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Perturbative and global anomalies in bosonic analogues of integer quantum Hall and topological insulator phases

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We study perturbative and global anomalies at the boundaries of bosonic analogues of integer quantum Hall (BIQH) and topological insulator (BTI) phases using a description of the boundaries of these phases in terms of a nonlinear sigma model (NLSM) with Wess-Zumino term. One of the main results of the paper is that these anomalies are robust against arbitrary smooth deformations of the target space of the NLSM which describes the phase, provided that the deformations also respect the symmetry of the phase. In the first part of the paper we discuss the perturbative $U(1)$ anomaly at the boundary of BIQH states in all odd (spacetime) dimensions. In the second part we study global anomalies at the boundary of BTI states in even dimensions. In a previous work [Phys. Rev. B 95, 035149 (2017)] we argued that the boundary of the BTI phase exhibits a global anomaly which is an analogue of the parity anomaly of Dirac fermions in three dimensions. Here we elevate this argument to a proof for the boundary of the two-dimensional BTI state by explicitly computing the partition function of the gauged NLSM describing the boundary. We then use the powerful equivariant localization technique to show that this global anomaly is robust against all smooth deformations of the target space of the NLSM which preserve the $U(1) \rtimes \mathbb{Z}_2$ symmetry of the BTI state. We also comment on the difficulties of generalizing this latter proof to higher dimensions. Finally, we discuss the expected low energy behavior of the boundary theories studied in this paper when the coupling constants are allowed to flow under the renormalization group.

I. INTRODUCTION

In the past few years it was realized that a powerful way to understand symmetry-protected topological (SPT) phases with symmetry group G in d (spacetime) dimensions is to study ‘t Hooft anomalies of $(d-1)$ -dimensional theories with global G -symmetry^{1–4}. A theory with global G -symmetry has a ‘t Hooft anomaly if it cannot be consistently coupled to a background gauge field A for the symmetry group G ⁵. It is often the case that an anomalous $(d-1)$ -dimensional theory can be realized in a gauge invariant manner at the boundary of a d -dimensional SPT phase. In that case, the anomaly of the boundary theory is canceled by the gauge variation of the bulk effective action for the SPT phase. This cancellation mechanism is known as *anomaly inflow*⁶. It is likely that all bulk-boundary correspondences in SPT phases can be understood through some version of the anomaly inflow mechanism, but perhaps involving global anomalies instead of the perturbative anomalies originally studied in Ref. 6.

It is clear from the discussion above that characterizing boundary anomalies offers a precise way to understand the bulk-boundary correspondence in SPT phases, topological insulators, and related systems. For example, the presence of a single chiral fermion at the edge of the $\nu = 1$ integer quantum Hall state in $2+1$ dimensions (and also the single chiral boson at the edge of the Laughlin states) can be understood very simply using anomaly inflow arguments^{7,8}. This chiral fermion is needed to cancel the gauge variation of the bulk Chern-Simons term⁹

$$S_{CS}[A] = \frac{1}{4\pi} \int_X A \wedge dA, \quad (1.1)$$

which describes the response of the integer quantum Hall state to an external electromagnetic field $A = A_\mu dx^\mu$. We also note that anomaly inflow has been discussed for analogs of the integer quantum Hall state in all odd spacetime dimensions⁹.

A related, but much more subtle, example of anomaly inflow occurs in time-reversal invariant, free-fermion topological insulators in $3+1$ dimensions^{10,11}. In Ref. 12 Witten has shown (among other results) that the bulk-boundary correspondence in this system can be understood very precisely in terms of the parity anomaly of a Dirac fermion with $U(1)$ and time-reversal symmetry in $2+1$ dimensions^{13–16}. The parity anomaly is intimately related to the Atiyah-Potodis-Singer index theorem^{17–19} for the Dirac operator on an even-dimensional manifold with boundary (see Ref. 15 for the relation), and this connection was a central theme in Ref. 12. The connection between the parity anomaly and the boundary theory of the topological insulator, and in particular the fact that the bulk and boundary together are gauge invariant, was also previously discussed in Ref. 20.

In a separate series of developments, bosonic analogues of the integer quantum Hall and topological insulator states were introduced and studied in detail in the SPT literature. The bosonic integer quantum Hall (BIQH) state is an SPT phase of bosons with $U(1)$ symmetry in $2+1$ dimensions^{21–31}. It is characterized by a Hall conductance which is an *even* integer (in units of $\frac{e^2}{h}$). On the other hand, the bosonic topological insulator (BTI) state is an SPT phase of bosons with $U(1)$ symmetry and \mathbb{Z}_2 time-reversal symmetry in $3+1$ dimensions^{32–35}. It is characterized by a bulk electromagnetic response of the “Chern character” type

$$S_{CC}[A] = \frac{\Theta}{8\pi^2} \int_X F \wedge F, \quad (1.2)$$

with coefficient $\Theta = 2\pi$. In a recent work, the present authors computed the electromagnetic response of generalizations of the BIQH and BTI states to *all* odd and even spacetime dimensions, respectively³⁶.

Given these separate developments, a natural next step would be to give a precise characterization of the anomalies at

the boundaries of the BIQH and BTI states. In Ref. 36 we initiated such a program. There we used a nonlinear sigma model (NLSM) description^{37–47} of the boundary of the BIQH state in odd dimensions to compute the perturbative $U(1)$ anomaly of the boundary theory. Our result implied that the electromagnetic response of the bulk of a BIQH state in $2m - 1$ dimensions is characterized by a Chern-Simons term⁷ with level $N_{2m-1} = (m!)k$, $k \in \mathbb{Z}$, where the value $k = 1$ represents the fundamental BIQH state.

In Ref. 36 we also argued that the boundary theory of the $2m$ -dimensional BTI state exhibits a bosonic analogue of the well-known parity⁷ anomaly of Dirac fermions in three dimensions. Our argument was based on a demonstration (again using a NLSM description) that the boundary of the BTI state can exhibit a \mathbb{Z}_2 symmetry-breaking electromagnetic response described by a Chern-Simons term with level $N_{2m-1} = \frac{m!}{2}$ for the external field A . Since this boundary response is *half* the response of the fundamental BIQH state in $2m - 1$ dimensions, we argued, by analogy with the case of a massless Dirac fermion (with Hall conductance = Chern-Simons level = $\frac{1}{2}$) on the surface of the $(3 + 1)$ -dimensional topological insulator^{10,11}, that the boundary theory of the BTI displays a bosonic analogue of the parity anomaly.

In this paper we continue this program of characterizing anomalies at the boundary of BIQH and BTI states. In the first part of the paper we revisit the perturbative $U(1)$ anomaly at the boundary of $(2m - 1)$ -dimensional BIQH states. In Ref. 36 we computed this anomaly by gauging the Wess-Zumino (WZ) term in an $O(2m)$ NLSM description of the boundary of the BIQH state. In any NLSM, the field is a map from spacetime to a manifold \mathcal{M} , known as the *target space* of the NLSM. In the $O(2m)$ NLSM the target space is just the $(2m - 1)$ -dimensional unit sphere S^{2m-1} , and the NLSM field \mathbf{n} is a $2m$ -component unit vector. This particular NLSM description possesses a $SO(2m)$ global symmetry, which is much larger than the $U(1)$ symmetry required to protect the BIQH state. One might then wonder if (perhaps) more realistic models of the BIQH boundary can be found which still possess the correct perturbative $U(1)$ anomaly, but have only the $U(1)$ global symmetry of the BIQH state. In this paper we show that a large family of such models do indeed exist by proving the following result.

Let \mathcal{M} be any $(2m - 1)$ -dimensional manifold which can be reached from S^{2m-1} by smooth deformations which preserve the $U(1)$ symmetry of the BIQH phase (i.e., we have a diffeomorphism $f : \mathcal{M} \rightarrow S^{2m-1}$ which is *equivariant* with respect to the $U(1)$ symmetry). Then a description of the boundary of the BIQH state using a NLSM with target space \mathcal{M} has *the same* perturbative $U(1)$ anomaly as the $O(2m)$ NLSM description.

In the second part of the paper we revisit the bosonic analogue of the parity anomaly at the boundary of the BTI states. In the simplest case of the BTI state in two spacetime dimensions we are able to compute the partition function of the gauged boundary theory exactly. The BTI state in two dimen-

sions has the symmetry group $G = U(1) \rtimes \mathbb{Z}_2$, where \mathbb{Z}_2 represents a unitary charge-conjugation symmetry. Our exact computation of the boundary partition function shows that the boundary of the BTI does indeed exhibit a bosonic analogue of the global anomaly of Dirac fermions in $0 + 1$ dimensions which also have $U(1)$ symmetry and \mathbb{Z}_2 charge-conjugation symmetry⁴⁸. We first compute this anomaly within the $O(3)$ NLSM description (with target space S^2) of the BTI boundary which we previously used in Ref. 36. Based on this calculation, one might again wonder if a more realistic model of the BTI boundary can be found which has the same global anomaly, but which possesses only the $G = U(1) \rtimes \mathbb{Z}_2$ symmetry of the BTI state and not the full $SO(3)$ symmetry of the $O(3)$ NLSM. We again show that such models do exist by proving the following result.

Let \mathcal{M} be *any* two-dimensional manifold which can be reached from S^2 by smooth deformations which preserve the full $G = U(1) \rtimes \mathbb{Z}_2$ symmetry of the BTI state (i.e., we have a diffeomorphism $f : \mathcal{M} \rightarrow S^2$ which is *equivariant* with respect to the action of the group G). Then a description of the boundary of the BTI state using a NLSM with target space \mathcal{M} has *the same* global anomaly as the $O(3)$ NLSM description.

To prove this result we use the powerful *equivariant localization* technique originally developed for the exact computation of certain phase space path integrals^{49–53}. Whereas for the perturbative anomaly we are able to extend our proof to any spacetime dimension, the calculation for global anomalies becomes challenging in higher dimensions and is not easily extendable. We comment on this difficulty later, and discuss possible alternative approaches.

As in our previous work³⁶, gauged WZ actions play a central role in the calculations in this paper. Gauging WZ actions, and also obstructions to gauging these actions (i.e., anomalies), have been discussed previously in Refs. 54–61. Since we consider two kinds of anomalies in this paper (perturbative and global), it is important for us to explain at the outset how exactly our anomalies are related to obstructions to gauging a WZ action. For the perturbative $U(1)$ anomalies that we study, the anomaly that we find is a direct result of the existence of an obstruction to gauging the WZ action. Therefore, these anomalies are already present at the level of the classical action for these theories. On the other hand, for the global anomalies that we study there is *no obstruction* to gauging the $U(1)$ symmetry of the WZ action. Instead, the anomaly is a completely quantum effect which stems from an inability to regulate the theory in such a way as to preserve both large $U(1)$ gauge invariance, and the additional discrete \mathbb{Z}_2 symmetry of the theory.

This paper is organized as follows. In Sec. II we analyze perturbative $U(1)$ anomalies at the even-dimensional boundary of BIQH states in generic odd spacetime dimensions. In Sec. III we analyze the global anomaly at the $(0 + 1)$ -dimensional boundary of the $(1 + 1)$ -dimensional BTI state. In Sec. IV we comment on the expected behavior of the bound-

ary theories studied in this paper under renormalization group flows. In Sec. V we present our conclusions. In Appendix A we review the form of the phase space path integral for Hamiltonian systems on a general phase space \mathcal{M} equipped with symplectic form ω . In Appendix B we give a brief introduction to the equivariant localization technique for phase space path integrals. Finally, in Appendix C we present the detailed calculations of the regularized determinants which appear in the expression (obtained from the equivariant localization technique) for the partition function of the BTI boundary.

II. PERTURBATIVE ANOMALIES IN BOSONIC INTEGER QUANTUM HALL STATES

In this section we study perturbative $U(1)$ anomalies at the boundary of a class of bosonic SPT phases in odd spacetime dimensions which are protected by the symmetry of the group $G = U(1)$. We refer to these phases as bosonic integer quantum Hall (BIQH) states. They are all higher-dimensional generalizations of the $(2+1)$ -dimensional BIQH state introduced in Ref. 21. Upon coupling to a background $U(1)$ gauge field $A = A_\mu dx^\mu$, the boundary of these states exhibits a perturbative $U(1)$ anomaly. For the BIQH phase in $2m-1$ dimensions, the anomaly of the boundary is such that it can be compensated by a bulk Chern-Simons (CS) term

$$S_{CS}[A] = \frac{N_{2m-1}}{(2\pi)^{m-1}m!} \int_X A \wedge (dA)^{m-1} \quad (2.1)$$

with the level N_{2m-1} of the CS term quantized in integer multiples of $m!$ (factorial). Here X denotes the $(2m-1)$ -dimensional bulk spacetime. We computed this anomaly in Ref. 36 using a NLSM description of the boundary theory of the BIQH state. Specifically, we modeled the boundary using an $O(2m)$ NLSM with Wess-Zumino (WZ) term, with a particular action of the group $U(1)$ on the NLSM field. The field in this model is a $2m$ -component unit vector $\mathbf{n} = (n^1, \dots, n^{2m})$, and so the target space of the $O(2m)$ NLSM is the $(2m-1)$ -dimensional unit sphere S^{2m-1} .

In this section we first recall the result of Ref. 36, and we also show that the anomaly computed there is well-defined in the sense that it is independent of a certain freedom in the specific form of the terms appearing in the gauged WZ action for the boundary theory. We then consider alternative descriptions of the BIQH state using NLSMs with a general target space \mathcal{M} , and we prove that if \mathcal{M} can be obtained from S^{2m-1} by smooth deformations which preserve the $U(1)$ symmetry of the BIQH state, then the anomaly of the NLSM theory with target space \mathcal{M} is *identical* to the anomaly of the $O(2m)$ NLSM theory. Later in the paper, in Sec. IV, we discuss the expected low energy behavior of the NLSMs discussed in this section.

The results of this section prove that the anomaly computed in Ref. 36 is robust against arbitrary smooth, symmetry-preserving deformations of the NLSM used to describe the boundary of the BIQH state. This is exactly what one hopes for in a model of an SPT phase: smooth, symmetry-preserving deformations of a model of an SPT phase should not affect the ability of that model to capture the universal properties of the

SPT phase, provided that the deformations do not take one across a phase boundary. We also note here that in Ref. 36 we gave a more general *gauge invariance* argument for the quantization of the level N_{2m-1} of the CS term describing the bulk response of the BIQH state. That argument also implies that the boundary anomaly is robust and independent of the specific details of any particular model of the boundary of the BIQH state. Therefore, the results of this section could have been anticipated from the gauge invariance argument in Ref. 36. However, it is also instructive to have an explicit proof of this invariance for the class of NLSM descriptions of the boundary considered here.

A. Review of $O(2m)$ NLSM calculation of the anomaly

We start by reviewing the calculation of the boundary anomaly of the BIQH state using the $O(2m)$ NLSM description. The boundary of the $(2m-1)$ -dimensional BIQH state can be described by an $O(2m)$ NLSM with WZ term. Let X_{bdy} denote the $(2m-2)$ -dimensional boundary spacetime. As we discussed above, the NLSM field $\mathbf{n} = (n^1, \dots, n^{2m})$ should be understood as a map $\mathbf{n} : X_{bdy} \rightarrow S^{2m-1}$ from the boundary spacetime X_{bdy} to the target space of the NLSM, which is just the unit sphere S^{2m-1} in this case.

The WZ term for the NLSM requires the following ingredients for its construction. First, we need the volume form ω_{2m-1} on S^{2m-1} . In terms of the coordinates n^a , $a = 1, \dots, 2m$, it takes the form

$$\omega_{2m-1} = \sum_{a=1}^{2m} (-1)^{a-1} n^a dn^1 \wedge \dots \wedge \overline{dn^a} \wedge \dots \wedge dn^{2m}, \quad (2.2)$$

where the overline means to omit that term from the wedge product. Next, we need an extension \mathcal{B} of the boundary spacetime X_{bdy} such that $\partial\mathcal{B} = X_{bdy}$, where $\partial\mathcal{B}$ denotes the boundary of \mathcal{B} . Finally, we need an extension $\tilde{\mathbf{n}}$ of the NLSM field \mathbf{n} into the bulk of \mathcal{B} such that $\tilde{\mathbf{n}}|_{\partial\mathcal{B}} = \mathbf{n}$. The extended field $\tilde{\mathbf{n}}$ should be viewed as a map $\tilde{\mathbf{n}} : \mathcal{B} \rightarrow S^{2m-1}$. Then the WZ term for the $O(2m)$ NLSM on the $(2m-2)$ -dimensional boundary spacetime X_{bdy} takes the form

$$S_{WZ}[\mathbf{n}] = \frac{2\pi k}{\mathcal{A}_{2m-1}} \int_{\mathcal{B}} \tilde{\mathbf{n}}^* \omega_{2m-1}, \quad (2.3)$$

where $k \in \mathbb{Z}$ is the level of the WZ term and $\mathcal{A}_{2m-1} = \text{Area}[S^{2m-1}] = \frac{2\pi^m}{(m-1)!}$. Here the notation $\tilde{\mathbf{n}}^* \omega_{2m-1}$ denotes the pullback of the volume form ω_{2m-1} on S^{2m-1} to the extended boundary spacetime \mathcal{B} via the map $\tilde{\mathbf{n}} : \mathcal{B} \rightarrow S^{2m-1}$.

