g, A), \end{aligned} \quad (19)$$

where $\mathcal{I}_{q_a}(M; g, A)$ is the index of the charge- q_a Dirac operator $i\hat{D} - q_a A$ on M endowed with a metric and a spin^c structure. Since any two-dimensional spin^c manifold M is bordant to a multiple of $\mathbb{CP}^1 \cong S^2$,⁴⁵ that is, $[M] = k_M \cdot [\mathbb{CP}^1]$ for some integer k_M , where $[\cdot]$ denotes the equivalence class under bordism, and the index

$\mathcal{I}_{q_a}(\cdot)$ is a bordism invariant, we have

$$\begin{aligned} \mathcal{I}_{q_a}(M; g, A) &= k_M \cdot \mathcal{I}_{q_a}(\mathbb{CP}^1; g_c, A_c) \\ &= k_M \cdot q_a \int_{S^2} c_1(F_c) \\ &= -k_M q_a, \end{aligned} \quad (20)$$

where $c_1(F_c)$ is the first Chern class of the field strength $F_c = dA_c$ of the spin^c connection A_c on \mathbb{CP}^1 equipped with the canonical spin^c structure arising from the holomorphic structure. Then, it is obvious that the anomaly-free condition for the theory on any M is given by

$$\exp\left(-2\pi i k_M \sum_a \nu_a q_a\right) = 1 \iff \sum_a \nu_a q_a \in \mathbb{Z}. \quad (21)$$

2. Chiral Anomaly: Pumping Argument

Note that the above discussions rely on the specific assumptions on the ground state: Fermi liquid (or Luttinger liquid). Here, we derive the same anomaly-free condition without assuming a particular ground state. This allows us to include the interacting bosonic case to the discussion. This is a reformulation of Oshikawa's argument².

We start from a length- L system of charge- q particle c_x with the Hamiltonian $H(\Phi)$ in which the flux Φ specifies the twisted boundary condition, $c_{x+L} = c_x e^{iq\Phi}$. To compare with the field theoretic discussion, the phase factor Φ can be generated by the uniform gauge field $A_x = \Phi/L$, i.e., the flux $\Phi = \oint A_x$ threaded into the circle.

Next we imagine that we introduce a flux insertion operator \hat{F} which winds Φ by 2π adiabatically, i.e., $\hat{F} : \Phi \rightarrow \Phi + 2\pi$. Then, the ground state $|GS\rangle$ of $H(0)$ will evolve into another state $\hat{F}|GS\rangle$ after the 2π -flux insertion. On assuming that $|GS\rangle$ is translation symmetric, $\hat{F}|GS\rangle$ is also translation symmetric since the flux insertion does not break the translation symmetry², i.e., $\hat{T}_1 \hat{F} = \hat{F} \hat{T}_1$.

Following the field theoretic discussions, we calculate the relative change in the momentum of $\hat{F}|GS\rangle$ to that of $|GS\rangle$. Thus we consider the eigenvalue of the translation symmetry.

Note that the momentum is a gauge-dependent quantity. Hence, to correctly compare the momentum of $\hat{F}|GS\rangle$ which is obtained after the 2π -flux insertion, we need to use the translation symmetry at the gauge $A_x = 2\pi/L$. This translation operator $\hat{T}(2\pi)$ can be obtained by the large gauge transformation of the usual translation operator $\hat{T}_1 : c_x \rightarrow c_{x+1}$.

$$\hat{T}_1(2\pi) = U \hat{T}_1 U^\dagger = \hat{T}_1 \exp(-2\pi i q \nu),$$

with $U = \exp[2\pi i \sum_x x \frac{\hat{n}_x}{L}]$.² Hence we have

$$\hat{T}_1(2\pi) \hat{F} |GS\rangle = e^{-2\pi i q \nu} \hat{T}_1 \hat{F} |GS\rangle = e^{iP_0} e^{-2\pi i q \nu} F |GS\rangle,$$

where we have used $\hat{T}_1 \hat{F} = \hat{F} \hat{T}_1$ and $\hat{T}_1 |GS\rangle = e^{iP_0} |GS\rangle$. Thus the momentum pumped into the system is $2\pi q\nu$. This nicely matches the chiral anomaly calculation. Note that when we have multiple species of particles, the phase factors from each species will add up. Hence, the triviality of the momentum pumping is $\sum_a q_a \nu_a = 0 \bmod \mathbb{Z}$.

It should be also noted that, within this pumping argument, we do not make an assumption about the statistics of the particles and the particular form of the ground states. Hence, the criteria is now independent of the statistics. Note also that the minimum transparent flux 2π is imposed by assuming the presence of the charge-1 particle in the spectrum, which may be gapped or gapless.

B. Extension of the LSMOH theorem: a stability analysis

From the comparison between the chiral anomaly calculation and the LSMOH theorem, we expect that when the anomaly free condition (18) is satisfied, it should be possible to gap the system trivially without symmetry breaking. This section is devoted to construct the symmetric insulators explicitly by using bosonization. More specifically, let us write $[\nu, q]$ to represent the "equivalence class" of the physical Hamiltonian of the charge- q particle, either fermionic or bosonic, system at the filling ν , under the consideration based on the (generalized) LSMOH theorem. Then, we prove the followings:

$$[\nu, q] = 0, \quad \text{if } \nu q \in \mathbb{Z}; \quad (22a)$$

$$[\nu_1 + \nu_2, q] = [\nu_1, q] \oplus [\nu_2, q]; \quad (22b)$$

$$[\nu, q_1 + q_2] = [\nu, q_1] \oplus [\nu, q_2]. \quad (22c)$$

Here we use \oplus to denote the direct sum of various systems. We also need to define the trivial class (phase), denoted as 0 above, as follows:

- (1) A system that can be gapped (with the consideration of interactions) in a symmetry-preserving fashion is trivial;
- (2) ("Stably-trivial" condition) If a system can be gapped, when coupled to some trivial systems of the first kind, in a symmetry-preserving fashion, then it is also trivial.

The properties (22a)-(22c) are naturally satisfied from the point of view of anomalies, as $[\nu, q]$ can actually be characterized by the chiral anomaly index $\nu q \bmod \mathbb{Z}$. (In this case, " \oplus " represents an usual addition of numbers in \mathbb{R}/\mathbb{Z} .) Nevertheless, here we perform a stability analysis to provide another evidence to conform the anomaly argument presented before.

It should be noted that the meaning of being trivial in the current context is different from the SPT context. First, for the non-perturbative stability of lattice systems, the trivial system is solely identified through their

chiral anomaly. The other part of the 't Hooft anomaly is not important. In other words, trivial systems in this context may accidentally have non-trivial \mathbb{Z}_N anomaly (the other part of the full 't Hooft anomaly). Second, here we use the full translation symmetry \mathbb{Z} , instead of the effective \mathbb{Z}_N with fixed N . Hence, trivial systems in the present context are different from trivial systems in the SPT context where we classify SPT boundaries with $U(1) \times \mathbb{Z}_N$ with fixed N . The different meaning of trivial states is also reflected in how we "add" (and "subtract") systems. In the SPT context, we are allowed to add only systems with vanishing full 't Hooft anomalies, which should be contrasted with our Condition (2). Also in the SPT context we do not add trivial systems with the "extended" symmetries than the symmetry of edge theories. In other words, the label N should not be treated as a fixed symmetry label, but as indicating a representation under the microscopic translation symmetry (rather than effective symmetry).

