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We discuss several bosonic topological phases in (3+1) dimensions enriched by a global $\mathbb{Z}_2$ symmetry, and gauging the $\mathbb{Z}_2$ symmetry. More specifically, following the spirit of the bulk-boundary correspondence, expected to hold in topological phases of matter in general, we consider boundary (surface) field theories and their orbifold. From the surface partition functions, we extract the modular $\mathcal{S}$ and $\mathcal{T}$ matrices and compare them with $(2+1)$d toplogical phase after dimensional reduction. As a specific example, we discuss topologically ordered phases in $(3+1)$ dimensions described by the BF topological quantum field theories, with abelian exchange statistics between point-like and loop-like quasiparticles. Once the $\mathbb{Z}_2$ charge conjugation symmetry is gauged, the $\mathbb{Z}_2$ flux becomes non-abelian excitation. The gauged topological phases we are considering here belong to the quantum double model with non-abelian group in $(3+1)$ dimensions.

CONTENTS

I. INTRODUCTION

Symmetry and topology intertwine in many phases of matter. Two prime examples are symmetry-protected topological (SPT) phases and symmetry-enriched topological (SET) phases. In SPT phases, symmetries play a crucial role, in that they are sharply distinct from trivial phases (i.e., product states) only in the presence of symmetries.[1–5] On the other hand, topologically ordered phases can be enriched by global symmetries. The seminal example is the charge fractionalization of Laughlin quasiparticles in the fractional quantum Hall effect. The relevant global symmetry in this case is $U(1)$ associated to particle number conservation. Varieties of SET phases have been discussed in the literature.[6–8]

A global symmetry in SPT or SET phases can be promoted to a local symmetry through gauging. Such "gauging" (in the bulk) or "orbifolding" (on the edge) is a useful tool to understand parent SPT and SET phases in $(2 + 1)$ dimensions.[9–12] Gauging an SPT phase leads to a topological phase; thus the SPT phase is the parent of this topological phase. The topological class of the parent SPT phase can be inferred from, and, in fact, has one-to-one correspondence with the topological order (i.e., properties of anyons) which arises by the gauging procedure. We can also gauge a global (discrete) symmetry $\mathcal{G}$ in an SET phase, thereby giving rise to a new topological phase. In particular, if the global symmetry acts on emergent excitations (anyons) by permuting the anyon labels[13] in the parent SET, then gauging these symmetries will lead to more interesting "twist liquids".[14–17]

The focus of this paper is to generalize the above idea to $(3 + 1)$ dimensions and discuss gauging/orbifolding global symmetries in (3+1)-dimensional bosonic topologically ordered phases. Starting from $(3 + 1)$-dimensional topologically ordered phases with Abelian topological order, which are described by (multi-component) BF theo-

ries, we gauge a global $\mathbb{Z}_2$ symmetry and show that the new topological phase is non-Abelian and related to a non-Abelian quantum double model.

Previous work has constructed line defects in $(3 + 1)$-dimensional topological phases.[18–20] These semi-classical defects are analogous to twist defects in $(2+1)$d topological phases and can twist the anyon labels.[21–28] In this paper, we will fully gauge the discrete global symmetries so that these topological defects will become fully deconfined loop-like excitations. Our method for determining the resulting $(3 + 1)$d topological phase relies on the bulk-boundary correspondence. We work with the $(2 + 1)$-dimensional surface theories of the bulk $(3 + 1)$d BF theories, and consider the $\mathbb{Z}_2$ orbifold thereof. As in the context of $(1 + 1)$-dimensional conformal field theories (CFT), orbifolding a CFT amounts to considering the partition functions in the presence of twisted boundary conditions.[29–32] By putting the surface theory on the spacetime torus T^3 with proper boundary conditions in time and two spatial directions, we derive the transformation properties of the twisted partition function under the mapping class group (the large diffeomorphisms) of T^3.[33,34] This procedure allows us to read off the modular $\mathcal{S}$ and $\mathcal{T}$ matrices, which encode the properties of anyons in the gauged surface theory. This, in turn, allows us to deduce the gauged bulk theory.

The rest of the paper is organized as follows: In Sec. II A, we briefly review orbifold CFTs in $(1 + 1)$d. We further elucidate this with a simple example in the $\mathbb{Z}_K$ quantum double model in Sec. II B. In Sec. III, we consider three different topological phases in $(3 + 1)$d and study the corresponding orbifolded surface theories. In Sec. V, we discuss the bulk non-Abelian topological phase that results from gauging $\mathbb{Z}_2$ symmetry and make a connection with the surface orbifold theory. We summarize our results in Sec. VI.

II. ORBIFOLDING THE $(1 + 1)$D BOUNDARY THEORY

A. Summary of orbifold CFT in $(1 + 1)$d

Topologically ordered phases in $(2 + 1)$d are often equipped (or enriched) with some global discrete symmetries. These symmetries may permute anyon labels, but leave the $\mathcal{S}$ and $\mathcal{T}$ matrices invariant. Gauging the discrete symmetry $\mathcal{G}$ will give rise to more interesting topological phases.

A way to understand this new topological phase is through studying the boundary theory. For a topological phase, the gapless boundary state can be described by a rational CFT $\mathcal{C}$. Gauging the global symmetry in the bulk corresponds to orbifolding the edge CFT by the symmetry. By orbifolding by symmetry $\mathcal{G}$, we project out the symmetry non-invariant states and, simultaneously, add some twist sectors to the Hilbert space. For the details of orbifold CFTs, see Ref. 29–32. The orbifold

CFT $\mathcal{C}/\mathcal{G}$ can be understood by calculating the character for each primary field. The characters are the partition functions under the symmetry projection,

$$\chi_n^h = \mathrm{Tr}_h(\mathcal{P}_n e^{-tH}) \qquad (1)$$

where $\mathcal{P}_n$ is a projection operator, and $h \in \mathcal{G}$ defines the twist in the spatial direction. Here, the projection operator $\mathcal{P}_n$ depends on the set of phase factors $\{\omega\}$ known as the discrete torsion phases. E.g., if $\mathcal{G}$ is an abelian $\mathbb{Z}_N$ symmetry, the projection operator is simply written as $\mathcal{P}_n = \sum_{k=0}^{N-1} \omega^{-nk} g^k / N$ where g is the generator of $\mathbb{Z}_N$ and ω is the k-th root of unity. More generally, if $\mathcal{G}$ is a non-Abelian group, we require g belongs to the centralizer subgroup of h containing all g that commutes with h, i.e., $gh = hg$. The character χ_n^h can be understood as a linear combination of partition functions with fixed twist h in the spatial direction and all allowed twists g in the time direction. This construction of the orbifold characters is a reminiscence of the minimal entangled states (MESs) in the corresponding $(2+1)$d bulk topologically ordered phase. MESs form a convenient basis of the degenerate ground states defined on the spatial 2-torus.

The characters form a complete basis for a reprsentation of the group of modular transformations. The modular $\mathcal{S}$ and $\mathcal{T}$ matrices for the orbifold CFT, which are identical to the $\mathcal{S}$ and $\mathcal{T}$ matrices for the bulk topological order, thus encode topological information of the topological phase.

One interesting example is the toric code model. It has a global duality symmetry which exchanges the charge e and flux m, while leaving $\psi = e \times m$ invariant. Gauging the $\mathbb{Z}_2$ duality symmetry leads to a non-Abelian topological phase. The calculation of the characters for the edge orbifold CFT shows that $\mathcal{S}$ and $\mathcal{T}$ matrices are equivalent to that for the Ising $\times$ $\overline{\text{Ising}}$ CFT and suggests that the bulk topological phase has nine quasi-particles, including the Ising-like anyon excitation.[35,36]

B. Orbifolding $\mathbb{Z}_2$ symmetry on the boundary of the $\mathbb{Z}_K$ quantum double model

Starting from an abelian topological phase, if we gauge a global symmetry G, the resulting non-abelian topological phase includes the following three types of excitations:[15] First, for the anyon in $\mathbf{a}$ which is invariant under symmetry G, they remain as excitations in the twist liquid. They can also couple with gauge charge and form a composite particle. These excitations are abelian excitations with zero gauge flux and called Type (1) excitations. Next, for anyons $\{\mathbf{a}_i\}$ which are not invariant under symmetry, they need to group together to form a superselection sector so that $\mathbf{a}_1 + \mathbf{a}_2 + \ldots$ is gauge invariant. Such Type (2) excitations are non-abelian quasi-particles with zero gauge flux. Finally, Type (3) excitations are the most interesting ones: They carry non-trivial gauge flux and correspond to non-abelian twist defects before gauging. From the boundary field theory point of view,

for these three types of excitations, we can construct the corresponding characters on the boundary.

We now work out an example explicitly in order to illuminate the general strategy: $\mathbb{Z}_2$ charge-conjugation symmetry in the $D(\mathbb{Z}_K)$ quantum double model. We here consider the case when K is an odd number. The detail for this method can be found in Ref. 37.

Our use of a gapless edge CFT deserves further comment. For a single-component Chern-Simons theory, there is a chiral gapless mode on the boundary, which is stable and cannot be gapped out. This is because the single component Chern-Simons theory is anomalous and requires a gapless edge mode on the boundary to compensate the anomaly in the bulk. Meanwhile, for the non-chiral $D(\mathbb{Z}_K)$ quantum double model without anomaly, the boundary CFT can be gapped out if we do not impose any symmetry. Although the gapless CFT is not stable, it does encode topological data in the bulk. Thus, we can use the "fine-tuned" gapless CFT as an intermediate step to study the bulk topological phase via the bulk boundary correspondence. This is also true for the $(3+1)d$ topological phase.

The $D(\mathbb{Z}_K)$ quantum double model has two fundamental quasi-particle excitations, e and m. All the quasi-particle excitations can be written as $e^a m^b$, where $0 \leq a, b < K$. e and m are self-bosons, and have non-trivial mutual braiding statistics with braiding phase $e^{2\pi i/K}$.

The $D(\mathbb{Z}_K)$ quantum double model has a global $\mathbb{Z}_2$ charge-conjugation symmetry which exchanges $e^a m^b$ and $e^{K-a} m^{K-b}$. If K is an odd number, there is no quasi-particle which is invariant under the charge-conjugation.

Once the $\mathbb{Z}_2$ symmetry is gauged, i.e., the global charge-conjugation symmetry is promoted to a local gauge symmetry, there is a $\mathbb{Z}_2$ bosonic charge j which satisfies the fusion rule $j \times j = 1$. On the other hand, the $\mathbb{Z}_2$ flux σ is a non-Abelian quasi-particle. σ can combine with $\mathbb{Z}_2$ charge to form the flux-charge composite quasi-particle, $\tau = \sigma \times j$. The original abelian anyons $e^a m^b$ will group together to form gauge invariant superselection sector $e^a m^b + e^{K-a} m^{K-b}$ with quantum dimension equal to 2.