The WZ term can be written in a more familiar form if we introduce a system of local coordinates $(s, x^0, \dots, x^{2m-3})$ on \mathcal{B} , where (x^0, \dots, x^{2m-3}) are a system of local coordinates on X_{bdy} , and where $s \in [0, 1]$ is a coordinate for the extra direction in \mathcal{B} . We choose boundary conditions on the extended field configuration such that $\tilde{\mathbf{n}}$ is equal to a trivial constant configuration at $s = 0$, and $\tilde{\mathbf{n}} = \mathbf{n}$ at $s = 1$. Hence, the physical boundary spacetime X_{bdy} is located at $s = 1$. In these coordinates the WZ term takes the more explicit form

$$S_{WZ}[\mathbf{n}] = \frac{2\pi k}{\mathcal{A}_{2m-1}} \int_0^1 ds \int d^{2m-2}x \epsilon_{a_1 \dots a_{2m}} \tilde{n}^{a_1} \partial_s \tilde{n}^{a_2} \partial_{x_0} \tilde{n}^{a_3} \dots \partial_{x_{2m-3}} \tilde{n}^{a_{2m}}, \quad (2.4)$$

where we sum over all indices which appear once as a subscript and once as a superscript (the standard summation notation). In addition to the WZ term, the action for the $O(2m)$ NLSM also includes a conventional kinetic term

$$S_{kin}[\mathbf{n}] = \frac{1}{2f} \int d^{2m-2}x (\partial_\mu \mathbf{n}) \cdot (\partial^\mu \mathbf{n}), \quad (2.5)$$

where f is a coupling constant with dimensions of $(mass)^{4-2m}$ (the power is equal to two minus the boundary spacetime dimension).

The action of the $U(1)$ symmetry that protects the BIQH state on the NLSM field is best described by first pairing the components of \mathbf{n} into m “bosons” $b_\ell = n^{2\ell-1} + i n^{2\ell}$, $\ell = 1, \dots, m$. Then, for the NLSM model of the BIQH phase, the $U(1)$ symmetry can be defined to act on these bosons as^{21,36}

$$U(1) : b_\ell \rightarrow e^{i\xi} b_\ell, \forall \ell. \quad (2.6)$$

Let us briefly explain the rationale for this choice of the $U(1)$ action. In the NLSM description of bosonic SPT phases from Ref. 37, the information about the symmetry group G is encoded in a homomorphism $\sigma : G \rightarrow SO(2m)$ (in the case of unitary symmetries which have trivial action on spacetime). The NLSM equipped with the homomorphism σ will describe a trivial phase if there exists a vector \mathbf{v} such that $\sigma(g)\mathbf{v} = \mathbf{v}$, $\forall g \in G$. This is because in this case it is possible to add a “Zeeman” term $\mathbf{n} \cdot \mathbf{v}$ to the NLSM action to drive the NLSM into a trivial direct product state in which \mathbf{n} is parallel or anti-parallel to \mathbf{v} at all points in space. Therefore, we must choose a homomorphism σ where no such vector \mathbf{v} exists if we want our NLSM to describe a nontrivial SPT phase with $U(1)$ symmetry. Mathematically, the problem is to embed $U(1) \cong SO(2)$ inside the maximal torus of $SO(2m)$ in such a way that no vector \mathbf{v} is fixed under the action of $\sigma(g)$ $\forall g \in U(1)$. The unique solution to this problem², modulo trivial permutations of the components n^a in the definition of the bosons b_ℓ , is the one in Eq. (2.6).

Next, we couple the NLSM describing the boundary of the BIQH state to a background $U(1)$ gauge field $A = A_\mu dx^\mu$, and attempt to construct an action which is invariant under the gauge transformation

$$\begin{aligned} b_\ell &\rightarrow e^{i\xi} b_\ell, \forall \ell \\ A &\rightarrow A + d\xi, \end{aligned} \quad (2.7)$$

where ξ is now a function of the boundary spacetime coordinates. This gauge transformation can be recast in a more geometric form using the vector field $\underline{v} = v^a \frac{\partial}{\partial n^a}$ which generates the action of the $U(1)$ symmetry on S^{2m-1} . Concretely, this means that under an infinitesimal $U(1)$ transformation, the coordinates on S^{2m-1} transform as

$$n^a \rightarrow n^a + \xi v^a. \quad (2.8)$$

For the $U(1)$ symmetry action defined in Eq. (2.6), the vector field \underline{v} takes the form

$$\underline{v} = \sum_{\ell=1}^m \left(-n^{2\ell} \frac{\partial}{\partial n^{2\ell-1}} + n^{2\ell-1} \frac{\partial}{\partial n^{2\ell}} \right). \quad (2.9)$$

This transformation of the coordinates also induces a transformation for general p -forms β on S^{2m-1} ,

$$\beta \rightarrow \beta + \mathcal{L}_{\underline{v}} \beta, \quad (2.10)$$

where $\mathcal{L}_{\underline{v}} = di_{\underline{v}} + i_{\underline{v}}d$ is the Lie derivative (acting on differential forms) along \underline{v} , and $i_{\underline{v}}$ is the interior multiplication by \underline{v} (d is the ordinary exterior derivative).

To simplify the presentation of the gauged WZ action it is best to work with a more compact notation. Let us define the normalized volume form $\alpha^{(2m-1)} = \frac{\omega_{2m-1}}{\mathcal{A}_{2m-1}}$ so that the WZ term can be written as

$$S_{WZ}[\mathbf{n}] = 2\pi k \int_B \mathbf{n}^* \alpha^{(2m-1)}. \quad (2.11)$$

The derivation of the gauged WZ action is somewhat technical, and so we refer the reader to Ref. 36 for details. In Ref. 36 we showed that the gauged WZ action for the $O(2m)$ NLSM takes the form

$$\begin{aligned} S_{WZ,gauged}[\mathbf{n}, A] = \\ S_{WZ}[\mathbf{n}] + 2\pi k \sum_{r=1}^{m-1} \int_{X_{bdy}} A \wedge F^{r-1} \wedge \mathbf{n}^* \alpha^{(2m-1-2r)}, \end{aligned} \quad (2.12)$$

where the $\alpha^{(2m-1-2r)}$ are a set of differential forms on S^{2m-1} of degree $2m-1-2r$, $r = 1, \dots, m-1$, which have a form that we now discuss.

First, for each $\ell = 1, \dots, m$, we define one-forms \mathcal{J}_ℓ and two-forms \mathcal{K}_ℓ on S^{2m-1} by

$$\mathcal{J}_\ell = n_{2\ell-1} dn_{2\ell} - n_{2\ell} dn_{2\ell-1} \quad (2.13a)$$

$$\mathcal{K}_\ell = dn_{2\ell-1} \wedge dn_{2\ell}. \quad (2.13b)$$

Then, for each $r = 0, \dots, m-1$, we define the forms $\Omega^{(r)}$ by

$$\Omega^{(r)} = \sum_{\ell_1, \dots, \ell_{m-r}=1}^m \mathcal{J}_{\ell_1} \wedge \mathcal{K}_{\ell_2} \wedge \dots \wedge \mathcal{K}_{\ell_{m-r}}. \quad (2.14)$$

In particular, $\Omega^{(r)}$ is a form of degree $2m-1-2r$ and the volume form can be expressed in terms of $\Omega^{(0)}$ as $\omega_{2m-1} = \frac{1}{(m-1)!} \Omega^{(0)}$. One can show that these forms obey the relation

$$i_{\underline{v}} \Omega^{(r)} = \frac{1}{2} d\Omega^{(r+1)}, \quad (2.15)$$

and this relation allows for the construction of the gauged WZ action. In terms of these forms, the forms $\alpha^{(2m-1-2r)}$ appearing in the gauged WZ action are given by

$$\alpha^{(2m-1-2r)} = \frac{1}{\mathcal{A}_{2m-1}} \frac{1}{(m-1)!} \frac{1}{2^r} \Omega^{(r)}. \quad (2.16)$$

This collection of forms obeys the set of equations

$$i_{\underline{v}} \alpha^{(2m-1-2r)} = d\alpha^{(2m-1-2r-2)}, \quad r = 0, \dots, m-2, \quad (2.17)$$

and

$$i_{\underline{v}} \alpha^{(1)} = \frac{1}{\mathcal{A}_{2m-1}} \frac{1}{(m-1)!} \frac{1}{2^{m-1}}. \quad (2.18)$$

Since $\mathcal{A}_{2m-1} = \frac{2\pi^m}{(m-1)!}$, we can rewrite the equations satisfied by the $\alpha^{(2m-1-2r)}$ as

$$i_{\underline{v}} \alpha^{(2m-1-2r)} = d\alpha^{(2m-1-2r-2)}, \quad r = 0, \dots, m-2, \quad (2.19a)$$

$$i_{\underline{v}} \alpha^{(1)} = \frac{1}{(2\pi)^m}. \quad (2.19b)$$

Under a $U(1)$ gauge transformation $b_\ell \rightarrow e^{i\xi} b_\ell$, $A \rightarrow A + d\xi$, the gauged WZ action for the $O(2m)$ NLSM transforms as

$$\delta_\xi S_{WZ, \text{gauged}}[\mathbf{n}, A] = k \int_{X_{bdy}} \xi \left(\frac{F}{2\pi} \right)^{m-1}. \quad (2.20)$$

In the $O(2m)$ NLSM description of the boundary of the BIQH state, this anomaly of the gauged WZ term implies that the topological electromagnetic response of the bulk of the BIQH state is described by a CS term with level $N_{2m-1} = -(m!)k$, i.e., the level must be an integer multiple of $m!$. By inspecting the individual terms in the gauged WZ action, one can see that the anomaly in Eq. (2.20) is completely determined by the value of $i_{\underline{v}} \alpha^{(1)}$ as shown in Eqs. (2.19). This is because Eq. (2.19a) guarantees that the transformation of the form $\alpha^{(2m-1-2r)}$ in the r^{th} term in Eq. (2.12) is canceled by the transformation of the gauge field A in the $(r+1)^{th}$ term. This means that the final anomaly only depends on the transformation of $\alpha^{(1)}$ in the $(m-1)^{th}$ term (i.e., the last term). It turns out that the equations which define the form $\alpha^{(1)}$ do not have a unique solution, and in the computation above we have chosen a particular solution. We now show that although there is an ambiguity in the choice of solution for $\alpha^{(1)}$, the anomaly of the gauged action is not affected by this ambiguity.

B. Uniqueness of the anomaly

In the previous subsection we showed that the anomaly of the $O(2m)$ NLSM with WZ term is completely determined by the one-form $\alpha^{(1)}$ which appears in the final term of the gauged WZ action, and we also mentioned that $\alpha^{(1)}$ is not unique. If we are to ascribe any physical meaning to the

anomaly computed in the last subsection, then we need to make sure that the anomaly is not affected by the ambiguity in the choice of the form $\alpha^{(1)}$. In this section we prove that the anomaly is well-defined even though the choice of $\alpha^{(1)}$ is not unique.

We start by precisely characterizing the ambiguity in the choice of the one-form $\alpha^{(1)}$. According to Eqs. (2.19), this form should satisfy the equation

$$i_{\underline{v}} \alpha^{(3)} = d\alpha^{(1)}. \quad (2.21)$$

However, for a given three-form $\alpha^{(3)}$, the solutions to this equation for $\alpha^{(1)}$ are not unique. To see this, let us fix a choice of $\alpha^{(3)}$ (and also $\alpha^{(5)}, \dots, \alpha^{(2m-3)}$) and suppose that we have two solutions $\alpha^{(1)}$ and $\tilde{\alpha}^{(1)}$ to Eq. (2.21). If we subtract the equation for $\alpha^{(1)}$ from the equation for $\tilde{\alpha}^{(1)}$ then we find that these two forms are related by the equation

$$d(\tilde{\alpha}^{(1)} - \alpha^{(1)}) = 0, \quad (2.22)$$

i.e., the difference $\tilde{\alpha}^{(1)} - \alpha^{(1)}$ is a closed form on S^{2m-1} . However, on the sphere S^{2m-1} all closed one-forms are also exact², which means that we have

$$\tilde{\alpha}^{(1)} - \alpha^{(1)} = d\gamma^{(0)} \quad (2.23)$$

for some function $\gamma^{(0)}$ on S^{2m-1} .

We now want to understand the possible dependence of the anomaly on the function $\gamma^{(0)}$ which parametrizes the ambiguity in the solution for $\alpha^{(1)}$. Therefore we should compare the gauged WZ action constructed using $\alpha^{(1)}$ with the gauged WZ action constructed using $\tilde{\alpha}^{(1)}$ (but keeping all other terms in the action the same). Let $S_{WZ, \text{gauged}}[\mathbf{n}, A]$ be the gauged WZ action constructed using the form $\alpha^{(1)}$, and let $\tilde{S}_{WZ, \text{gauged}}[\mathbf{n}, A]$ be the gauged WZ action constructed from the form $\tilde{\alpha}^{(1)}$. These actions differ by a single term

$$\begin{aligned} \tilde{S}_{WZ, \text{gauged}}[\mathbf{n}, A] - S_{WZ, \text{gauged}}[\mathbf{n}, A] &= 2\pi k \int_{X_{bdy}} A \wedge F^{m-2} \wedge \mathbf{n}^* d\gamma^{(0)} \\ &= 2\pi k \int_{X_{bdy}} \mathbf{n}^* \gamma^{(0)} F^{m-1}, \end{aligned} \quad (2.24)$$

where we rearranged the forms and performed an integration by parts to derive the second equality. Under a gauge transformation this difference transforms as

$$\begin{aligned} \delta_\xi \tilde{S}_{WZ, \text{gauged}}[\mathbf{n}, A] - \delta_\xi S_{WZ, \text{gauged}}[\mathbf{n}, A] &= 2\pi k \int_{X_{bdy}} \mathbf{n}^* (\mathcal{L}_{\xi \underline{v}} \gamma^{(0)}) F^{m-1}. \end{aligned} \quad (2.25)$$

However, since $\gamma^{(0)}$ is a *function*, we have

$$\begin{aligned} \mathcal{L}_{\xi \underline{v}} \gamma^{(0)} &= d(\xi i_{\underline{v}} \gamma^{(0)}) + \xi i_{\underline{v}} d\gamma^{(0)} \\ &= \xi i_{\underline{v}} d\gamma^{(0)} \\ &= \xi \mathcal{L}_{\underline{v}} \gamma^{(0)}, \end{aligned} \quad (2.26)$$

where we used the fact that $i_{\underline{v}}\gamma^{(0)} = 0$. Then the difference of gauge transformations reduces to

$$\begin{aligned} \delta_\xi \tilde{S}_{WZ,gauged}[\mathbf{n}, A] - \delta_\xi S_{WZ,gauged}[\mathbf{n}, A] \\ = 2\pi k \int_{X_{bdy}} \xi \mathbf{n}^*(\mathcal{L}_{\underline{v}}\gamma^{(0)}) F^{m-1}. \end{aligned} \quad (2.27)$$

We can now make the following observation. The gauged action $S_{WZ,gauged}[\mathbf{n}, A]$ constructed using $\alpha^{(1)}$ from Eq. (2.16) still possesses *global* $U(1)$ symmetry and, in particular, is invariant under the transformation $b_\ell \rightarrow e^{i\xi} b_\ell$ for an infinitesimal *constant* parameter ξ . However, the above considerations show that under the same infinitesimal $U(1)$ transformation, the gauged action $\tilde{S}_{WZ,gauged}[\mathbf{n}, A]$ constructed from $\tilde{\alpha}^{(1)}$ will transform as

$$\delta_\xi \tilde{S}_{WZ,gauged}[\mathbf{n}, A] = 2\pi k \xi \int_{X_{bdy}} \mathbf{n}^*(\mathcal{L}_{\underline{v}}\gamma^{(0)}) F^{m-1}. \quad (2.28)$$

Now even if the gauged WZ action cannot be made to be invariant under $U(1)$ gauge transformations, we should still require it to be invariant under *global* $U(1)$ transformations. Therefore we must demand that for any alternative solution $\tilde{\alpha}^{(1)}$ to Eq. (2.21), the function $\gamma^{(0)}$ relating this form to $\alpha^{(1)}$ from Eq. (2.16) should satisfy

$$\mathcal{L}_{\underline{v}}\gamma^{(0)} = 0, \quad (2.29)$$

i.e., this function should be invariant under the action of the $U(1)$ symmetry on S^{2m-1} . Then, since we have the relation $\mathcal{L}_{\underline{v}}\gamma^{(0)} = \xi \mathcal{L}_{\underline{v}}\gamma^{(0)}$ for any function $\gamma^{(0)}$ and any spacetime-dependent ξ , we immediately find that the anomaly of the gauged WZ action is not sensitive to the ambiguity in the choice of $\alpha^{(1)}$. In other words, the requirement that the gauged WZ action should still possess *global* $U(1)$ symmetry is enough to ensure that the anomaly of the gauged action is well-defined and independent of the ambiguity in the choice of $\alpha^{(1)}$.

C. Deforming the target space

Now that we know that the anomaly in Eq. (2.20) is well-defined, we can move on and study how deformations of the target space of the NLSM might affect the anomaly. Recall that we previously derived this anomaly using the $O(2m)$ NLSM with target space S^{2m-1} . In this subsection we show that this anomaly is not affected by arbitrary smooth, symmetry-preserving deformations of the target space of the NLSM. The notion of a smooth, symmetry-preserving deformation of the target space can be formulated precisely in terms of diffeomorphisms which are *equivariant* with respect to the symmetry action, as we discuss below.