Our result for symmetric insulators, $\sum_a \nu_a q_a = 0 \bmod \mathbb{Z}$, and the following field theory discussion have some resemblance to the recently-discovered lattice homotopy argument.¹⁷ The statement in the lattice homotopy argument is that, as far as the lattice symmetry in concern is not changed (in our case, it is the translation symmetry), the system can be deformed into a simpler lattice by adding the symmetry charges from each lattice sites. For example, imagine a system of two spin- $\frac{1}{2}$'s per unit cell, e.g., two lattice sites in the unit cell, and we want to impose the translation symmetry only. Then, as far as the translation symmetry is concerned, we can deform the lattice so that the two lattice sites are sitting on top of each other, and combine the two spin- $\frac{1}{2}$ into a single spin-1 or spin-0 object. Then we know that the integral spin inside the unit cell gives a trivial insulator. In our case, we are adding up the electric charges of particles per unit cell. Then the criteria we obtained is equivalent to having an integral charge per unit cell. Hence, our anomaly seems to be the manifestation of the lattice homotopy for the translation symmetry case.

a. Purely Fermionic Case: To show (22a), (22b) and (22c), we go through a few steps. In addition, we assume that there is a charge-1 particle in the spectrum. We first warm up with the fermionic case. For this case, we use the subscript F to represent the fermion, i.e., $[\nu, q]_F$.

Step 0. For the property (22a), it is easy to show that $[\nu, q]_F = 0$ when $\nu \in \mathbb{Z}$, as one can introduce a conventional backscattering term to have a symmetric gapped ground state. When $\nu = k/q \notin \mathbb{Z}$, let us consider the low-energy theory

$$H = \int dx \Psi_1^\dagger (-i\partial_x) \sigma^z \Psi_1. \quad (23)$$

Here the Ψ_1 is the charge- q fermion field at the filling k/q with the translation symmetry

$$trans : \Psi_1 \rightarrow e^{i\pi k/q\sigma^z + i\vec{k}} \Psi_1. \quad (24)$$

Without losing generality, we can take the center of the momentum $\vec{k} = 0$ for this fermionic system.

To gap out the system, we couple the system to other two systems at integral filling which consist of particles carrying unit charge; both of these systems are trivial. We then find possible gapping potentials. For example, we consider

$$[k/q, q]_F \oplus [0, 1]_F \oplus [0, 1]_F, \quad (25)$$

for which the corresponding Hamiltonian is given by

$$H = \int dx \sum_{a=1}^3 \Psi_a^\dagger (-i\partial_x) \sigma^z \Psi_a, \quad (26)$$

where $\Psi_{2,3}$ are the fermion fields from the $[0, 1]_F$ sectors. Under the translations, $trans : \Psi_{2,3} \rightarrow \Psi_{2,3}$ because they are at zero filling. Next we use the bosonization representation to write $\Psi_1 \sim (e^{i\phi_1}, e^{i\phi_2})$, $\Psi_2 \sim (e^{i\phi_3}, e^{i\phi_4})$ and so on, where ϕ_i are properly compactified bosonic fields. We can then construct bosonic fields $\Phi_j \sim \exp(i\vec{l}_j \cdot \vec{\phi})$ with $\vec{\phi} = (\phi_1, \phi_2, \dots, \phi_6)$ and

$$\begin{aligned} \vec{l}_1 &= (1, 1, -q, 0, 0, -q), \\ \vec{l}_2 &= (q, -q, 2, -2, 0, 0), \\ \vec{l}_3 &= (0, 0, 1, -1, -1, 1). \end{aligned} \quad (27)$$

By adding the interaction term $\propto -\sum_j (\mu_j \Phi_j + h.c.)$, with large enough μ_j , which will condense the bosons Φ_j , we obtain a gapped ground state which is symmetric. We have thus shown $[k/q, q] = [k/q, q] \oplus [0, q] \oplus [0, q] = 0$. This confirms Eq. (22a).

Step 1. Independence to the center of momentum: For the gappability conditions, we can show that the center of momentum (i.e., the center between the left-mover's and right-mover's momenta) is not important. Although the center of momentum is 0 when the lattice model has accidental parity symmetry, it needs not to be so. Here, we will show that the center of momentum can be forgotten for those conditions and can be taken to be 0 without losing generality.

To show this, we need to show that the charge- q fermionic system $[\nu, q, \vec{k}]_F$ (with the third index to represent the center of the momentum) at filling ν with the center of momentum at $\vec{k} \in (-\pi, \pi)$ is equivalent to another charge- q fermionic system $[\nu, q, 0]_F$ at filling ν with vanishing center of momentum. For this, we show that

$$[\nu, q, \vec{k}]_F = [\nu, q, 0]_F. \quad (28)$$

This can be shown by constructing a symmetric gap for the coupled two fermionic systems, $[\nu, q, 0]_F$ and $[1 - \nu, q, \vec{k}]_F$ with arbitrary \vec{k} .

For the coupled system, $[\nu, q, 0]_F$ and $[1 - \nu, q, \vec{k}]_F$ with arbitrary \vec{k} , let us consider the Hamiltonian

$$H = \int dx \sum_a \Psi_a^\dagger (-i\partial_x) \sigma^z \Psi_a \quad (29)$$

with $U(1)_{\delta\phi} : \Psi_a \rightarrow e^{iq\delta\phi} \Psi_a$. The translation symmetry is encoded as

$$\begin{aligned} trans : \quad \Psi_1 &\rightarrow e^{i\pi\nu\sigma^z} \Psi_1, \\ \Psi_2 &\rightarrow e^{i\pi(1-\nu)\sigma^z + i\vec{k}} \Psi_2. \end{aligned} \quad (30)$$

To construct the symmetric gap, we use the stably-trivial condition and include the following two trivial charge- q fermion fields Ψ_3 and Ψ_4 which transform under translation as

$$\begin{aligned} trans : \quad \Psi_3 &\rightarrow e^{-i\pi\sigma^z} \Psi_3, \\ \Psi_4 &\rightarrow e^{i\vec{k}} \Psi_4. \end{aligned} \quad (31)$$

Note that Ψ_3 is at filling $\nu = 1$ and Ψ_4 at $\nu = 0$ and thus they are trivial. To gap out $a = 1, 2, 3, 4$ all together in a symmetric fashion, we need to consider the gapping potentials generated by the following vectors

$$\begin{aligned} \vec{l}_1 &= (1, 0, 1, 0, 0, -1, 0, -1), \\ \vec{l}_2 &= (0, 1, 0, 1, -1, 0, -1, 0), \\ \vec{l}_3 &= (1, 1, 0, 0, -1, -1, 0, 0), \\ \vec{l}_4 &= (0, 0, 1, 1, 0, 0, -1, -1). \end{aligned} \quad (32)$$

The bosons to condense are given by $\Phi_j \sim \exp(i\vec{l}_j \cdot \vec{\phi})$, where $\vec{\phi} = (\phi_1, \phi_2, \dots, \phi_8)$ with the bosonization representation $\Psi_1 \sim (e^{i\phi_1}, e^{i\phi_2})$, $\Psi_2 \sim (e^{i\phi_3}, e^{i\phi_4})$ and so on. We can show that the condensed bosons do not break any symmetry.

Hence, with this, for the given filling ν and charge q , we can now take the center of momentum to be zero, i.e., $\vec{k} = 0$, to study the symmetric gappability. So, from here and on, we drop the index \vec{k} from $[\nu, q, \vec{k}]_F$ to represent systems and simply write $[\nu, q]_F$.

Step 2. To verify the property (22b), we show that $[\nu_1, q]_F \oplus [\nu_2, q]_F \oplus [\nu_1 + \nu_2, q]_F^{-1}$ is trivial, where $[\nu, q]_F^{-1} = [-\nu, q]_F$ is the inverse of a phase $[\nu, q]_F$ (as $[\nu, q]_F \oplus [-\nu, q]_F$ can be trivially gapped). A simple way to do this is by coupling it to a trivial phase, say, $[0, q]_F$; that is, we instead consider

$$[\nu_1, q]_F \oplus [\nu_2, q]_F \oplus [\nu_1 + \nu_2, q]_F^{-1} \oplus [0, q]_F \quad (33)$$

and examine its stability. In fact, a set of null vectors (in the above order of the bosonized fields) can be chosen as

$$\begin{aligned} \vec{l}_1 &= (1, 0, 1, 0, 0, -1, 0, -1), \\ \vec{l}_2 &= (0, 1, 0, 1, -1, 0, -1, 0), \\ \vec{l}_3 &= (1, 1, -1, -1, 0, 0, 0, 0), \\ \vec{l}_4 &= (0, 0, 0, 0, 1, 1, -1, -1), \end{aligned} \quad (34)$$

such that the ground state is symmetry invariant. This confirms Eq. (22b).