Let us now take a look at the gauging procedure from the boundary field theory point of view. The relevant boundary theory is described by the following Lagrangian density

$$\mathcal{L} = \frac{1}{4\pi} (\partial_t \phi^I \mathbf{K}_{IJ} \partial_x \phi^J + \partial_x \phi^I V_{IJ} \partial_x \phi^J), \quad (2)$$

where (t, x) are the spacetime coordinates of the boundary; $\mathbf{K} = K\sigma_x$, $\vec{\Phi} = (\phi^1, \phi^2)$ is a two-component boson, and V is a symmetric and positive definite matrix that accounts for the interaction on the edge and is non-universal. This model has K^2 characters and there is a choice of interaction V_{IJ} for which they take the form

$$B_{a,b}^K(\tau) = \frac{1}{|\eta(\tau)|^2} \sum_{s,t \in \mathbb{Z}} q^{\frac{1}{4K}(Ks+a+Kt+b)^2} \bar{q}^{\frac{1}{4K}(Kt+a-Ks-b)^2} \quad (3)$$

character χ	d_χ	h_χ	$\mathcal{N}$
$\chi_I = \frac{1}{2}B_{0,0}^K + \frac{1}{2}\left|\frac{\eta}{\theta_2}\right|$	1	0	1
$\chi_j = \frac{1}{2}B_{0,0}^K - \frac{1}{2}\left|\frac{\eta}{\theta_2}\right|$	1	0	1
$\chi_{a,b} = \frac{1}{2}B_{a,b}^K + \frac{1}{2}B_{K-a,K-b}^K$	2	$\frac{ab}{K}$	$\frac{K^2-1}{2}$
$\chi_\sigma = \frac{1}{2}\left|\frac{\eta}{\theta_4}\right| + \frac{1}{2}\left|\frac{\eta}{\theta_3}\right|$	K	0	1
$\chi_\tau = \frac{1}{2}\left|\frac{\eta}{\theta_4}\right| - \frac{1}{2}\left|\frac{\eta}{\theta_3}\right|$	K	$\frac{1}{2}$	1

TABLE I. The quantum dimensions d_χ, conformal dimensions h_χ, and the number $\mathcal{N}$ of characters χ from orbifolding the charge-conjugate $\mathbb{Z}_2$ symmetry of Eq. (A1) when K is odd. The conformal dimensions h_χ are defined mod $\mathbb{Z}$. For $\chi_{a,b}$, if $a \neq b$, we require $0 \leq a < b \leq K$ here. If $a = b$, we require $a \leq (K-1)/2$.

where $a, b \in \mathbb{Z}$ mod K are the anyon labels, τ is the modular parameter of the spacetime torus, $q = \exp(2\pi i \tau)$, and $\eta(\tau)$ is the Dedekind eta function. The details of the calculation can be found in Ref. 38 and is also summarized in Appendix A.

The boundary theory Eq.(2) is invariant under the $\mathbb{Z}_2$ charge-conjugation symmetry

$$\phi^{1,2} \to -\phi^{1,2}. \quad (4)$$

Once orbifolded by the $\mathbb{Z}_2$ symmetry, $\phi^{1,2}$ can become $-\phi^{1,2}$ when the coordinates are taken around the time and spatial directions on the $(1+1)d$ torus. Therefore orbifolding introduces anti-periodic boundary conditions in the x and t directions. The partition function with twisted boundary condition is labelled by $Z^{\mu\nu}$, where $\mu, \nu = 0, \frac{1}{2}$ represents untwisted and twisted boundary condition in, respectively, the time and space directions. The twisted partition functions are given by[32]

$$Z^{\frac{1}{2},0} = \left|\frac{\eta}{\theta_2}\right|, \quad Z^{0,\frac{1}{2}} = \left|\frac{\eta}{\theta_4}\right|, \quad Z^{\frac{1}{2},\frac{1}{2}} = \left|\frac{\eta}{\theta_3}\right|, \quad (5)$$

where $\theta_{2,3,4}$ are Jacobi theta functions defined by

$$\theta_2 = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2}, \quad \theta_3 = \sum_{n \in \mathbb{Z}} q^{\frac{n^2}{2}}, \quad \theta_4 = \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{n^2}{2}}. \quad (6)$$

One can readily check that under modular transformations,

$$Z^{\frac{1}{2},0} \overset{\mathcal{S}}{\longleftrightarrow} Z^{0,\frac{1}{2}}, \quad Z^{\frac{1}{2},\frac{1}{2}} \overset{\mathcal{S}}{\longleftrightarrow} Z^{\frac{1}{2},\frac{1}{2}},$$
$$Z^{\frac{1}{2},0} \overset{\mathcal{T}}{\longleftrightarrow} Z^{\frac{1}{2},0}, \quad Z^{0,\frac{1}{2}} \overset{\mathcal{T}}{\longleftrightarrow} Z^{\frac{1}{2},\frac{1}{2}}. \quad (7)$$

We use these twist blocks and the original characters $B_{a,b}^K$ to construct the characters for the orbifold CFT and the result is shown in Table I. χ_I is the character for the vacuum and χ_j is the character for $\mathbb{Z}_2$ charge, $\chi_{a,b}$ is the superselection sector, χ_σ is the character for $\mathbb{Z}_2$ flux and χ_τ is the character for charge-flux composite. From the

table we can see that the quantum dimension for a $\mathbb{Z}_2$ flux σ is K, indicating that it is a non-Abelian quasi-particle. The $\mathcal{S}$ matrix is

$$
\mathcal{S} = \frac{1}{\mathcal{D}}
\begin{pmatrix}
1 & 1 & 2 & K & K \\
1 & 1 & 2 & -K & -K \\
2 & 2 & 4\cos[\frac{2\pi}{K}(ab'+ba')] & 0 & 0 \\
K & -K & 0 & K & -K \\
K & -K & 0 & -K & K
\end{pmatrix}
\tag{8}
$$

where the total quantum dimension is $\mathcal{D} = 2K$. The topologica spin $e^{2\pi i h}$ of anyonic excitations can be read off from the (eigenvalues of the) $\mathcal{T}$ matrix.

A trivial example is K = 1. Before gauging, the bulk has no topological order. After gauging the $\mathbb{Z}_2$ symmetry, the $\mathcal{S}$ and $\mathcal{T}$ matrix is the same as that for the toric code model, indicating the original phase is not a symmetry protected topological phase (SPT).[9] The twist fields σ, τ have quantum dimension equal to one, and correspond to the Abelian $\mathbb{Z}_2$ flux in the bulk. For K $\geq$ 3, the gauged system has non-Abelian topological order in the bulk. This non-abelian topological order can be described by the $D(\mathcal{D}_K)$ quantum double model, where $\mathcal{D}_K$ is the dihedral group of order 2K.[16]

III. GAUGING $\mathbb{Z}_2$ CHARGE-CONJUGATION SYMMETRY IN (3+1)D $\mathbb{Z}_K$ GAUGE THEORIES

In this section, we discuss gauging discrete symmetries in (3+1)d topologically ordered phases. A specific example we consider in this section is the topological $\mathbb{Z}_K$ gauge theory. It is the long wavelength limit of the deconfined phase of the $\mathbb{Z}_K$ gauge theory. A convenient description of the topological $\mathbb{Z}_K$ gauge theory is given by the single-component $(3+1)$d BF theory, which is defined by the following action

$$
S_{bulk} = \frac{K}{2\pi} \int_{\mathcal{M}} b \wedge da \tag{9}
$$

where $\mathcal{M}$ is a (3+1)d spacetime manifold; a and b are a one- and two-form, respectively; K is an integral parameter ("level"). This action describes the simplest topological phase in $(3+1)$d; its fundamental excitations are a particle excitation e and a loop excitation m. They have non-trivial mutual braiding statistics with the braiding phase $e^{2\pi i/K}$.

The BF theory (9) has $\mathbb{Z}_2$ charge-conjugation symmetry:

$$
b \to -b, \quad a \to -a, \tag{10}
$$

and our goal in this section is to gauge this symmetry. We will show that the resulting gauge theory has a non-Abelian topological order.

Similar to the (2+1)d case, after gauging $\mathbb{Z}_2$ symmetry, there will be a $\mathbb{Z}_2$ charge j which is a bosonic particle, and a $\mathbb{Z}_2$ flux which, in (3+1)d, is a vortex-line excitation.

The original excitations $e^a m^b$ in the BF theory are not $\mathbb{Z}_2$ symmetry invariant and will be grouped together. We will denote them simply as $e^a m^b + e^{K-a} m^{K-b}$.

Our approach here is to generalize orbifolds of $(1+1)$d edge theories discussed in Sec. II, and discuss orbifolds of $(2+1)$ gapless boundary theories. In particular, we define a set of quantities which are analogous to the characters defined for $(1+1)$d CFTs. These characters are constructed by applying a projection operator on the partition function, which is equivalent to imposing a twisted boundary condition in the time direction. The characters form a complete basis under $SL(3,\mathbb{Z})$, the mapping class group of T^3, and the $\mathcal{T}$ matrix takes a diagonal form. By studying the characters on the boundary, we can extract information about the non-Abelian bulk topological order.

A. The BF surface theory

Our starting point is thus the boundary of the BF theory at level K, which can be described by the following Lagrangian density[34,39]

$$
\mathcal{L} = \frac{K}{2\pi}(\epsilon_{ij}\partial_i\zeta_j)(\partial_t\varphi) \\
- \frac{1}{2\lambda_1}(\epsilon_{ij}\partial_i\zeta_j)^2 - \frac{1}{2\lambda_2}G^{ij}\partial_i\varphi\partial_j\varphi, \tag{11}
$$

where $i,j = x,y$, φ is a scalar, and ζ_i is a one-form field (the temporal component of ζ is gauge-fixed to zero for convenience). We fix the coupling constant λ_1 and λ_2 according to

$$
\frac{(2\pi)^2}{K^2\lambda_1\lambda_2} = 1, \quad \lambda_1 = \frac{1}{K}, \quad \lambda_2 = \frac{(2\pi)^2}{K}. \tag{12}
$$

for convenience. The boson field φ is compact and satisfy $\varphi \equiv \varphi + 2\pi$. Hence, physical observables are exponentials

$$
\exp[im\varphi(t,x,y)], \quad m \in \mathbb{Z}. \tag{13}
$$

The winding number of φ is quantized in the absence of bulk quasiparticles, according to

$$
\oint dx^i \partial_i\varphi = 2\pi N_i, \quad N_i \in \mathbb{Z}, \tag{14}
$$

where $i = 1,2$ and i is not summed on the right hand side. On the other hand, the gauge field ζ_i is compact, meaning that physical observables are Wilson loops,

$$
\exp\left(im\int_C dx^i \zeta_i(t,x,y)\right), \quad m \in \mathbb{Z}, \tag{15}
$$

where C is a closed loop on $\partial\Sigma = T^2$. (Since the different components of ζ_i commute with each other, path-ordering is unecessary.) The flux associated to ζ_i is quantized, in the absence of bulk quasiparticles, according to

$$
\int dxdy\, \epsilon_{ij}\partial_i\zeta_j = 2\pi N_0 \tag{16}
$$

where N_0 is an integer.

The surface theory (11) is put on a flat spacetime three-torus T^3, and G^{ij} represents the spatial part of the metric. For the properties of the the flat T^3, and our parameterization of the metric, see Appendix C. Our flat three-torus is parameterized by six real parameters, $R_{0,1,2}$ and α, β, γ. For example, $R_{0,1,2}$ are the periods in t, x, and y directions, respectively. The mapping class group of T^3, $SL(3,\mathbb{Z})$, is generated by two transformations which we call U_1 and U_2. Any $SL(3,\mathbb{Z})$ transformation can be written as $U_1^{n_1} U_2^{n_2} U_1^{n_3} \cdots$.[40] We further decompose U_1 as $U_1 = U_1' M$, where U_1' corresponds to a 90° rotation in the $\tau - x$ plane and M is a 90° rotation in the $x - y$ plane. The transformation U_1' can be viewed as the modular S transformation on the $\tau - x$ plane. Similarly, U_2 can be introduced as the modular T^{-1} transformation on the $\tau - x$ plane. Combined together, U_1' and U_2, which transform the modular parameter $\tau = \alpha + iR_0/R_1$ as $\tau \to -1/\tau$ and $\tau \to \tau - 1$, respectively, generate the $SL(2,\mathbb{Z})$ subgroup of $SL(3,\mathbb{Z})$ group. Further combined with M, they generate the whole $SL(3,\mathbb{Z})$ group. In the following, we denote $U_1' M$ by $\mathcal{S}$ and U_2 by $\mathcal{T}^{-1}$.