In the NLSM description of the BIQH state the target space S^{2m-1} of the $O(2m)$ NLSM is equipped with an action of the group $U(1)$. For any $g \in U(1)$ let us write $g \cdot \mathbf{n}$ to denote the image of the point $\mathbf{n} \in S^{2m-1}$ under the action of the

group element g . As we discussed above, the $U(1)$ action on S^{2m-1} is generated by the vector field \underline{v} in the sense that $n^a \rightarrow n^a + \xi v^a$ under an infinitesimal $U(1)$ transformation parametrized by ξ . Now suppose that \mathcal{M} is another $(2m-1)$ -dimensional manifold with the following properties.

- (1) There is a $U(1)$ action on \mathcal{M} generated by a vector field \underline{w} .
- (2) There exists a Riemannian metric on \mathcal{M} for which \underline{w} is a Killing vector.
- (3) There exists a diffeomorphism $f : \mathcal{M} \rightarrow S^{2m-1}$ which is *equivariant* with respect to the $U(1)$ action, i.e.,

$$g \cdot f(\mathbf{m}) = f(g \cdot \mathbf{m}), \quad \forall \mathbf{m} \in \mathcal{M}, \quad \forall g \in U(1). \quad (2.30)$$

Intuitively, these properties imply that the manifold \mathcal{M} also has a $U(1)$ symmetry, and that it can be reached from S^{2m-1} (or vice-versa) by smooth deformations which respect the $U(1)$ symmetry. We now show that for any such manifold \mathcal{M} the NLSM with target space \mathcal{M} , WZ term at level k , and $U(1)$ action generated by \underline{w} possesses the exact same perturbative $U(1)$ anomaly as the $O(2m)$ NLSM with WZ term at level k .

Before presenting the proof, we first discuss some consequences of the three properties of the map f . First, properties (1) and (2) together imply that we can construct a WZ term for the NLSM with target space \mathcal{M} with the property that the WZ term is invariant under the $U(1)$ transformation generated by \underline{w} (we construct the WZ term using the volume form on \mathcal{M} determined by its $U(1)$ -symmetric Riemannian metric). Next, the first part of property (3), namely the fact that $f : \mathcal{M} \rightarrow S^{2m-1}$ is a diffeomorphism, implies that the de Rham cohomology groups of \mathcal{M} and S^{2m-1} are identical. In addition, the fact that f is a diffeomorphism implies that the *degree* of f , defined via the equation

$$\begin{aligned} \frac{1}{\mathcal{A}_{2m-1}} \int_{\mathcal{M}} f^* \omega_{2m-1} &= \deg[f] \frac{1}{\mathcal{A}_{2m-1}} \int_{S^{2m-1}} \omega_{2m-1} \\ &= \deg[f], \end{aligned} \quad (2.31)$$

is equal to plus or minus one, $\deg[f] = \pm 1$ (see Ch. VI of Ref. 62 for the definition of the degree of a smooth map). Intuitively this means that the map f “wraps” \mathcal{M} around S^{2m-1} only once. This has to be the case since f is injective (f is invertible so it is both injective and surjective). In what follows we assume $\deg[f] = 1$ so that f is orientation-preserving. This then implies that

$$f^* \left(\frac{\omega_{2m-1}}{\mathcal{A}_{2m-1}} \right) = \frac{\omega_{\mathcal{M}}}{\mathcal{A}_{\mathcal{M}}}, \quad (2.32)$$

where $\omega_{\mathcal{M}}$ is the volume form on \mathcal{M} determined by its Riemannian metric, and $\mathcal{A}_{\mathcal{M}} = \int_{\mathcal{M}} \omega_{\mathcal{M}}$ is the area of \mathcal{M} .

Next, properties (1) and (3) together imply that

$$\underline{v} = f_* \underline{w}, \quad (2.33)$$

i.e., the vector field \underline{v} which generates the $U(1)$ action on S^{2m-1} is equal to the pushforward, via the map f , of the vector field \underline{w} that generates the $U(1)$ action on \mathcal{M} . This can be

verified by expanding out both sides of Eq. (2.30) for an element $g \in U(1)$ which is close to the identity. This property implies the following relation, which is central to the proof in this section. If α is a differential form on S^{2m-1} , then we have

$$i_{\underline{w}}(f^*\alpha) = f^*(i_{\underline{v}}\alpha). \quad (2.34)$$

This relation implies that the action of interior multiplication commutes with the action of taking the pullback, provided that we use $i_{\underline{w}}$ when acting on forms on \mathcal{M} and $i_{\underline{v}}$ when acting on forms on S^{2m-1} . Again, this relation holds because under our assumptions the vector field \underline{v} is equal to the pushforward of \underline{w} by the map f .

Now let us consider an alternative description of the boundary of a BIQH state in terms of a NLSM with target space \mathcal{M} , where \mathcal{M} satisfies the three properties stated above. The field in this NLSM theory, which we denote by \mathbf{m} , is a map from the boundary spacetime to the manifold \mathcal{M} , $\mathbf{m} : X_{bdy} \rightarrow \mathcal{M}$. We also assume that the transformation of the NLSM field \mathbf{m} under the $U(1)$ symmetry of the BIQH state is determined by the $U(1)$ action on \mathcal{M} generated by the vector field \underline{w} . For example, under an infinitesimal $U(1)$ transformation parametrized by ξ we have $m^a \rightarrow m^a + \xi w^a$, $\forall a$. The WZ term for this NLSM is constructed in the same way as for the NLSM with target space S^{2m-1} . We start with a volume form $\omega_{\mathcal{M}}$ on \mathcal{M} which we assume is obtained from a $U(1)$ -symmetric Riemannian metric on \mathcal{M} ² (which exists by our assumption (2) above). We denote the normalized volume form on \mathcal{M} by $\beta^{(2m-1)} = \frac{\omega_{\mathcal{M}}}{\mathcal{A}_{\mathcal{M}}}$, where $\mathcal{A}_{\mathcal{M}} = \int_{\mathcal{M}} \omega_{\mathcal{M}}$. We also need an extension $\tilde{\mathbf{m}}$ of the NLSM field \mathbf{m} into the extended boundary spacetime \mathcal{B} such that $\tilde{\mathbf{m}}|_{\partial\mathcal{B}} = \mathbf{m}$. In terms of these quantities, the WZ term for the NLSM with target space \mathcal{M} can be written in the compact form

$$S_{WZ}[\mathbf{m}] = 2\pi k \int_{\mathcal{B}} \tilde{\mathbf{m}}^* \beta^{(2m-1)}. \quad (2.35)$$

We can now attempt to couple $S_{WZ}[\mathbf{m}]$ to the gauge field A and study the perturbative anomaly of the gauged action. We find that the gauged WZ term for the NLSM with target space \mathcal{M} takes the form

$$S_{WZ,gauged}[\mathbf{m}, A] = S_{WZ}[\mathbf{m}] + 2\pi k \sum_{r=1}^{m-1} \int_{X_{bdy}} A \wedge F^{r-1} \wedge \mathbf{m}^* \beta^{(2m-1-2r)}, \quad (2.36)$$

where the forms $\beta^{(2m-1-2r)}$ on \mathcal{M} are obtained by pulling back the forms $\alpha^{(2m-1-2r)}$ on S^{2m-1} which appear in the gauged WZ action for the $O(2m)$ NLSM,

$$\beta^{(2m-1-2r)} = f^* \alpha^{(2m-1-2r)}. \quad (2.37)$$

The explicit form of $\alpha^{(2m-1-2r)}$ was given above in Eq. (2.16). Using Eq. (2.34) and the fact that the pullback operation commutes with the exterior derivative, we find that

the forms $\beta^{(2m-1-2r)}$ for $r = 0, 1, \dots, m-1$, obey the set of equations

$$i_{\underline{w}} \beta^{(2m-1-2r)} = d\beta^{(2m-1-2r-2)}, \quad r = 0, \dots, m-2, \quad (2.38a)$$

$$i_{\underline{w}} \beta^{(1)} = \frac{1}{(2\pi)^m}. \quad (2.38b)$$

These equations are identical to Eqs. (2.19) but with \underline{v} replaced by \underline{w} and $\alpha^{(2m-1-2r)}$ replaced by $\beta^{(2m-1-2r)}$. The form of these equations implies that the NLSM theory with target space \mathcal{M} has the exact same perturbative $U(1)$ anomaly as the $O(2m)$ NLSM with target space S^{2m-1} . In addition, our argument for the uniqueness of the anomaly from the previous subsection also applies to the theory with target space \mathcal{M} . This follows from the fact that the de Rham cohomology groups of \mathcal{M} are identical to those of S^{2m-1} as a consequence of our assumption (3). Therefore we have shown that the perturbative $U(1)$ anomaly at the boundary of the BIQH state is robust against arbitrary smooth, symmetry-preserving deformations of the target space of the NLSM used to describe the BIQH state.

III. GLOBAL ANOMALIES IN BOSONIC TOPOLOGICAL INSULATOR STATES

In this section we study global anomalies at the boundary of a class of bosonic SPT phases which exist in even spacetime dimensions and are protected by the symmetry of the group $G = U(1) \rtimes \mathbb{Z}_2$. We refer to these phases as bosonic topological insulator (BTI) phases. They are generalizations to all even-dimensional spacetimes of the BTI phase introduced in Ref. 32. Note also that the system of bosons studied in Ref. 63 can be considered to be an example of a $(1+1)$ -dimensional BTI state according to our definition. In all cases the $U(1)$ symmetry represents a physical charge conservation symmetry, however, the character of the \mathbb{Z}_2 symmetry depends on the specific dimension of spacetime. Let the bulk spacetime dimension be $2m$ for a positive integer m . Then for m odd the \mathbb{Z}_2 symmetry is a unitary charge-conjugation symmetry, while for m even the \mathbb{Z}_2 symmetry is an anti-unitary time-reversal symmetry.

In Ref. 36 we argued that the boundary theory of the $2m$ -dimensional BTI state exhibits a bosonic analogue of the parity anomaly of a Dirac fermion in odd dimensions. Our argument was based on the form of the gauged WZ action in an $O(2m+1)$ NLSM description of the boundary of these phases. Specifically, we showed that if the NLSM field on the boundary of the BTI condensed in such a way as to break the \mathbb{Z}_2 symmetry but preserve the $U(1)$ symmetry of the BTI phase, then the boundary would exhibit a BIQH response with *half-quantized* CS coefficient $N_{2m-1} = \frac{m!}{2}$. We then argued by analogy with the free fermion topological insulator^{10,11} that this half-quantized BIQH response indicated that the boundary of the BTI phase displays a bosonic analogue of the parity anomaly.

In this section we make this reasoning precise in the special case of the BTI state in $1+1$ spacetime dimensions. In this case we are able to compute the boundary partition function exactly, and the global anomaly can be seen clearly from our exact result. We start by reviewing the form of the $O(3)$ NLSM action which describes the $(0+1)$ -dimensional boundary of this BTI state, including the form of the gauged WZ action which describes the boundary theory coupled to the external gauge field A ³⁶. We then explicitly compute the boundary partition function and show that it cannot retain both the \mathbb{Z}_2 symmetry of the BTI and large $U(1)$ gauge invariance, i.e., the boundary theory possesses a *global anomaly* in the \mathbb{Z}_2 symmetry of the BTI state. We then consider arbitrary smooth, symmetry-preserving deformations of the target space of the NLSM used to describe the BTI, and we use the powerful *equivariant localization* (EL) technique to show that the boundary partition function and the global anomaly are robust against such deformations of the model. We also note here that the global anomaly computed in this section is very similar to the global anomaly computed in Ref. 48 for a single Dirac fermion in $(0+1)$ -dimensions with $U(1)$ symmetry and unitary \mathbb{Z}_2 charge-conjugation symmetry.

A. The BTI state in $1+1$ dimensions and its $O(3)$ NLSM description

The BTI state in $1+1$ dimensions is an SPT phase of bosons with symmetry group $G = U(1) \rtimes \mathbb{Z}_2$, where $U(1)$ represents charge conservation and \mathbb{Z}_2 is a unitary charge-conjugation (or particle-hole) symmetry. The semi-direct product “ \rtimes ” indicates that the $U(1)$ and \mathbb{Z}_2 symmetries do not commute with each other. The physical signature of the BTI state is that a fractional charge of $\pm \frac{1}{2}$ (in units of the boson charge) is bound at an interface between the BTI state and the vacuum (or a trivial state). One possible model for the bulk of the BTI state is an $O(3)$ NLSM with theta term and coefficient $\theta = 2\pi k$, $k \in \mathbb{Z}$ ³⁷. The boundary of the BTI state is then described by the same NLSM but with a WZ term at level k . In $1+1$ dimensions SPT phases with the symmetry group $G = U(1) \rtimes \mathbb{Z}_2$ have a \mathbb{Z}_2 classification, meaning that there is only a single nontrivial phase^{37,64}. This single nontrivial phase is the BTI state. Within the NLSM description, the NLSM for any odd k represents the nontrivial BTI state, while the model for any even k represents the trivial state.

In the $O(3)$ NLSM description the field is a unit vector field \mathbf{n} with components n^a , $a = 1, 2, 3$. The target space of the $O(3)$ NLSM is the unit two-sphere S^2 . As in Sec. II, the action of the symmetry group $G = U(1) \rtimes \mathbb{Z}_2$ of the BTI on the NLSM field is best expressed by first combining n^1 and n^2 into the “boson” field $b = n^1 + in^2$. Then for the BTI state, the action of G on the NLSM field is given by (see Sec. IV.D

of Ref. 37)

$$U(1) : b \rightarrow e^{i\xi} b, \quad (3.1)$$

and

$$\mathbb{Z}_2 : b \rightarrow b^* \quad (3.2a)$$

$$n^3 \rightarrow -n^3. \quad (3.2b)$$

Since the \mathbb{Z}_2 symmetry is unitary, the transformation $b \rightarrow b^*$ is equivalent to $n^1 \rightarrow n^1$, $n^2 \rightarrow -n^2$. We can interpret b as the field which annihilates a boson of charge 1, and n^3 can be interpreted as the deviation of the boson density from a non-zero constant value.

The theta term and the WZ term for the $O(3)$ NLSM are both expressed in terms of the volume form ω_2 on S^2 ,

$$\omega_2 = n^1 dn^2 \wedge dn^3 - n^2 dn^1 \wedge dn^3 + n^3 dn^1 \wedge dn^2. \quad (3.3)$$

In what follows we use $\mathcal{A}_2 = 4\pi$ to denote the surface area of S^2 (and $\int_{S^2} \omega_2 = \mathcal{A}_2$). In this article we are only interested in the boundary theory of the BTI, and so we focus our attention on the WZ term. The boundary theory lives in one spacetime dimension. To make our discussion as precise as possible, we take the time coordinate (the only coordinate here) to lie in the interval $t \in [0, T)$, and we impose periodic boundary conditions in the time direction. This makes our one-dimensional spacetime into a circle of circumference T . Let us denote the one-dimensional spacetime by S_T^1 (the circle of circumference T). Constructing the WZ term requires extending the spacetime into a two-dimensional spacetime \mathcal{B} such that $\partial\mathcal{B} = S_T^1$. We use $\tilde{\mathbf{n}}$ to denote the extension of the NLSM field \mathbf{n} into the bulk of \mathcal{B} , and we require that $\tilde{\mathbf{n}}|_{\partial\mathcal{B}} = \mathbf{n}$. Using \mathcal{B} and the extension $\tilde{\mathbf{n}}$ of \mathbf{n} , the WZ term takes the form

$$S_{WZ}[\mathbf{n}] = \frac{2\pi k}{\mathcal{A}_2} \int_{\mathcal{B}} \tilde{\mathbf{n}}^* \omega_2, \quad (3.4)$$

where $\tilde{\mathbf{n}}^* \omega_2$ denotes the pullback of ω_2 to \mathcal{B} via the map $\tilde{\mathbf{n}} : \mathcal{B} \rightarrow S^2$, and k is the level of the WZ term (the same integer k determines the coefficient $\theta = 2\pi k$ of the theta term describing the bulk of the SPT phase).

The complete $O(3)$ NLSM action describing the boundary of the BTI takes the form

$$S_{bdy}[\mathbf{n}] = \int_0^T dt \frac{1}{2f_{bdy}} [(\partial^t b)^*(\partial_t b) + (\partial^t n^3)(\partial_t n^3)] + S_{WZ}[\mathbf{n}], \quad (3.5)$$

where f_{bdy} is a boundary coupling constant and $\partial^t = \partial_t$ for our choice of the signature of the spacetime metric (we use a “mostly minus” Minkowski metric). We can now consider coupling the boundary theory to an external $U(1)$ gauge field $A = A_t dt$. In Ref. 36, we showed that the properly gauged boundary action has the form

$$S_{bdy,gauged}[\mathbf{n}, A] = \int_0^T dt \frac{1}{2f_{bdy}} [(D^t b)^*(D_t b) + (\partial^t n^3)(\partial_t n^3)] + S_{WZ,gauged}[\mathbf{n}, A], \quad (3.6)$$

where

$$S_{WZ,gauged}[\mathbf{n}, A] = S_{WZ}[\mathbf{n}] + \frac{2\pi k}{\mathcal{A}_2} \int_0^T dt n^3 A_t, \quad (3.7)$$

and $D_t = \partial_t - iA_t$ ($\partial_t = \partial^t$, $A_t = A^t$, etc., for our choice of signature). The action for the fully gauged boundary theory is invariant under $U(1)$ gauge transformations

$$b \rightarrow e^{i\xi} b \quad (3.8a)$$

$$A \rightarrow A + d\xi, \quad (3.8b)$$

and \mathbb{Z}_2 transformations

$$b \rightarrow b^* \quad (3.9a)$$

$$n^3 \rightarrow -n^3 \quad (3.9b)$$

$$A \rightarrow -A. \quad (3.9c)$$

B. Boundary partition function and global anomaly

We now study the partition function for the gauged boundary theory of the BTI in the topological limit $f_{bdy} \rightarrow \infty$. In this limit we keep only the low energy information about the boundary theory, including possible anomalies. The partition function

$$Z[A] = \int [d\mathbf{n}] e^{iS_{bdy,gauged}[\mathbf{n}, A]} \quad (3.10)$$

of the gauged boundary theory can be evaluated very simply in this limit, as we now discuss. First, in the limit $f_{bdy} \rightarrow \infty$ the path integral we need to evaluate is

$$Z[A] = \int [d\mathbf{n}] e^{iS_{WZ,gauged}[\mathbf{n}, A]}, \quad (3.11)$$

where $S_{WZ,gauged}[\mathbf{n}, A]$ is the gauged WZ action from Eq. (3.7). The path integral measure appearing here has the precise definition

$$[d\mathbf{n}] = \prod_{t \in [0, T]} \omega_2(t), \quad (3.12)$$

where $\omega_2(t)$ denotes the volume form for a copy of S^2 located at the point t in spacetime, and we integrate over all field configurations with periodic boundary conditions in time.

