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Surfaces and Slabs of Fractional Topological Insulator Heterostructures

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Fractional topological insulators (FTI) are electronic topological phases in $(3 + 1)$ dimensions enriched by time reversal (TR) and charge $U(1)$ conservation symmetries. We focus on the simplest series of fermionic FTI, whose bulk quasiparticles consist of deconfined partons that carry fractional electric charges in integral units of $e^* = e/(2n + 1)$ and couple to a discrete \mathbb{Z}_{2n+1} gauge theory. We propose massive symmetry preserving or breaking FTI surface states. Combining the long-ranged entangled bulk with these topological surface states, we deduce the novel topological order of quasi- $(2 + 1)$ dimensional FTI slabs as well as their corresponding edge conformal field theories.

I. INTRODUCTION

Conventional topological insulators (TI)^{1–4} are time reversal (TR) and charge $U(1)$ symmetric electronic band insulators in three dimensions that host massless surface Dirac fermions. The topologically protected surface Dirac fermion can acquire a single-body ferromagnetic or superconducting mass by breaking TR or charge $U(1)$ symmetry respectively. Alternatively it can acquire a many-body interacting mass while preserving both symmetries, and exhibit long-ranged entangled surface topological order^{5–8}. On the other hand, fractional topological insulators (FTI)^{9?–14} are long-range entangled topologically ordered electronic phases in $(3 + 1)$ dimensions outside of the single-body mean-field band theory description. They carry TR and charge $U(1)$ symmetries, which enrich its topological order (TO) in the sense that a symmetric surface must be anomalous and cannot be realized non-holographically by a true $(2 + 1)$ D system. In this article, we describe the topological properties of various massive surface states and quasi- $(2 + 1)$ D slabs of a series of FTI. In particular, we focus on the quasiparticle (QP) structure.

In particular, we consider FTI that support deconfined fermionic parton excitations ψ in the bulk, each carrying a fractional electric charge of $e^* = e/(2n + 1)$. The electronic QP decomposes as $\psi_{el} \sim \psi_1 \dots \psi_{2n+1}$. The $(3 + 1)$ D TO is based on a discrete \mathbb{Z}_{2n+1} gauge theory¹¹. The theory supports electrically neutral string-like gauge flux Φ , so that a monodromy quantum phase of $e^{2\pi i g/(2n+1)}$ is obtained each time ψ orbits around it. In other words, ψ carries the gauge charge g . The integer g and $2n + 1$ are relatively prime so that all local QP must be combinations of the electronic QP ψ_{el} and must carry integral electric charges and trivial gauge charges.

Generalizing the surface state of a conventional TI, the surface of a FTI hosts massless Dirac partons coupling with a \mathbb{Z}_{2n+1} gauge theory. Unlike its non-interacting counterpart whose gapless Dirac surface state is symmetry protected in the single-body picture, a FTI is strongly interacting to begin with and there is no topological reason for its surface state to remain gapless. In this article we focus on three types of gapped surface states – ferromagnetic surfaces (FS) that break TR, superconducting surfaces (SCS) that break charge $U(1)$, and symmetric surfaces which generalize the \mathcal{T} -Pfaffian surface state of a conventional TI and is denoted by \mathcal{T} -Pf*. The topological order for FTI slab with these surfaces are discussed in section II, section III and section IV respectively. In section V, we discuss, using an anyon condensation picture, the gluing of a pair of \mathcal{T} -Pfaffian surfaces. We conclude in section VI with remarks on a complementary way to understand these topological order¹⁶.

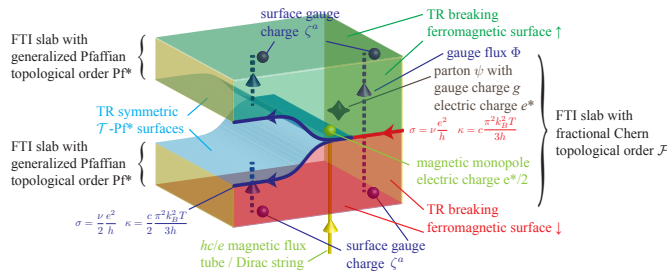


FIG. 1. Summary of the QP and gauge flux content in FTI slabs. A pair of Pf* FTI slabs are merged into a fractional Chern FTI slab \mathcal{F} by gluing the two TR symmetric \mathcal{T} -Pf* surfaces. Directed bold lines on the front surface are chiral edge modes of the Pf* and \mathcal{F} FTI slabs.

We focus on a series of fermionic FTI, labeled by integers n , whose magneto-electric response is characterized by the θ -angle $\theta = \pi/(2n + 1)$ (modulo $2\pi/(2n + 1)$) that associates an electric charge of $e^*/2 = e/(2(2n + 1))$ to each magnetic monopole¹⁵, for e the electric charge of the elec-

II. FERROMAGNETIC HETEROSTRUCTURE

We begin with a slab that has opposite TR breaking FS. In the FS, in addition to the single-body Dirac mass m for the surface parton, the \mathbb{Z}_{2n+1} gauge sector also shows TR breaking signature. The \mathbb{Z}_{2n+1} gauge theory is only present inside the FTI, and when a flux line Φ terminates at the surface, the TR breaking boundary condition confines an electrically neutral surface gauge QP, denoted by ζ^a , with gauge charge a at the flux-surface junction (see fig. 1). This gauge flux-charge composite, referred as a dyon $\delta = \Phi \times \zeta^a$, carries fractional

spin $h_\delta = a/(2n+1)$ because a 2π -rotation about the normal axis braids a gauge charges around Φ and results in the monodromy quantum phase of $e^{2\pi i a/(2n+1)}$. **TR** conjugates all quantum phases so, $a \not\equiv 0$ modulo $2n+1$ breaks **TR**.