Step 3. Finally, the property (22c) can be derived in a similar way. We consider the following combination:

$$[\nu, q_1]_F \oplus [\nu, q_2]_F \oplus [\nu, q_1 + q_2]^{-1} \oplus [-\nu, 0]_F, \quad (35)$$

where the extra term $[-\nu, 0]$ is a trivial phase, as it can be trivially gapped. Then a set of null vectors (in the above order) for gapping such a system can be chosen as

$$\begin{aligned}\vec{l}_1 &= (1, 0, 1, 0, 0, -1, 0, -1), \\ \vec{l}_2 &= (0, 1, 0, 1, -1, 0, -1, 0), \\ \vec{l}_3 &= (1, -1, -1, 1, 0, 0, 0, 0), \\ \vec{l}_4 &= (0, 0, 0, 0, 1, -1, -1, 1),\end{aligned}\quad (36)$$

This confirms (22c).

b. Boson-Fermion Conversion: Next we prove $[\nu, q]_B = [\nu, q]_F$ where the subscript B represents that the system is made of bosons. To show this, we show

$$[\nu, q]_B \oplus [\nu, q]_F^{-1} = [\nu, q]_B \oplus [1 - \nu, q]_F = 0. \quad (37)$$

The low-energy theory of the bosons $[\nu, q]_B$ is given by

$$\begin{aligned}\Phi(x) &\sim e^{i\phi}, \\ \rho(x) &\sim \frac{1}{\pi} \partial_x \theta + \sum_{n \neq 0} \rho_n e^{in(2k_F x + 2\theta)},\end{aligned}\quad (38)$$

with $2k_F = 2\pi\nu$, and $[\partial_x \theta(x), \phi(x')] = i\pi\delta(x - x')$ (θ is compactified as $\theta \equiv \theta + \pi$, i.e., $e^{2i\theta}$ is the smallest local field involving θ). Here $\Phi(x)$ represents the fundamental local boson field.

Without losing generality, we can take the low-energy theory of $[1 - \nu, q]_F$ whose center of momentum is zero.

$$\begin{aligned}trans: \quad \phi_1 &\rightarrow \phi_1 + \pi(1 - \nu), \\ \phi_2 &\rightarrow \phi_2 - \pi(1 - \nu),\end{aligned}\quad (39)$$

where the fermions are written as $\Psi = (e^{i\phi_1}, e^{i\phi_2})$ with the usual kinetic term. Now the symmetric gapping potentials are given by the following two operators $B_i \sim \exp(i\vec{l}_i \cdot \vec{\phi})$ such that

$$\begin{aligned}\vec{\phi} &= (\phi, \theta, \phi_1, \phi_2), \\ \vec{l}_1 &= (-2, 0, 1, 1), \quad \vec{l}_2 = (0, 2, 1, -1).\end{aligned}\quad (40)$$

The vectors \vec{l}_i are chosen so that $\vec{l}_i \cdot K^{-1} \cdot \vec{l}_j = 0$ with $K^{-1} = (\frac{1}{2}\sigma_x) \oplus \sigma_z$.

Hence, with this conversion between fermions and bosons, we can now ignore the distinction between fermions and bosons in terms of the symmetric gappability. Thus, when the filling ν and the charge q are given, we can in general take the charge- q fermionic system to represent $[\nu, q]$ to investigate the gappability conditions. Hence, from here and on, we can forget about the statistics for the symmetric insulator conditions (22a), (22b), and (22c).

c. Novel Symmetric Insulators: Now, using the above results, we present a few novel symmetric insulator states, which are beyond the conventional LSMOH theorem. Note that in the conventional LSMOH theorem, usually a single-species of charge-1 particles are considered (the important exception to this is the Kondo system, where the mixture of the spin and fermion is considered).

(i) We have a trivial symmetric insulator of charge-3 fermion at the filling $\nu = 1/3$. This insulator requires the help from the charge-1 trivial fermionic systems.

The gapping potentials are inherently multi-fermion operators, which make the insulator beyond the conventional Slater insulator.

(ii) We have a trivial symmetric insulator of the mixed system of charge-1 boson at $\nu = 1/2$ and charge-1 fermion at $\nu = 1/2$. This system may be realizable in the optical lattice system.^{46,47}

(iii) We have a trivial symmetric insulator of charge-2 boson at half-filling. This insulator needs the help from the charge-1 fermion systems at the integer fillings. This is related to the superconductors of the fermions, where the Cooper pair of the fermion is bound to the boson and is turned into a neutral boson.

Microscopic lattice Hamiltonians for realizing these insulators as well as their higher-dimensional analogues will be left for the future studies.

IV. \mathbb{Z}_N ANOMALY AND LOCAL STABILITY

Having discussed the physics of the chiral anomaly in relation to the LSMOH theorem, we now investigate the physics of the first index in Eq. (14), the \mathbb{Z}_N anomaly, assuming that the continuum theory (11) is derived from the low-energy effective theory of a (1+1)d lattice fermion model. This is the part of the full 't Hooft anomaly, which is unrelated to the LSMOH theorem. They can be non-zero even when the LSMOH theorem does not enforce the gaplessness. In other words, the theory with the non-zero first index only (i.e., with the vanishing chiral anomaly) can be gapped symmetrically in the lattice system. This is in sharp contrast with SPT boundaries, where the both \mathbb{Z}_N and the chiral anomalies (the full 't Hooft anomaly) signal the non-perturbative stability. The difference can be traced back to using the effective translation symmetry $G_{eff} = \mathbb{Z}_N$ instead of the full translation symmetry $G = \mathbb{Z}$. When G_{eff} is properly extended by using the full symmetry G , then the \mathbb{Z}_N anomaly can be completely gone and the system can be gapped symmetrically.

This is best illustrated in the following example of the double copies of the low-energy fermions $U(1) \times (G_{eff} = \mathbb{Z}_2)$.

$$H = \int dx \sum_{a=1,2} \Psi_a^\dagger (-i\partial_x) \sigma^z \Psi_a \quad (41)$$

with $\mathbb{Z}_2 : \Psi_a \rightarrow e^{i\pi(\sigma^z - 1)/2} \Psi_a$. In terms of Eq. (13), we have $2\pi \frac{s_R}{N} = \pi$ and $\frac{s_L}{N} = 0$. Given the natural ($G_{eff} = \mathbb{Z}_2$)-ness of the low-energy symmetry, we first take $N = 2$, i.e., $s_R = 1$ and $s_L = 0$. For this state,

the first index is non-zero while the second index (chiral anomaly) vanishes. This non-trivial first index implies that we cannot find the symmetric insulator phase within the $U(1) \times G_{eff}$ symmetry.

However, the stability enforced by the \mathbb{Z}_N anomaly index is not non-perturbative in the lattice system as we saw in the section of the LSMOH theorem: we can always find a way to gap out the spectrum without breaking symmetries when the chiral anomaly is absent. Indeed, if we allow to extend $(G_{eff} = \mathbb{Z}_2) \rightarrow \mathbb{Z}_4$, which is still the subgroup in \mathbb{Z} , then $N = 4$ with $s_R = 2$ and $s_L = 0$. This generates the completely trivial anomalies, i.e., it is labeled by the 't Hooft anomaly $(0,0)$, and hence it can be gapped without breaking the symmetry $U(1) \times \mathbb{Z}_4$ (for filling $\nu = \frac{1}{N}$, by extending the symmetry $\mathbb{Z}_N \rightarrow \mathbb{Z}_{N^2}$, we can remove the \mathbb{Z}_N anomaly completely for any N). Note that, in the way of such a symmetry extension, we usually add some extra degrees of freedom (e.g. additional systems of electrons), which transform under an exact $U(1) \times \mathbb{Z}_4$ symmetry group and have vanishing 't Hooft anomaly, to the original system, such that they can help gap the whole (combined) system. The idea here is similar to the construction of gapped boundary states of SPT phases via symmetry extension.⁴⁸

That being said, we may ask what this \mathbb{Z}_N anomaly means to the low-energy theory of the fermionic lattice system. The first thing to note is that, the non-zero \mathbb{Z}_N anomaly implies that the given theory cannot be gapped within the quadratic term because the theory must be realizable as the non-trivial SPT boundary. The non-zero \mathbb{Z}_N anomaly implies that the system is perturbatively stable. Furthermore, we can explicitly show that the electronic lattice systems which can be gapped by adding quadratic terms do not possess \mathbb{Z}_N anomaly, see Appendix C for detail. This implies that to gap out the theory with the \mathbb{Z}_N anomaly, we need to include non-perturbative ingredients to the theory, e.g., a help from the extra trivial gapless modes and interactions beyond the terms quadratic in fermions (see, for example, the gapping potentials in the section of LSMOH theorem IIIB). Thus the non-zero \mathbb{Z}_N anomaly provides a perturbative stability of the given low-energy fermionic theory. (Note that the converse is not true. Even when the \mathbb{Z}_N anomaly is absent, the system may be perturbatively stable.)