We can also use a gauge transformation to simplify the form of the coupling to the gauge field A . In one spacetime dimension the gauge field one-form $A = A_t dt$ has only one component. Since our spacetime is a circle, which has first cohomology group $H^1(S^1, \mathbb{R}) = \mathbb{R}$, we can decompose a generic A_t as

$$A_t = \bar{A}_t + \partial_t \lambda, \quad (3.13)$$

where

$$\bar{A}_t := \frac{1}{T} \int_0^T dt A_t \quad (3.14)$$

represents the nontrivial part of A , and $\partial_t \lambda$ represents the exact part of A (here λ is some function of t). The exact part of A can be removed from the action via a small $U(1)$ gauge transformation, which are those gauge transformations $A \rightarrow A + \partial_t \xi$ with the function ξ satisfying $\xi(0) = \xi(T)$. Large $U(1)$ gauge transformations are those transformations which send $\bar{A}_t \rightarrow \bar{A}_t + \frac{2\pi n}{T}$, for any $n \in \mathbb{Z}$, and they will play an important role in the discussion of the global anomaly in this theory later in this section. The upshot of all of this is that we can replace the coupling to A_t in the gauged WZ action with a coupling to the *constant* gauge field \bar{A}_t .

We now move on to the calculation of the partition function $Z[A]$. We compute the partition function by observing that it is identical to the phase space path integral for a spin of magnitude $J = \frac{k}{2}$ (or $\frac{|k|}{2}$ for negative k) in a constant magnetic field B pointing in the 3-direction, with the magnitude of the magnetic field given in terms of the gauge field A by $B = -\bar{A}_t$. To prove this we now briefly review the form of the phase space path integral for spin. At this point we recommend that the reader skim through Appendix A where we review the phase space path integral expression for the partition function of a quantum mechanical system obtained by quantizing a general classical system defined on a phase space \mathcal{M} equipped with a symplectic form ω and Hamiltonian function H .

The classical mechanics of a spin $J = \frac{1}{2}, 1, \frac{3}{2}, \dots$, is described by a phase space $\mathcal{M} = S^2$ equipped with the symplectic form $\omega = J\omega_2$, where ω_2 is the volume form on S^2 from Eq. (3.3). It is convenient to work in spherical coordinates (ϕ, θ) on S^2 . In this system of coordinates the components of the NLSM field \mathbf{n} take the form

$$n^1 = \sin(\theta) \cos(\phi) \quad (3.15a)$$

$$n^2 = \sin(\theta) \sin(\phi) \quad (3.15b)$$

$$n^3 = \cos(\theta), \quad (3.15c)$$

and we have

$$\omega = J \sin(\theta) d\theta \wedge d\phi. \quad (3.16)$$

Using the definition Eq. (A4) for the Poisson bracket one can check that

$$\{n^a, n^b\} = \frac{1}{J} \sum_c \epsilon^{abc} n^c, \quad (3.17)$$

so that the spin components S^a are given in terms of n^a by

$$S^a = J n^a. \quad (3.18)$$

The spin components then obey the Poisson algebra

$$\{S^a, S^b\} = \sum_c \epsilon^{abc} S^c. \quad (3.19)$$

We can now see that replacing the Poisson bracket with a commutator according to the rule $\{\cdot, \cdot\} \rightarrow -i[\cdot, \cdot]$ will give the usual commutation relations for spin in quantum mechanics.

Now let us assume that the dynamics of the spin system are specified by a Hamiltonian H . Then the phase space path integral representing the partition function $\text{tr}_J[e^{-iHT}]$, where

the trace is taken in the spin J representation of $SU(2)$, takes the form

$$\text{tr}_J[e^{-iHT}] = \int [d\phi d\theta] \left[\prod_{t \in [0, T)} J \sin(\theta(t)) \right] e^{iS[\phi, \theta]} . \quad (3.20)$$

where

$$S[\phi, \theta] = \int_0^T dt [\vartheta_\phi \partial_t \phi + \vartheta_\theta \partial_t \theta - H(\theta, \phi)] . \quad (3.21)$$

Here ϑ_ϕ and ϑ_θ are the components of the symplectic potential ϑ , which is defined locally on the phase space by the relation $\omega = d\vartheta$ (Eq. (A8) in Appendix A). Then, since

$$(\vartheta_\phi \partial_t \phi + \vartheta_\theta \partial_t \theta) dt = \mathbf{n}^* \vartheta , \quad (3.22)$$

we can rewrite the first term in this action using an extension \mathcal{B} of the spacetime S_T^1 and an extension $\tilde{\mathbf{n}}$ of the field configuration (satisfying $\tilde{\mathbf{n}}|_{\partial \mathcal{B}} = \mathbf{n}$). We have

$$\begin{aligned} \int_{S_T^1} \mathbf{n}^* \vartheta &= \int_{\mathcal{B}} \tilde{\mathbf{n}}^* \omega \\ &= J \int_{\mathcal{B}} \tilde{\mathbf{n}}^* \omega_2 , \end{aligned} \quad (3.23)$$

where the first line follows from Stokes' theorem. If we choose the Hamiltonian to be

$$H = BS^3 = BJn^3 , \quad (3.24)$$

which is the Hamiltonian for a spin in a constant magnetic field of magnitude B and pointing in the 3-direction, then the action becomes

$$S[\phi, \theta] = J \int_{\mathcal{B}} \tilde{\mathbf{n}}^* \omega_2 - J \int_{S_T^1} n^3 B . \quad (3.25)$$

We can now compare the path integral for a spin in a magnetic field to our path integral in Eq. (3.11) for the partition function of the boundary of the BTI state. Using the fact that $\mathcal{A}_2 = 4\pi$, we find that these path integrals are identical if we make the identifications

$$J = \frac{k}{2} \quad (3.26a)$$

$$B = -\bar{A}_t . \quad (3.26b)$$

More precisely, the path integrals are not identical but differ by the infinite constant factor

$$\prod_{t \in [0, T)} J , \quad (3.27)$$

but we can give a more careful definition of the path integral measure for the partition function of the gauged boundary theory for the BTI by including this factor. Using all of this in-

formation we then find that

$$\begin{aligned} Z[A] &= \text{tr}_{\frac{k}{2}} [e^{iS^3 \bar{A}_t T}] \\ &= \sum_{j=-\frac{k}{2}}^{\frac{k}{2}} e^{ij \bar{A}_t T} \\ &= \frac{\sin \left[\frac{\bar{A}_t T}{2} (k+1) \right]}{\sin \left[\frac{\bar{A}_t T}{2} \right]} . \end{aligned} \quad (3.28)$$

Note that in deriving this formula we assumed that $k > 0$. For $k < 0$ one just needs to replace k with $|k|$. For the discussion below it is useful to decompose the gauge field as $\bar{A}_t = \frac{2\pi\ell}{T} + \bar{a}_t$ for some $\ell \in \mathbb{Z}$ and $\bar{a}_t \in (0, \frac{2\pi}{T})$, and to then rewrite $Z[A]$ in terms of ℓ and \bar{a}_t ,

$$Z[A] = (-1)^{k\ell} \frac{\sin \left[\frac{\bar{a}_t T}{2} (k+1) \right]}{\sin \left[\frac{\bar{a}_t T}{2} \right]} . \quad (3.29)$$

It is important to observe that the factor $(-1)^{k\ell}$ is nontrivial for odd k . This minus sign is related to the global anomaly in this theory for odd k , as we now discuss.

For any level k the partition function $Z[A]$ respects the \mathbb{Z}_2 symmetry of the BTI state, i.e., we have

$$Z[-A] = Z[A] . \quad (3.30)$$

However, for odd k the partition function is not invariant under a large $U(1)$ gauge transformation,

$$\bar{A}_t \rightarrow \bar{A}_t + \frac{2\pi}{T} , \quad (3.31)$$

which is equivalent to the transformation $\ell \rightarrow \ell + 1$ if we decompose the gauge field as $\bar{A}_t = \frac{2\pi\ell}{T} + \bar{a}_t$. Instead, for odd k the partition function $Z[A]$ changes sign under this transformation. We can try to fix this large gauge invariance issue by modifying the partition function to

$$\tilde{Z}[A] = Z[A] e^{\pm \frac{k}{2} \bar{A}_t T} . \quad (3.32)$$

This is equivalent to adding the local counterterm $\pm \frac{1}{2} \int_0^T dt A_t$ to the original boundary action, which is a $(0+1)$ -dimensional Chern-Simons term with fractional level $\pm \frac{1}{2}$. Note, however, that adding this counterterm spoils the invariance of the partition function under the action of the \mathbb{Z}_2 symmetry. Therefore we find that although the gauged action $S_{WZ, \text{gauged}}[\mathbf{n}, A]$ for the BTI boundary has large $U(1)$ gauge invariance and \mathbb{Z}_2 symmetry, the partition function $Z[A]$ for the boundary theory only has both of these symmetries when k is even.

This is a classic sign of a *global anomaly* in the \mathbb{Z}_2 symmetry: for odd k we can quantize the theory in such a way as to keep either the \mathbb{Z}_2 symmetry or large $U(1)$ gauge invariance, but not both. Physically, this anomaly is related to the fact that for odd k the boundary of the BTI has states with half-integer (i.e., fractional) charge. In addition, the fact that the presence or absence of the anomaly depends only on the parity of

k (even or odd) is due to the aforementioned \mathbb{Z}_2 classification of bosonic SPT phases with $G = U(1) \rtimes \mathbb{Z}_2$ symmetry in $1+1$ dimensions (the theories with odd k all represent the nontrivial BTI state, while the theories with even k all represent the trivial phase). As we discussed above, the anomaly here is very similar to the global anomaly computed in Ref. 48 for a Dirac fermion in $0+1$ dimensions with $U(1)$ and \mathbb{Z}_2 symmetry. In addition, a similar anomaly in the (purely bosonic) $(0+1)$ -dimensional theory of a particle on a ring was discussed recently in Appendix D of Ref. 65.

C. Deforming the target space

In the previous subsection we showed that, at least within the $O(3)$ NLSM description, the boundary of the $(1+1)$ -dimensional BTI phase exhibits a global anomaly in the \mathbb{Z}_2 symmetry of the BTI phase. However, our derivation of the anomaly seemed to rely on the specific geometry of the target space S^2 of the $O(3)$ NLSM. Specifically, our derivation used the fact that the partition function for the BTI boundary was equivalent to a phase space path integral for a spin in a magnetic field. In addition, since $U(1) \rtimes \mathbb{Z}_2$ is a subgroup of $SO(3)$, the anomaly we derived is closely related to the global $SO(3)$ anomaly of the $O(3)$ NLSM with WZ term in $0+1$ dimensions (see, for example, the discussion in Sec. 1.2 of Ref. 66). Our calculation then shows that the $U(1) \rtimes \mathbb{Z}_2$ subgroup of $SO(3)$ is also anomalous in this theory.

In the rest of this section we show that the boundary anomaly of the BTI state is not affected by any smooth deformation of the target space S^2 of the $O(3)$ NLSM which also preserves the $U(1) \rtimes \mathbb{Z}_2$ symmetry of the BTI phase. In other words, we break the $SO(3)$ symmetry of the model down to $U(1) \rtimes \mathbb{Z}_2$, and we show that the anomaly still exists in these less symmetric theories.

In this subsection we describe the geometry of such deformed target spaces, and then we construct models of the BTI boundary using WZ terms for NLSMs with these deformed target spaces. We also show how to properly gauge these WZ actions. In the next subsection we use the equivariant localization (EL) technique to compute the partition function for these models, and we show that all such models have a partition function which is identical to Eq. (3.28). Thus, we find that the boundary anomaly is completely unaffected by smooth, symmetry-preserving deformations of the target space of the NLSM.

As stated above, we consider descriptions of the BTI using NLSMs with a target space \mathcal{M} that can be obtained from the target space S^2 of the $O(3)$ NLSM by smooth deformations which preserve the $G = U(1) \rtimes \mathbb{Z}_2$ symmetry of the BTI phase. As in Sec. II, we can characterize such spaces \mathcal{M} precisely through the notion of a diffeomorphism which is *equivariant* with respect to the symmetry of the BTI phase. The target space of the $O(3)$ NLSM is S^2 , and the NLSM description of the BTI phase includes an action of the group $G = U(1) \rtimes \mathbb{Z}_2$ on S^2 . This action was shown explicitly in Eq. (3.1) and Eqs. (3.2). Let us assume that the manifold \mathcal{M} is also equipped with an action of the group G . Then a dif-

feomorphism $f : \mathcal{M} \rightarrow S^2$ is equivariant with respect to G if

$$f(g \cdot \mathbf{m}) = g \cdot f(\mathbf{m}), \quad \forall g \in G, \quad \forall \mathbf{m} \in \mathcal{M}. \quad (3.33)$$

This is the correct mathematical notion corresponding to the intuitive idea of a manifold which can be obtained from S^2 by smooth, symmetry-preserving deformations.

The spaces \mathcal{M} which are related to S^2 in this way can be realized as surfaces of revolution in \mathbb{R}^3 which are symmetric under rotation about the z -axis (this guarantees $U(1)$ symmetry), and which are also invariant under reflection $z \rightarrow -z$ through the x - y plane². The latter condition guarantees that \mathcal{M} possesses the \mathbb{Z}_2 symmetry of the BTI phase. These spaces \mathcal{M} are completely specified by a parametric curve $(r(\sigma), z(\sigma))$, where $r(\sigma)$ is the distance of the surface from the z -axis in \mathbb{R}^3 at the height $z(\sigma)$, and $\sigma \in [a, b]$ is a parameter used to specify the curve. If we think of $(r(\sigma), z(\sigma))$ as, say, a curve in the x - z plane (replace r with x), then we can imagine constructing the full surface \mathcal{M} by rotating the curve about the z -axis in \mathbb{R}^3 . We can then choose coordinates on \mathcal{M} to be (σ, ϕ) , where ϕ is the usual azimuthal angle in spherical or cylindrical coordinates in \mathbb{R}^3 . Finally, in order for this construction to produce a smooth manifold (with no conical singularities at the top and bottom), we require that $\frac{dz}{d\sigma} = 0$ at the top and bottom of the curve. This is equivalent to the condition

$$\frac{\partial_\sigma z(\sigma)|_{\sigma=a,b}}{\partial_\sigma r(\sigma)|_{\sigma=a,b}} = 0, \quad (3.34)$$

or just

$$\partial_\sigma z(\sigma)|_{\sigma=a,b} = 0, \quad (3.35)$$

assuming that $\partial_\sigma r(\sigma)$ does not vanish at $\sigma = a, b$.

In principle we can use any parametrization of the surface, but the most convenient choice is a parametrization $(r(s), z(s))$ in terms of the arc length s along the curve, where

$$s(\sigma) = \int_a^\sigma d\sigma' \sqrt{(\partial_{\sigma'} r(\sigma'))^2 + (\partial_{\sigma'} z(\sigma'))^2}. \quad (3.36)$$

We define $L = s(b)$ to be the total length of the curve. In the coordinate system (s, ϕ) , the metric on \mathcal{M} takes the form

$$g = ds \otimes ds + [r(s)]^2 d\phi \otimes d\phi, \quad (3.37)$$

and the volume form is

$$\omega_{\mathcal{M}} = r(s) ds \wedge d\phi. \quad (3.38)$$

The total area of the target space is then $\mathcal{A}_{\mathcal{M}} = 2\pi \int_0^L ds r(s)$. In addition, the “unit speed” property $(\partial_s r(s))^2 + (\partial_s z(s))^2 = 1$ of the arc length parametrization, combined with the restriction $\partial_s z(s)|_{s=0,L} = 0$, implies that $\partial_s r(s)|_{s=0} = 1$ and $\partial_s r(s)|_{s=L} = -1$. The signs here follow from the fact that the width of the surface \mathcal{M} *increases* from zero near $s = 0$ and *decreases* back to zero at $s = L$. We also assume that $z(s) = -z(L - s)$ so that \mathcal{M} is symmetric under reflection through the $z = 0$ plane in \mathbb{R}^3 .