The surface theory (11) can be studied in the presence of the following twisted boundary conditions:

$$\varphi(t, x + 2\pi R_1, y) = \varphi(t, x, y) + 2\pi \left(N_1 + \frac{n_1}{K} \right),$$

$$\varphi(t, x, y + 2\pi R_2) = \varphi(t, x, y) + 2\pi \left(N_2 + \frac{n_2}{K} \right),$$

$$\int dx dy \, \epsilon_{ij} \partial_i \zeta_j = 2\pi \left(N_0 + \frac{n_0}{K} \right), \tag{17}$$

where $N_{0,1,2} \in \mathbb{Z}$, and $n_{0,1,2} = 0, \ldots, K-1$. We denote the corresponding partition functions as

$$Z^{n_0 n_1 n_2} = Z_{zero}^{n_0, n_1, n_2} Z_{osc} \tag{18}$$

where $Z_{zero}^{n_0, n_1, n_2}$ and Z_{osc} will be explained later. From the bulk point of view, these (twisted) boundary conditions correspond to insertion of Wilson loop and Wilson surfaces, i.e., bulk excitations. The bulk-boundary correspondence implies that, by studying the partition functions of the surface theories in the presence of these boundary conditions, in particular, their transformation law under $SL(3,\mathbb{Z})$, one can extract properties of bulk quasiparticles.[41]. The details for the calculation of surface partition functions can be found in Ref. 34, where the bulk-(gapless) boundary correspondence is also discussed. Here we directly write down the partition functions for the surface. The zero mode part is

$$Z_{zero}^{n_0 n_1 n_2} = \sum_{N_{0,1,2} \in \mathbb{Z}} \exp \left\{ - \frac{\pi K^2 \tau_2}{2r^2 R_2} \tilde{N}_0^2 - 2r^2 \pi R_2 \tau_2 \left[\tilde{N}_1 + \beta \tilde{N}_2 \right]^2 - \frac{2r^2 \pi R_0 R_1}{R_2} \tilde{N}_2^2 \right.$$

$$\left. + 2\pi i \tau_1 K \tilde{N}_0 \left[\tilde{N}_1 + \beta \tilde{N}_2 \right] + 2\pi i \gamma K \tilde{N}_0 \tilde{N}_2 \right\}, \tag{19}$$

where $2r^2 = K$, and we have introduced the notation

$$\tilde{N}_\mu := N_\mu + n_\mu / K. \tag{20}$$

For the oscillator part,

$$Z_{osc} = \left| \frac{1}{\eta(\tau)} \right|^2 \prod_{s_2 \in \mathbb{Z}^+} \Theta_{[\beta s_2, \gamma s_2]}^{-1} \left(\tau, \frac{R_1}{R_2} s_2 \right), \tag{21}$$

where $\Theta_{[a,b]}(\tau, m)$ is the massive theta function (see Appendix B). The total partition function for each sector is $Z^{n_0 n_1 n_2} = Z_{zero}^{n_0 n_1 n_2} Z_{osc}$. The modular $\mathcal{S}$ and $\mathcal{T}$ matrices are given by

$$\mathcal{S}_{n_i, n_i'} = \frac{1}{K} \delta_{n_1, n_2'} e^{-\frac{2\pi i}{K}(n_0' n_2 - n_0 n_1')},$$

$$\mathcal{T}_{n_i, n_i'} = \delta_{n_0, n_0'} \delta_{n_1, n_1'} \delta_{n_2, n_2'} e^{\frac{2\pi i}{K} n_0 n_1}. \tag{22}$$

B. Gauging $\mathbb{Z}_2$ symmetry in the surface theory

In terms of the surface theory (11), the $\mathbb{Z}_2$ charge conjugation symmetry is implemented as

$$\varphi \to -\varphi, \quad \zeta \to -\zeta. \tag{23}$$

We now gauge the $\mathbb{Z}_2$ symmetry. We consider the cases of odd and even level K separately.

First, we include the twisted sectors obtained by twisting boundary conditions in t, x, and y directions. The partition functions for these sectors are denoted by $V^{\mu, \nu, \lambda}$ where $\mu, \nu, \lambda = 0$ $(1/2)$ represents the untwisted (twisted) boundary condition, for t, x and y directions, respectively. The partition functions with twisted boundary condition in x, y, t directions can be readily computed

character χ	d_χ	h_χ	$\mathcal{N}$
$\chi_I = \frac{1}{2}Z_{0,0,0} + \frac{1}{2}V^{\frac{1}{2},0,0}$	1	0	1
$\chi_j = \frac{1}{2}Z_{0,0,0} - \frac{1}{2}V^{\frac{1}{2},0,0}$	1	0	1
$\chi_{a,b,c} = \frac{1}{2}Z_{a,b,c} + \frac{1}{2}Z_{K-a,K-b,K-c}$	2	$\frac{ab}{K}$	$\frac{K^3-1}{2}$
$\chi_{\sigma_x} = \frac{1}{2}V^{0,\frac{1}{2},0} + \frac{1}{2}V^{\frac{1}{2},\frac{1}{2},0}$	K	0	1
$\chi_{\tau_x} = \frac{1}{2}V^{0,\frac{1}{2},0} - \frac{1}{2}V^{\frac{1}{2},\frac{1}{2},0}$	K	$\frac{1}{2}$	1
$\chi_{\sigma_y} = \frac{1}{2}V^{0,0,\frac{1}{2}} + \frac{1}{2}V^{\frac{1}{2},0,\frac{1}{2}}$	K	0	1
$\chi_{\tau_y} = \frac{1}{2}V^{0,0,\frac{1}{2}} - \frac{1}{2}V^{\frac{1}{2},0,\frac{1}{2}}$	K	0	1
$\chi_{\sigma_{xy}} = \frac{1}{2}V^{0,\frac{1}{2},\frac{1}{2}} + \frac{1}{2}V^{\frac{1}{2},\frac{1}{2},\frac{1}{2}}$	K	0	1
$\chi_{\tau_{xy}} = \frac{1}{2}V^{0,\frac{1}{2},\frac{1}{2}} - \frac{1}{2}V^{\frac{1}{2},\frac{1}{2},\frac{1}{2}}$	K	$\frac{1}{2}$	1

TABLE II. The characters for the single-component BF theory after gauging the charge-conjugate $\mathbb{Z}_2$ symmetry when K is odd: The quantum dimensions d_χ, spin statistics $\theta_\chi = e^{2\pi i h_\chi}$, and the number $\mathcal{N}$ of characters χ (h_χ are defined only modulo $\mathbb{Z}$.)

and are given, respectively, by

$$V^{0,\frac{1}{2},0} = \left| \frac{\eta(\tau)}{\theta_4(\tau)} \right| \prod_{s_2>0} \Theta^{-1}_{[\frac{1}{2}+\beta s_2,\gamma s_2]}\left(\tau, \frac{R_1}{R_2}s_2\right),$$

$$V^{0,0,\frac{1}{2}} = \prod_{s_2>0} \Theta^{-1}_{[\beta s_2-\frac{1}{2},\gamma(s_2-\frac{1}{2})]}\left(\tau, \frac{R_1}{R_2}(s_2-\frac{1}{2})\right),$$

$$V^{\frac{1}{2},0,0} = \left| \frac{\eta(\tau)}{\theta_2(\tau)} \right| \prod_{s_2>0} \Theta^{-1}_{[\beta s_2,\gamma s_2+\frac{1}{2}]}\left(\tau, \frac{R_1}{R_2}s_2\right), \quad (24)$$

(s_2 is an integer). All the other twisted sectors can be obtained by considering modular transformations:

$$V^{[\mu],[\nu],[\lambda]} \xrightarrow{U_1'} V^{[\nu],[\mu],[\lambda]},$$
$$V^{[\mu],[\nu],[\lambda]} \xrightarrow{M} V^{[\mu],[\lambda],[\nu]},$$
$$V^{[\mu],[\nu],[\lambda]} \xrightarrow{U_2} V^{[\mu+\nu],[\nu],[\lambda]}, \quad (25)$$

where $[a] = a \mod \mathbb{Z}$. In total, there are 7 sectors with twisted boundary conditions. (To obtain this result, we note several properties of the massive theta functions $\Theta_{[a.b]}(\tau, m)$, listed in Appendix B.)

We use the twisted sectors $V^{\mu,\nu,\lambda}$ and $Z_{a,b,c}$ to build up the characters of the gauged $(2+1)$d surface theory. The result is listed in Table II. Here, the subscript h of χ_h denotes the type of bulk excitations. As mentioned previously, the characters are constructed by inserting projection operators in the partition function. This is similar to the construction of the minimal entangled state (MES) defined for $(3+1)$d topological phases.[41,42] By summing up the multiplicity $\mathcal{N}$ for χ_h, we find that there are $(K^3+15)/2$ characters in total, the same as the bulk ground state degeneracy on T^3. In Table II, d_χ denotes the quantum dimension for the character and can be obtained from the first column (row) of U_1' matrix ($\mathcal{S}$ matrix for T^2). Under the transformation $\mathcal{T}$, the character will pick up a phase $\exp(2\pi i h_\chi)$, where h_χ encodes information related to $(3+1)$d analogue of topological spins.

As in the $(1+1)$d edge theory, here we have the vacuum I, $\mathbb{Z}_2$ boson j and superselection sector a, b, c. σ_x corresponds to the bulk $\mathbb{Z}_2$ flux excitation which leaves a twist in the x direction on the boundary. The character χ_{σ_x} includes partition functions with twisted boundary condition in the x direction. τ_x can be understood as combining σ_x with $\mathbb{Z}_2$ charge j and therefore is a flux-charge composite excitation. χ_{τ_x} is also linear combination of partition functions with twist boundary in x direction. χ_{σ_y} and χ_{τ_y} are the characters with twisted boundary condition in y direction. $\chi_{\sigma_{xy}}$ is twisted in both the x and y directions.

We now consider the U_1' matrix which, after dimensional reduction, is the $\mathcal{S}$ matrix for a $(1+1)$d CFT. The U_1' matrix encodes braiding information about bulk excitations,

$$U_1' = \frac{1}{2K} \begin{pmatrix} 1 & 1 & 2 & K & K & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & -K & -K & 0 & 0 & 0 & 0 \\ 2 & 2 & 4\cos[\frac{2\pi}{K}(ab'+ba')] & 0 & 0 & 0 & 0 & 0 & 0 \\ K & -K & 0 & K & -K & 0 & 0 & 0 & 0 \\ K & -K & 0 & -K & K & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & K & K & K & K \\ 0 & 0 & 0 & 0 & 0 & K & K & -K & -K \\ 0 & 0 & 0 & 0 & 0 & K & -K & K & -K \\ 0 & 0 & 0 & 0 & 0 & K & -K & -K & K \end{pmatrix} \quad (26)$$

From the first column (row) in the above matrix, we can extract the "quantum dimension" of the bulk excitation d_χ shown in Table II. When K = 1, the characters for the twist sectors have $d_\chi = 1$. In this case, it is easy to verify that U_1' matrix and h_χ match up with that for the ordinary $(3+1)$d $\mathbb{Z}_2$ gauge theory (the $(3+1)$d toric code model). When K > 1, the characters for the twist sectors have $d_\chi > 1$, suggesting that they are non-Abelian excitations. We will discuss these non-Abelian braiding statistics later in Sec. V.

We can similarly calculate the characters when K is even. Unlike in the case of K odd, there are several excitations in the original BF theory that are invariant under $\mathbb{Z}_2$ symmetry operation. They will remain in the gauged topological phase and can couple with the $\mathbb{Z}_2$ boson to form composite excitations. They will also provide species labels for σ and τ; the characters are shown in Table III. There are $K^3/2 + 60$ characters in total, which describes the ground state degeneracy in the bulk topological phase on T^3.