The 1D interface between two **TR** conjugate **FS** domains hosts a fractional chiral channel. For example, the interface between two **FS** domains with opposite ferromagnetic orientations on the surface of a conventional **TI** bounds a chiral Dirac channel^{17–19}, where electrons propagate only in the forward direction. Alternatively, a **TI** slab with opposite **TR** breaking **FS** is topologically identical to a quasi-(2+1)D Chern insulator^{20,21} and supports a chiral Dirac edge mode. Similarly, in the **FTI** case, the low-energy content of the fractional chiral channel between a pair of **TR** conjugate **FS** domains can be inferred by the edge mode of a **FTI** slab with **TR** breaking **FS** that is topologically identical to a quasi-(2+1)D fractional Chern insulator^{22–25} or fractional quantum Hall (FQH) state²⁶. The chiral (1+1)D channel is characterized by two response quantities^{27–35} – the differential electric conductance $\sigma = dI/dV = \nu e^2/h$ that relates the changes of electric current and potential, and the differential thermal conductance $\kappa = dI_T/dT = c(\pi^2 k_B^2/3h)T$ that relates the changes of energy current and temperature. In the slab geometry, they are equivalent to the Hall conductance $\sigma = \sigma_{xy}$, $\kappa = \kappa_{xy}$. $\nu = N_e/N_\phi$ is also referred to as the filling fraction of the **FTI** slab and associates the gain of electric charge (in units of e) to the addition of a magnetic flux quantum hc/e . $c = c_R - c_L$ is the chiral central charge of the conformal field theory (CFT)³⁶ that effectively describes the low-energy degrees of freedom of the fractional chiral channel.

Since the top and bottom surfaces of the **FTI** slab are **TR** conjugate, their parton Dirac masses m and gauge flux-charge ratio a have opposite signs. The anyon content is generated by the partons and gauge dyons. When a gauge flux passes through the entire slab geometry from the bottom to the top surface, it associates with total $2a$ gauge charges at the two surface junctions. We denote this dyon by $\gamma = \Phi \times \zeta^{2a}$, which corresponds to an electrically neutral anyon in the slab with spin $h_\gamma = 2a/(2n+1)$. If a is relatively prime with $2n+1$, the primitive dyon generates the chiral Abelian topological field theory $\mathbb{Z}_{2n+1}^{(2a)}$ ^{37,38}, which consists of the dyons γ^m , for $m = 0, \dots, 2n$, with spins $h_{\gamma^m} = 2am^2/(2n+1)$ modulo 1 and fusion rules $\gamma^m \times \gamma^{m'} = \gamma^{m+m'}$, $\gamma^{2n+1} = \gamma^0 = 1$. In particular, when $a = -1$, γ^n now has spin $-2n^2/(2n+1) \equiv n/(2n+1)$ modulo 1, which is identical to that of the fundamental **QP** of the $SU(2n+1)$ Chern-Simons theory at level 1^{37,38}. This identifies the Abelian theories $\mathbb{Z}_{2n+1}^{(-2)} \cong \mathbb{Z}_{2n+1}^{(n)} = SU(2n+1)_1$, which has chiral central charge $c_{\text{neutral}} = 2n$.

The **FTI** slab also supports fractionally charged partons ψ , each carrying a gauge charge g . The electrically charged sector can be decoupled from the neutral $\mathbb{Z}_{2n+1}^{(2a)}$ sector by combining each parton with a specific number

of dyons $\lambda = \psi \times \gamma^{-n^2 u g}$, where $ua + v(2n+1) = 1$ for some integer u, v , so that the combination is local (i.e. braids trivially) with any dyons γ^m . λ has fractional electric charge $q_\lambda = e^*$ and spin $h_\lambda = 1/2 + n^3 u g^2/(2n+1)$ modulo 1. The $\langle \text{charge} \rangle$ sector consists of the fractional Abelian **QP** products λ^m , where $\lambda^{2n+1} \sim \psi^{2n+1} \sim \psi_{\text{el}}$ corresponds to the local electronic **QP**. In particular, when $a = -1$ and $g = -2$, $h_\lambda = 1/2(2n+1)$ and therefore λ behaves exactly like the Laughlin **QP** of the **FQH** state $U(1)_{(2n+1)/2}$ with filling fraction $\nu = 1/(2n+1)$ and chiral central charge $c_{\text{charge}} = 1$. Combining the neutral and charge sectors, the **FTI** slab with **TR** breaking **FS** has the decoupled tensor product **TO**

$$\mathcal{F} = \langle \text{charge} \rangle \otimes \mathbb{Z}_{2n+1}^{(2a)}, \quad (1)$$

and in the special case when $a = -1$ and $g = -2$, it is identical to the Abelian state $U(1)_{(2n+1)/2} \otimes SU(2n+1)_1$, which has a total central charge $c = 2n+1$. In general, the filling fraction and chiral central charge are not definite and are subject to surface reconstruction. For instance, the addition of $2N$ electronic Dirac fermions per surface modifies the two response quantities by an equal amount $\nu \rightarrow \nu + 2N$, $c \rightarrow c + 2N$.