We can now construct a model for the boundary of the BTI using the NLSM with target space \mathcal{M} . We denote the NLSM field by $\mathbf{m} = (m^1, m^2)$, with components $m^1 = s$ and $m^2 = \phi$. In the low energy (topological) limit the boundary action contains only a WZ term for \mathbf{m} . As usual, to construct this term we require an extension \mathcal{B} of the boundary spacetime S_T^1 , and an extension $\tilde{\mathbf{m}}$ of the NLSM field \mathbf{m} into the bulk of \mathcal{B} . Then the WZ action describing the low energy physics of the boundary is

$$S_{WZ}[\mathbf{m}] = \frac{2\pi k}{\mathcal{A}_{\mathcal{M}}} \int_{\mathcal{B}} \tilde{\mathbf{m}}^* \omega_{\mathcal{M}}, \quad (3.39)$$

where $k \in \mathbb{Z}$ is the level of the WZ term. We choose the $U(1)$ and \mathbb{Z}_2 symmetries of the BTI state to act on the components of the field \mathbf{m} as

$$U(1) : \phi \rightarrow \phi + \xi, \quad (3.40)$$

and

$$\mathbb{Z}_2 : \phi \rightarrow -\phi \quad (3.41a)$$

$$s \rightarrow L - s. \quad (3.41b)$$

This action of the \mathbb{Z}_2 symmetry is the generalization to the target space \mathcal{M} of the \mathbb{Z}_2 action on S^2 from Eqs. (3.2).

The next step is to gauge the $U(1)$ symmetry by coupling the boundary WZ action to the gauge field $A = A_t dt$. One can check that the action

$$S_{WZ,gauged}[\mathbf{m}, A] = S_{WZ}[\mathbf{m}] - \frac{2\pi k}{\mathcal{A}_{\mathcal{M}}} \int_0^T dt f(s(t)) A_t \quad (3.42)$$

will be invariant under the gauge transformation $\phi \rightarrow \phi + \xi$, $A \rightarrow A + d\xi$, if the function $f(s)$ satisfies the first order differential equation

$$\partial_s f(s) = r(s). \quad (3.43)$$

This equation has the simple solution $f(s) = C + \int_0^s ds' r(s')$, where C is an as yet undetermined constant. However, since we require the gauged action to be invariant under the charge-conjugation operation

$$\mathbb{Z}_2 : \phi \rightarrow -\phi \quad (3.44a)$$

$$s \rightarrow L - s \quad (3.44b)$$

$$A \rightarrow -A, \quad (3.44c)$$

we find that this constant is fixed to take the value $C = -\frac{\mathcal{A}_{\mathcal{M}}}{4\pi}$. Therefore the function $f(s)$ appearing in the gauged boundary action is given by

$$f(s) = \int_0^s ds' r(s') - \frac{\mathcal{A}_{\mathcal{M}}}{4\pi}. \quad (3.45)$$

In particular we have

$$f(L) = -f(0) = \frac{\mathcal{A}_{\mathcal{M}}}{4\pi}, \quad (3.46)$$

which will be needed for the calculation of the partition function in the next subsection.

D. Boundary partition function and global anomaly for all target spaces

We now turn to the evaluation of the partition function $Z[A]$ for the NLSM with target space \mathcal{M} and action given by Eq. (3.42) using the equivariant localization (EL) technique. We give a brief introduction to the EL technique in Appendix B, and in Appendix C we show how to calculate the Pfaffians which appear in the final expression for $Z[A]$. Therefore, in this section we only outline the calculation and present the result. The final result for the partition function turns out to be *completely identical* to the partition function of Eq. (3.28) which we derived for the special case of the $O(3)$ NLSM with target space S^2 . The mechanism which underlies the EL technique allows us to understand why this is the case. First, the EL technique applied to our particular problem yields the result that the partition function depends only on field configurations \mathbf{m} near the points on \mathcal{M} which are fixed by the $U(1)$ action. These are just the two points $s = 0$ and $s = L$ at the bottom and the top of \mathcal{M} . The value of the gauged WZ action at these two points is actually *independent* of the specific choice of the target space \mathcal{M} (see Eqs. (3.52) below). Therefore we find that since the partition function only receives contributions from field configurations near $s = 0$ and $s = L$, and since the action at those two points is independent of the details of \mathcal{M} , the partition function $Z[A]$ is independent of the specific details of the target space \mathcal{M} . The discussion here is meant to be heuristic, and so we now move on to a more detailed presentation of the calculation.

We start by rewriting the gauged WZ action for the NLSM in a way which makes the problem of computing the partition function of this theory look like a phase space path integral for a dynamical system with phase space \mathcal{M} . The reason for this is that the EL technique, in its original formulation, applies to phase space path integrals. To achieve this goal we first recall that we can use a small $U(1)$ gauge transformation to replace the gauge field A_t with its time average \bar{A}_t in the gauged WZ action. Next, we rewrite the gauged WZ action as

$$S_{WZ,gauged}[\mathbf{m}, A] = \int_{\mathcal{B}} \tilde{\mathbf{m}}^* \omega - \int_0^T dt H(\mathbf{m}), \quad (3.47)$$

where we defined

$$\omega = \frac{2\pi k}{\mathcal{A}_{\mathcal{M}}} \omega_{\mathcal{M}} \quad (3.48a)$$

$$H(\mathbf{m}) = \frac{2\pi k}{\mathcal{A}_{\mathcal{M}}} f(s) \bar{A}_t. \quad (3.48b)$$

We can now see that the path integral for $Z[A]$ is equivalent to a phase space path integral (see our Appendix A for a review) for a dynamical system described by the triple (\mathcal{M}, ω, H) , with the symplectic form ω and Hamiltonian H defined by Eqs. (3.48). The Hamiltonian H and the symplectic form ω are related via the equation $dH = -i_{\underline{v}}\omega$, where the vector field \underline{v} is given by

$$\underline{v} = \bar{A}_t \partial_{\phi}. \quad (3.49)$$

This vector field is clearly proportional to the vector field ∂_{ϕ} which generates the action of the $U(1)$ part of the symmetry

group $G = U(1) \rtimes \mathbb{Z}_2$ of the BTI on the target space \mathcal{M} of the NLSM. The classical equations of motion for this system are

$$\dot{s} = 0 \quad (3.50a)$$

$$\dot{\phi} = \bar{A}_t. \quad (3.50b)$$

These equations say that (classically) each point on \mathcal{M} revolves around the z -axis in \mathbb{R}^3 with a period $\frac{2\pi}{\bar{A}_t}$. In the notation of Appendix B the classical equations of motion can be rewritten as $V_S^a[\mathbf{m}(t); t] = 0$, $a = 1, 2$, where $V_S^a[\mathbf{m}(t); t] = \dot{m}^a(t) - v^a(\mathbf{m}(t))$ and v^a are the components of the vector field \underline{v} from Eq. (3.49).

We are now almost ready to apply the EL results from Appendix B to compute the partition function. First, let us assume that $T \neq \frac{2\pi n}{\bar{A}_t}$ for any $n \in \mathbb{Z}$. This means that the only T -periodic solutions to the classical equations of motion for the dynamical system defined by (\mathcal{M}, ω, H) are the *constant* solutions $s = 0$ and $s = L$. Therefore, the set $L\mathcal{M}_S$ of T -periodic solutions to the classical equations of motion (defined in Eq. (B18)) has only these two elements, and the final result for the partition function $Z[A]$ only involves contributions from field configurations close to these solutions. Using the EL technique we find that the partition function can be expressed only in terms of contributions from $s = 0$ and $s = L$ as

$$Z[A] \sim \frac{e^{iS_{WZ, \text{gauged}}[\mathbf{m}, A]_{s=0}}}{\text{Pf}[\mathcal{O}]_{s=0}} + \frac{e^{iS_{WZ, \text{gauged}}[\mathbf{m}, A]_{s=L}}}{\text{Pf}[\mathcal{O}]_{s=L}}, \quad (3.51)$$

where the operator \mathcal{O} is defined in Eq. (B21) of Appendix B. The value of the gauged WZ action at these two solutions is

$$S_{WZ, \text{gauged}}[\mathbf{m}, A]_{s=0} = \frac{k}{2} \bar{A}_t T \quad (3.52a)$$

$$S_{WZ, \text{gauged}}[\mathbf{m}, A]_{s=L} = -\frac{k}{2} \bar{A}_t T. \quad (3.52b)$$

Remarkably, these expressions do not depend on the area $\mathcal{A}_{\mathcal{M}}$, or any other details, of the target space \mathcal{M} . We now turn to the evaluation of the Pfaffians appearing in the denominators in Eq. (3.51).

To calculate the Pfaffians (which by Eq. (B21) depend on the derivatives of the vector field \underline{v}), we first need to express \underline{v} in a system of local coordinates (x, y) near the points $s = 0$ and $s = L$ of the space \mathcal{M} . The coordinate system (s, ϕ) is singular at these two points (ϕ is undefined there) and so it cannot be used for an analysis of the space near these two points. Near $s = 0$ we choose coordinates $x = \frac{2\pi k}{\bar{A}_{\mathcal{M}}} s \cos(\phi)$, $y = \frac{2\pi k}{\bar{A}_{\mathcal{M}}} s \sin(\phi)$, and near $s = L$ we choose coordinates $x = -\frac{2\pi k}{\bar{A}_{\mathcal{M}}} (L - s) \cos(\phi)$, $y = \frac{2\pi k}{\bar{A}_{\mathcal{M}}} (L - s) \sin(\phi)$. This choice of coordinates has the virtue that the symplectic form ω takes the Darboux form $\omega = dx \wedge dy$ at both $s = 0$ and $s = L$. To derive this result we had to use the important property that $\partial_s r(s)|_{s=0} = -\partial_s r(s)|_{s=L} = 1$. For these choices of coordinates the vector field \underline{v} takes the form

$$\underline{v} = \bar{A}_t (x \partial_y - y \partial_x) \quad (3.53)$$

near $s = 0$, and the form

$$\underline{v} = -\bar{A}_t (x \partial_y - y \partial_x) \quad (3.54)$$

near $s = L$.

Using the definition of $\text{Pf}[\mathcal{O}]$ in terms of a fermion path integral from Eq. (B23) of Appendix B, we find that $\text{Pf}[\mathcal{O}]$ at the points $s = 0$ and $s = L$ is given formally by the determinant of a one-dimensional Dirac operator. More precisely, after expanding the path integral in Fourier modes we find that

$$\begin{aligned} \text{Pf}[\mathcal{O}]_{s=0} &= -\bar{A}_t \prod_{m>0} \left(\frac{2\pi m}{T} + \bar{A}_t \right) \left(-\frac{2\pi m}{T} + \bar{A}_t \right) \\ &= -\det[-i\partial_t + \bar{A}_t], \end{aligned} \quad (3.55)$$

and

$$\text{Pf}[\mathcal{O}]_{s=L} = -\det[-i\partial_t - \bar{A}_t]. \quad (3.56)$$

The operators

$$\mathcal{D}_{\pm} := -i\partial_t \pm \bar{A}_t \quad (3.57)$$

are equivalent to one-dimensional Dirac operators for a fermion in one spacetime dimension coupled to the external field $A = A_t dt$ ⁴⁸. As we discussed in Appendix B, the overall sign of these Pfaffians is ambiguous, since we are free to alter the order of factors in the definition of the path integral measure. Therefore at this point we are free to choose a particular definition of the path integral measure such that

$$\text{Pf}[\mathcal{O}]_{s=0} = \det[\mathcal{D}_+] \quad (3.58)$$

$$\text{Pf}[\mathcal{O}]_{s=L} = \det[\mathcal{D}_-]. \quad (3.59)$$

These determinants still require proper regularization, and we now turn to a discussion of this issue.

We choose to regularize these determinants using *zeta* and *eta* function methods (see Appendix C for details). To motivate the definition of the regularized determinants in terms of zeta and eta functions, we first consider the following (non-rigorous) manipulations of a definition of these determinants in terms of an infinite product of their eigenvalues. We are also careful to point out any ambiguities which arise in defining the determinants in this way. Let $\lambda_m^{(\pm)} = \frac{2\pi m}{T} \pm \bar{A}_t$, $m \in \mathbb{Z}$, be the eigenvalues of the operator \mathcal{D}_{\pm} . Formally, we have

$$\begin{aligned} \det[\mathcal{D}_{\pm}] &= \prod_{m \in \mathbb{Z}} \lambda_m^{(\pm)} \\ &= \prod_{m \in \mathbb{Z}} |\lambda_m^{(\pm)}| \text{sgn}(\lambda_m^{(\pm)}). \end{aligned} \quad (3.60)$$

So far we encounter no difficulties. However, the next step is to express the sign of the eigenvalues as

$$\text{sgn}(\lambda_m^{(\pm)}) = e^{i\frac{\pi}{2}(1 - \text{sgn}(\lambda_m^{(\pm)}))}. \quad (3.61)$$

But this step is ambiguous because we could just as well have written

$$\text{sgn}(\lambda_m^{(\pm)}) = e^{i\frac{(2p+1)\pi}{2}(1 - \text{sgn}(\lambda_m^{(\pm)}))} \quad (3.62)$$

for any integer p . For now we work with the most general expression for $\text{sgn}(\lambda_m^{(\pm)})$, which involves an arbitrary integer p . Later in this section we show how the value of p can be fixed by a minimal number of physical assumptions on the properties of the partition function $Z[A]$.

Continuing with our manipulations, we find that the determinant can be expressed formally as

$$\det[\mathcal{D}_\pm] = \left(\prod_{m \in \mathbb{Z}} |\lambda_m^{(\pm)}| \right) e^{i \frac{(2p+1)\pi}{2} \sum_{m \in \mathbb{Z}} (1 - \text{sgn}(\lambda_m^{(\pm)}))}. \quad (3.63)$$

We now use zeta and eta function methods to make sense of the different terms in this expression. Before we start, we again decompose \bar{A}_t as $\bar{A}_t = \frac{2\pi\ell}{T} + \bar{a}_t$, for some $\ell \in \mathbb{Z}$ and $\bar{a}_t \in (0, \frac{2\pi}{T})$. To start with the regularization, we first use zeta function regularization to define the product over the magnitude of all the eigenvalues $\lambda_m^{(\pm)}$. We carry out this calculation in Appendix C and we find that

$$\left(\prod_{m \in \mathbb{Z}} |\lambda_m| \right)_{reg} = 2 \sin \left(\frac{\bar{a}_t T}{2} \right). \quad (3.64)$$

Next, we define the sum $\sum_{m \in \mathbb{Z}} 1$ as

$$\left(\sum_{m \in \mathbb{Z}} 1 \right)_{reg} = 1 + 2\zeta(0) = 0, \quad (3.65)$$

where $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is the Riemann zeta function and we used $\zeta(0) = -\frac{1}{2}$. Finally, we define

$$\left(\sum_{m \in \mathbb{Z}} \text{sgn}(\lambda_m^{(\pm)}) \right)_{reg} = \eta_{\pm}(0), \quad (3.66)$$

where $\eta_{\pm}(0)$ is the analytic continuation to $s = 0$ of the eta function $\eta_{\pm}(s)$ of the operator \mathcal{D}_{\pm} (see Appendix C for details). We calculate $\eta_{\pm}(0)$ in Appendix C and we find that

$$\eta_{\pm}(0) = \pm 1 \mp \frac{\bar{a}_t T}{\pi}. \quad (3.67)$$

Putting this all together, we find that the regularized determinants of \mathcal{D}_{\pm} are given by

$$\begin{aligned} \det[\mathcal{D}_{\pm}]_{reg} &= 2 \sin \left(\frac{\bar{a}_t T}{2} \right) e^{-i \frac{(2p+1)\pi}{2} (\pm 1 \mp \frac{\bar{a}_t T}{\pi})} \\ &= 2(\mp i)^{2p+1} \sin \left(\frac{\bar{a}_t T}{2} \right) e^{\pm i \frac{(2p+1)}{2} \bar{a}_t T}, \end{aligned} \quad (3.68)$$

where p was the arbitrary integer which appeared when we tried to rewrite $\text{sgn}(\lambda_m^{(\pm)})$ as an exponential. We then find that the partition function for our quantum mechanical system coupled to the external field $A = A_t dt$ evaluates to

$$Z[A] = (-1)^{k\ell+p+1} \frac{\sin \left[\frac{\bar{a}_t T}{2} (k - 2p - 1) \right]}{\sin \left(\frac{\bar{a}_t T}{2} \right)}. \quad (3.69)$$

The next step is to determine which choice of p gives the correct partition function. To do this, we will impose the following two conditions on the value of $Z[A = 0]$ (the partition function in zero external field). Physically, the value of $Z[A = 0]$ is the dimension of the Hilbert space of our quantum mechanical system. Therefore it makes sense to impose the following two conditions on $Z[A = 0]$.

1. For $k = 0$, we require $Z[A = 0] = 1$, since $k = 0$ gives a trivial theory with action equal to zero. The dimension of the Hilbert space of this theory should be equal to one.
2. For $k \neq 0$, $Z[A = 0]$ should be a positive number.