V. (3+1)D CHIRAL ANOMALY AND WEYL SEMIMETALS

Given the relation between the (1+1)d chiral anomaly and the LSMOH theorem in (1+1)d, we now ask if the (3+1)d chiral anomaly also contain any stability information. Here we will show that the (3+1)d (abelian) chiral anomaly provides the local stability by considering the (3+1)-dimensional relativistic semimetals^{28,30-33} and chiral anomaly^{35-37,43,44,49,50} captured by the triangular $G-U(1)-U(1)$ diagram with G being unitary spatial

symmetry. For example, in the Weyl semimetal, G is the translation symmetry. $U(1)$ is the external non-dynamical electromagnetic gauge field.

To illustrate that the (3+1)d chiral anomaly does not give non-perturbative stability related to the filling and the translation, we take a specific two-band model with the Bloch Hamiltonian³²

$$\begin{aligned} \mathcal{H}(\mathbf{k}) = & \sin(k_x)\sigma^x + \sin(k_y)\sigma^y \\ & + M(\cos(k_z) - \cos(Q))\sigma^z \\ & + m(2 - \cos(k_x) - \cos(k_y))\sigma^z. \end{aligned} \quad (42)$$

This model has the two Weyl points at $k_z = \pm Q$. Remarkably, the Weyl points appear at zero energy, which makes the system exactly at half-filling for any Q . Since the system is spinful, the half-filling means that there is one electron per unit cell, and thus it can be trivially gapped while preserving the translation and charge conservation symmetries. Indeed, by changing $Q \rightarrow 0$, we can achieve such a symmetric trivial insulator within the Bloch Hamiltonian (42) at half-filling. However, this process involves the change in the dispersion from the relativistic dispersion to the non-relativistic dispersion, which is a non-perturbative process seen from the low-energy Weyl fermion Hamiltonian.

On the other hand, in the low-energy limit, the two Weyl points,

$$H = \int d^3x \Psi^\dagger \tau^z \sigma \cdot k \Psi, \quad (43)$$

cannot be gapped while keeping the translation symmetry along z

$$T_z = \exp(iQ\tau^z). \quad (44)$$

(The other two translation symmetries along x and y directions are trivial in this fine-tuned model.) Given the allowed symmetric insulator with translation and the filling, any stability condition of the Weyl semimetal must be only local in the parameter space. This local stability then can be captured by the chiral anomaly $T_z - U(1) - U(1)$ diagram:

$$\delta S = \int d^4x \frac{2Q}{16\pi^2} \epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho}, \quad (45)$$

which can be obtained by the change in the path integral measure by the translation symmetry in the presence of the electromagnetic gauge field.

There is another reason why the non-trivial chiral anomaly (45) may not be equivalent to the filling-constrained gapless-ness of the original lattice model (42). We essentially show that the number of electrons relevant for the physical (semi-classical) picture^{35-37,51} of the chiral anomaly deviates from that of the original lattice problem. For this, we count the number of electrons in the system in the presence and in the absence of the magnetic field for the Weyl semimetal.

First, in the absence of the magnetic field, we note that the spectrum has the (accidental) particle-hole symmetry, i.e., $E = \pm|E(k)|$. See (C) of Fig. 1 for $k_x = k_y = 0$ band structure. We now count the number of the states below the chemical potential $\mu = 0$. Then, there is a single band below the chemical potential. Hence, there is a single filled state per each momentum \vec{k} . The total number of electrons in the system thus equals to the number of the allowed momentum. The spacing between the momentum along a -direction ($a = x, y, z$) is $\frac{2\pi}{L_a}$ and the each momentum spans from $-\pi$ to π . Thus, the number of the filled states are:

$$N_e = \prod_{a=x,y,z} \left(\frac{2\pi}{2\pi/L_a} \right) = L_x L_y L_z, \quad \nu = \frac{N_e}{L_x L_y L_z} = 1.$$

Hence there is one electron per unit cell.

Next, if we apply the magnetic field along z direction, then the band structure is changed into the series of the Landau levels. At zero chemical potential, there is a chiral mode passing through each Weyl point. See Fig. 1(D). There chiral modes are equivalent to the “0th Landau level” of the Weyl fermions. Now, let us count the number of the filled states in this case.

We start with counting with the fully filled bands. Given a momentum k_z , there are $N_{f-LL} \in \mathbb{Z}$ filled Landau levels with N_{f-LL} varying with the strength of the magnetic field. Then, each k_z has the following number of the filled states

$$\begin{aligned} N_{e:f-LL} &= \left(N_{f-LL} \times \frac{L_x L_y}{2\pi l_B^2} \right) \times \frac{2\pi}{2\pi/L_z} \\ &= N_{f-LL} \frac{L_x L_y L_z}{2\pi l_B^2}. \end{aligned} \quad (46)$$

Here $l_B^2 = 1/B$, in which B is the magnetic field strength. Here, to impose the periodic boundary condition along x and y , the number of the states inside the Landau level, or $\frac{L_x L_y}{2\pi l_B^2}$, must be integral, and hence $N_{e:f-LL}$ is also integral. We next count the number of electrons in the 0th Landau level. The momentum k_z inside the filled 0th Landau level expands from $-Q$ to Q , (see (D) of Fig. 1) and hence the counting gives

$$N_{e:0-thLL} = \frac{L_x L_y}{2\pi l_B^2} \times \frac{2Q}{2\pi/L_z} = \frac{L_x L_y L_z}{2\pi l_B^2} \frac{2Q}{2\pi}. \quad (47)$$

Now we take a filling, which is the number of electrons divided by the volume,

$$\nu = \frac{N_e}{L_x L_y L_z} = \frac{1}{2\pi l_B^2} \left(N_{f-LL} + \frac{2Q}{2\pi} \right), \quad (48)$$

where $N_e = N_{e:f-LL} + N_{e:0-thLL}$. Obviously, this depends on the separation between the Weyl points $2Q$ and the magnetic field B , and it is not necessarily $\nu = 1$. Hence, the (3+1)d chiral anomaly, which can be faithfully understood from this Landau level physics, loses information about the microscopic filling of the original

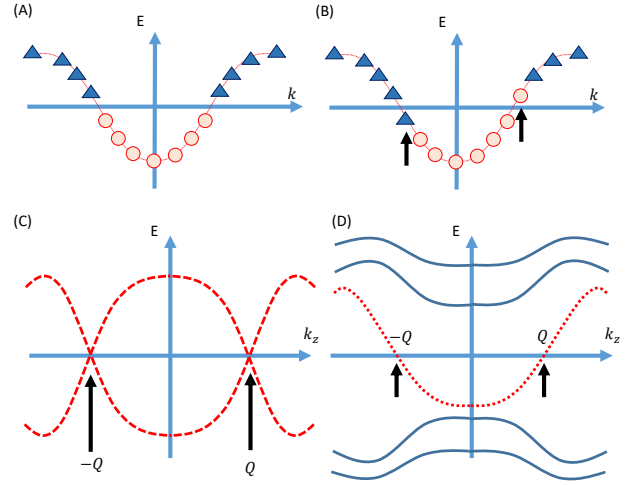


FIG. 1. Semi-classical illustrations of Anomaly. (A) (1+1)d metallic state. (B) On adiabatic insertion of the flux by 2π , one state at the left is pumped to the right. Equivalently, the momentum labeling each state is shifted by $\frac{2\pi}{L}$. (C) Spectrum of the Weyl semimetal in cubic lattice. Number of the state below the chemical potential $\mu = 0$ is precisely $L_x \times L_y \times L_z$, which is equivalent to the number of the electrons. (D) On applying the magnetic field, the band structure is changed.

model, which is crucial for the existence of a trivial insulator allowed by the LSMOH theorem.