III. SUPERCONDUCTING HETEROSTRUCTURE

Next we move on to superconducting heterostructures. We begin with the fractional Chern **FTI** slab \mathcal{F} in (1) and introduce weak superconducting pairing, perhaps induced by proximity with a bulk superconductor, without closing the bulk energy gap. In the simplest scenario, this condenses all parton pairs ψ^{2m} , which form a *Lagrangian subgroup*³⁹ – a maximal set of mutually local bosons – containing the Cooper pair $\psi_{\text{el}}^2 = \psi^{2(2n+1)}$. Since the parton pair ψ^2 carries gauge charge $2g$, which is relatively prime with $2n+1$, the condensate confines all non-trivial dyons γ^m , which are non-local and have non-trivial monodromy with ψ^2 . As the neutral sector $\mathbb{Z}_{2n+1}^{(2a)}$ is killed by pairing, the superconducting **FTI** slab with **TR** conjugate **FS** has a trivial fermionic **TO**. It however still carries chiral fermionic edge modes with the same chiral central charge $c_{\mathcal{F}}$. On the other hand, these fermionic channels also live along the line interface between **TR** conjugate ferromagnetic domains on the surface of a weakly superconducting **FTI**. When the line interface hits a **TR** symmetric **SCS** island (c.f. fig. 1 by replacing the **T-Pf*** surfaces by **SCS**), these chiral channels split and divide along the pair of **SCS** -**FS** line interfaces. Both of these channels are electrically neutral as charge $U(1)$ symmetry is broken by the superconductor, and each of them carries half of the energy current of \mathcal{F} and has chiral central charge $c_{\mathcal{F}}/2$. For example, the **SCS** -**FS** heterostructure on a conventional **TI** surface holds a chiral Majorana channel with $c = 1/2$ along

the line tri-junction^{17,18}. In the specific fractional case when $a = -1$ and $g = -2$, each **SCS** -**FS** line interfaces holds $2n + 1$ chiral Majorana fermions and is described by the Wess-Zumino-Witten $SO(2n + 1)_1$ **CFT** with the central charge $c = (2n + 1)/2$. Analogous to the conventional superconducting **TI** surface⁴⁰, **SCS** of **FTI** supports a zero energy Majorana bound state (MBS) at a vortex core. Now that the condensate consists of parton pairs, vortices are quantized with the magnetic flux $hc/2e^* = (2n + 1)hc/2e$. Each pair of **MBS** forms a two-level system distinguished by parton fermion parity.

IV. GENERALIZED T-PFAFFIAN* SURFACE STATE

Lastly, we describe the **T-Pf*** surface state that preserves both **TR** and charge $U(1)$ symmetries of the **FTI**. Generalizing the \mathcal{T} -Pfaffian symmetric gapped surface state of a conventional **TI** described in Ref.⁷, the **FTI** version – referred here as \mathcal{T} -Pfaffian* – consists of the Abelian surface anyons $\mathbb{1}_j$ and Ψ_j , for j even, and the non-Abelian Ising-like anyons Σ_j , for j odd. The index j corresponds to the fractional electric charge $q_j = je/4(2n + 1)$. The surface anyons satisfy the fusion rules

$$\begin{aligned} \mathbb{1}_j \times \mathbb{1}_{j'} &= \Psi_j \times \Psi_{j'} = \mathbb{1}_{j+j'}, & \mathbb{1}_j \times \Psi_{j'} &= \Psi_{j+j'}, \\ \Psi_j \times \Sigma_{j'} &= \Sigma_{j+j'}, & \Sigma_j \times \Sigma_{j'} &= \mathbb{1}_{j+j'} + \Psi_{j+j'}, \end{aligned} \quad (2)$$

and the spin statistics

$$h_{\mathbb{1}_j} = h_{\Psi_j} - \frac{1}{2} = \frac{j^2}{16}, \quad h_{\Sigma_j} = \frac{j^2 - 1}{16} \quad \text{modulo } 1 \quad (3)$$

so that $\mathbb{1}_j, \Psi_j$ are bosonic, fermionic or semionic, and Σ_j are bosonic or fermionic. The fermion Ψ_4 is identical to the super-selection sector of the bulk parton ψ , which is local with respect to all surface anyons and can escape from the surface and move into the bulk. **TR** symmetry acts on the surface anyons the same way it acts on those in the \mathcal{T} -Pfaffian state for conventional **TI**^{7,16}. For example, the parton combinations $\psi^{2j+1} = \Psi_{8j+4}$ (and $\psi^{2j} = \mathbb{1}_{8j}$) are Kramers doublet fermions (resp. Kramers singlet bosons), while Ψ_{8j} ($\mathbb{1}_{8j+4}$) are Kramers singlet fermions (resp. Kramers doublet bosons). Moreover, for identical reasons as in the conventional **TI** case, the **T-Pf*** state is anomalous and can only be supported holographically on the surface of a topological bulk. For instance, the bosonic **TO** of the **T-Pf*** state after gauging fermion parity would necessarily violate **TR** symmetry. We notice in passing that there are alternative surface **TO** that generalizing those in Ref.^{5,6}. However we will only focus on the **T-Pf*** state in this article.