In terms of ℓ and \bar{a}_t , the limit $\bar{A}_t \rightarrow 0$ is taken by first setting $\ell = 0$, and then taking $\bar{a}_t \rightarrow 0$. In this limit we find

$$Z[A = 0] = (-1)^{p+1} (k - 2p - 1). \quad (3.70)$$

The first condition implies that p satisfies the equation

$$1 = (-1)^p (2p + 1). \quad (3.71)$$

This equation has the two solutions $p = 0$ and $p = -1$. For these two solutions for p , we find that $Z[A = 0]$ at any k takes the form

$$Z[A = 0] = \begin{cases} -k + 1 & , p = 0 \\ k + 1 & , p = -1 \end{cases}. \quad (3.72)$$

We see that in order to satisfy condition two, we must pick $p = -1$ for $k > 0$ and $p = 0$ for $k < 0$. In this way we find that for all k , the partition function is given by

$$Z[A] = (-1)^{k\ell} \frac{\sin \left[\frac{\bar{a}_t T}{2} (|k| + 1) \right]}{\sin \left(\frac{\bar{a}_t T}{2} \right)}, \quad (3.73)$$

which is identical to the answer we computed for the $O(3)$ NLSM. Therefore we find that for any two-dimensional target space \mathcal{M} which respects the symmetries of the BTI phase, the NLSM description of the BTI using the target space \mathcal{M} has the same global anomaly as the $O(3)$ NLSM description. This result also implies that a large class of bosonic theories in $0+1$ dimensions with $U(1) \rtimes \mathbb{Z}_2$ symmetry share the same global anomaly as a Dirac fermion in $0+1$ dimensions with the same symmetry⁴⁸.

IV. RENORMALIZATION GROUP FLOWS AND THE FATE OF OUR MODELS AT LOW ENERGIES

In this section we briefly comment on the expected low energy behavior of the boundary theories discussed in this article. Recall that the basic models we consider are NLSMs with a WZ term. On a d -dimensional spacetime X_{bdy} (which we imagine to lie at the boundary of an SPT phase), we can construct a WZ term for a NLSM with target space \mathcal{M} if

$\dim[\mathcal{M}] = d + 1$. In addition to the WZ term, the NLSM action will also contain an ordinary kinetic term

$$S_{kin}[\mathbf{m}] = \frac{1}{2f} \int_{X_{bdy}} d^d x G_{ab}(\mathbf{m}) \partial_\mu m^a \partial^\mu m^b, \quad (4.1)$$

where $\mathbf{m} : X_{bdy} \rightarrow \mathcal{M}$ is the NLSM field, and $G_{ab}(\mathbf{m})$ is the Riemannian metric on \mathcal{M} (compare with Eq. (2.5) for the case of a spherical target space). If we assume that the NLSM field \mathbf{m} is dimensionless, then the coupling constant f has dimensions of $(\text{mass})^{2-d}$. Equivalently, the inverse $\frac{1}{f}$ of the coupling constant has dimensions of $(\text{mass})^{d-2}$. We now consider the consequences of this fact for the low energy behavior of the theories discussed in this paper. We focus on the case where $d \geq 2$ since for $d = 1$ our theory is not a quantum field theory but just an ordinary quantum mechanical system.

For simplicity, we first consider the case where the target space \mathcal{M} is the sphere S^{d+1} and so the NLSM field is a $(d+2)$ -component unit vector \mathbf{n} . In the absence of the WZ term (i.e., for a WZ term with level $k = 0$) then for $d = 2$ the renormalization group (RG) flow is towards the disordered ($f \rightarrow \infty$) phase at all scales⁶⁷⁻⁷⁰. In this limit the theory is massive and the ground state (or vacuum state) possesses the full $O(d+2) = O(4)$ symmetry of the action (the ground state transforms as a singlet under the action of the $O(4)$ symmetry). When the WZ term is turned on, a stable fixed point appears at a finite value of the coupling f ⁷¹, and this fixed point is actually the $SU(2)_k$ Wess-Zumino-Witten conformal field theory. To see this we note that the four-component unit vector field of the $O(4)$ NLSM is equivalent to a 2×2 $SU(2)$ matrix field. Explicitly, if $\mathbf{n} = (n^1, \dots, n^4)$, then one possible mapping to the matrix field U is $U = n^4 \mathbb{I} + i \sum_{a=1}^3 n^a \sigma^a$, where σ^a for $a = 1, 2, 3$, are the Pauli matrices. In addition, the $U(1)$ symmetry that we are interested in in this paper is realized as a right (or left, depending on the mapping from \mathbf{n} to U) $U(1)$ symmetry of the $SU(2)_k$ theory, and this symmetry is well-known to be anomalous^{56,57}.

For the case of $d > 2$ the coupling constant f is dimensionful and one expects (by a simple power-counting argument) that the theory flows towards the ordered phase $f \rightarrow 0$ and so the $O(d+2)$ symmetry of the theory is spontaneously broken at low energies. In fact, for the theory without a topological term, a double perturbation expansion in f and $\epsilon = d - 2$ reveals the existence of an unstable fixed point at a finite value f_1 of the (suitably rescaled) coupling f ^{68,69}. If this computation can be trusted, then below this fixed point the theory flows to $f \rightarrow 0$ and the symmetry is spontaneously broken, while above the fixed point the theory flows to a (presumably) disordered strong-coupling ($f \rightarrow \infty$) phase in which the $O(d+2)$ symmetry is restored⁷. Since in the $d = 2$ case turning on the WZ term introduces a stable fixed point at a finite value of the coupling, some authors have recently proposed a scenario for $d > 2$ in which the introduction of the WZ term introduces a stable fixed point at a finite value $f = f_2$ of the coupling constant, with $f_2 > f_1$, where f_1 is the location of the unstable fixed point (see Figure 2a of Ref. 73). This possibility was first raised in Ref. 73, and it has been pursued recently in Ref. 74 using a combination of several perturbation

expansions. Both of these works consider the case of $d = 3$ spacetime dimensions⁷.

What can we deduce about our boundary theories from this discussion? Let us first consider the case for BIQH states. Recall that these boundary theories were $O(2m)$ NLSMs with WZ term in spacetime dimension $2m - 2$, and also NLSMs with deformed target spaces \mathcal{M} which still possessed a $U(1)$ symmetry. We first discuss the case $m > 2$, so that the boundary spacetime dimension is larger than two. In this case, the conclusion which is supported by the most evidence is that the $U(1)$ symmetry of these theories is spontaneously broken in the ground state. In this case the symmetry-broken theory will possess a gapless Goldstone mode. Interestingly, this gapless mode will still couple to the external field $A = A_\mu dx^\mu$ and it is this Goldstone mode which exhibits the anomaly in the symmetry-broken theory. For example, if we consider a general target space \mathcal{M} with $U(1)$ symmetry, and we add a potential to the action which is minimized along the $U(1)$ orbit of a particular point on \mathcal{M} , then the low energy theory will possess a gapless Goldstone mode corresponding to motion around this orbit⁷.

It is helpful to see an explicit example of this kind in order to appreciate the fact that the Goldstone mode really does exhibit the anomaly. Let us take the $O(2m)$ NLSM with WZ term at level k and introduce a potential into the action which is minimized when $|b_1|^2 = 1$ and all other $b_\ell = 0$ ($\ell = 2, \dots, m$). In the symmetry-broken vacuum we then have $b_1 = e^{i\varphi_{vac}}$ for some constant φ_{vac} . If we expand around this vacuum by setting $b_1 = e^{i\varphi_{vac} + i\varphi}$ then the gauged NLSM action with WZ term reduces to an action for the gapless Goldstone mode φ coupled to A . This action takes the explicit form

$$S[\varphi] = \frac{1}{2f} \int d^{2m-2} x (\partial_\mu \varphi - A_\mu)(\partial^\mu \varphi - A^\mu) - \frac{k}{(2\pi)^{m-1}} \int_{X_{bdy}} d\varphi \wedge A \wedge F^{m-2}. \quad (4.2)$$

It is now easy to see that under a $U(1)$ gauge transformation $\varphi \rightarrow \varphi + \chi$, $A \rightarrow A + d\chi$, the action for the Goldstone mode φ has the same anomaly as the original $O(2m)$ NLSM. From this analysis we can conclude that even when the $U(1)$ symmetry of the BIQH state is spontaneously broken in the boundary theory, the boundary theory will still possess the same perturbative $U(1)$ anomaly as the original NLSM that we started with.

In the case of $m = 2$ (boundary spacetime dimension equal to two) the situation is more interesting. As we noted above, if we preserve the full $O(4)$ symmetry of the theory, then our theory flows at low energies to the $SU(2)_k$ Wess-Zumino-Witten conformal field theory. On the other hand, we can introduce some $O(4)$ -breaking but $U(1)$ -preserving anisotropy into the theory to set $|b_1|^2 = 1$ and $b_2 = 0$ (or vice-versa). In this case we end up with a free boson theory of the form

$$S[\varphi] = \frac{1}{2f} \int d^2 x (\partial_\mu \varphi - A_\mu)(\partial^\mu \varphi - A^\mu) - \frac{k}{2\pi} \int_{X_{bdy}} d\varphi \wedge A, \quad (4.3)$$

where we have $b_1 = e^{i\varphi}$. In this case, however, φ should not be interpreted as a Goldstone boson as we do not have spontaneous symmetry breaking in this dimension. The $SU(2)_k$ theory has a central charge of $c = \frac{3k}{k+2} \geq 1$ (see, for example, Ref. 75) so it can and will flow to the free boson theory with central charge $c = 1$ when perturbations which break the $O(4)$ symmetry down to $U(1)$ are introduced (this flow is consistent with Zamolodchikov's c-theorem⁷⁶). Note that if we preserve the $U(1)$ symmetry, then the boundary theory cannot be gapped out since we always need some gapless degrees of freedom to saturate the anomaly. Finally, we remark that in the $k = 1$ case, the free boson theory is actually equivalent to the $SU(2)_1$ theory for a particular value of the coupling f . However, marginal perturbations which break the $O(4)$ symmetry down to $U(1)$ will in general tune f away from this special value.

We close this section with a few words about the boundary theories of BTI states. These boundary theories occur in odd spacetime dimensions $2m - 1$, and they lie at the boundary of a BTI state in $2m$ dimensions. We have already analyzed the case $m = 1$ in detail in Sec. III. In this case the boundary is just a quantum mechanical system and there are no subtleties involved in assessing the fate of the system at low energies. For the case of $m > 1$, the most likely scenario is that these boundary theories spontaneously break the $U(1) \rtimes \mathbb{Z}_2$ symmetry of the BTI state. As we noted in Ref. 36, because of the way the $U(1)$ symmetry in our models acts on S^{2m} (the target space of the NLSM in this case), in the BTI case it is possible to break the \mathbb{Z}_2 symmetry while preserving the $U(1)$ symmetry. In this way we were able to show that the boundary of the BTI state can exhibit a \mathbb{Z}_2 symmetry-breaking electromagnetic response, and we found that this response is given by a CS term for A_μ with level $\frac{m!}{2}$. We then argued, based on this evidence, that the boundary theories of the BTI state exhibit a bosonic analogue of the parity anomaly.

For $m > 1$ it is still an open problem to exhibit this bosonic analogue of the parity anomaly in a concrete way (e.g., at the level of the partition function). The most interesting case is $m = 2$ in which the boundary spacetime dimension is $d = 3$. Here we can list three possibilities for the fate of the boundary theory at low energies. First, as noted above, the boundary could break part or all of the symmetry group $U(1) \rtimes \mathbb{Z}_2$ of the BTI state. Second, the results of Refs. 73 and 74 indicate that a gapless conformal field theory preserving the full $U(1) \rtimes \mathbb{Z}_2$ symmetry may be possible. Finally, since the anomaly in this case is global and not perturbative, there is the possibility that the boundary theory can flow to a topological quantum field theory whose partition function (in the presence of the external field A_μ) exhibits the anomaly. In this last case all other degrees of freedom at the boundary become gapped and decouple from the topological quantum field theory which describes only the ground state sector of the boundary theory. We comment more on this last possibility in Sec. V.

V. DISCUSSION AND CONCLUSION

In this paper we continued the program, initiated in Ref. 36, of characterizing the anomalies at the boundary of BIQH and BTI states in all odd and even dimensions, respectively. In Sec. II we revisited the perturbative $U(1)$ anomaly at the boundary of BIQH states. There we proved that the target space \mathcal{M} of the NLSM describing the boundary theory of these states can be subjected to arbitrary smooth, symmetry-preserving deformations without affecting the anomaly. In Sec. III we revisited the global anomaly at the boundary of BTI states. In Ref. 36 we gave an argument that the boundary of the BTI state exhibits a bosonic analogue of the parity anomaly of Dirac fermions in odd dimensions. In this paper we elevated this argument to a proof for the case of the $(0+1)$ -dimensional boundary of the $(1+1)$ -dimensional BTI state. In that case we also used the equivariant localization technique to prove that the global anomaly of the BTI boundary is robust against arbitrary smooth, symmetry-preserving deformations of the target space of the NLSM used to describe this state.

From a fundamental point of view, perhaps the most important result in this paper is our concrete demonstration, at the level of the partition function, of an analogue of the parity anomaly in a purely bosonic system. Indeed, our result in Sec. III is a direct bosonic analogue of the results of Ref. 48 on global anomalies of fermions in $0 + 1$ dimensions. In the context of SPT phases, our results in this paper also imply that the universal properties of an SPT phase can be captured by a much wider range of models than the NLSMs with spherical target space originally considered in Refs. 37 and 38. The results of this paper lead us to conjecture that an SPT phase in $D + 1$ dimensions with symmetry group G , which would be described by an $O(D + 2)$ NLSM in the approach of Refs. 37 and 38, can be modeled using an NLSM with *any* target space \mathcal{M} related to S^{D+1} by a diffeomorphism which is equivariant with respect to the action of the group G . Note that this conjecture only applies to SPT phases for which an NLSM description *exists*. This does not seem to be the case for all SPT (or short-range entangled) phases, for example the “E8” state in $2 + 1$ dimensions and the “beyond cohomology” state with time-reversal symmetry in $3 + 1$ dimensions^{22,32}.

An ambitious goal for future work would be to present a concrete demonstration, again at the level of the partition function, of an analogue of the parity anomaly in a $(2 + 1)$ -dimensional bosonic model with $U(1)$ and \mathbb{Z}_2 symmetry, where \mathbb{Z}_2 now represents time-reversal. A precise understanding of global anomalies in $(2 + 1)$ -dimensional bosonic systems would also be extremely useful in the search for new dualities in quantum field theory in $2 + 1$ dimensions^{77–83}. A crucial check on any proposed duality is that the two theories which are conjectured to be dual to each other must have the same ‘t Hooft anomalies when coupled to various external fields.

An interesting candidate for a $(2 + 1)$ -dimensional bosonic model displaying a bosonic analogue of the parity anomaly is the $O(5)$ NLSM with WZ term, and with the $U(1) \rtimes \mathbb{Z}_2$ symmetry of the BTI state acting in the manner described in Ref. 36. In Ref. 36 we already gave several pieces of evidence

which suggest that this model displays a bosonic analogue of the parity anomaly. The first piece of evidence was our computation of the time-reversal breaking electromagnetic response of this model, which we already mentioned above. However, we also gave a second argument which was based on the demonstration that there is a certain composite vortex excitation in this model with fermionic statistics (an observation which goes back to Refs. 32 and 84), and such an excitation should not exist in a purely bosonic model which is not anomalous.

The $O(5)$ NLSM with WZ term may be tractable analytically in the topological limit in which the coupling constant f_{bdy} of the NLSM is sent to infinity (i.e., if one considers the model with only the topological term). This would correspond to the third possibility that we raised at the end of Sec. IV: the boundary theory could flow to a topological quantum field theory whose partition function exhibits the anomaly. It may even be the case that a more sophisticated version of the equivariant localization technique can be used to calculate the partition function of the $O(5)$ NLSM with WZ term in the topological limit and properly coupled to an external $U(1)$ gauge field as described in Sec. VI of Ref. 36. However, there are several difficulties which must be surmounted before one can apply any kind of equivariant localization technique to this problem. The main problem is that one needs to find a hidden supersymmetry in this problem which can be exploited in order to establish the localization of the path integral. In the $(0+1)$ -dimensional case this supersymmetry followed, at least partially⁷, from the fact that the path integral measure could be exponentiated by introducing a set of *real* Grassmann-valued (i.e., fermionic) fields $\eta^a(t)$ into the problem. This could only be done with real fermionic fields because the target spaces of the $(0+1)$ -dimensional NLSMs that we studied were all symplectic manifolds. On the other hand, the target space S^4 of the $O(5)$ NLSM is not symplectic. Therefore one can only exponentiate the path integral measure by introducing complex fermionic fields. Currently, we are not aware of a generalization of the equivariant localization techniques of Refs. 49–52 which starts by exponentiating the path integral measure by introducing complex fermions, but such a generalization may still be possible. We leave a detailed investigation of this to future work.

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Appendix A: Classical mechanics and phase space path integral for general Hamiltonian systems

In this appendix we review the symplectic geometry formulation of classical Hamiltonian mechanics, closely following the discussion in Ch. 11 of Ref. 85. We use this formalism in Sec. III of the paper to aid in the evaluation of the partition function for a gauged NLSM with WZ term which describes the $(0+1)$ -dimensional boundary of the BTI state in $1+1$ dimensions. The symplectic geometry formulation of Hamiltonian mechanics is a geometric formulation in terms of a phase space \mathcal{M} (a closed, orientable, smooth manifold) equipped with a symplectic form ω . We take \mathcal{M} to have dimension $2n$, where n is an integer greater than or equal to one. The symplectic form ω is a closed, non-degenerate two-form on \mathcal{M} . In a system of local coordinates m^a on \mathcal{M} , in which $\omega = \frac{1}{2}\omega_{ab}(\mathbf{m})dm^a \wedge dm^b$, the non-degeneracy condition is equivalent to the condition that the components $\omega_{ab}(\mathbf{m})$ are the elements of an invertible matrix. We use the notation $\mathbf{m} = (m^1, \dots, m^{2n})$ to refer to the entire collection of phase space coordinates, and we use Latin indices near the beginning of the alphabet to label the components of general tensor fields on \mathcal{M} . We also use the notation $\partial_a \equiv \frac{\partial}{\partial m^a}$ and $\dot{m}^a \equiv \frac{dm^a}{dt}$ in what follows.