The new $(3+1)$d topological phase which we have obtained by gauging the $\mathbb{Z}_2$ symmetry is the $D(\mathcal{D}_K)$ quantum double model, where $\mathcal{D}_K$ is the dihedral group of order 2K. The excitations for this quantum double model are labeled by $(\mathcal{C}, \rho)$, where $\mathcal{C}$ denotes the conjugacy class and ρ denotes the irreducible representation of the normalizer group for each conjugacy class. When K is odd, there are in total $\frac{n+3}{2} + \frac{n-1}{2} \times n + 2 = \frac{n^2+7}{2}$ excitations. When K is even, there are in total $(\frac{n}{2}+3) \times 2 + (\frac{n}{2}-1) \times n + 2 \times 4 = \frac{n^2+28}{2}$ excitations. These results are consistent with the calculation for the boundary orbifold theory. For instance, when K = 3, $\mathcal{D}_3$ is equivalent to the symmetric group S_3. There are eight excitations in the $D(S_3)$ quantum double model, which

character χ	d_χ	h_χ	$\mathcal{N}$
$\chi_I = \frac{1}{2}Z_{0,0,0} + V^{\frac{1}{2},0,0}$	1	0	1
$\chi_j = \frac{1}{2}Z_{0,0,0} - V^{\frac{1}{2},0,0}$	1	0	1
$\chi^i_{\frac{K}{2},0,0} = \frac{1}{2}Z_{\frac{K}{2},0,0}$	1	0	2
$\chi^i_{0,\frac{K}{2},0} = \frac{1}{2}Z_{0,\frac{K}{2},0}$	1	0	2
$\chi^i_{\frac{K}{2},\frac{K}{2},0} = \frac{1}{2}Z_{\frac{K}{2},\frac{K}{2},0}$	1	$\frac{K}{4}$	2
$\chi^i_{0,\frac{K}{2},\frac{K}{2}} = \frac{1}{2}Z_{0,\frac{K}{2},\frac{K}{2}}$	1	0	2
$\chi^i_{\frac{K}{2},0,\frac{K}{2}} = \frac{1}{2}Z_{\frac{K}{2},0,\frac{K}{2}}$	1	0	2
$\chi^i_{0,0,\frac{K}{2}} = \frac{1}{2}Z_{0,0,\frac{K}{2}}$	1	0	2
$\chi^i_{\frac{K}{2},\frac{K}{2},\frac{K}{2}} = \frac{1}{2}Z_{\frac{K}{2},\frac{K}{2},\frac{K}{2}}$	1	$\frac{K}{4}$	2
$\chi_{a,b,c} = \frac{1}{2}Z_{a,b,c} + \frac{1}{2}Z_{K-a,K-b,K-c}$	2	$\frac{ab}{K}$	$\frac{K^3-8}{2}$
$\chi^m_{\sigma_x} = \frac{1}{2}V^{0,\frac{1}{2},0} + \frac{1}{2}V^{\frac{1}{2},\frac{1}{2},0}$	$\frac{K}{2}$	0	8
$\chi^m_{\tau_x} = \frac{1}{2}V^{0,\frac{1}{2},0} - \frac{1}{2}V^{\frac{1}{2},\frac{1}{2},0}$	$\frac{K}{2}$	$\frac{1}{2}$	8
$\chi^m_{\sigma_y} = \frac{1}{2}V^{0,0,\frac{1}{2}} + \frac{1}{2}V^{\frac{1}{2},0,\frac{1}{2}}$	$\frac{K}{2}$	0	8
$\chi^m_{\tau_y} = \frac{1}{2}V^{0,0,\frac{1}{2}} - \frac{1}{2}V^{\frac{1}{2},0,\frac{1}{2}}$	$\frac{K}{2}$	0	8
$\chi^m_{\sigma_{xy}} = \frac{1}{2}V^{0,\frac{1}{2},\frac{1}{2}} + \frac{1}{2}V^{\frac{1}{2},\frac{1}{2},\frac{1}{2}}$	$\frac{K}{2}$	0	8
$\chi^m_{\tau_{xy}} = \frac{1}{2}V^{0,\frac{1}{2},\frac{1}{2}} - \frac{1}{2}V^{\frac{1}{2},\frac{1}{2},\frac{1}{2}}$	$\frac{K}{2}$	$\frac{1}{2}$	8

TABLE III. The characters for the single-component BF theory after gauging the charge-conjugate $\mathbb{Z}_2$ symmetry when K is even: The quantum dimensions d_χ, spin statistics $\theta_\chi = e^{2\pi i h_\chi}$, and the number $\mathcal{N}$ of characters χ (h_χ are defined only modulo $\mathbb{Z}$.)

agree with the above counting. Apart from the vacuum sector and two charge (particle) excitations, all the other five excitations are flux or flux-charge composite excitations with non-Abelian fusion rules. These fusion rules have been discussed in Ref. [41] and will not be analyzed further here.

More generically, for all (3+1)d Abelian topological phases considered in this paper, after gauging the $\mathbb{Z}_2$ symmetry, the new topological phases can always be described by the quantum double model with group $\widetilde{\mathcal{G}} = \mathcal{G} \rtimes \mathbb{Z}_2$, where $\mathcal{G}$ is the original abelian group. This is because the $\mathbb{Z}_2$ symmetry acts on both the charge and flux excitations independently and in the same way. If $\mathcal{G} = \mathbb{Z}_K$, i.e., the $\mathbb{Z}_K$ gauge theory, the gauged model is the $D(\mathcal{D}_K)$ quantum double model.

IV. GAUGING $\mathbb{Z}_2$ SYMMETRY IN (3+1)D $\mathbb{Z}_K \times \mathbb{Z}_K$ GAUGE THEORIES

In this section, we consider ordinary $\mathbb{Z}_K \times \mathbb{Z}_K$ topological gauge theory in (3+1) dimensions with only particle-loop braiding statistics. The excitations in this model can be denoted by $e_1^{a_1} m_1^{b_1} e_2^{a_2} m_2^{b_2}$, where e_1, m_1, e_2, m_2 are the fundamental excitations of this model. This model can be described by the two-component BF theory with both components at level K.

The topological $\mathbb{Z}_K \times \mathbb{Z}_K$ gauge theory has a $\mathbb{Z}_2$ sym-

metry which exchanges

$$e_1, \ m_1 \leftrightarrow e_2, \ m_2. \tag{27}$$

Gauging this symmetry will lead to a non-Abelian topological phase.

The surface theory of the topological $\mathbb{Z}_K \times \mathbb{Z}_K$ gauge theory can be described by taking two copies of (11). The surface partition function can then be written down as

$$W^{l_0,l_1,l_2}_{n_0,n_1,n_2} = Z_{n_0,n_1,n_2} Z_{l_0,l_1,l_2} \tag{28}$$

where $n_i, l_i \in \mathbb{Z}_K$ and $i = 0, 1, 2$.

After orbifolding the $\mathbb{Z}_2$ symmetry, the partition function with anti-periodic boundary condition in the t direction is

$$Y^{\frac{1}{2},0,0}_{n_0,n_1,n_2} = V^{\frac{1}{2},0,0} Z^{2K}_{2n_0,2n_1,2n_2} \tag{29}$$

where $V^{[\mu],[\nu],[\lambda]}$ is defined in the previous section and $Z^{2K}_{a,b,c}$ represents the surface partition function for $\mathbb{Z}_{2K}$ gauge theory model. The term $Z^{2K}_{2n_i}$ is obtained by identifying n_i and l_i in $Z_{n_0,n_1,n_2} Z_{l_0,l_1,l_2}$, which is imposed by the $\mathbb{Z}_2$ symmetry.

The other twisted partition functions $Y^{\mu,\nu,\lambda}_{n_0,n_1,n_2}$ can be obtained, starting from $Y^{\frac{1}{2},0,0}_{n_0,n_1,n_2}$, by applying modular transformations.

Schematically,

$$\begin{aligned}
Y^{\frac{1}{2},0,0}_{m_0,m_1,m_2} &\xrightarrow{U'_1} Y^{0,\frac{1}{2},0}_{n_0,n_1,n_2}, \\
Y^{0,\frac{1}{2},0}_{m_0,m_1,m_2} &\xrightarrow{U_2} Y^{\frac{1}{2},\frac{1}{2},0}_{n_0,n_1,n_2}, \\
Y^{0,\frac{1}{2},0}_{m_0,m_1,m_2} &\xrightarrow{M} Y^{0,0,\frac{1}{2}}_{n_0,n_1,n_2}, \\
Y^{\frac{1}{2},\frac{1}{2},0}_{m_0,m_1,m_2} &\xrightarrow{M} Y^{\frac{1}{2},0,\frac{1}{2}}_{n_0,n_1,n_2}, \\
Y^{\frac{1}{2},0,\frac{1}{2}}_{m_0,m_1,m_2} &\xrightarrow{U'_1} Y^{0,\frac{1}{2},\frac{1}{2}}_{n_0,n_1,n_2}, \\
Y^{0,\frac{1}{2},\frac{1}{2}}_{m_0,m_1,m_2} &\xrightarrow{U'_1} Y^{\frac{1}{2},\frac{1}{2},\frac{1}{2}}_{n_0,n_1,n_2}.
\end{aligned} \tag{30}$$

The explicit form of other sectors are obtained by requiring them to be invariant (up to a phase) under U'_1 transformation. They are given by

$$Y^{0,\frac{1}{2},0}_{m_0,m_1,m_2} = V^{0,\frac{1}{2},0} \sum_{p,q} Z^{2K}_{m_0+pK,m_1+qK,2m_2},$$

$$\begin{aligned}
Y^{\frac{1}{2},\frac{1}{2},0}_{m_0,m_1,m_2} = V^{\frac{1}{2},\frac{1}{2},0} &\times \big[Z^{2K}_{m_0,m_1,2m_2} \\
&+ (-1)^{m_1} Z^{2K}_{m_0+K,m_1,2m_2} + (-1)^{m_0} Z^{2K}_{m_0,m_1+K,2m_2} \\
&+ (-1)^{m_0+m_1+K} Z^{2K}_{m_0+K,m_1+K,2m_2} \big],
\end{aligned}$$

$$Y^{0,0,\frac{1}{2}}_{m_0,m_1,m_2} = V^{0,0,\frac{1}{2}} \sum_{p,q} Z^{2K}_{m_0+pK,2m_1,m_2+qK},$$

$$Y^{\frac{1}{2},0,\frac{1}{2}}_{m_0,m_1,m_2} = V^{\frac{1}{2},0,\frac{1}{2}} \sum_{p,q} Z^{2K}_{m_0+pK,2m_1,m_2+qK},$$

$$Y^{0,\frac{1}{2},\frac{1}{2}}_{m_0,m_1,m_2} = V^{0,\frac{1}{2},\frac{1}{2}} \sum_{p,q} Z^{2K}_{m_0+pK,2m_1,m_2+qK},$$

$$Y^{\frac{1}{2},\frac{1}{2},\frac{1}{2}}_{m_0,m_1,m_2} = V^{\frac{1}{2},\frac{1}{2},\frac{1}{2}} \sum_{p,q} Z^{2K}_{m_0+pK,2m_1,m_2+qK}, \tag{31}$$