The **FTI** slab with a **TR** symmetric **T-Pf*** top surface and a **TR** breaking bottom **FS** carries a novel quasi- $(2 + 1)$ D **TO**. Its topological content consists of the fractional partons coupled with the \mathbb{Z}_{2n+1} gauge theory in the bulk and the **T-Pf*** surface state (see fig. 1). All surface anyons are confined to the **TR** symmetric surface except the parton combinations $\psi^{2j+1} = \Psi_{8j+4}$ and

$\psi^{2j} = \mathbb{1}_{8j}$. Moreover, the **TR** breaking boundary condition confines a gauge **QP** ζ^a per gauge flux Φ ending on the **FS**. On the other hand, there is no gauge charge associate with a gauge flux ending on the **T-Pf*** surface because of **TR** symmetry. Thus a gauge flux passing through the entire slab corresponds to the dyon $\delta = \Phi \times \zeta^a$ with spin $h_\delta = a/(2n + 1)$ modulo 1. The **T-Pf*** state couples non-trivially to the \mathbb{Z}_{2n+1} gauge theory as the parton $\psi = \Psi_4$ carries a gauge charge g . The general surface anyons X_j , for $X = \mathbb{1}, \Psi, \Sigma$, must carry the gauge charge $z(j) \equiv n^2 g j$ modulo $2n + 1$ and associate to the monodromy quantum phase $e^{2\pi i z(j)/(2n+1)}$ when orbiting around the dyon δ . For instance, as $2n \equiv -1$ modulo $2n + 1$, $z(4j) \equiv g j$ counts the gauge charge of the parton combination ψ^j .

The **TO** of this **FTI** slab is therefore generated by combinations of the **T-Pf*** anyons and the dyon δ . We denote the composite anyon by

$$\tilde{X}_{j,z} = X_j \otimes \delta^{z+n^3 u g j}, \quad (4)$$

where $X = \mathbb{1}, \Psi$ for j even or Σ for j odd, $z = 0, \dots, 2n$ modulo $2n + 1$, and $ua + v(2n + 1) = 1$. They satisfy the fusion rules

$$\begin{aligned} \tilde{\mathbb{1}}_{j,z} \times \tilde{\mathbb{1}}_{j',z'} &= \tilde{\Psi}_{j,z} \times \tilde{\Psi}_{j',z'} = \tilde{\mathbb{1}}_{j+j',z+z'}, \\ \tilde{\mathbb{1}}_{j,z} \times \tilde{\Psi}_{j',z'} &= \tilde{\Psi}_{j+j',z+z'}, & \tilde{\Psi}_{j,z} \times \tilde{\Sigma}_{j',z'} &= \tilde{\Sigma}_{j+j',z+z'}, \\ \tilde{\Sigma}_{j,z} \times \tilde{\Sigma}_{j',z'} &= \tilde{\mathbb{1}}_{j+j',z+z'} + \tilde{\Psi}_{j+j',z+z'}. \end{aligned} \quad (5)$$

They follow the spin statistics

$$\begin{aligned} h(\tilde{\mathbb{1}}_{j,z}) &= h(\tilde{\Psi}_{j,z}) - \frac{1}{2} = h(\tilde{\Sigma}_{j,z}) + \frac{1}{16} \\ &= \frac{j^2}{16} + \frac{az^2 - n^6 u g^2 j^2}{2n + 1} \quad \text{modulo } 1. \end{aligned} \quad (6)$$

The j, z indices in (4) are defined in a way so that $\tilde{X}_{j,0}$ are local with respect to the dyons $\delta^z = \tilde{\mathbb{1}}_{0,z}$ and decoupled from the dyon sector $\mathbb{Z}_{2n+1}^{(a)}$. The **T-Pf*** surface anyons belong to the subset $X_j = \tilde{X}_{j,-n^3 u g j}$, which is a maximal sub-category that admits a **TR** symmetry. The electronic **QP** belongs to the super-selection sector $\psi_{\text{el}} = \tilde{\Psi}_{4(2n+1),0}$, which is local with respect to all anyons. If one gauges fermion parity and includes anyons that associate -1 monodromy phase with ψ_{el} , i.e. if one includes $\tilde{\mathbb{1}}_{j,z}, \tilde{\Psi}_{j,z}$ for j odd and $\tilde{\Sigma}_{j,z}$ for j even, the $\langle \text{Ising} \rangle$ sector generated by $1 = \tilde{\mathbb{1}}_{0,0}$, $f = \tilde{\Psi}_{0,0}$, $\sigma = \tilde{\Sigma}_{0,0}$ is local with and decoupled from the $\langle \text{charge} \rangle_{\text{Pf}^*}$ sector generated by $\tilde{\mathbb{1}}_{j,0}$. The **TO** of the **FTI** slab thus takes the decoupled tensor product form after gauging fermion parity

$$\text{Pf}^* = \langle \text{charge} \rangle_{\text{Pf}^*} \otimes \langle \overline{\text{Ising}} \rangle \otimes \mathbb{Z}_{2n+1}^{(a)}. \quad (7)$$

Gauging fermion parity is not the focus of this article. Nevertheless, we notice in passing that there are inequivalent ways of fermion parity gauging, and in order for the Pf^* theory to have the appropriate central charge, (7) needs to be modified by a neutral Abelian $SO(2n)_1$

sector¹⁶. However, the tensor product (7) is sufficient and correct to describe the fermionic **TO** of the **FTI** slab (with global ungauged fermion parity) by restricting to super-selection sectors $\tilde{X}_{j,z}$ that are local with respect to the electronic **QP** ψ_{el} . We refer to this fermionic **TO** as a generalized Pfaffian state.

