Model for continuous thermal metal to insulator transition

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A model for continuous thermal Metal to Insulator Transition

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We propose a $d-$dimensional interacting Majorana fermion model with quenched disorder, which gives us a continuous quantum phase transition between a diffusive thermal metal phase with a finite entropy density to an insulator phase with zero entropy density. This model is based on coupled Sachdev-Ye-Kitaev model clusters, and hence has a controlled large-N limit. The metal-insulator transition is accompanied by a spontaneous time-reversal symmetry breaking. We perform controlled calculations to show that the energy diffusion constant jumps to zero discontinuously at the metal-insulator transition, while the time-reversal symmetry breaking order parameter increases continuously.

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I. INTRODUCTION

The metal to insulator transition (MIT) is a fairly old subject. In a noninteracting fermion system with quenched disorder, MIT can occur by tuning the Fermi surface across the mobility edge of the single particle spectrum[1]; in a system with interacting electrons, MIT can be driven by the competition between interaction and kinetic energy, *i.e.* the so-called Mott transition[2], and the Mott insulator phase often develops a Landau order parameter which spontaneously breaks certain symmetry of the system, though a more interesting possibility is that the Mott insulator phase close to the Mott transition is a spin liquid phase without any Landau order parameter[3–9].

In recent years, MIT between highly excited states with finite energy density have attracted enormous interests and efforts. This type of MIT is driven by the interplay between interaction and quenched disorder. On one side of the transition, the state obeys the eigenstate thermalization hypothesis (ETH)[10,11], and the many-body eigenstate is extended (either in the real space or the fock space) with a volume-law entanglement entropy, while the other side of the transition is a many-body localized (MBL) state[12–16] which is a product of eigenstates of localized conserved quantities[17,18] and hence only possesses boundary-law entanglement entropy. Great numerical and analytical efforts have been made towards understanding the nature of the ETH-MBL transition[14,16,19–34]. Due to the difficulty of directly studying the ETH-MBL transition in a generic nonintegrable many-body Hamiltonian, studies based on many-body wave functions instead of Hamiltonians have also been pursued[35,36].

In this work we construct a many-body Hamiltonian which gives us a quantum MIT at its ground state. The basic degrees of freedom in our model are Majorana fermions, which only transport energy rather than electric charge. The thermal metallic phase of our model has a finite entropy density and energy diffusion constant; while the insulator phase has zero entropy density and diffusion. The main advantage of our model is that it is accessible analytically with controlled methods, if the fermion flavor number N on every site (cluster) is taken to be large. If we take the large$-N$ limit, and the infrared limit, the diffusion constant would jump discontinuously to zero at the transition. But just like the model discussed in Ref. 37, a time-reversal symmetry breaking order parameter would increase continuously from the MIT. The MIT in our model is controlled by a tuning parameter in the Hamitonian, rather than the ratio between the sizes of two clusters of different nature, like the pase transition considered in a two-cluster model in Ref. 38

This paper is organized as follows: in section II we will first derive the saddle point equations of our model, and the mean field solution of the saddle point equations demonstrates that a quantum phase transition occurs by tuning one parameter in our model; in section III we perform a large$-q$ calculation to demonstrate that the transition we construct is between a thermal metal and insulator with finite and zero energy diffusion constants respectively; in section IV we use the effective action of quasi Goldstone modes and scaling argument to verify the results in section III.