character χ	d_χ	h_χ	$\mathcal{N}$
$\chi^0_{n_0,n_1,n_2} = \frac{1}{2}W^{n_0,n_1,n_2}_{n_0,n_1,n_2} + \frac{1}{2}Y^{\frac{1}{2},0,0}_{n_0,n_1,n_2}$	1	$\frac{2n_0n_1}{K}$	K^3
$\chi^1_{n_0,n_1,n_2} = \frac{1}{2}W^{n_0,n_1,n_2}_{n_0,n_1,n_2} - \frac{1}{2}Y^{\frac{1}{2},0,0}_{n_0,n_1,n_2}$	1	$\frac{2n_0n_1}{K}$	K^3
$\chi^{l_0,l_1,l_2}_{n_0,n_1,n_2} = \frac{1}{2}W^{l_0,l_1,l_2}_{n_0,n_1,n_2} + \frac{1}{2}W^{n_0,n_1,n_2}_{l_0,l_1,l_2}$	2	$\frac{n_0n_1+l_0l_1}{K}$	$\frac{K^6-K^3}{2}$
$\chi^{m_0,m_1,m_2}_{\sigma_x} = \frac{1}{4}Y^{0,\frac{1}{2},0}_{m_0,m_1,m_2} + \frac{1}{4}Y^{\frac{1}{2},\frac{1}{2},0}_{m_0,m_1,m_2}$	K	$\frac{m_0m_1}{2K}$	K^3
$\chi^{m_0,m_1,m_2}_{\tau_x} = \frac{1}{4}Y^{0,\frac{1}{2},0}_{m_0,m_1,m_2} - \frac{1}{4}Y^{\frac{1}{2},\frac{1}{2},0}_{m_0,m_1,m_2}$	K	$\frac{K+m_0m_1}{2K}$	K^3
$\chi^{m_0,m_1,m_2}_{\sigma_y} = \frac{1}{4}Y^{0,0,\frac{1}{2}}_{m_0,m_1,m_2} + \frac{1}{4}Y^{\frac{1}{2},0,\frac{1}{2}}_{m_0,m_1,m_2}$	K	$\frac{m_0m_1}{K}$	K^3
$\chi^{m_0,m_1,m_2}_{\tau_y} = \frac{1}{4}Y^{0,0,\frac{1}{2}}_{m_0,m_1,m_2} - \frac{1}{4}Y^{\frac{1}{2},0,\frac{1}{2}}_{m_0,m_1,m_2}$	K	$\frac{m_0m_1}{K}$	K^3
$\chi^{m_0,m_1,m_2}_{\sigma_{xy}} = \frac{1}{4}Y^{0,\frac{1}{2},\frac{1}{2}}_{m_0,m_1,m_2} + \frac{1}{4}Y^{\frac{1}{2},\frac{1}{2},\frac{1}{2}}_{m_0,m_1,m_2}$	K	$\frac{m_0m_1}{K}$	K^3
$\chi^{m_0,m_1,m_2}_{\tau_{xy}} = \frac{1}{4}Y^{0,\frac{1}{2},\frac{1}{2}}_{m_0,m_1,m_2} - \frac{1}{4}Y^{\frac{1}{2},\frac{1}{2},\frac{1}{2}}_{m_0,m_1,m_2}$	K	$\frac{K+2m_0m_1}{2K}$	K^3

TABLE IV. The characters for the topological $\mathbb{Z}_K\times\mathbb{Z}_K$ theory in $(3+1)$d after gauging the $\mathbb{Z}_2$ symmetry: The quantum dimensions d_χ, spin statistics $\theta_\chi = e^{2\pi i h_\chi}$, and the number $\mathcal{N}$ of characters χ (h_χ are defined only modulo $\mathbb{Z}$.)

character χ	d_χ	h_χ	$\mathcal{N}$
$\chi^0_{n_0,n_1} = \frac{1}{2}W^{n_0,n_1}_{n_0,n_1} + \frac{1}{2}Y^{\frac{1}{2},0}_{n_0,n_1}$	1	$\frac{2n_0n_1}{K}$	K^2
$\chi^1_{n_0,n_1} = \frac{1}{2}W^{n_0,n_1}_{n_0,n_1} - \frac{1}{2}Y^{\frac{1}{2},0}_{n_0,n_1}$	1	$\frac{2n_0n_1}{K}$	K^2
$\chi^{l_0,l_1,l_2}_{n_0,n_1} = \frac{1}{2}W^{l_0,l_1}_{n_0,n_1} + \frac{1}{2}W^{n_0,n_1}_{l_0,l_1}$	2	$\frac{n_0n_1+l_0l_1}{K}$	$\frac{K^4-K^2}{2}$
$\chi^{m_0,m_1}_{\sigma} = \frac{1}{4}Y^{0,\frac{1}{2}}_{m_0,m_1} + \frac{1}{4}Y^{\frac{1}{2},\frac{1}{2}}_{m_0,m_1}$	K	$\frac{m_0m_1}{2K}$	K^2
$\chi^{m_0,m_1}_{\tau} = \frac{1}{4}Y^{0,\frac{1}{2}}_{m_0,m_1} - \frac{1}{4}Y^{\frac{1}{2},\frac{1}{2}}_{m_0,m_1}$	K	$\frac{1}{2}+\frac{m_0m_1}{2K}$	K^2

TABLE V. The quantum dimensions d_χ, spin statistics $\theta_\chi = e^{2\pi i h_\chi}$ and number $\mathcal{N}$ of characters χ from orbifolding $\mathbb{Z}_2$ symmetry of boundary of the $\mathbb{Z}_K\times\mathbb{Z}_K$ gauge theory in $(2+1)$d. This table matches up with Table IV after dimensional reduction.

where $p,q=0,1$ and $0\le m_i < K$. $W^{l_0l_1l_2}_{n_0n_2n_2}$ and $Y^{\nu\nu\lambda}_{m_0m_1m_2}$ can be properly combined to construct the characters. The result is summarized in Table IV. It is also instructive to consider the dimensional reduction. After dimensional reduction, the complete results for the characters on the $(1+1)$d edge are shown in Table V.

Finally, by noting the transformation properties of the characters under U'_1 listed in Appendix D, we read off the U'_1 matrix:

$$U'_1 = \frac{1}{\mathcal{D}}\begin{pmatrix} e^{\frac{2\pi i}{K}(2n_0n'_1+2n_1n'_0)} & e^{\frac{2\pi i}{K}(2n_0n'_1+2n_1n'_0)} & 2e^{\frac{2\pi i}{K}(n_0(n'_1+l'_1)+n_1(n'_0+l'_0))} & Ke^{\frac{2\pi i}{K}(n_0m'_1+n_1m'_0)} & Ke^{\frac{2\pi i}{K}(n_0m'_1+n_1m'_0)} \\ e^{\frac{2\pi i}{K}(2n_0n'_1+2n_1n'_0)} & e^{\frac{2\pi i}{K}(2n_0n'_1+2n_1n'_0)} & 2e^{\frac{2\pi i}{K}(n_0(n'_1+l'_1)+n_1(n'_0+l'_0))} & -Ke^{\frac{2\pi i}{K}(n_0m'_1+n_1m'_0)} & -Ke^{\frac{2\pi i}{K}(n_0m'_1+n_1m'_0)} \\ 2e^{\frac{2\pi i}{K}(n'_0(n_1+l_1)+n'_1(n_0+l_0))} & 2e^{\frac{2\pi i}{K}(n'_0(n_1+l_1)+n'_1(n_0+l_0))} & 2e^{\frac{2\pi i}{K}(n_0n'_1+n_1n'_0+l_0l'_1+l_1l'_0)} & 0 & 0 \\ Ke^{\frac{2\pi i}{K}(n'_0m_1+n'_1m_0)} & -Ke^{\frac{2\pi i}{K}(n'_0m_1+n'_1m_0)} & 0 & \frac{K}{2}Pe^{\frac{\pi i}{K}(m_0m'_1+m_1m'_0)} & -\frac{K}{2}Pe^{\frac{\pi i}{K}(m_0m'_1+m_1m'_0)} \\ Ke^{\frac{2\pi i}{K}(n'_0m_1+n'_1m_0)} & -Ke^{\frac{2\pi i}{K}(n'_0m_1+n'_1m_0)} & 0 & -\frac{K}{2}Pe^{\frac{\pi i}{K}(m_0m'_1+m_1m'_0)} & \frac{K}{2}PKe^{\frac{\pi i}{K}(m_0m'_1+m_1m'_0)} \end{pmatrix}$$

(32)

where $\mathcal{D}=2K^2$ and $P = \left[1+(-1)^{m_1+m'_1}+(-1)^{m_0+m'_0}(1+(-1)^{m_1+m'_1+K})\right]$.

A. Orbifolding phases with non-trivial three-loop braiding statistics

For the $\mathbb{Z}_K\times\mathbb{Z}_K$ twisted gauge theory with non-trivial three loop braiding statistics[42–44], we can also construct the surface partition function and calculate $\mathcal{S}$ and $\mathcal{T}$ matrices.[34] In this case, the quantum numbers n_0 and l_0 are shifted according to

$$n_0 \to \widetilde{n}_0 = n_0 + (l_1n_2 - l_2n_1)/K,$$
$$l_0 \to \widetilde{l}_0 = l_0 + (n_1l_2 - n_2l_1)/K. \tag{33}$$

This model also has the $\mathbb{Z}_2$ exchange symmetry, which switches n_i and l_i. Therefore we can gauge the $\mathbb{Z}_2$ symmetry. After orbifolding this symmetry on the surface, for the characters $\chi^0_{n_0,n_1,n_2}$ with $n_i = l_i$, they are still the same as for the untwisted $\mathbb{Z}_K\times\mathbb{Z}_K$ gauge theory. The character $\chi^{l_i}_{n_i}$ (in the third row of Table VI) is modified slightly, where $\widetilde{W}$ denotes the surface partition function for topological phase with three-loop braiding statistics defined in Ref. 34 and the quantum numbers n_0, l_0 are replaced by $\widetilde{n}_0$ and $\widetilde{l}_0$. The characters for twisted sectors are still the same as before. The complete result is shown in Table VI. The U'_1 matrix is very similar to the ordinary $\mathbb{Z}_K\times\mathbb{Z}_K$ gauge theory defined in Eq.(32) and the only difference is that for the braiding phase between $\chi^{l_i}_{n_i}$, n_0 and l_0 are replaced by $\tilde{n}_0$ and $\tilde{l}_0$.

V. PHYSICS IN THE BULK

In this section, we will study the bulk physics and discuss the non-Abelian braiding statistics of loop excitations.

a. *Twist defects in (2+1)-dimensional topological phases* Before we discuss the $(3+1)$d case, it is instructive to review briefly the $(2+1)$d case.[21–27] For the $D(\mathbb{Z}_K)$ quantum double model, we can introduce a twofold twist defect which exchanges Abelian excitations $e^a m^b$ and $e^{K-a}m^{K-b}$. The twist defect is a point-like defect with

character χ	d_χ	h_χ	$\mathcal{N}$
$\chi^0_{n_0,n_1,n_2} = \frac{1}{2} W^{n_0,n_1,n_2}_{n_0,n_1,n_2} + \frac{1}{2} Y^{\frac{1}{2},0,0}_{n_0,n_1,n_2}$	1	$\frac{2n_0 n_1}{K}$	K^3
$\chi^1_{n_0,n_1,n_2} = \frac{1}{2} W^{n_0,n_1,n_2}_{n_0,n_1,n_2} - \frac{1}{2} Y^{\frac{1}{2},0,0}_{n_0,n_1,n_2}$	1	$\frac{2n_0 n_1}{K}$	K^3
$\chi^{l_0,l_1,l_2}_{n_0,n_1,n_2} = \frac{1}{2} \widehat{W}^{l_0,l_1,l_2}_{\tilde{n}_0,n_1,n_2} + \frac{1}{2} \widehat{W}^{\tilde{n}_0,n_1,n_2}_{\tilde{l}_0,l_1,l_2}$	2	$\frac{\tilde{n}_0 n_1 + \tilde{l}_0 l_1}{K}$	$\frac{K^6-K^3}{2}$
$\chi^{m_0,m_1,m_2}_{\sigma_x} = \frac{1}{4} Y^{0,\frac{1}{2},0}_{m_0,m_1,m_2} + \frac{1}{4} Y^{\frac{1}{2},\frac{1}{2},0}_{m_0,m_1,m_2}$	K	$\frac{m_0 m_1}{2K}$	K^3
$\chi^{m_0,m_1,m_2}_{\tau_x} = \frac{1}{4} Y^{0,\frac{1}{2},0}_{m_0,m_1,m_2} - \frac{1}{4} Y^{\frac{1}{2},\frac{1}{2},0}_{m_0,m_1,m_2}$	K	$\frac{1}{2} + \frac{m_0 m_1}{2K}$	K^3
$\chi^{m_0,m_1,m_2}_{\sigma_y} = \frac{1}{4} Y^{0,0,\frac{1}{2}}_{m_0,m_1,m_2} + \frac{1}{4} Y^{\frac{1}{2},0,\frac{1}{2}}_{m_0,m_1,m_2}$	K	$\frac{m_0 m_1}{K}$	K^3
$\chi^{m_0,m_1,m_2}_{\tau_y} = \frac{1}{4} Y^{0,0,\frac{1}{2}}_{m_0,m_1,m_2} - \frac{1}{4} Y^{\frac{1}{2},0,\frac{1}{2}}_{m_0,m_1,m_2}$	K	$\frac{m_0 m_1}{K}$	K^3
$\chi^{m_0,m_1,m_2}_{\sigma_{xy}} = \frac{1}{4} Y^{0,\frac{1}{2},\frac{1}{2}}_{m_0,m_1,m_2} + \frac{1}{4} Y^{\frac{1}{2},\frac{1}{2},\frac{1}{2}}_{m_0,m_1,m_2}$	K	$\frac{m_0 m_1}{K}$	K^3
$\chi^{m_0,m_1,m_2}_{\tau_{xy}} = \frac{1}{4} Y^{0,\frac{1}{2},\frac{1}{2}}_{m_0,m_1,m_2} - \frac{1}{4} Y^{\frac{1}{2},\frac{1}{2},\frac{1}{2}}_{m_0,m_1,m_2}$	K	$\frac{1}{2} + \frac{m_0 m_1}{K}$	K^3