V. GLUING T-PFAFFIAN* SURFACES

The chiral channel \mathcal{F} in (1) between a pair of **TR** conjugate **FS** domains divides into a pair of fermionic Pf^* in (7) at a junction where the two **FS** domains sandwich a **TR** symmetric $\mathcal{T}\text{-Pf}^*$ surface domain (see fig. 1). Conservation of charge and energy requires the filling fractions and chiral central charges to equally split, i.e. $2\nu_{\text{Pf}^*} = \nu_{\mathcal{F}}$ and $2c_{\text{Pf}^*} = c_{\mathcal{F}}$. For instance, in the prototype case when $a = -1$ and $g = -2$, $\nu_{\text{Pf}^*} = 1/2(2n+1)$ and $c_{\text{Pf}^*} = (2n+1)/2$. Similar to the aforementioned \mathcal{F} case, these quantities are subjected to surface reconstruction $\nu \rightarrow \nu + N$, $c \rightarrow c + N$.

In addition to the response quantities, the **TO** of \mathcal{F} for the **FTI** slab with **TR** conjugate **FS** is related to that of the fermionic Pf^* by a *relative tensor product*

$$\mathcal{F} = \text{Pf}^* \boxtimes_b \text{Pf}^*. \quad (8)$$

This can be understood by juxtaposing the **TR** symmetric surfaces of a pair of Pf^* **FTI** slabs and condensing surface bosonic anyon pairs on the two $\mathcal{T}\text{-Pf}^*$ surfaces. This anyon condensation^{41–43} procedure effectively glues the two **FTI** slabs together along the **TR** symmetric surfaces (see fig. 1). The relative tensor product \boxtimes_b involves first taking a decoupled tensor product \otimes when the two Pf^* **FTI** slabs are put side by side. Among the **TR** symmetric surface anyons in $(\mathcal{T}\text{-Pf}^*)^A \otimes (\mathcal{T}\text{-Pf}^*)^B$ where A, B refers to the two slabs, we condense the collection of electrically neutral bosonic pairs

$$b = \left\{ \begin{array}{l} \mathbb{1}_{4j}^A \mathbb{1}_{-4j}^B, \Psi_{4j}^A \Psi_{-4j}^B, \mathbb{1}_{4j+2}^A \Psi_{-4j-2}^B, \\ \Psi_{4j+2}^A \mathbb{1}_{-4j-2}^B, \Sigma_{2j+1}^A \Sigma_{-2j-1}^B \end{array} \right\}. \quad (9)$$

All anyons that are non-local with respect to and braid non-trivially around any of the bosons in b are confined. This includes all anyon combinations $\tilde{X}_{j_a, z_a}^A \tilde{X}_{j_b, z_b}^B$ where the dyon numbers $z_a + n^3 u g j_a$ and $z_b + n^3 u g j_b$ disagree modulo $2n+1$. Physically, this ensures gauge fluxes must continue through both A and B slabs, or equivalently all gauge monopoles at the interface are confined as they signify imbalances of gauge fluxes through the two slabs. The new deconfined dyon $\gamma = \tilde{\mathbb{1}}_{0,1}^A \tilde{\mathbb{1}}_{0,1}^B$ consist of a gauge flux that pass continuously across both slabs with gauge **QP** ζ^a on each of the remaining top and bottom **TR** breaking surfaces. A deconfined anyon thus splits into a dyon component γ^z and a surface component in $(\mathcal{T}\text{-Pf}^*)^A \otimes (\mathcal{T}\text{-Pf}^*)^B$. Within the surface part, all combinations that involve only Σ^A or only Σ^B are confined by the $\Psi_0^A \Psi_0^B$ condensate. Other confined

anyons include $\mathbb{1}_{j_a}^A \mathbb{1}_{j_b}^B$, $\Psi_{j_a}^A \Psi_{j_b}^B$, $\mathbb{1}_{j_a+2}^A \Psi_{j_b-2}^B$, $\Psi_{j_a+2}^A \mathbb{1}_{j_b-2}^B$ and $\Sigma_{j_a \pm 1}^A \Sigma_{j_b \mp 1}^B$ for $j_a \not\equiv j_b$ modulo 8. The remaining deconfined Ising pair splits into simpler Abelian components

$$\Sigma_{j_a \pm 1}^A \Sigma_{j_b \mp 1}^B = S_{j_a \pm 1, j_b \mp 1}^+ + S_{j_a \pm 1, j_b \mp 1}^-, \quad (10)$$

where each S^\pm carries the same spin as the original Ising pair but differs the other by a unit fermion $S^\pm \times \Psi^{A/B} = S^\mp$. In general the two Abelian components are non-local with respect to each other. For instance, the **TR** symmetric surface anyons S^+ and S^- are mutually semionic when $j_a = j_b = 0$. We choose to include S^+ in the condensate b in (9) while confining S^- . Furthermore, the condensate identifies the deconfined anyons that are different up to bosons in b .