We now compare this with the 1d chiral anomaly. In the 1d case, we apply only an electric field adiabatically to encode the pumping associated with the anomaly and this effectively is encoded through $k \rightarrow k + A(t)$, in which $A(t)$ is the time-dependent gauge field varying from 0 to $2\pi/L$. This process does not change the band structure but only the momentum is shifted. Hence we shift the state from left end to the right end as in (A) and (B) of Fig. 1 after inserting 2π flux. During the process, the filling is not changed and we can directly access the information of the filling and thus directly to the LSM theorem.

For the Dirac semimetals, e.g., distorted spinel, Na_3Bi and Cd_3As_2 ,^{28–30} where the accidental band crossings are protected by spatial symmetries, we can also find a similar $G-U(1)-U(1)$ chiral anomaly. This can be found in Appendix D.

VI. CONCLUSIONS AND OUTLOOKS

In this paper, we have compared the physics of the LSMOH theorem and the boundaries of strong SPT phases from the perspective of quantum anomalies. We have shown that the same form of the effective theory of the edge of the SPT state can be constructed within the lower-dimensional lattice models. Hence, the no-go theorem for the boundary of the SPTs is circumvented by encoding some on-site symmetry in the strong SPT as

the non-on-site translation symmetry in the corresponding lattice model.

From the connection, we further clarify the implications of the anomalies on the stabilities of the gaplessness in the two systems. Though the two systems have the identical low-energy theory with the effective symmetry, the anomalies are different in the two systems. The central distinctions between the edge of the SPT and the lattice systems are originated from the non-on-site-ness of the translation symmetry and also from the effective reduction of the translation symmetry.

By viewing the LSMOH theorem as the anomaly, we have expanded the LSMOH theorem to the case of the multi-charge and multi-species problems and constructed several exotic symmetric insulators.

Finally, we also briefly discussed the (3+1)d chiral anomaly and have shown that they provide local stability of topological semimetals.

There are several directions to extend the studies here.

An obvious direction is to include time-reversal symmetry and other spatial symmetries.^{17,52} There are several extensions of the LSMOH theorem, i.e., obstructions to construct a symmetric trivial insulator, by including time-reversal and several crystalline symmetries. It would be desirable to interpret these extensions in the language of anomalies.

Next, given the connection between SPT boundaries and the lattice systems, another interesting direction is to clarify, if any, the distinction between fermionic and bosonic systems in lattice models. Note that, on SPT boundaries, fermions and bosons are fundamentally different. This can be seen from the fact that the SPT classifications of interacting fermion systems assumes spin structures, which the bosons are not sensitive to. Note that, in the lattice systems, we know that spin-statistics connection is not required, and thus naively we do not expect to have much distinctions for the no-go conditions of trivial symmetric insulators between the fermions and the bosons in the lattice models. However, from the lights of the physics of SPT phases, it would be interesting how far the bosons and fermions are identical or different in the lattice systems.

ACKNOWLEDGMENTS

Authors thank helpful discussion with Joel Moore, Akira Furusaki, Masaki Oshikawa, Takahiro Morimoto, Eun-Gook Moon, Sunghin Lee, and Sheng-Jie Huang and acknowledge the financial supports from the National Science Foundation grant DMR-1455296 (S.R.), Brain Korea 21 PLUS Project of the Korea government (G.Y.C.) and NRF of Korea under Grant No. 2016R1A5A1008184 (G.Y.C.). G.Y.C. also acknowledges the support from Korea Institute for Advanced Study (KIAS) grant funded by the Korea government (MSIP).

Appendix A: Derivation of the two anomaly indices in (14)

In this appendix, we compute the two indices in (14) that characterize the 't Hooft anomaly of (1+1)d non-chiral fermions in any $(\text{spin}-\frac{1}{2})$ representation of $\text{Spin}^c(2) \times \mathbb{Z}_N$.

Let $\xi = \{\xi_I, \xi_{II}\}$ be a couple of (2+1)d massive Dirac fermions that have masses with opposite signs and transform in a representation R_{s_R, s_L}^q of $U(1)_Q \times \mathbb{Z}_N$, where q , the $U(1)$ charge, is an odd integer (as we are considering $\text{Spin}^c(d)$ groups) and $s_{R/L} \in \mathbb{Z}_N$ are charges associated to the \mathbb{Z}_N transformation

$$\xi_I \rightarrow e^{2\pi i s_R/N} \xi_I, \quad \xi_{II} \rightarrow e^{2\pi i s_L/N} \xi_{II}. \quad (\text{A1})$$

When ξ is defined on an open three-manifold, the (standard) boundary state of ξ would consist of a set of (1+1)d non-chiral fermions $\{\xi_I|_{\text{bdy}} := \psi_R, \xi_{II}|_{\text{bdy}} := \psi_L\}$ transforming in a representation of $U(1)_Q \times \mathbb{Z}_N$ with the same $U(1)$ charge and \mathbb{Z}_N transformation as the bulk fermions.

The boundary theory including both ψ_R and ψ_L is free of perturbative gravitational and $U(1)$ gauge anomalies. However, there might be a global anomaly, associated with the whole $\text{Spin}^c(2) \times \mathbb{Z}_N$ group, when the boundary theory is coupled to a background \mathbb{Z}_N gauge field (and a spin^c structure). Such an anomaly is characterized by the topological-quantum-field-theory (TQFT) partition function of the bulk fermions ξ – in the large-mass limit – on any closed three-dimensional spin^c manifold with a \mathbb{Z}_N gauge bundle – an element of the equivariant spin^c cobordism group $\Omega_{\text{Spin}^c}^3(B\mathbb{Z}_N)$. (To be more precise, the TQFT partition function is an element of the torsion subgroup of $\Omega_{\text{Spin}^c}^3(B\mathbb{Z}_N)$. In the case here, however, this torsion subgroup is exactly $\Omega_{\text{Spin}^c}^3(B\mathbb{Z}_N)$ itself.)

The partition function of ξ , when formulated on a closed three-manifold M endowed with a spin^c structure and a background \mathbb{Z}_N gauge field, is presented by a complex number $e^{2\pi i \eta_{\text{Spin}^c}(M, R_{s_R, s_L}^q)}$, which is a cobordism invariant.¹² Here $\eta_{\text{Spin}^c}(M, R_{s_R, s_L}^q)$ is the eta-invariant of the Dirac operator acting on ξ in the representation R_{s_R, s_L}^q . Since any such three-manifold M is bordant to a linear combination, with integral coefficients, of the lens space $L(3; 1, 1) := S^3/\tau(1, 1)$ with the natural spin^c structure (inherited from S^3) and \mathbb{Z}_N structure (by the identification $\pi_1(L(3; 1, 1)) = \mathbb{Z}_N$), where $\tau(1, 1) = e^{2\pi i/N} \oplus e^{2\pi i/N}$ is a representation of \mathbb{Z}_N in $U(2)$ and its action on the three-sphere $S^3 \subset \mathbb{C}^2$ is fixed-point free, and the “twisted lens space” $\tilde{L}(3; 1, 1)$, a three-dimensional lens space with the same \mathbb{Z}_N structure as $L(3; 1, 1)$ and with the spin^c structure twisted by the complex line bundle which corresponds to the representation $e^{2\pi i/N}$,^{41,42} we have

$$\eta_{\text{Spin}^c}(M, R_{s_R, s_L}^q) \equiv a\eta_{\text{Spin}^c}(L(3; 1, 1), R_{s_R, s_L}^q) + b\eta_{\text{Spin}^c}(\tilde{L}(3; 1, 1), R_{s_R, s_L}^q) \pmod{\mathbb{Z}} \quad (\text{A2})$$

for some integers a, b . Therefore, the 't Hooft anomaly $\alpha_{R_{s_R, s_L}^q}$ of the representation

R_{s_R, s_L}^q of $U(1)_Q \times \mathbb{Z}_N$ can be represented by the two indices $(\eta_{\text{Spin}^c}(L(3; 1, 1), R_{s_R, s_L}^q) \bmod \mathbb{Z}, \eta_{\text{Spin}^c}(\tilde{L}(3; 1, 1), R_{s_R, s_L}^q) \bmod \mathbb{Z})$.