To start, for any function f on phase space we define an associated vector field \underline{v}_f by the equation

$$df = -i_{\underline{v}_f} \omega, \quad (\text{A1})$$

where $i_{\underline{v}}\omega = v^a \omega_{ab} dm^b$ denotes interior multiplication of the form ω by the vector field \underline{v} . The components of \underline{v}_f then take the form

$$v_f^a = \omega^{ab} \partial_b f, \quad (\text{A2})$$

where ω^{ab} are the elements of a matrix which is the inverse of the matrix with elements ω_{ab} , i.e.,

$$\omega^{ab} \omega_{bc} = \delta^a_c. \quad (\text{A3})$$

We see that the symplectic two-form ω must be non-degenerate for this to work.

The Poisson bracket of two functions f and g on phase space is then defined by⁸

$$\{f, g\} = i_{\underline{v}_g} i_{\underline{v}_f} \omega. \quad (\text{A4})$$

In a system of local coordinates the Poisson bracket has the form

$$\{f, g\} = \omega^{ab} \partial_b f \partial_a g. \quad (\text{A5})$$

For a given Hamiltonian function H , Hamilton's equations are equivalent to the single equation

$$dH = -i_{\underline{v}_H} \omega, \quad (\text{A6})$$

where \underline{v}_H is the vector field whose components are the time derivatives of the phase space coordinates,

$$\underline{v}_H = \dot{m}^a \partial_a. \quad (\text{A7})$$

Finally, in each coordinate patch on \mathcal{M} we can write

$$\omega = d\vartheta, \quad (\text{A8})$$

where the one-form $\vartheta = \vartheta_a(\mathbf{m})dm^a$ is known as the *symplectic potential*.

Next, we review the form of the phase space path integral for the partition function $Z(T) = \text{tr}[e^{-iHT}]$ of the quantum mechanical system obtained via quantization of the classical system defined by the triple (\mathcal{M}, ω, H) . Here the trace is over the Hilbert space of the quantum mechanical system. As is reviewed in Sec. 4.1 of Ref. 53, the phase space path integral for $Z(T)$ takes the form

$$Z(T) = \int_{L\mathcal{M}} [d^{2n}\mathbf{m}] \left[\prod_{t \in [0, T)} \text{Pf}[\omega_{ab}(\mathbf{m}(t))] \right] e^{iS[\mathbf{m}]}, \quad (\text{A9})$$

where the action appearing in the exponential is

$$S[\mathbf{m}] = \int_0^T dt [\vartheta_a(\mathbf{m})\dot{m}^a - H(\mathbf{m})]. \quad (\text{A10})$$

The path integral is taken over all field configurations $m^a(t)$ with periodic boundary conditions $m^a(0) = m^a(T)$ on the interval $[0, T)$. The space of all such configurations is known as the *loop space* $L\mathcal{M}$ of the phase space manifold \mathcal{M} . In addition, $[d^{2n}\mathbf{m}]$ denotes a flat measure on phase space at all points in time. The nontrivial geometry of the phase space is taken into account by the insertion of

$$\prod_{t \in [0, T)} \text{Pf}[\omega_{ab}(\mathbf{m}(t))] \quad (\text{A11})$$

into the path integral. This factor can be understood by noting that the $2n$ -form $\frac{\omega^n}{n!}$ provides a natural volume form (the Liouville measure) on \mathcal{M} , and also by making use of the formula $\frac{\omega^n}{n!} = \text{Pf}[\omega_{ab}]dm^1 \wedge \dots \wedge dm^{2n}$.

The first term in the action can also be recast into a form which is very similar to a WZ term. Let us denote the interval $[0, T)$ with periodic boundary conditions by S_T^1 , the circle of circumference T . This circle is the spacetime that our quantum mechanical system evolves on. To write the first term in the action in a WZ form, we first introduce a two-dimensional manifold \mathcal{B} which has S_T^1 as its boundary, $\partial\mathcal{B} = S_T^1$. Then we choose an extension $\tilde{\mathbf{m}}$ of the field configuration \mathbf{m} into the bulk of \mathcal{B} such that $\tilde{\mathbf{m}}|_{\partial\mathcal{B}} = \mathbf{m}$. We can now use Stokes' theorem to rewrite the first term in $S[\mathbf{m}]$ as

$$\begin{aligned} \int_0^T dt \vartheta_a(\mathbf{m})\dot{m}^a &= \int_{S_T^1} \mathbf{m}^* \vartheta \\ &= \int_{\mathcal{B}} \tilde{\mathbf{m}}^* d\vartheta \\ &= \int_{\mathcal{B}} \tilde{\mathbf{m}}^* \omega. \end{aligned} \quad (\text{A12})$$

In this form the term $\int_0^T dt \vartheta_a(\mathbf{m})\dot{m}^a$ appearing in the action looks very similar to a WZ term, in the sense that it involves (i) an extended spacetime \mathcal{B} , (ii) an extension $\tilde{\mathbf{m}}$ of the field configuration \mathbf{m} into \mathcal{B} , and (iii) the integral over \mathcal{B} of the pullback of a *closed* form on \mathcal{M} .

Appendix B: A brief introduction to equivariant localization for phase space path integrals

In this appendix we give a brief review of the *equivariant localization* (EL) technique for the evaluation of certain phase space path integrals of the form of Eq. (A9) from Appendix A. We use the EL technique in Sec. III to evaluate the partition function for a gauged NLSM with WZ term which describes the $(0+1)$ -dimensional boundary of a BTI state in $1+1$ dimensions. Our presentation in this appendix is based on the discussion in Sec. 4 of Ref. 53. We also give a brief discussion on how one can define the Pfaffians of infinite-dimensional operators which appear in the formulas obtained by applying the EL technique.

The EL technique for phase space path integrals can be thought of as an infinite-dimensional generalization of the finite-dimensional integration formulas derived in Refs. 86–88. In this paper we only use the simplest version of the EL technique. The path integral formula which follows from this particular version of the EL technique is sometimes referred to as the “WKB” localization formula. This basic version of the EL method and several generalizations of it (in particular the “Niemi-Tirkkonen” formula) were developed in Refs. 49–52. Stone’s paper⁸⁹ on a hidden supersymmetry in the quantum mechanics of spin can be seen as a herald for the developments on the EL technique for phase space path integrals which followed soon after. The application of the EL technique to systems with a two-dimensional phase space, which is the case of interest in this paper, was considered in detail in Ref. 90. Finally, some issues related to the regularization of determinants and Pfaffians appearing in the EL formulas were greatly clarified by Miettinen in Ref. 91.

In the context of the EL technique, the word “localization” refers to the fact that although the path integral in question ostensibly gets contributions from all possible field configurations, the final result only depends on contributions from a very small subset of these configurations. Thus, the integral “localizes” to a sum or, in some cases, a finite-dimensional integral over this subset of all field configurations. The word “equivariant” refers to the fact that the mechanism responsible for the localization of the integral is best understood in terms of the *equivariant cohomology* of the manifold that one is integrating over⁸⁸. In the case of the phase space path integrals considered here this turns out to be the $U(1)$ -equivariant cohomology of the infinite-dimensional *loop space* $L\mathcal{M}$ of the classical phase space \mathcal{M} .

The basic idea of the EL technique is as follows. First, to apply the EL technique we need to start with a classical system possessing a $U(1)$ symmetry. It turns out that this $U(1)$ symmetry “lifts” to a supersymmetry of the phase space path integral. This supersymmetry is then used to construct a one parameter family of equivalent path integrals parametrized by $\lambda \in [0, \infty)$, with the original path integral of interest corresponding to $\lambda = 0$. The supersymmetry guarantees that the path integral at any value of λ is equivalent to the original path integral. Therefore, the original path integral can be computed by taking the opposite limit $\lambda \rightarrow \infty$. In this limit the path integral simplifies dramatically, getting contributions

only (in the cases considered here) from the field configurations which correspond to *time-independent* solutions to the classical equations of motion. One says that the path integral *localizes* onto these configurations. We now outline the main ideas behind the EL technique in more detail, closely following Sec. 4 of Ref. 53.

To start, we assume that it is possible to define an action of the group $U(1)$ on the phase space \mathcal{M} . Let $\underline{v} = v^a(\mathbf{m})\partial_a$ be the vector field which generates the $U(1)$ action, in the sense that under a $U(1)$ transformation by the small angle ξ the phase space coordinates transform as $m^a \rightarrow m^a + \xi v^a$. On \mathcal{M} there is a Hamiltonian function $H(\mathbf{m})$ which is naturally associated with this vector field, and which is determined by \underline{v} and ω via the equation

$$dH = -i_{\underline{v}}\omega. \quad (\text{B1})$$

Note that this is just Eq. (A1) with the function f taken to be the Hamiltonian. We choose this specific Hamiltonian to describe the dynamics of the system that we consider in what follows. With this choice of Hamiltonian, the action for our dynamical system will also have a $U(1)$ symmetry. Finally, we will need a Riemannian metric $g_{ab}(\mathbf{m})$ on \mathcal{M} which is invariant under the $U(1)$ action generated by \underline{v} . This is equivalent to the requirement that \underline{v} is a Killing vector for the metric, i.e., g_{ab} and v^a should satisfy the Killing equation

$$v^c \partial_c g_{ab} + g_{ac} \partial_b v^c + g_{bc} \partial_a v^c = 0, \forall a, b. \quad (\text{B2})$$

The path integral in Eq. (A9) involves an integration over the loop space $L\mathcal{M}$ of \mathcal{M} , which is spanned by the T -periodic functions $m^a(t)$ which, for each t , represent a point on \mathcal{M} . We now introduce an additional set of Grassmann-valued fields $\eta^a(t)$ which also obey periodic boundary conditions. The space of these new fields is equivalent to the loop space of $\Lambda^1\mathcal{M}$, the vector space of one-forms on \mathcal{M} , and this space is denoted by $L\Lambda^1\mathcal{M}$. The interpretation in terms of $\Lambda^1\mathcal{M}$ is due to the fact that at each time t the anticommuting fields $\eta^a(t)$ can be regarded as a basis of one-forms on \mathcal{M} . Using the rules for integration over real Grassmann variables, the new fields $\eta^a(t)$ can be used to rewrite $Z(T)$ in the form

$$Z(T) = \int_{L\mathcal{M} \otimes L\Lambda^1\mathcal{M}} [d^{2n}\mathbf{m}][d^{2n}\boldsymbol{\eta}] e^{i(S[\mathbf{m}] + \Omega[\mathbf{m}, \boldsymbol{\eta}])}, \quad (\text{B3})$$

where we defined

$$\Omega[\mathbf{m}, \boldsymbol{\eta}] = \frac{1}{2} \int_0^T dt \omega_{ab}(\mathbf{m}(t)) \eta^a(t) \eta^b(t), \quad (\text{B4})$$

and where the integration is now over the “super loop space” $L\mathcal{M} \otimes L\Lambda^1\mathcal{M}$. One should compare Eq. (B3) with the original expression Eq. (A9) for $Z(T)$.

Using the Grassmann-valued fields we can define the operators

$$d_L = \int_0^T dt \eta^a(t) \frac{\delta}{\delta m^a(t)}, \quad (\text{B5})$$

and

$$i_S = \int_0^T dt V_S^a[\mathbf{m}(t); t] \frac{\delta}{\delta \eta^a(t)}, \quad (\text{B6})$$

where

$$V_S^a[\mathbf{m}(t); t] = \dot{m}^a(t) - v^a(\mathbf{m}(t)). \quad (\text{B7})$$

The quantities $V_S^a[\mathbf{m}(t); t]$ can be interpreted as the components of a vector field

$$\underline{V}_S = \int_0^T dt V_S^a[\mathbf{m}(t); t] \frac{\delta}{\delta m^a(t)} \quad (\text{B8})$$

on the loop space. To understand the physical significance of the components $V_S^a[\mathbf{m}(t); t]$, note that the classical equations of motion for the system under consideration are

$$\frac{\delta S[\mathbf{m}]}{\delta m^a(t)} = \omega_{ab}(\mathbf{m}(t)) V_S^b[\mathbf{m}(t); t] = 0, \forall a. \quad (\text{B9})$$

Since ω is non-degenerate, the classical equations of motion are equivalent to the equations $V_S^a[\mathbf{m}(t); t] = 0, \forall a$. The operator d_L can be interpreted as an exterior derivative on $L\mathcal{M}$, and i_S has the interpretation of interior multiplication by the loop space vector field \underline{V}_S .

In terms of these operators we now define the loop space equivariant exterior derivative

$$Q_S = d_L + i_S. \quad (\text{B10})$$

The square of this operator can be interpreted as a loop space Lie derivative (acting on loop space differential forms) along the loop space vector field \underline{V}_S ,

$$\mathcal{L}_S \equiv Q_S^2 = d_L i_S + i_S d_L. \quad (\text{B11})$$

Some algebra shows that

$$Q_S(S[\mathbf{m}] + \Omega[\mathbf{m}, \boldsymbol{\eta}]) = 0, \quad (\text{B12})$$

which means that the integrand in the path integral is equivariantly closed (i.e., closed under the action of the equivariant exterior derivative). To prove this relation one needs to use the fact that ω is closed as an ordinary two-form on \mathcal{M} , and also Eq. (B9) relating \underline{V}_S to the classical equations of motion.

The closure of the integrand can be interpreted in terms of a supersymmetry (SUSY) of this system which is generated by the “supercharge” Q_S . In particular, Eq. (B12) implies that the path integral for $Z(T)$ is invariant under the SUSY transformation

$$\delta_\epsilon m^a(t) = \epsilon Q_S m^a(t) \quad (\text{B13a})$$

$$\delta_\epsilon \eta^a(t) = \epsilon Q_S \eta^a(t), \quad (\text{B13b})$$

where ϵ is a constant Grassmann parameter. An explicit calculation gives $Q_S m^a(t) = \eta^a(t)$ and $Q_S \eta^a(t) = V_S^a[\mathbf{m}(t); t]$, so we know the exact form that this SUSY transformation takes. The next step towards establishing localization of the path integral is to use the supersymmetry to deform the path integral by adding a suitably chosen SUSY-exact term to the integrand. To this end, we modify $Z(T)$ to

$$Z(T, \lambda) = \int_{L\mathcal{M} \otimes L\Lambda^1\mathcal{M}} [d^{2n}\mathbf{m}][d^{2n}\boldsymbol{\eta}] e^{i(S[\mathbf{m}] + \Omega[\mathbf{m}, \boldsymbol{\eta}] - \lambda Q_S \Psi[\mathbf{m}, \boldsymbol{\eta}])}, \quad (\text{B14})$$

where $\Psi[\mathbf{m}, \boldsymbol{\eta}]$ is some functional of \mathbf{m} and $\boldsymbol{\eta}$ which will be required to satisfy

$$Q_S^2 \Psi[\mathbf{m}, \boldsymbol{\eta}] = 0. \quad (\text{B15})$$

If we can find such a functional $\Psi[\mathbf{m}, \boldsymbol{\eta}]$, then we can show that $Z(T, \lambda)$ is actually independent of λ by the following manipulations. We compute (we suppress the arguments of the different terms for brevity)

$$\begin{aligned} \frac{dZ(T, \lambda)}{d\lambda} &= - \int [d^{2n} \mathbf{m}] [d^{2n} \boldsymbol{\eta}] Q_S \Psi e^{i(S+\Omega) - \lambda Q_S \Psi} \\ &= - \int [d^{2n} \mathbf{m}] [d^{2n} \boldsymbol{\eta}] Q_S \left[\Psi e^{i(S+\Omega) - \lambda Q_S \Psi} \right] \\ &= 0. \end{aligned} \quad (\text{B16})$$

The second line follows from the first since the argument of the exponential is annihilated by Q_S (and this requires that $Q_S^2 \Psi = 0$). Finally, the third line follows from the second due to an infinite-dimensional version of the statement that the integral of a total derivative is zero. In the infinite-dimensional case this is only true if the path integral measure is invariant under the action of Q_S , but that is the case here. An alternative explanation of the λ -independence of this integral, which uses a Ward identity associated with the symmetry generated by Q_S , can be found in Ref. 53.

The arguments from the last paragraph show that the original partition function $Z(T)$ is equal to the deformed partition function $Z(T, \lambda)$ for any value of λ . The final step in establishing the localization of $Z(T)$ is to pick a particular functional $\Psi[\mathbf{m}, \boldsymbol{\eta}]$ such that the $\lambda \rightarrow \infty$ limit of $Z(T, \lambda)$ becomes easy to evaluate. There are various choices for such a $\Psi[\mathbf{m}, \boldsymbol{\eta}]$, but the choice which leads to the WKB localization formula is

$$\Psi[\mathbf{m}, \boldsymbol{\eta}] = \int_0^T dt g_{ab}(\mathbf{m}(t)) V_S^a[\mathbf{m}(t); t] \eta^b(t). \quad (\text{B17})$$

One can check that this functional satisfies $Q_S^2 \Psi[\mathbf{m}, \boldsymbol{\eta}] = 0$, but the derivation relies on the fact that \underline{v} is a Killing vector for the metric g_{ab} .