TABLE VI. The quantum dimensions d_χ, spin statistics $\theta_\chi = e^{2\pi i h_\chi}$ and number $\mathcal{N}$ of characters χ from orbifolding $\mathbb{Z}_2$ symmetry of surface of $\mathbb{Z}_K \times \mathbb{Z}_K$ twisted gauge theory in $(3+1)$d. Here, $\tilde{n}_0 = n_0 + (l_1 n_2 - l_2 n_1)/K$ and $\tilde{l}_0 = l_0 + (n_1 l_2 - n_2 l_1)/K$.

a branch cut emanating from it. In Fig. 1, we depict several pairs of twist defects, with branch cuts extending between them. When an anyon $e^a m^b$ ($e^{K-a} m^{K-b}$) is dragged around the twist defect, it is transformed to $e^{K-a} m^{K-b}$ ($e^a m^b$) when it passes through the branch cut. Once the $\mathbb{Z}_2$ charge-conjugation symmetry is gauged, twist defects, where were treated above as non-dynamical objects, are deconfined $\mathbb{Z}_2$ flux excitations.[14–16] These $\mathbb{Z}_2$ fluxes can leave twists on the boundary, which correspond to the twist fields σ or τ on the boundary discussed in the previous section. As we will see below, these $\mathbb{Z}_2$ fluxes are non-Abelian excitations.

The non-Abelian braiding statistics of these twist defects (before gauging) can be studied by calculating the fundamental unitary exchange operations, called B-symbols.[25–27] Each B-move represents a counterclockwise permutation of a pair of adjacent defects, which can result in a transformation of different ground states. The B-operations can be generated by a sequence of F and R-moves, and evaluated exactly once we specify the splitting space of the twist defects.[27] Here we show that twist defects are non-Abelian objects by directly deforming Wilson loop operators around a pair of twist defects shown in Fig. 1. Let us consider a system with $2N$ twist defects aligned on a line. The Hilbert space can be characterized by the non-contractible Wilson loop operators around the neighboring twist defects (Fig. 1 (a)). The Wilson loop operators are represented as the powers of fundamental e-loop and m-loop operators. It is important to note that the neighboring Wilson loop operators do not commute with each other due to the intersections (highlighted by brown dots). Therefore, the Hilbert space can be spanned by the eigenstates of Wilson loops $\{W_1, W_3, \ldots, W_{2N-1}\}$ where W_{2j-1} can be either e_{2j-1}
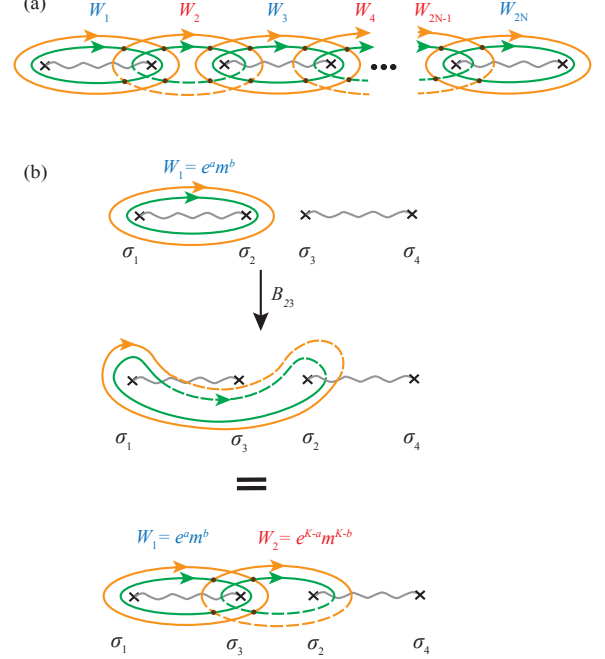

FIG. 1. (a) The Wilson loop operator defined on a pair of adjacent twist defects. The green loop is for the e-loop and orange loop is for m-loop. The solid orange and green lines denote the Wilson loop $e^a m^b$ and the dotted lines represent $e^{K-m} m^{K-b}$. (b) The braiding process between σ_2 and σ_3.

or m_{2j-1}:

$$e_{2j-1}|n_1,m_1,\ldots,n_j,m_j,\ldots,n_N,m_N\rangle$$
$$= e^{\frac{4\pi i m_j}{K}}|n_1,m_1,\ldots,n_j,m_j,\ldots,n_N,m_N\rangle,$$
$$m_{2j-1}|n_1,m_1,\ldots,n_j,m_j,\ldots,n_N,m_N\rangle$$
$$= e^{\frac{-4\pi i n_j}{K}}|n_1,m_1,\ldots,n_j,m_j,\ldots,n_N,m_N\rangle. \quad (34)$$

On the other hand, Wilson loops $\{W_2, W_4, \ldots, W_{2N}\}$ act on the basis states as

$$e_{2j}|n_1,m_1,\ldots,n_j,m_j,\ldots,n_N,m_N\rangle$$
$$= |n_1,m_1,\ldots,n_j+1,m_j,\ldots,n_N,m_N\rangle,$$
$$m_{2j}|n_1,m_1,\ldots,n_j,m_j,\ldots,n_N,m_N\rangle$$
$$= |n_1,m_1,\ldots,n_j,m_j+1,\ldots,n_N,m_N\rangle. \quad (35)$$

The non-commutative algebra between neighboring Wilson loop operators leads to non-Abelian braiding of twist defects.[26,45] In Fig. 1 (b), we consider four twist defects with $W_1 = e^a m^b, W_2 = \mathbb{I}$. If we exchange the twist defects σ_2 and σ_3, the new branch cuts are connecting (σ_1, σ_3) and (σ_2, σ_4). The Wilson loop W_1 deforms to $W_1 W_2$ up to some phase with $W_2 = e^{K-a} m^{K-b}$ and
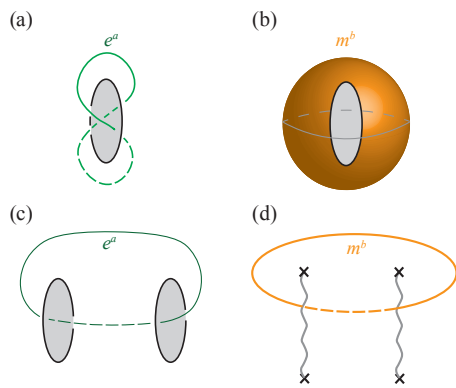
FIG. 2. Wilson loop or surface operators in the presence of a twist defect. (a): A Wilson loop operator threading through the branch sheet of a twist defect. A twist defect is a loop (line)-like object with a branch sheet (represented by a shaded region). The Wilson loop must intersect the branch surface twice in order to close: the charge e^a is conjugated to e^{K-a} at the first intersection and then back to e^a at the second intersection. (b): A Wilson surface (sphere) operator wrapping around a twist defect loop. The surface operator cannot shrink to nothing since there is a loop living inside. (c): A Wilson loop operator threading through the branch sheets of two twist defects. (d): A Wilson surface operator threading though the branch sheets of twist defects. Here, for pictorial simplicity, the dimensionally-reduced configuration is shown.

the original state $|0,0\rangle$ changes to $|a,b\rangle$. Fig. 1 (b) gives a physical picture of this non-abelian braiding process. More careful and rigorous study can be found by calculating the fundamental unitary exchange operations.[25–27]

b. *Twist defects in (3+1)-dimensional topological phases* Similarly, we can understand the non-Abelian braiding of twist defects in three spatial dimensions. As shown in Fig. 2, the twist defect has a loop configuration in three spatial dimensions and has a branch sheet attached to it. Unlike in $(2+1)$d, this defect loop does not need to pair up with another twist defect since it is equivalent to a pair of extended defect lines that wrap around a non-contractible circle on T^3. For any excitation $e^a m^b$, as it goes through the branch sheet, it will become $e^{K-a} m^{K-b}$.

For a system with a finite number of twist defects, the Hilbert space is labeled by the non-contractible Wilson loop and surface operators as shown in Fig. 2. In Figs. 2 (a) and (b), we show the Wilson loop and surface operators defined in the presence of a single defect loop, while in (c) and (d), the Wilson operators are defined for a pair of loops. By counting these Wilson operators, we find that the quantum dimension of a defect loop is K^2. Therefore the extended defect line has quantum dimension K and matches up with that for the twist field on the boundary theory in Table II.

Here we shall use (b) and (c) to construct a subspace of the total Hilbert space and show that these loop excitations have non-Abelian braiding statistics. As shown in
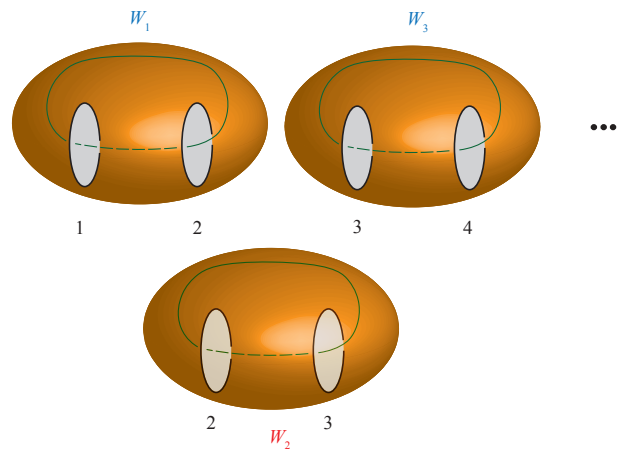
FIG. 3. A subset of Wilson loop or Wilson surface operators defined in the presence of twist defects.
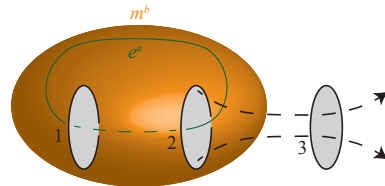
FIG. 4. Braiding process between loop 2 and loop 3.

Fig. 3, the Wilson operators W_{2i-1} and W_{2j} are defined on a pair of defect loops. The adjacent Wilson operators do not commute with each other. As in the 2d case, the Hilbert space can be generated by acting with $\{W_{2j}\}$ operators on basis states in which $\{W_{2j-1}\}$ is diagonal. The braiding process of loop 2 and 3 is defined in Fig. 4. This exchange process can be better understood if we look at the dimensionally-reduced version in Fig. 5. Similar to Fig. 1 (b), W_1 under this process deforms into W_1W_2, suggesting that a defect loop is a non-Abelian object. Once $\mathbb{Z}_2$ symmetry is fully gauged, the defect loops will be the intrinsic non-Abelian $\mathbb{Z}_2$ flux excitations. These are loops in $D(\mathcal{D}_K)$ that carry flux equal to the conjugacy class of reflections in the dihedral group $\mathcal{D}_K$. They fuse non-trivially because the composition of two reflections can be either the identity or a rotation; braiding transforms the system within the state space of these different fusion outcomes. For the $\mathbb{Z}_K \times \mathbb{Z}_K$ gauge theory (both with and without non-trivial three-loop braiding), using similar method, we can also show that the twist defect loop/$\mathbb{Z}_2$ flux excitations are non-Abelian objects.