The value of the eta-invariant on the lens space modulo integers can be evaluated using the Todd polynomials $Td_m(x)$, where m is a nonnegative integer.⁴² (For $m = 0, 1, 2$, $Td_0(x) = 1$, $Td_1(x) = \frac{1}{2} \sum_i x_i$, $Td_2(x) = \frac{1}{12} [\sum_{i < j} x_i x_j + (\sum_i x_i)^2]$.) Defining an auxiliary function $Td_k(u; x) := \sum_{m+n=k} u^m Td_n(x)/m!$, we have⁴²

$$\begin{aligned} & \eta_{\text{Spin}^c}(L(3; 1, 1), R_{s_R, s_L}^q) \\ &= -\frac{1}{N} [Td_2(s_R + q - 1; N, 1, 1) - Td_2(q - 1; N, 1, 1)] \\ &+ \frac{1}{N} [Td_2(s_L + q - 1; N, 1, 1) - Td_2(q - 1; N, 1, 1)] \bmod \mathbb{Z} \\ &\equiv -\frac{1}{2N} [s_R^2 - s_L^2 + (N + 2q)(s_R - s_L)] \bmod \mathbb{Z} \end{aligned} \quad (\text{A3})$$

and

$$\begin{aligned} & \eta_{\text{Spin}^c}(\tilde{L}(3; 1, 1), R_{s_R, s_L}^q) \\ &\equiv -\frac{1}{N} [Td_2(s_R + 2q - 1; N, 1, 1) - Td_2(2q - 1; N, 1, 1)] \\ &+ \frac{1}{N} [Td_2(s_L + 2q - 1; N, 1, 1) - Td_2(2q - 1; N, 1, 1)] \bmod \mathbb{Z} \\ &\equiv -\frac{1}{2N} [s_R^2 - s_L^2 + (N + 4q)(s_R - s_L)] \bmod \mathbb{Z}. \end{aligned} \quad (\text{A4})$$

One can further simplify these two indices by a group isomorphism – as $e^{2\pi i \eta_{\text{Spin}^c}}$ for all representations of $U(1)_Q \times \mathbb{Z}_N$ form a subgroup of $\Omega_{\text{Spin}^c}^3(B\mathbb{Z}_N)$ – as

$$\alpha_{R_{s_R, s_L}^q} \cong \left(\frac{s_R^2 - s_L^2}{\epsilon_N \cdot N} \bmod \mathbb{Z}, \frac{(s_R - s_L)q}{N} \bmod \mathbb{Z} \right), \quad (\text{A5})$$

where $\epsilon_N = 1$ for odd N and $\epsilon_N = 2$ for even N .

It is straightforward to generalize the above result for multiple-flavors Dirac fermions in a generic representation $R = \oplus_a R_{s_{R,a}, s_{L,a}}^{q_a}$:

$$\begin{aligned} \alpha_R \cong & \left(\sum_a \frac{s_{R,a}^2 - s_{L,a}^2}{\epsilon_N \cdot N} \bmod \mathbb{Z}, \right. \\ & \left. \sum_a \frac{(s_{R,a} - s_{L,a})q_a}{N} \bmod \mathbb{Z} \right). \end{aligned} \quad (\text{A6})$$

Expressed in terms of the fillings $\nu_a = (k_{F,R}^a - k_{F,L}^a)/2\pi = (s_{R,a} - s_{L,a})/N$, the two indices in (A6) become

$$\begin{aligned} & \sum_a \nu_a \cdot \frac{s_{R,a} + s_{L,a}}{\epsilon_N} \bmod \mathbb{Z}, \\ & \sum_a \nu_a q_a \bmod \mathbb{Z}. \end{aligned} \quad (\text{A7})$$

Appendix B: Chiral anomaly from bosonization

Let us again start from the low-energy Hamiltonian

$$H = \int dx \Psi^\dagger(x) (-i\partial_x) \sigma_z \Psi(x),$$

where $\Psi(x) = (\psi_\uparrow(x), \psi_\downarrow(x))^T$. We first need to implement the twisted boundary condition by $U(1)_Q$ on the circular edge $x \sim x + L$ as

$$\Psi(x) = e^{-i\Phi} \Psi(x + L). \quad (\text{B1})$$

We call the resulting “ground state” in the presence of this boundary condition as $|\Phi\rangle$.

A convenient way to construct and study $|\Phi\rangle$ is to use (abelian) bosonization. By bosonization, we represent the fermionic operators as $\psi_\uparrow \sim e^{i\phi_\uparrow}$ and $\psi_\downarrow \sim e^{i\phi_\downarrow}$ with the following commutators

$$[\phi_{\sigma'}(x'), \partial_x \phi_\sigma(x)] = 2\pi i \cdot \text{sgn}(\sigma) \delta_{\sigma, \sigma'} \delta(x - x'), \quad (\text{B2})$$

where $\sigma, \sigma' = \uparrow, \downarrow$ and $\text{sgn}(\uparrow) = +1$, $\text{sgn}(\downarrow) = -1$. Correspondingly, the densities of the ψ_\uparrow and ψ_\downarrow fermions are given by $\rho_\uparrow(x) = \frac{1}{2\pi} \partial_x \phi_\uparrow(x)$ and $\rho_\downarrow(x) = -\frac{1}{2\pi} \partial_x \phi_\downarrow(x)$. The conserved charge and the momentum can be constructed as $Q = Q_\uparrow + Q_\downarrow$, $P = \nu\pi(Q_\uparrow - Q_\downarrow)$, where $Q_\uparrow = \int \rho_\uparrow(x) dx$ and $Q_\downarrow = \int \rho_\downarrow(x) dx$.

The ground state $|\Phi\rangle$ in the presence of the twisted boundary condition $\Psi(x) = e^{-i\Phi} \Psi(x + L)$ obeys

$$(\psi_\sigma(x) - e^{-i\Phi} \psi_\sigma(x + L)) |\Phi\rangle = 0. \quad (\text{B3})$$

By the standard operator-state correspondence in CFT,^{53–55} we can represent such state by

$$\begin{aligned} |\Phi\rangle &= \lim_{\tau \rightarrow -\infty} V_\Phi(\tau) |0\rangle, \\ V_\Phi(\tau) &\sim e^{i\frac{\Phi}{2\pi}(\phi_\uparrow(\tau) - \phi_\downarrow(\tau))}, \end{aligned} \quad (\text{B4})$$

where $|0\rangle$ is the ground state of the untwisted sector. Now the (relative) quantum number carried by $|\Phi\rangle$ can be directly read off from the operator V_{ϕ_Q} because $[Q, V_\Phi] |0\rangle = Q V_\Phi |0\rangle = Q |\Phi\rangle$, where we have used $Q |0\rangle = 0$. On the other hand, $[Q, V_\Phi] = 0$ from the direct computation of the commutator. Hence, $|\Phi\rangle$ does not carry any charge. On the other hand, one verifies $[P, V_\Phi] = \Phi \nu$ and $|\Phi\rangle$ carries the momentum $P = \Phi \nu$. Hence, when $\Phi = 2\pi$, the state has momentum $2\pi \nu$ relative to the untwisted sector $\Phi = 0$. This is consistent with the field theory calculations in Sec. III A 1.