$$\begin{aligned} \mathbb{1}_{j_a}^A \mathbb{1}_{j_b}^B &\equiv \Psi_{j_a}^A \Psi_{j_b}^B \equiv \Psi_{j_a+2}^A \mathbb{1}_{j_b-2}^B \equiv \mathbb{1}_{j_a+2}^A \Psi_{j_b-2}^B \\ &\equiv S_{j_a \pm 1, j_b \mp 1}^\pm \equiv \mathbb{1}_{j_a \pm 4}^A \mathbb{1}_{j_b - 4}^B \end{aligned} \quad (11)$$

for $j_a \equiv j_b \pmod{8}$ and j_a, j_b both even. Eq.(B2) are just parton combinations. For instance, $\psi^A = \Psi_4^A \mathbb{1}_0^B \equiv \mathbb{1}_4^A \Psi_4^B = \psi^B$ are now free to move inside both **FTI** slabs after gluing. The **TO** after the gluing is generated by the partons and dyons, which behave identically as those in \mathcal{F} of (1). This proves (8). The anyon condensation gluing of the pair of $\mathcal{T}\text{-Pf}^*$ states preserves symmetries for the same reason it does for the conventional **TI** case^{7,16}.

It is worth noting that a magnetic monopole can be mimicked by a magnetic flux tube / Dirac string (with flux quantum hc/e) that originates at the **TR** symmetric surface interface and passes through one of the two **FTI** slab, say the A slab. In the prototype $a = -2$ and $g = -1$, the filling fraction $\nu_{\text{Pf}^*} = 1/2(2n+1)$ of the quasi-2D slab ensures, according to the Laughlin argument²⁷, that the monopole associates to the fractional charge $q = 1/2(2n+1)$, which is carried by the confined $\mathcal{T}\text{-Pf}^*$ surface anyons $\mathbb{1}_2^A$ or Ψ_2^A . This surface condensation picture therefore provides a simple verification of the Witten effect¹⁵ for $\theta = \pi/(2n+1)$.

Lastly, we noticed that in the band insulator case for $n = 0$, \mathcal{F} in (1) reduces to the Chern insulator or the lowest Landau level (LLL), and Pf^* in (7) is simply the particle-hole (PH) symmetric Pfaffian state^{???}. The **PH** symmetry is captured by the relative tensor product (8), which can be formally rewritten into

$$\text{Pf}^* = \mathcal{F} \boxtimes \overline{\text{Pf}^*} \quad (12)$$

by putting Pf^* on the other side of the equation. Here, the tensor product is relative with respect to some collection of condensed bosonic pairs, and $\overline{\text{Pf}^*}$ is the **TR** conjugate of Pf^* . (12) thus equates Pf^* with its **PH** conjugate, which is obtained by subtracting itself from the **LLL**. In the fractional case with $n > 0$, (12) suggests a generalized **PH** symmetry for Pf^* , whose **PH** conjugate is the subtraction of itself from the **FQH** state \mathcal{F} .

VI. CONCLUSION

To conclude, we studied gapped **FTI** surface states with (i) **TR** breaking order, (ii) charge $U(1)$ breaking order, as well as (iii) symmetry preserving **\mathcal{T} -Pf*** topological order. We focused on **FTI** that supported fractionally charged partons coupling with a discrete \mathbb{Z}_{2n+1} gauge theory. We characterized the fractional interface channels sandwiched between different gapped surface domains by describing their charge and energy response, namely the differential electric and thermal conductance. The low-energy **CFT** for these fractional interface channels corresponded to the **TO** of quasi-(2+1)D **FTI** slabs with the corresponding gapped top and bottom surfaces. In particular a **FTI** slab with **TR** conjugate ferromagnetic surfaces behaved like a fractional Chern insulator with **TO** (1), and in the particular case when $a = -1$ and $g = -2$, its charge sector was identical to that of the Laughlin $\nu = 1/(2n+1)$ **FQH** state. Combining the **TR** symmetric **\mathcal{T} -Pf*** surface with the **FTI** bulk as well as the opposite **TR** breaking surface, this **FTI** slab exhibited a generalized Pfaffian **TO** (7). Furthermore, we demonstrated the gluing of a pair of parallel **\mathcal{T} -Pf*** surfaces, which are supported by two **FTI** on both sides. It was captured by an anyon condensation picture that killed the **\mathcal{T} -Pf*** **TO** and left behind deconfined partons and confined gauge and magnetic monopoles in the bulk.

a. Acknowledgement In the following work Ref. ¹⁶, we also construct the **\mathcal{T} -Pf*** state of the **FTI** from the field theoretic duality approach. J.C.Y.T. is supported by NSF Grant No. DMR-1653535. G.Y.C. acknowledges the support from Korea Institute for Advanced Study (KIAS) grant funded by the Korea government (MSIP) and Grant No. 2016R1A5A1008184 under NRF of Korea.