VI. CONCLUSION

In this paper, we gauge the $\mathbb{Z}_2$ symmetry in various Abelian topological phases in $(3 + 1)$ dimensions. By
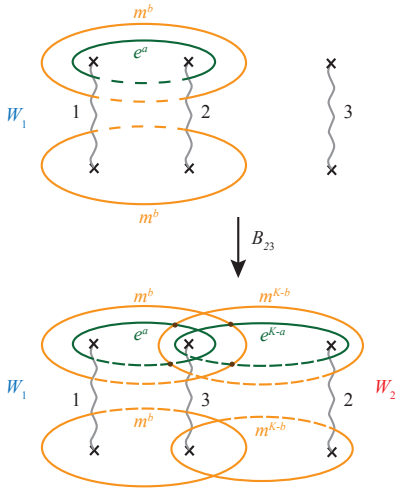
FIG. 5. Braiding process between loop 2 and loop 3 after dimensional reduction.

making use of the bulk-boundary correspondence, we discuss the orbifold theory on the $(2 + 1)$d surface state. We calculate the partition function on the $(2 + 1)$d torus with twisted boundary conditions and group them into characters. We further study how the characters transform under modular the $\mathcal{S}$ and $\mathcal{T}$ transformations which characterize the braiding statistics of particle and loops excitations in the bulk. Based on the topological data obtained on the boundary, we further analyze the defect loops/$\mathbb{Z}_2$ flux excitations in the bulk and use the Wilson loop algebra to show that these loop defects are non-Abelian.

Recently, it was shown that Abelian topological phases in $(3 + 1)$ dimensions, such as the $\mathbb{Z}_K$ and $\mathbb{Z}_K \times \mathbb{Z}_K$ gauge theories discussed here, have flux line excitations carrying Cheshire charge, topological charge that cannot be localized to a point on the flux line or measured locally[46]. Since many properties can be deduced from those of the parent Abelian theory, the gauging procedure discussed here may be an entry point for exploring the properties of Cheshire charge in non-Abelian topological phases in $(3 + 1)$ dimensions.

All the $(3 + 1)$d topological phases in this paper, obtained by gauging the $\mathbb{Z}_2$ symmetry, can be described by quantum double models with a non-Abelian group and their dimensional reductions are $(2+1)$d quantum double models. In the future, it would be interesting to explore non-Abelian topological phases in $(3+1)$d that go beyond quantum double models.

Appendix A: Edge CFT of $(2 + 1)$d abelian topological phase

The effective edge theory of an abelian topological phase can be described by the N-component bosonic field theory:

$$\mathcal{L} = \frac{1}{4\pi} \left(\partial_t \vec{\Phi}^T(x) \mathbf{K} \partial_x \vec{\Phi}(x) - \partial_x \vec{\Phi}^T(x) \mathbf{V} \partial_x \vec{\Phi}(x) \right), \tag{A1}$$

where the bosonic field $\vec{\Phi}(x)$ is a compact variable

$$\vec{\Phi}(x) \equiv \vec{\Phi}(x) + 2\pi \vec{n}, \tag{A2}$$

and $\mathbf{V}$ is a symmetric and positive definite matrix that accounts for the interaction on the edge. Unlike the K-matrix, the information encoded in $\mathbf{V}$ is non-universal.

Here we follow the method used in Ref. 12 to quantize this theory and construct the characters explicitly. For each component of the bosonic field $\vec{\Phi}$, the canonical commutation relation is given by

$$[\phi^I(x), \partial_x \phi^J(x')] = 2\pi i (\mathbf{K}^{-1})^{IJ} \delta(x - x') \tag{A3}$$

The K-matrix can be diagonalized and written as $\mathbf{K} = U^T \eta U$, where η is a signature matrix with ± 1 in its diagonal entries. We can define a new multi-component boson operator $\vec{\varphi}(x) = U \vec{\Phi}(x)$ so that Eq. (A1) takes the form

$$\mathcal{L} = \frac{1}{4\pi} \left(\partial_t \vec{\varphi}^T(x) \eta \vec{\varphi}(x) - \partial_x \vec{\varphi}^T(x) \partial_x \vec{\varphi}(x) \right), \tag{A4}$$

where we assume $(U^{-1})^T \mathbf{V} U^{-1} = I$ for simplicity. In term of the new boson operator $\vec{\varphi}(x)$, the compactification condition reads

$$\vec{\varphi}(x) \equiv \vec{\varphi}(x) + 2\pi U \vec{n} \tag{A5}$$

Each component satisfies the commutation relation:

$$[\varphi^I(x), \partial_x \varphi^J(x')] = 2\pi i \eta^{IJ} \delta(x - x') \tag{A6}$$

The mode expansion for $\vec{\varphi}(x)$ is given by

$$\varphi^I(t, x) = \varphi_0^I - (t - \eta^{II}x) p^I$$
$$+ i \sum_{n \neq 0} b_n^I e^{-in(t - \eta^{II}x)}, \tag{A7}$$

where $[\varphi_0^I, p^I] = i\delta^{IJ}$ and $[b_n^I, b_m^J] = \frac{1}{m}\delta^{IJ}\delta_{n+m}$. The Hamiltonian and the total momentum are

$$H_0 = \frac{1}{4\pi}\int_0^{2\pi}dx\,\partial_x\vec{\varphi}^T(x)\partial_x\vec{\varphi}(x)$$
$$= \frac{1}{2}\vec{p}^T\vec{p} - \frac{1}{24}\mathrm{tr}\,(\eta\eta) + \sum_{n=1}^{\infty}n^2\vec{b}_{-n}^T\vec{b}_n,$$

$$P_0 = \frac{1}{4\pi}\int_0^{2\pi}dx\,\partial_x\vec{\varphi}^T(x)\eta\partial_x\vec{\varphi}(x)$$
$$= \frac{1}{2}\vec{p}^T\eta\vec{p} - \frac{1}{24}\mathrm{tr}(\eta) + \sum_{n=1}^{\infty}n^2\vec{b}_{-n}^T\eta\vec{b}_n \qquad (A8)$$

where $\mathrm{Tr}(\eta)$ gives the chiral central charge. Since $\vec{p}$ is the momentum conjugate to the zero modes $\vec{\varphi}_0$, the compactification condition of $\vec{\varphi}(x)$ leads to the quantization condition for $\vec{p}$: $\vec{p} = (U^{-1})^T\vec{m}$ where $m^I \in \mathbb{Z}$. Notice that $\vec{m}$ can be written as $\vec{m} = \mathbf{K}\vec{\Lambda} + \vec{\lambda}$, where $\vec{\Lambda}$ and $\vec{\lambda}$ are integer valued vectors. Here, $\vec{\lambda}$ lives in the unit cell of the anyon lattice Γ^* and characterizes different twist boundary conditions for $\vec{\varphi}(t,x)$: From Eq. (A7), we read off

$$\varphi^I(t,x+2\pi) = \varphi^I(t,x) + 2\pi\eta^{II}p^I$$
$$= \varphi^I(t,x) + 2\pi\eta^{II}\left[(U^{-1})^T\vec{m}\right]^I$$
$$= \varphi^I(t,x) + 2\pi[U\vec{\Lambda}]^I$$
$$+ 2\pi\eta^{II}[(U^{-1})^T\vec{\lambda}]^I \qquad (A9)$$

The part $2\pi\eta^{II}[(U^{-1})^T\vec{\lambda}]^I$ represents the twisted boundary condition. On the other hand, for $\vec{\lambda} = 0$, $\vec{p} = \eta U\vec{m}$, this corresponds to the untwisted boundary condition. Physically, $\vec{\lambda}$'s represent different bulk excitations in the (2+1)d topological phase. When there is a quasi-particle excitation in the bulk, the edge theory is subject to the corresponding twist boundary condition.

The partition functions of the edge theory on the spacetime torus under the twist boundary conditions form the characters. In total, there are $|\det(\mathbf{K})|$ characters. The partition function for each character $[\lambda]$ is given by

$$\chi_\lambda(\tau) = \mathrm{Tr}_\lambda\left[e^{2\pi i\tau_1 P_0}e^{-2\pi\tau_2 H_0}\right] \qquad (A10)$$

where $\tau = \tau_1 + i\tau_2$. Under the T transformation,

$$\chi_\lambda(\tau+1) = e^{2\pi i\vec{\lambda}^T\mathbf{K}^{-1}\vec{\lambda}/2}\chi_\lambda(\tau). \qquad (A11)$$

where we neglect the overall phase $-2\pi c/24$. Under the $\mathcal{S}$ transformation,

$$\chi_\lambda(-1/\tau) = \sum_{\lambda'}\frac{1}{\sqrt{|\det(\mathbf{K})|}}e^{-2\pi i\vec{\lambda}^T\mathbf{K}^{-1}\vec{\lambda}'}\chi_{\lambda'}(\tau). \qquad (A12)$$

Let us take a simple example given by $\mathbf{K} = \mathrm{K}\sigma_x$, which describes the $D(\mathbb{Z}_\mathrm{K})$ quantum double model. This K-matrix can be diagonalized as $\mathbf{K} = U^T\eta U$, where $U = \sqrt{\frac{\mathrm{K}}{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ and $\eta = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. After diagonalizing the K-matrix, there are two new bosonic fields, φ_1 and φ_2, with the zero mode momentum

$$p^1 = \sqrt{\frac{1}{2\mathrm{K}}}(\mathrm{K}s+a+\mathrm{K}t+b),$$
$$p^2 = \sqrt{\frac{1}{2\mathrm{K}}}(\mathrm{K}s+a-\mathrm{K}t-b), \qquad (A13)$$

where $\vec{\Lambda}^T = (s,t) \in \mathbb{Z}^2$, $0 \le a < \mathrm{K}$ and $0 \le b < \mathrm{K}$. The characters for the $D(\mathbb{Z}_\mathrm{K})$ quantum double model are labeled by a and b, and given by

$$B_{a,b}^\mathrm{K}(\tau) = \frac{1}{|\eta(\tau)|^2}\sum_{s,t}q^{\frac{1}{4\mathrm{K}}(\mathrm{K}s+a+\mathrm{K}t+b)^2}\bar{q}^{\frac{1}{4\mathrm{K}}(\mathrm{K}s+a-\mathrm{K}t-b)^2}. \qquad (A14)$$

There are K^2 characters in total. Under T and S transformations, these characters are transformed as

$$B_{a,b}^\mathrm{K}(\tau+1) = e^{2\pi i\frac{ab}{\mathrm{K}}}B_{a,b}^\mathrm{K}(\tau),$$
$$B_{a,b}^\mathrm{K}(-1/\tau) = \frac{1}{\mathrm{K}}\sum_{a',b'}B_{a',b'}^\mathrm{K}e^{-2\pi i\frac{a'b+b'a}{\mathrm{K}}}. \qquad (A15)$$

Appendix B: Theta functions

The Dedekind eta function $\eta(\tau)$ is defined by

$$\eta(\tau) := e^{\frac{\pi i\tau}{12}}\prod_{n=1}^{\infty}(1-q^n), \quad q := e^{2\pi i\tau}. \qquad (B1)$$

The massive theta function $\Theta_{[a,b]}(\tau, m)$ is defined by

$$\Theta_{[a,b]}(\tau, m) \equiv e^{4\pi\tau_2 \Delta(m,a)} \prod_{n\in\mathbb{Z}} \left| 1 - e^{-2\pi\tau_2\sqrt{m^2+(n+a)^2}+2\pi i \tau_1(n+a)+2\pi i b} \right|^2 \tag{B2}$$

where

$$\Delta(m,a) \equiv \frac{1}{2}\sum_{n\in\mathbb{Z}} \sqrt{m^2+(n+a)^2} - \frac{1}{2}\int_{-\infty}^{\infty} dk(m^2+k^2)^{1/2} \tag{B3}$$

The massive theta functions $\Theta_{[a.b]}(\tau, m)$ satisfy

$$\begin{aligned}
\Theta_{[a,b]}(\tau, m) &= \Theta_{[-a,-b]}(\tau, m) = \Theta_{[a+p,b+q]}(\tau, m), \\
\Theta_{[a,b]}(\tau+1, m) &= \Theta_{[a,b+a]}(\tau, m), \\
\Theta_{[a,b]}(-1/\tau, m|\tau|) &= \Theta_{[b,-a]}(\tau, m),
\end{aligned} \tag{B4}$$

where $p, q \in \mathbb{Z}$.