Appendix C: Vanishing \mathbb{Z}_N anomaly in Perturbatively-gappable Electronic Systems

Here we prove that trivial electronic systems, which can be perturbatively gapped without breaking the translation and charge $U(1)$ symmetries, must have vanishing \mathbb{Z}_N anomaly.

Imagine that we have N_f species of electrons $a = 1, 2, \dots, N_f$ at filling $\nu_a = \frac{p_a}{n}$ with $p_a \in \mathbb{Z}$ such that $\sum_{a=1}^{N_f} p_a = z \cdot n$ with $z, n \in \mathbb{Z}$ (so that the chiral anomaly vanishes). Furthermore, by some fine-tuning, translation symmetry is realized as \mathbb{Z}_N symmetry, so that

$$k_{R,a} = 2\pi \frac{s_{R,a}}{N}, \quad k_{L,a} = 2\pi \frac{s_{L,a}}{N}, \quad (\text{C1})$$

with $s_{R,a}$ and $s_{L,a}$ taking their values in \mathbb{Z} .

Now when this system can be gapped perturbatively, i.e., in the quadratic level, we must have the backscattering term which respects the translation symmetry. Hence, we can order the momentum in the following way

$$k_{L,1} = k_{R,2}, \quad k_{L,2} = k_{R,3}, \quad \dots, \quad k_{R,1} = k_{L,N_f}. \quad (\text{C2})$$

With this, we can now show that

$$\sum_a \nu_a \frac{(s_{R,a} + s_{L,a})}{\epsilon_N} = 0 \quad \text{mod } \mathbb{Z} \quad (\text{C3})$$

To see this, we note that the momenta are labeled as following

$$\begin{aligned} s_{R,1} &= \bar{s}, & s_{L,1} &= \bar{s} + \frac{p_1}{n}N, \\ s_{R,2} &= \bar{s} + \frac{p_1}{n}N, & s_{L,2} &= \bar{s} + \frac{p_1 + p_2}{n}N, \\ \dots & & \dots & \\ s_{R,N_f} &= \bar{s} + \frac{\sum_{a=1}^{N_f-1} p_a}{n}N, & s_{L,N_f} &= \bar{s} + \frac{\sum_{a=1}^{N_f} p_a}{n}N \end{aligned} \quad (\text{C4})$$

with $p_a \frac{N}{n} \in \mathbb{Z}$ for all a (to keep the translation symmetry as \mathbb{Z}_N) and $\bar{s} \in \mathbb{Z}$.

Now the \mathbb{Z}_N anomaly in this system is

$$\frac{1}{\epsilon_N} (2\bar{s}N_f + zN + 2zNN_f - 2 \sum_{a=1}^{N_f} ap_a \frac{N}{n}) \in \mathbb{Z}, \quad (\text{C5})$$

where we have used $p_a \frac{N}{n} \in \mathbb{Z}$ and so $2 \sum_{a=1}^{N_f} ap_a \frac{N}{n} \in 2\mathbb{Z}$. Thus it has vanishing \mathbb{Z}_N anomaly.

Hence the electronic system which can be perturbatively gapped (equivalently, which can be gapped by the quadratic terms) has a vanishing chiral and \mathbb{Z}_N anomaly.

Appendix D: Dirac Semimetals and Chiral Anomaly

Here we show that the Dirac semimetals^{28–30} have the G - $U(1)$ - $U(1)$ chiral anomaly, which is a manifestation of the local stability given by the spatial symmetries.

Here we note a few points about the spatial symmetry G . In the Dirac semimetal, we distinguish the two symmetry groups: H to realize the relativistic Dirac spectrum in a lattice model, and G for prohibiting the relativistic mass terms to the Dirac spectrum. When seen

from the low-energy theory, some elements in H may be superfluous and are not required for the stability. In general, G is a subgroup of H .

Typically, H must contain (i) inversion and (ii) time-reversal symmetries to guarantee the four-fold degeneracy at the band crossings of Dirac fermions. Furthermore, they accompany (symmorphic or non-symmorphic) rotational symmetries. Otherwise, the dispersions may be gapped or deformed away from the relativistic Dirac spectrum, e.g., line-nodal spectrum. However, as soon as we get to the relativistic spectrum and concentrate only on the relativistic mass-gap deformation, some of the symmetries in H is not necessary for stabilizing the Dirac semimetal. Hence, G can be smaller than H .

For the known materials of Dirac semimetals, we can show that G can be generated by only a few orientation-preserving space groups inside H , and consider G - $U(1)$ - $U(1)$ anomaly.

1. Dirac semimetal: disorted spinel

It has a single Dirac point at the zone boundary. It is an accidental band crossing, not related to the Lieb-Schultz-Mattis theorem. The low-energy Hamiltonian in the chiral basis is given as

$$H = \int d^3k \Psi_{\mathbf{k}}^\dagger \tau^z \boldsymbol{\sigma} \cdot \mathbf{k} \Psi_{\mathbf{k}} \quad (\text{D1})$$

with two-fold rotation C_2 in xy -plane, inversion P , and time-reversal symmetry \mathcal{T} . They are given as following:

$$C_2 = \tau_z \sigma_z, \quad P = \tau_y, \quad \mathcal{T} = i\sigma_y \tau_z \mathcal{K}. \quad (\text{D2})$$

To keep the relativistic Dirac spectrum, all the three symmetries are required. However, within the relativistic theory, $C_2 \propto \tau^z$ is enough to remove the relativistic mass terms. Obviously, it is captured by the C_2 - $U(1)$ - $U(1)$ chiral anomaly.

$$\delta S = \int d^4x \frac{1}{16\pi} \epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho} \quad (\text{D3})$$

We may extend the symmetry group G to be generated by C_2 and P , which will be isomorphic to D_8 .

2. Dirac semimetal: Na₃Bi and Cd₃As₂

They have two Dirac points on the k_z axis, symmetric under the rotations. They are “accidental band crossings”, not related to the LSM theorem. The low-energy Hamiltonian in the chiral basis is given as

$$H = \int d^3k \Psi_{\mathbf{k}}^\dagger \mu^0 \otimes \sigma^z (\boldsymbol{\tau} \cdot \mathbf{k}) \Psi_{\mathbf{k}} \quad (\text{D4})$$

in which μ^a is the Pauli matrix acting on the “valley” index. Symmetries are: inversion P , time-reversal symmetry \mathcal{T} , and 3-fold rotation for Na₃Bi (4-fold rotation

for Cd_3As_2). They are given by

$$\mathcal{T} = \mu^x \sigma^z i \tau^y \mathcal{K}, \quad P = \mu^y \sigma^y. \quad (\text{D5})$$

Translation along z -direction is given by

$$T_z = \exp(iQ\mu^z) \quad (\text{D6})$$

where $(0, 0, \pm Q)$ are the positions of the Dirac points.

The symmorphic 3-fold rotation C_3 for Na_3Bi is

$$C_3 = \exp\left(i\frac{\pi}{3}\sigma_z \otimes \mu^z\right) \otimes \exp\left(i\frac{2\pi}{3}\tau^z\right) \quad (\text{D7})$$

The symmorphic 4-fold rotation C_4 for Cd_3As_2 is

$$C_4 = \mu^z \otimes \sigma^z \otimes \tau^z \exp\left(-i\frac{\pi}{4}\tau^z\right). \quad (\text{D8})$$

The “stability” statements involve rotation, inversion and time-reversal. In particular, inversion and time-reversal are invoked to guarantee the four-fold degeneracy at the zero energy (not about the gapless-ness).

- *Anomaly*: Now the relativistic gapless-ness is guaranteed if we impose T_z and C_n . However, T_z and C_n are not anomalous in g - $U(1)$ - $U(1)$ diagram in which g is generated by composing T_z and C_n .