Appendix A: Abelian Chern-Simons theory of dyons

The fractional topological insulator slab with time-reversal conjugate surfaces has anyons which are dyons and partons. The neutral sector consist of only dyons. A dyon γ is composed of a number of \mathbb{Z}_{2n+1} gauge charge on each surface associated with an unit gauge flux through the bulk. The dyons γ^m where $m = 0, 1, \dots, 2n$, with $1 = \gamma^0$ being the vacuum, form the anyon content of an Abelian topological state denoted as $\mathbb{Z}_{2n+1}^{(2a)}$. They have spins $h_{\gamma^m} = \frac{2am^2}{2n+1}$ modulo 1 and satisfy the \mathbb{Z}_{2n+1} fusion rule $\gamma^m \times \gamma^{m'} = \gamma^{[m+m']}$, where $[m+m']$ is the remainder between 0 and $2n$ when dividing $m+m'$ by $2n+1$. For the case when $a = -1$, the Abelian topological theory becomes $\mathbb{Z}_{2n+1}^{(-2)}$, which is actually identical to $\mathbb{Z}_{2n+1}^{(n)}$. This is because the dyon $\mathbf{e} = \gamma^n$ has spin $\frac{-2n^2}{2n+1} \equiv \frac{n}{2n+1}$ modulo 1. The collection $\{\mathbf{e}^l : l = 0, 1, \dots, 2n\}$ is of 1-1 correspondence with $\{\gamma^m : m = 0, 1, \dots, 2n\}$. For instance $\gamma = \mathbf{e}^{-2} = \mathbf{e}^{2n-1}$. At the same time, $\mathbb{Z}_{2n+1}^{(n)} = \{\mathbf{e}^l : l = 0, 1, \dots, 2n\}$ is the anyon content of

the Abelian Chern-Simons $SU(2n+1)_1$ theory with Lagrangian density $\mathcal{L}_{2+1} = \frac{1}{4\pi} \int_{2+1} K_{IJ} \alpha^I \wedge d\alpha^J$, where α^I for $I = 1, \dots, 2n$ are $U(1)$ gauge fields, and

$$K_{SU(2n+1)} = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & & \\ & & & \ddots & \\ & & & & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix} \quad (\text{A1})$$

is the Cartan matrix of $SU(2n+1)$.

Appendix B: Anyon Condensation

Here we will elaborate how to glue the two **TR** symmetric surfaces of a pair of **Pf*** **FTI** slabs and condense surface bosonic anyon pairs on the two **TPf** surfaces. As before we take the decoupled tensor product of the anyons in two **Pf*** **TO**, denoted $(\text{Pf}^*)^A \otimes (\text{Pf}^*)^B$ where A, B refers to the two slabs. Then we choose a set of bosons that braid trivially around each other.

First notice that dyon combinations $\gamma^z \equiv \tilde{\mathbb{1}}_{0,z}^A \tilde{\mathbb{1}}_{0,z}^B$ are not confined. A particle with charge “j” has gauge charge $n^2 g j$, so our neutral pairs have gauge charge $n^2 g j \times -n^2 g j$. Thus the braiding phase with these dyons is $zn^2 g j - zn^2 g j = 0$.

Our parton should continuously move from slab A to slab B, so we should condense $\Psi_4^A \Psi_{-4}^B$, the parton creation annihilation operator. Anything that braids with it is confined. We can derive braiding statistics with the ribbon formula, $\theta_{A,B} = h_{A \times B} - h_A - h_B$. The braiding phase from the anyon combination $\tilde{X}_{j_a, z_a}^A \tilde{X}_{j_b, z_b}^B$ around $\Psi_4^A \Psi_{-4}^B$ is the same as $(\delta^A)^{z_a + n^3 u g j_a} (\delta^B)^{z_b + n^3 u g j_b}$ around $\Psi_4^A \Psi_{-4}^B$. The parton carries “g” gauge charge so this phase is $g(z_a + n^3 u g j_a - z_b - n^3 u g j_b)$. This is zero if the dyon number $z + n^3 u g j$ is equal on the A and B particle. This ensures gauge fluxes must continue through both A and B slabs, i.e. confines gauge magnetic monopoles. This means that we are left with combinations $X_{j_a}^A X_{j_b}^B \gamma^z$. It also identifies $\Psi_4^A \Psi_{-4}^B$ with the vacuum, which identifies

$$\begin{aligned} \mathbb{1}_{j_a}^A \mathbb{1}_{j_b}^B \gamma^z &\equiv \Psi_{j_a+4}^A \Psi_{j_b-4}^B \gamma^z \equiv \mathbb{1}_{j_a+8}^A \mathbb{1}_{j_b-8}^B \gamma^z \\ \mathbb{1}_{j_a}^A \Psi_{j_b}^B \gamma^z &\equiv \Psi_{j_a+4}^A \mathbb{1}_{j_b-4}^B \gamma^z \equiv \mathbb{1}_{j_a+8}^A \Psi_{j_b-8}^B \gamma^z \\ \Sigma_{j_a}^A \Sigma_{j_b}^B \gamma^z &\equiv \Sigma_{j_a+4}^A \Sigma_{j_b-4}^B \gamma^z \\ \mathbb{1}_{j_a}^A \Sigma_{j_b}^B \gamma^z &\equiv \Psi_{j_a+4}^A \Sigma_{j_b-4}^B \gamma^z \equiv \mathbb{1}_{j_a+8}^A \Sigma_{j_b-8}^B \gamma^z \\ &\equiv \Psi_{j_a+12}^A \Sigma_{j_b-12}^B \gamma^z \end{aligned}$$