Using this particular choice of $\Psi[\mathbf{m}, \boldsymbol{\eta}]$, one can now show that the path integral $Z(T)$ localizes to a sum over contributions from the field configurations in the set

$$L\mathcal{M}_S = \{ \mathbf{m}(t) \in L\mathcal{M} \mid V_S^a[\mathbf{m}(t); t] = 0, \forall a \}, \quad (\text{B18})$$

which is the set of all T -periodic solutions to the classical equations of motion. To motivate this, we simply note that the bosonic term in $Q_S \Psi[\mathbf{m}, \boldsymbol{\eta}]$ is

$$\int_0^T dt g_{ab}(\mathbf{m}(t)) V_S^a[\mathbf{m}(t); t] V_S^b[\mathbf{m}(t); t]. \quad (\text{B19})$$

Now $Q_S \Psi[\mathbf{m}, \boldsymbol{\eta}]$ appears in the exponential of the path integral multiplied by a factor of $-\lambda$, which means that in the limit $\lambda \rightarrow \infty$, this term becomes a delta function which restricts the path integral to only those field configurations where $V_S^a[\mathbf{m}(t); t] = 0$.

The final result of the EL calculation is the formula

$$\begin{aligned} Z(T) &= \lim_{\lambda \rightarrow \infty} Z(T, \lambda) \\ &\sim \sum_{\bar{\mathbf{m}}(t) \in L\mathcal{M}_S} \frac{e^{iS[\bar{\mathbf{m}}]}}{\text{Pf}[\mathcal{O}]_{\mathbf{m}(t)=\bar{\mathbf{m}}(t)}}, \end{aligned} \quad (\text{B20})$$

where the infinite-dimensional operator \mathcal{O} has matrix elements

$$\begin{aligned} \mathcal{O}_{ab}(t, t') &= \frac{\delta V_S^c[\mathbf{m}(t'); t']}{\delta \mathbf{m}^a(t)} \delta_{cb} \\ &= \delta_{ab} \partial_{t'} \delta(t - t') - \partial_a v^c(\mathbf{m}(t)) \delta_{cb} \delta(t - t'), \end{aligned} \quad (\text{B21})$$

and where the notation “ \sim ” indicates equivalence up to infinite products of constant (but λ -independent) factors. The final formula Eq. (B20) is famously equivalent to the stationary-phase approximation to $Z(T)$, but where the sum is taken over *all* T -periodic solutions of the classical equations of motion, and not just the solution which minimizes the action. In favorable cases there are a finite number of solutions $\bar{\mathbf{m}}(t) \in L\mathcal{M}_S$, and the partition function reduces to a sum of finitely many terms. In addition, the Pfaffians appearing in this expression can be computed using standard regularization techniques (see, for example, Ref. 91), as we discuss in Appendix C for the examples considered in this paper.

We note here that there is a typo in the presentation of this formula in several original references on the EL technique. The formula presented here is the correct one and it can be found in this form in Eq. 3.13 of Ref. 50 and Eq. 13 of Ref. 91, for example. Note, however, that we present this formula in terms of an operator \mathcal{O} which has all indices down, $\mathcal{O}_{ab}(t, t')$. We find that this presentation makes more sense since typically one considers the Pfaffian of an antisymmetric bilinear form \mathcal{O}_{ab} and not a linear operator \mathcal{O}^a_b which happens to be antisymmetric. In addition, in the infinite-dimensional case one needs to also properly define the Pfaffian, and with the index structure that we have chosen it is possible to define this Pfaffian in terms of a fermion path integral as we now discuss.

The Pfaffian of a $2n \times 2n$ antisymmetric matrix \mathcal{O}_{ab} is a well-defined object, in the sense that there is an explicit formula for it. One way of computing the Pfaffian is by Grassmann integration. If η^a , $a = 1, \dots, 2n$, are a set of $2n$ real Grassmann variables, then we have

$$\text{Pf}[\mathcal{O}] = \int d^{2n} \boldsymbol{\eta} e^{-\frac{1}{2} \eta^a \mathcal{O}_{ab} \eta^b}, \quad (\text{B22})$$

provided that we define the measure as $d^{2n} \boldsymbol{\eta} = d\eta^1 \dots d\eta^{2n}$. We therefore propose that in the infinite-dimensional case one should define the Pfaffian of the operator \mathcal{O} via the fermionic path integral

$$\text{Pf}[\mathcal{O}] = \int [d^{2n} \boldsymbol{\eta}] e^{-\frac{1}{2} \int_0^T dt \int_0^T dt' \eta^a(t) \mathcal{O}_{ab}(t, t') \eta^b(t')}, \quad (\text{B23})$$

where $\eta^a(t)$ are the Grassmann-valued fields with periodic boundary conditions that we considered earlier in this section.

We can then evaluate the integral by expanding the fields in Fourier modes as

$$\eta^a(t) = \sum_{m \in \mathbb{Z}} \eta_m^a \frac{e^{i \frac{2\pi m t}{T}}}{\sqrt{T}}, \quad (\text{B24})$$

where the Fourier coefficients η_m^a are ordinary Grassmann numbers. We also need to define the path integral measure. One possible definition is (we specialize to $n = 1$ here)

$$[d^2 \eta] = d\eta_0^1 d\eta_0^2 \prod_{m>0} d\eta_{-m}^1 d\eta_m^1 d\eta_{-m}^2 d\eta_m^2, \quad (\text{B25})$$

however, the definition of the measure is ambiguous because different orderings of the terms will lead to answers which differ by an overall sign. This ambiguity is not important at this stage however, because we will eventually need to regulate the result of the path integral in order to make sense of it. We consider the careful regularization of this integral for specific examples in Sec. III of the main text and in Appendix C.

Appendix C: Evaluation of Determinants

In this appendix we compute the amplitude and phase of the regularized determinants $\det[\mathcal{D}_\pm]_{reg}$ which are needed for the calculation of the partition function $Z[A]$ for the gauged boundary theory of the BTI state in Sec. III of this paper. We use zeta and eta functions (to be defined below) to regularize the magnitude and phase, respectively, of these determinants. The application of zeta and eta function methods to the regularization of determinants appearing in the context of EL calculations was discussed in detail by Miettinen in Ref. 91. In particular, Miettinen showed that by defining the phase of the regularized determinant using the *eta invariant* of the operator in question, the character formula for $SU(2)$ (equivalent to the partition function for a spin in a constant magnetic field) could be obtained directly from an EL path integral calculation, without the need to correct the final answer by hand using a so-called “Weyl shift”.

In Sec. III we showed that the expression for the determinant of \mathcal{D}_\pm could be manipulated into the form

$$\det[\mathcal{D}_\pm] = \left(\prod_{m \in \mathbb{Z}} |\lambda_m^{(\pm)}| \right) e^{i \frac{(2p+1)\pi}{2} \sum_{m \in \mathbb{Z}} (1 - \text{sgn}(\lambda_m^{(\pm)}))}, \quad (\text{C1})$$

where p was an arbitrary integer. We remind the reader that $\mathcal{D}_\pm = -i\partial_t \pm \bar{A}_t$, and the eigenvalues of \mathcal{D}_\pm are $\lambda_m^{(\pm)} = \frac{2\pi m}{T} \pm \bar{A}_t$, $m \in \mathbb{Z}$. In Sec. III we also showed that a regularization of the infinite sum $\sum_{m \in \mathbb{Z}} 1$ using the Riemann zeta function allowed us to reduce this expression to

$$\det[\mathcal{D}_\pm] = \left(\prod_{m \in \mathbb{Z}} |\lambda_m^{(\pm)}| \right) e^{-i \frac{(2p+1)\pi}{2} \sum_{m \in \mathbb{Z}} \text{sgn}(\lambda_m^{(\pm)})}. \quad (\text{C2})$$

In this appendix we show how zeta and eta function methods can be used to carefully define the amplitude and phase in this formal expression for the determinant of \mathcal{D}_\pm .

We start with the calculation of the amplitude $\prod_{m \in \mathbb{Z}} |\lambda_m^{(\pm)}|$. To be concrete, we first assume that $\bar{A}_t \in (0, \frac{2\pi}{T})$. In this case we have

$$\prod_{m \in \mathbb{Z}} |\lambda_m^{(\pm)}| = \bar{A}_t \prod_{m>0} \left[\left(\frac{2\pi m}{T} \right)^2 - (\bar{A}_t)^2 \right]. \quad (\text{C3})$$

To regularize the product on the right-hand side of this equation we first note that the ratio

$$\prod_{m>0} \left[\frac{\left(\frac{2\pi m}{T} \right)^2 - (\bar{A}_t)^2}{\left(\frac{2\pi m}{T} \right)^2} \right] = \frac{\sin\left(\frac{\bar{A}_t T}{2}\right)}{\frac{\bar{A}_t T}{2}}, \quad (\text{C4})$$

is a completely well-defined quantity. To compute this ratio we used the infinite product formula for the sine function,

$$\sin(x) = x \prod_{m=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 m^2} \right). \quad (\text{C5})$$

The product $\prod_{m>0} \left(\frac{2\pi m}{T} \right)^2$ in the denominator on the left-hand side of Eq. (C4) can be interpreted as $\det'[-i\partial_t]$, where the prime indicates the determinant without the contribution from the zero mode. We can use zeta function regularization⁹² to assign a finite value to this determinant.

To apply zeta function regularization we first define a differential operator \mathcal{P} with eigenvalues $\left(\frac{2\pi m}{T} \right)^2$, $m > 0$. We then define the *spectral zeta function* for this operator as

$$\zeta_{\mathcal{P}}(s) = \sum_{m>0} \left(\frac{2\pi m}{T} \right)^{-2s}, \quad (\text{C6})$$

which is well-defined for $\text{Re}[s] > \frac{1}{2}$. Then the regularized version of the determinant of \mathcal{P} is defined as

$$\det[\mathcal{P}]_{reg} = e^{-\zeta'_{\mathcal{P}}(0)}, \quad (\text{C7})$$

where $\zeta'_{\mathcal{P}}(0)$ is the analytic continuation of $\zeta'_{\mathcal{P}}(s)$ to $s = 0$ (and the prime denotes a derivative with respect to s). In this case the spectral zeta function $\zeta_{\mathcal{P}}(s)$ is related to the ordinary Riemann zeta function $\zeta(s)$ by

$$\zeta_{\mathcal{P}}(s) = \left(\frac{T}{2\pi} \right)^{2s} \zeta(2s), \quad (\text{C8})$$

which means that

$$\zeta'_{\mathcal{P}}(0) = 2 \ln \left(\frac{T}{2\pi} \right) \zeta(0) + 2\zeta'(0). \quad (\text{C9})$$

Using the well-known values $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2} \ln(2\pi)$, we find that $\zeta'_{\mathcal{P}}(0) = -\ln(T)$, so that

$$\det[\mathcal{P}]_{reg} = T. \quad (\text{C10})$$

Then, in view of the ratio Eq. (C4), we define

$$\begin{aligned} \left(\prod_{m \in \mathbb{Z}} |\lambda_m^{(\pm)}| \right)_{reg} &= \bar{A}_t \det[\mathcal{P}]_{reg} \frac{\sin\left(\frac{\bar{A}_t T}{2}\right)}{\frac{\bar{A}_t T}{2}} \\ &= 2 \sin\left(\frac{\bar{A}_t T}{2}\right). \end{aligned} \quad (\text{C11})$$

More generally, suppose that \bar{A}_t lies in the open interval $(\frac{2\pi\ell}{T}, \frac{2\pi\ell+2\pi}{T})$ for some $\ell \in \mathbb{Z}$. In this case it is convenient to decompose \bar{A}_t as

$$\bar{A}_t = \frac{2\pi\ell}{T} + \bar{a}_t, \quad (\text{C12})$$

where $\bar{a}_t \in (0, \frac{2\pi}{T})$. If we now repeat the amplitude calculation above for this case then we find that

$$\begin{aligned} \left(\prod_{m \in \mathbb{Z}} |\lambda_m^{(\pm)}| \right)_{\text{reg}} &= (-1)^\ell 2 \sin\left(\frac{\bar{A}_t T}{2}\right) \\ &= (-1)^\ell 2 \sin\left(\pi\ell + \frac{\bar{a}_t T}{2}\right) \\ &= 2 \sin\left(\frac{\bar{a}_t T}{2}\right). \end{aligned} \quad (\text{C13})$$

We now move on to the computation of the phase of the regularized determinants. First, for a complex number s with sufficiently large and positive real part, the *eta function* $\eta_\pm(s)$ of the Dirac operator \mathcal{D}_\pm is defined by¹⁷

$$\eta_\pm(s) = \sum_{m \in \mathbb{Z}} \text{sgn}(\lambda_m^{(\pm)}) |\lambda_m^{(\pm)}|^{-s}, \quad (\text{C14})$$

where we use the convention that $\text{sgn}(0) = 1$. This expression has a well-defined analytic continuation to $s = 0$, known as the *eta invariant*, and we use this analytic continuation to define the regularized phase of the determinant in question via the formula

$$\left(\sum_{m \in \mathbb{Z}} \text{sgn}(\lambda_m^{(\pm)}) \right)_{\text{reg}} = \eta_\pm(0). \quad (\text{C15})$$

We focus our attention on the calculation of the eta invariant for \mathcal{D}_+ . The calculation for \mathcal{D}_- is very similar.

First, recall that we are assuming that \bar{A}_t lies in an open interval between two eigenvalues of $-i\partial_t$. This guarantees that the operators \mathcal{D}_\pm do not possess any zero modes. In this case each term in $\eta_\pm(s)$ can be differentiated with respect to \bar{A}_t , since the value of $\text{sgn}(\lambda_m^{(\pm)})$ does not vary as we move \bar{A}_t within this open interval. After taking the derivative, we find that (focusing on the case of \mathcal{D}_+)

$$\frac{d\eta_+(s)}{d\bar{A}_t} = -s \zeta_{\mathcal{D}_+^2}\left(\frac{s+1}{2}\right), \quad (\text{C16})$$

where $\zeta_{\mathcal{D}_+^2}(s)$ is the spectral zeta function for \mathcal{D}_+^2 , the square of the Dirac operator \mathcal{D}_+ . This formula is in fact just a special case of the general formula in Proposition 2.10 of Ref. 19. Taking the $s \rightarrow 0$ limit then gives

$$\frac{d\eta_+(0)}{d\bar{A}_t} = - \lim_{s \rightarrow 0} s \zeta_{\mathcal{D}_+^2}\left(\frac{s+1}{2}\right). \quad (\text{C17})$$

The spectral zeta function $\zeta_{\mathcal{D}_+^2}(s)$ has a first order pole at $s = \frac{1}{2}$, which is due to the fact that the leading part of \mathcal{D}_+^2 is $-\partial_t^2$ (i.e., the dominant part of the eigenvalues of \mathcal{D}_+^2 for large m is the piece $(\frac{2\pi m}{T})^2$). It then follows from Eq. (C17) that $\frac{d\eta_+(0)}{d\bar{A}_t}$ is equal to *minus* the residue of $\zeta_{\mathcal{D}_+^2}(s)$ at $s = \frac{1}{2}$. This residue can be computed using the heat kernel expansion for \mathcal{D}_+^2 , and the residue turns out to be equal to the residue of the spectral zeta function for $-\partial_t^2$ at $s = \frac{1}{2}$, which is easier to compute. From these considerations we find that

$$\frac{d\eta_+(0)}{d\bar{A}_t} = -\frac{T}{\pi}, \quad (\text{C18})$$

and then an integration with respect to \bar{A}_t gives

$$\eta_+(0) = C_+ - \frac{\bar{A}_t T}{\pi}, \quad (\text{C19})$$

where C_+ is as yet undetermined constant.

The value of the constant C_+ can be fixed uniquely by requiring the eta invariant to vanish when \bar{A}_t lies halfway between two eigenvalues of $-i\partial_t$ (symmetry dictates that $\eta_+(s)$ for any s should vanish in this case). Let us assume that $\bar{A}_t \in (\frac{2\pi\ell}{T}, \frac{2\pi\ell+2\pi}{T})$ for some $\ell \in \mathbb{Z}$. Then we require $\eta_+(0)$ to vanish when $\bar{A}_t = \frac{2\pi}{T}(\ell + \frac{1}{2})$, which fixes $C_+ = 2\ell + 1$. Therefore the eta invariant is given in this case by

$$\eta_+(0) = 2\ell + 1 - \frac{\bar{A}_t T}{\pi}. \quad (\text{C20})$$

For the Dirac operator \mathcal{D}_- , and still assuming that $\bar{A}_t \in (\frac{2\pi\ell}{T}, \frac{2\pi\ell+2\pi}{T})$, all of the signs are reversed. We then find that for $\bar{A}_t \in (\frac{2\pi\ell}{T}, \frac{2\pi\ell+2\pi}{T})$, the eta invariants of the operators \mathcal{D}_\pm are

$$\eta_\pm(0) = \pm(2\ell + 1) \mp \frac{\bar{A}_t T}{\pi}. \quad (\text{C21})$$

As in Eq. (C12), it is convenient to again write $\bar{A}_t = \frac{2\pi\ell}{T} + \bar{a}_t$ with $\bar{a}_t \in (0, \frac{2\pi}{T})$. Then in terms of \bar{a}_t , the eta invariants for \mathcal{D}_\pm take the form

$$\eta_\pm(0) = \pm 1 \mp \frac{\bar{a}_t T}{\pi}. \quad (\text{C22})$$

We see that the eta invariant only depends on the value of \bar{A}_t modulo $\frac{2\pi}{T}$.

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