To see the anomaly structure carefully, we introduce the $U(1)$ valley gauge field a_μ such that $\mu^z = +1$ fermion carries the charge- Q and $\mu^z = -1$ fermion carries the charge- $(-Q)$, i.e., the covariant derivative of the fermions is $D_\mu = \partial_\mu - iA_\mu - iQ\mu^z a_\mu$. Now it is straightforward to compute the triangle diagram in the presence of the field strength of A_μ and a_μ , i.e., C_n - A - a , e.g., for C_3 case is

$$\mathcal{L} = \frac{2\pi/3}{16\pi^2} \times Q \times \epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} f_{\lambda\rho}, \quad (\text{D9})$$

where f is the field strength of a_μ . Note that a_μ is the “gauge field” by gauging on-site version of the translation. Hence, this anomaly can be thought of as C_3 -“ T_z ”- $U(1)$, where “ T_z ” is the on-site version of the translation.

* The first two authors contributed equally to the work.

- ¹ E. Lieb, T. Schultz, and D. Mattis, *Annals of Physics* **16**, 407 (1961).
- ² M. Oshikawa, *Physical review letters* **84**, 1535 (2000).
- ³ M. B. Hastings, *Physical review b* **69**, 104431 (2004).
- ⁴ X. Chen, Z.-C. Gu, and X.-G. Wen, *Physical review b* **83**, 035107 (2011).
- ⁵ M. Z. Hasan and C. L. Kane, *Rev. Mod. Phys.* **82**, 3045 (2010).
- ⁶ X.-L. Qi and S.-C. Zhang, *Rev. Mod. Phys.* **83**, 1057 (2011).
- ⁷ T. Senthil, *Annual Review of Condensed Matter Physics* **6**, 299 (2015).
- ⁸ G. Y. Cho, J. C. Teo, and S. Ryu, *Physical Review B* **89**, 235103 (2014).
- ⁹ C.-T. Hsieh, O. M. Sule, G. Y. Cho, S. Ryu, and R. G. Leigh, *Physical Review B* **90**, 165134 (2014).
- ¹⁰ C.-T. Hsieh, G. Y. Cho, and S. Ryu, *Physical Review B* **93**, 075135 (2016).
- ¹¹ G. Y. Cho, C.-T. Hsieh, T. Morimoto, and S. Ryu, *Physical Review B* **91**, 195142 (2015).
- ¹² E. Witten, *arXiv preprint arXiv:1508.04715* (2015).
- ¹³ X. Chen, F. J. Burnell, A. Vishwanath, and L. Fidkowski, *Physical Review X* **5**, 041013 (2015).
- ¹⁴ S. Ryu, J. E. Moore, and A. W. Ludwig, *Physical Review B* **85**, 045104 (2012).
- ¹⁵ A. Kapustin and R. Thorngren, *ArXiv e-prints* (2014), *arXiv:1403.0617 [hep-th]*.
- ¹⁶ S. C. Furuya and M. Oshikawa, *Physical Review Letters* **118**, 021601 (2017).
- ¹⁷ H. C. Po, H. Watanabe, C.-M. Jian, and M. P. Zaletel, *arXiv preprint arXiv:1703.06882* (2017).
- ¹⁸ H. Song, S.-J. Huang, L. Fu, and M. Hermele, *Physical Review X* **7**, 011020 (2017).
- ¹⁹ M. Cheng, M. Zaletel, M. Barkeshli, A. Vishwanath, and P. Bonderson, *Physical Review X* **6**, 041068 (2016).

- ²⁰ R. Thorngren and D. V. Else, *arXiv preprint arXiv:1612.00846* (2016).
- ²¹ D. T. Son, *Physical Review X* **5**, 031027 (2015).
- ²² S. D. Geraedts, M. P. Zaletel, R. S. Mong, M. A. Metlitski, A. Vishwanath, and O. I. Motrunich, *Science* **352**, 197 (2016).
- ²³ C. Wang and T. Senthil, *Physical Review X* **5**, 041031 (2015).
- ²⁴ M. A. Metlitski and A. Vishwanath, *arXiv preprint arXiv:1505.05142* (2015).
- ²⁵ D. F. Mross, J. Alicea, and O. I. Motrunich, *Phys. Rev. Lett.* **117**, 016802 (2016).
- ²⁶ C. Wang and T. Senthil, *Physical Review B* **93**, 085110 (2016).
- ²⁷ G. 'tHooft, “Recent development in gauge theories, eds. g. 'tHooft et. al.” (1980).
- ²⁸ S. M. Young, S. Zaheer, J. C. Teo, C. L. Kane, E. J. Mele, and A. M. Rappe, *Physical review letters* **108**, 140405 (2012).
- ²⁹ B.-J. Yang and N. Nagaosa, *Nature communications* **5** (2014).
- ³⁰ B.-J. Yang, T. Morimoto, and A. Furusaki, *Physical Review B* **92**, 165120 (2015).
- ³¹ A. Burkov and L. Balents, *Physical review letters* **107**, 127205 (2011).
- ³² K.-Y. Yang, Y.-M. Lu, and Y. Ran, *Physical Review B* **84**, 075129 (2011).
- ³³ G. Y. Cho, *arXiv preprint arXiv:1110.1939* (2011).
- ³⁴ A. Kapustin, R. Thorngren, A. Turzillo, and Z. Wang, *ArXiv e-prints* (2014), *arXiv:1406.7329 [cond-mat.str-el]*.
- ³⁵ H. B. Nielsen and M. Ninomiya, *Physics Letters B* **130**, 389 (1983).
- ³⁶ S. L. Adler, *Physical Review* **177**, 2426 (1969).
- ³⁷ J. S. Bell and R. Jackiw, *Il Nuovo Cimento A* (1965-1970) **60**, 47 (1969).
- ³⁸ C.-M. Jian, Z. Bi, and C. Xu, *arXiv preprint*

- arXiv:1705.00012 (2017).
- ³⁹ C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 226801 (2005).
- ⁴⁰ Here $\Omega_{\text{Spin}^c}^3(B\mathbb{Z}_N) = \Omega_{\text{Spin}^c}^{3,\text{tors}}(B\mathbb{Z}_N)$, since the three-dimensional spin^c cobordism group $\Omega_{\text{Spin}^c}^3(pt)$ is trivial.
- ⁴¹ A. Bahri and P. Gilkey, Pacific J. Math. **128**, 1 (1987).
- ⁴² P. Gilkey, *The Geometry of Spherical Space Form Groups* (World Scientific, 1989).
- ⁴³ T. Araki, T. Kobayashi, J. Kubo, S. Ramos-Sánchez, M. Ratz, and P. K. Vaudrevange, Nuclear physics B **805**, 124 (2008).
- ⁴⁴ H. Ishimori, T. Kobayashi, H. Ohki, Y. Shimizu, H. Okada, and M. Tanimoto, Progress of Theoretical Physics Supplement **183**, 1 (2010).
- ⁴⁵ To be more precise, any two-dimensional spin^c manifold M , together with its given spin^c structure, is bordant to a multiple of \mathbb{CP}^1 with a (canonical) spin^c structure.
- ⁴⁶ K. Günter, T. Stöferle, H. Moritz, M. Köhl, and T. Esslinger, Physical review letters **96**, 180402 (2006).
- ⁴⁷ M. Lewenstein, L. Santos, M. Baranov, and H. Fehrmann, Physical review letters **92**, 050401 (2004).
- ⁴⁸ J. Wang, X.-G. Wen, and E. Witten, arXiv preprint arXiv:1705.06728 (2017).
- ⁴⁹ M. Creutz, Reviews of Modern Physics **73**, 119 (2001).
- ⁵⁰ T. Araki, Progress of Theoretical Physics **117**, 1119 (2007).
- ⁵¹ S. Parameswaran, T. Grover, D. Abanin, D. Pesin, and A. Vishwanath, Physical Review X **4**, 031035 (2014).
- ⁵² H. Watanabe, H. C. Po, A. Vishwanath, and M. Zaletel, Proceedings of the National Academy of Sciences **112**, 14551 (2015).
- ⁵³ J. G. Polchinski, *String theory* (Cambridge university press, 2003).
- ⁵⁴ P. H. Ginsparg, arXiv preprint hep-th/9108028 **63** (1988).
- ⁵⁵ P. Mathieu and D. Senechal, *Conformal field theory* (New York: Springer, 1997).