Next we choose the fermion pair $\Psi_0^A \times \Psi_0^B$. Notice Σ braids with Ψ , so anything with just one Σ is confined. This brings the identification to

$$\begin{aligned}
\mathbb{1}_{j_a}^A \mathbb{1}_{j_b}^B \gamma^z &\equiv \mathbb{1}_{j_a+4j}^A \mathbb{1}_{j_b-4j}^B \gamma^z \equiv \Psi_{j_a+4j}^A \Psi_{j_b-4j}^B \gamma^z \\
\mathbb{1}_{j_a}^A \Psi_{j_b}^B \gamma^z &\equiv \mathbb{1}_{j_a+4j}^A \Psi_{j_b-4j}^B \gamma^z \equiv \Psi_{j_a+4j}^A \mathbb{1}_{j_b-4j}^B \gamma^z \\
\Sigma_{j_a}^A \Sigma_{j_b}^B \gamma^z &\equiv \Sigma_{j_a+4j}^A \Sigma_{j_b-4j}^B \gamma^z
\end{aligned}$$

Next we can condense $\Psi_2^A \mathbb{1}_{-2}^B$, which when braided around $\mathbb{1}_{j_a}^A \mathbb{1}_{j_b}^B$ or $\Psi_{j_a}^A \mathbb{1}_{j_b}^B$ gives $4(j_a - j_b)/16$ which is not confined if $j_a - j_b = 0 \bmod 4$. For $\Sigma_{j_a}^A \Sigma_{j_b}^B$ gives $4(j_a - j_b)/16 + 1/2$ which is not confined if $j_a - j_b = 2 \bmod 4$. The identification is now

$$\begin{aligned}
\mathbb{1}_{j_a}^A \mathbb{1}_{j_b}^B \gamma^z &\equiv \mathbb{1}_{j_a+4j}^A \mathbb{1}_{j_b-4j}^B \gamma^z \equiv \Psi_{j_a+4j}^A \Psi_{j_b-4j}^B \gamma^z \\
&\equiv \mathbb{1}_{j_a+2}^A \Psi_{j_b-2}^B \gamma^z \equiv \mathbb{1}_{j_a+2+4j}^A \Psi_{j_b-2-4j}^B \gamma^z \\
&\equiv \Psi_{j_a+2+4j}^A \mathbb{1}_{j_b-2-4j}^B \gamma^z \\
\Sigma_{j_a}^A \Sigma_{j_b}^B \gamma^z &\equiv \Sigma_{j_a+2j}^A \Sigma_{j_b-2j}^B \gamma^z
\end{aligned}$$

Our $\Sigma\Sigma$ pairs now split into simpler abelian compo-

nents

$$\Sigma_{j_a \pm 1}^A \Sigma_{j_b \mp 1}^B = S_{j_a \pm 1, j_b \mp 1}^+ + S_{j_a \pm 1, j_b \mp 1}^-, \quad (\text{B1})$$

where each S^\pm carries the same spin as the original Ising pair but differs from each other by a unit fermion $S^\pm \times \Psi^{A/B} = S^\mp$. S^+ and S^- normally have non-trivial mutual monodromy. We choose to condense the electrically neutral $S_{1,-1}^+$ and its multiples, while confining $S_{1,-1}^-$. This means $\Sigma_1^A \Sigma_{-1}^B$ is condensed. The Σ pair around $\mathbb{1}_{j_a}^A \mathbb{1}_{j_b}^B$ gives a phase of $2(j_a - j_b)/16$ which is zero if $j_a - j_b = 0 \bmod 8$. The Σ pair around $\mathbb{1}_{j_a}^A \Psi_{j_b}^B$ gives a phase of $2(j_a - j_b)/16 + 1/2$ which is zero if $j_a - j_b = 4 \bmod 8$. The Σ pair around $\Sigma_{j_a}^A \Sigma_{j_b}^B$ gives a phase of $2(j_a - j_b)/16 \pm 1/4$ which is zero if $j_a - j_b = 2$ or $6 \bmod 8$.

This then completes the full condensate, and we have the identification

$$\begin{aligned}
\mathbb{1}_{j_a}^A \mathbb{1}_{j_b}^B \gamma^z &\equiv \Psi_{j_a,z}^A \Psi_{j_b,z}^B \gamma^z \equiv \Psi_{j_a+2}^A \mathbb{1}_{j_b-2}^B \gamma^z \\
&\equiv \mathbb{1}_{j_a+2}^A \Psi_{j_b-2,z}^B \gamma^z \equiv S_{j_a \pm 1, j_b \mp 1}^\pm \gamma^z \\
&\equiv \mathbb{1}_{j_a+4}^A \mathbb{1}_{j_b-4}^B \gamma^z
\end{aligned} \quad (\text{B2})$$

for $j_a \equiv j_b \bmod 8$ and j_a, j_b both even. This ends up being just the multiples of the parton $\mathbb{1}_0^A \Psi_4^B$ together with the dyons γ^z . Together they generate the theory \mathcal{F} of a FTI slab with two conjugate TR breaking surfaces.

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