Appendix C: Modular transformations on T^3

In this appendix, we collect some necessary ingredients relating to a flat three-torus T^3 and its mapping class group.[33] A flat three-torus is parameterized by six real parameters, $R_{0,1,2}$ and α, β, γ. For a flat three-torus T^3, the dreibein can be factorized as

$$e^A{}_\mu = \begin{pmatrix} R_0 & 0 & 0 \\ 0 & R_1 & 0 \\ 0 & 0 & R_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\alpha & 1 & 0 \\ -\gamma & -\beta & 1 \end{pmatrix} = \begin{pmatrix} R_0 & 0 & 0 \\ -\alpha R_1 & R_1 & 0 \\ -\gamma R_2 & -\beta R_2 & R_2 \end{pmatrix}, \tag{C1}$$

and its inverse is given by

$$e_A{}^\mu = \begin{pmatrix} \frac{1}{R_0} & \frac{\alpha}{R_0} & \frac{\alpha\beta+\gamma}{R_0} \\ 0 & \frac{1}{R_1} & \frac{\beta}{R_1} \\ 0 & 0 & \frac{1}{R_2} \end{pmatrix}, \tag{C2}$$

such that $e^A{}_\mu e_A{}^\nu = \delta_\mu{}^\nu$ and $e^A{}_\mu e_B{}^\mu = \delta^A{}_B$. Here R_0, R_1, and R_2 are the radii for the directions τ, x, and y, and α, β, and γ are related to the angles between directions τ and x, x and y, and τ and y, respectively. The Euclidean metric is then given by

$$\begin{aligned}
g_{\mu\nu} &= e^A{}_\mu e^B{}_\nu \delta_{AB} \\
&= \begin{pmatrix} R_0^2 + \alpha^2 R_1^2 + \gamma^2 R_2^2 & -\alpha R_1^2 + \beta\gamma R_2^2 & -\gamma R_2^2 \\ -\alpha R_1^2 + \beta\gamma R_2^2 & R_1^2 + \beta^2 R_2^2 & -\beta R_2^2 \\ -\gamma R_2^2 & -\beta R_2^2 & R_2^2 \end{pmatrix},
\end{aligned} \tag{C3}$$

and the line element is

$$\begin{aligned}
ds^2 &= g_{\mu\nu} d\theta^\mu d\theta^\nu \\
&= R_0^2 (d\theta^0)^2 + R_1^2 (d\theta^1 - \alpha d\theta^0)^2 + R_2^2 (d\theta^2 - \beta d\theta^1 - \gamma d\theta^0)^2,
\end{aligned} \tag{C4}$$

where $0 \le \theta^\mu \le 2\pi$ are angular variables.

The group $SL(3, \mathbb{Z})$ is generated by two transformations:

$$U_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{C5}$$

An $SL(3,\mathbb{Z})$ transformation acts as follows on the dreibein and metric:

$$e^A{}_\mu \xrightarrow{L} (eL^T)^A{}_\mu = L_\mu{}^\rho e^A{}_\rho,$$
$$e^{\star\mu}_A \xrightarrow{L} (e^\star L^{-1})_A{}^\mu = e^{\star\rho}_A (L^{-1})_\rho{}^\mu,$$
$$g_{\mu\nu} \xrightarrow{L} (LgL^T)_{\mu\nu} = L_\mu{}^\rho L_\nu{}^\sigma g_{\rho\sigma}, \tag{C6}$$

for any $SL(3,\mathbb{Z})$ element $L = U_1^{n_1} U_2^{n_2} U_1^{n_3} \cdots$. Under the U_2 transformation, the metric transforms according to

$$g_{\mu\nu} \xrightarrow{U_2} (U_2 g U_2^T)_{\mu\nu} = \begin{pmatrix} R_0^2 + (\alpha-1)^2 R_1^2 + (\gamma+\beta)^2 R_2^2 & -(\alpha-1)R_1^2 + \beta(\gamma+\beta)R_2^2 & -(\gamma+\beta)R_2^2 \\ -(\alpha-1)R_1^2 + \beta(\gamma+\beta)R_2^2 & R_1^2 + \beta^2 R_2^2 & -\beta R_2^2 \\ -(\gamma+\beta)R_2^2 & -\beta R_2^2 & R_2^2 \end{pmatrix}, \tag{C7}$$

which corresponds to the changes

$$\alpha \to \alpha - 1, \quad \gamma \to \gamma + \beta, \tag{C8}$$

while R_0, R_1, R_2, and β are unchanged.

On the other hand, the less trivial generator U_1 can be decomposed as

$$U_1 = U_1' M, \quad U_1' = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \tag{C9}$$

where U_1' corresponds to the 90° rotation in the $\tau - x$ plane and M is the 90° rotation in the $x - y$ plane. The generator U_1' acts on the metric as

$$g_{\mu\nu} \xrightarrow{U_1'} (U_1' g U_1'^T)_{\mu\nu} = \begin{pmatrix} R_1^2 + \beta^2 R_2^2 & \alpha R_1^2 - \beta\gamma R_2^2 & \beta R_2^2 \\ \alpha R_1^2 - \beta\gamma R_2^2 & R_0^2 + \alpha^2 R_1^2 + \gamma^2 R_2^2 & -\gamma R_2^2 \\ \beta R_2^2 & -\gamma R_2^2 & R_2^2 \end{pmatrix}, \tag{C10}$$

which corresponds to the changes

$$R_0 \to R_0/|\tau|, \quad R_1 \to R_1|\tau|, \quad \tau_1 \to -\tau_1/|\tau|^2, \quad \gamma \to -\beta, \quad \beta \to \gamma \quad \text{(while } R_2 \text{ is unchanged)}, \tag{C11}$$

where we have introduced

$$\tau \equiv \alpha + i r_{01}, \quad r_{01} \equiv R_0/R_1. \tag{C12}$$

Observe also that under $R_0 \to R_0/|\tau|$ and $R_1 \to R_1|\tau|$, $\tau_2 \to \tau_2/|\tau|^2$. Hence, U_1' induces $\tau \to -1/\tau$.

Finally, the transformation M acts on the metric as

$$g_{\mu\nu} \xrightarrow{M} (MgM^T)_{\mu\nu} = \begin{pmatrix} R_0^2 + \alpha^2 R_1^2 + \gamma^2 R_2^2 & \gamma R_2^2 & -\alpha R_1^2 + \beta\gamma R_2^2 \\ \gamma R_2^2 & R_2^2 & \beta R_2^2 \\ -\alpha R_1^2 + \beta\gamma R_2^2 & \beta R_2^2 & R_1^2 + \beta^2 R_2^2 \end{pmatrix}. \tag{C13}$$

The two transformations U_1' and U_2 correspond respectively to modular S and T^{-1} transformations in the $\tau - x$ plane, generating the $SL(2,\mathbb{Z})$ subgroup of $SL(3,\mathbb{Z})$ group. Combined with M, they generate the whole $SL(3,\mathbb{Z})$ group. In the following, we denote $U_1' M$ by $\mathcal{S}$ and U_2 by $\mathcal{T}^{-1}$.

Appendix D: Transformation properties of the characters in the gauged $\mathbb{Z}_K \times \mathbb{Z}_K$ gauge theory

In this appendix, we list the transformation properties of the characters of the topological $\mathbb{Z}_K \times \mathbb{Z}_K$ gauge theory after gauging the $\mathbb{Z}_2$ symmetry. From these transformation properties, one can construct the $\mathcal{S}$ matrix.

$$U_1' \chi_{n_i}^0 = \frac{1}{2K^2} \sum_{n_0', n_1'} e^{\frac{2\pi i}{K}(2n_0 n_1' + 2n_1 n_0')} (\chi_{n_0', n_1', n_2}^0 + \chi_{n_0', n_1', n_2}^1) + \frac{1}{K^2} \sum_{n_0', n_1', l_0', l_1'} e^{\frac{2\pi i}{K}[n_0(n_1'+l_1') + n_1(n_0'+l_0')]} \chi_{n_0', n_1', n_2}^{l_0', l_1', l_2}$$
$$+ \frac{1}{2K} \sum_{m_0', m_1'} e^{\frac{2\pi i}{K}(n_0 m_1' + n_1 m_0')} (\chi_{\sigma_x, m_1', n_2}^{m_0'} + \chi_{\tau_x, m_1', n_2}^{m_0'}),$$

$$U_1' \chi_{n_i}^{l_i} = \frac{1}{K^2} \sum_{n_0', n_1'} e^{\frac{2\pi i}{K}[n_0'(n_1+l_1) + n_1'(n_0+l_0)]} (\chi_{n_{0,1}', n_2}^0 + \chi_{n_{0,1}', n_2}^1) + \frac{1}{K^2} \sum_{n_0', n_1', l_0', l_1'} e^{\frac{2\pi i}{K}(n_0 n_1' + n_1 n_0' + l_0 l_1' + l_1 l_0')} \chi_{n_0', n_1', n_2}^{l_0', l_1', l_2},$$

$$U_1' \chi_{\sigma_x}^{m_i} = \frac{1}{2K} \sum_{m_0', m_1'} e^{\frac{2\pi i}{K}(m_0 m_1' + m_1 m_0')} (\chi_{m_{0,1}', m_2}^0 - \chi_{m_{0,1}', m_2}^1)$$
$$+ \frac{1}{4K} \sum_{m_0', m_1'} \left[1 + (-1)^{m_1 + m_1'} + (-1)^{m_0 + m_0'}(1 + (-1)^{m_1 + m_1' + K}) \right] e^{\frac{\pi i}{K}(m_0 m_1' + m_1 m_0')} (\chi_{\sigma_x, m_1', m_2}^{m_0'} - \chi_{\tau_x, m_1', m_2}^{m_0'}),$$

$$U_1' \chi_{\sigma_y}^{m_i} = \frac{1}{2K} \sum_{m_0', m_1'} e^{\frac{2\pi i}{K}(m_0 m_1' + m_1 m_0')} \left(\chi_{\sigma_y, m_1', m_2}^{m_0'} + \chi_{\tau_y, m_1', m_2}^{m_0'} \right) + \frac{1}{2K} \sum_{m_0', m_1'} e^{\frac{2\pi i}{K}(m_0 m_1' + m_1 m_0')} \left(\chi_{\sigma_y, m_1', m_2}^{m_0'} + \chi_{\tau_y, m_1', m_2}^{m_0'} \right),$$

$$U_1' \chi_{\sigma_{xy}}^{m_i} = \frac{1}{2K} \sum_{m_0', m_1'} e^{\frac{2\pi i}{K}(m_0 m_1' + m_1 m_0')} \left(\chi_{\sigma_{xy}, m_1', m_2}^{m_0'} + \chi_{\tau_{xy}, m_1', m_2}^{m_0'} \right) + \frac{1}{2K} \sum_{m_0', m_1'} e^{\frac{2\pi i}{K}(m_0 m_1' + m_1 m_0')} \left(\chi_{\sigma_{xy}, m_1', m_2}^{m_0'} - \chi_{\tau_{xy}, m_1', m_2}^{m_0'} \right).$$

$$\text{(D1)}$$

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