II. THE MODEL AND SADDLE POINT SOLUTIONS

We first consider the following Hamiltonian for a one dimensional chain (it is trivial to generalize our model to higher dimensions, and the calculations and conclusions in this work are insensitive to the spatial dimension), with a

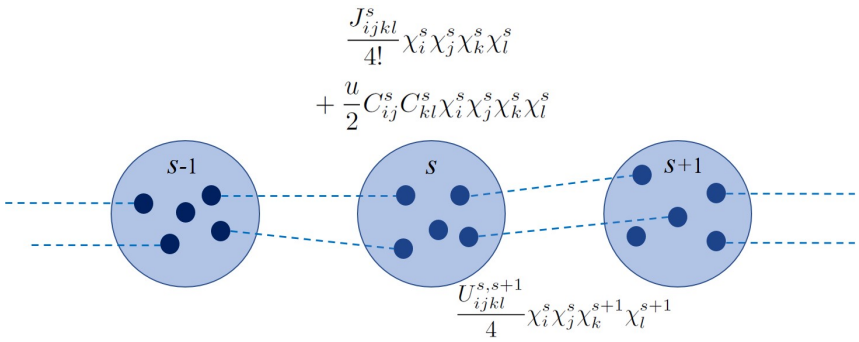

FIG. 1: The schematic structure of our model Eq. 1.

large number (N) of Majorana fermions defined on each site (cluster) s:

$$H_{\text{chain}} = \sum_{s=1}^{L} \frac{J^s_{ijkl}}{4!}\chi^s_i\chi^s_j\chi^s_k\chi^s_l + \frac{u}{2}C^s_{ij}C^s_{kl}\chi^s_i\chi^s_j\chi^s_k\chi^s_l + \frac{U^{s,s+1}_{ijkl}}{4}\chi^s_i\chi^s_j\chi^{s+1}_k\chi^{s+1}_l. \tag{1}$$

J^s, C^s and $U^{s,s+1}$ are all independent random variables which have zero mean and obey the Gaussian distribution:

$$N^3\overline{(J^s_{ijkl})^2} = 3!J^2, \quad N^2\overline{(C^s_{ij})^2} = C^2, \quad N^3\overline{(U^{s,s+1}_{ijkl})^2} = U^2. \tag{2}$$

The structure of this model is schematically shown in Fig. 1

Several limits of this model have been understood:

1. When $U = 0$, this model becomes decoupled sites with the first two terms of the Hamiltonian. The first term is the ordinary Sachdev-Ye-Kitaev model (SYK_4) with Majorana fermions[39–41], whose ground state in the large$-N$ limit is a non-fermi liquid with power-law fermion Green's function after disorder average, and the scaling dimension of fermion χ_j is $[\chi_j] = 1/4$[47]. It was shown recently[37] that if we treat the u term as a perturbation on the non-fermi liquid state described by the ordinary SYK_4 model (the first term of Eq. 1), it is marginally relevant (irrelevant) when $u > 0$ ($u < 0$). And for $u > 0$, the u term will lead to spontaneous time-reversal symmetry breaking and generate a SYK_2 term with a random two-body interaction, at an exponentially low energy scale. This seemingly zero dimensional system is capable of having its own phase transition because it has infinite degrees of freedom in the large$-N$ limit. The spontaneously generated SYK_2 term will dominate the physics in the infrared, and the SYK_4 term becomes irrelevant.

2. When $u = 0$, this model becomes the coupled SYK clusters studied in Ref. 42. A complex fermion version of this case was studied later[43]. Both the on-site J^s term and the inter-cluster $U^{s,s+1}$ term contribute on the equal footing. The entire $1d$ system is a diffusive thermal metal, with a finite energy diffusion constant, and a "butterfly velocity" due to the chaotic nature of the SYK_4 model.

In this work we treat the onsite u term as a perturbation on the coupled SYK cluster chain. It is straightforward to show that, the renormalization group flow of the u term is almost identical to the single cluster case discussed in Ref. 37, *i.e.* when $u < 0$, it can be ignored in the infrared limit because it is marginally irrelevant; while when $u > 0$, it becomes marginally relevant. The saddle point equation that we will discuss next demonstrates that in the latter case the system again spontaneously generates an onsite SYK_2 random two-body interaction term, which dominates all the low energy physics. If we start with an onsite SYK_2 term, then both the J^s and $U^{s,s+1}$ terms are irrelevant, and the system becomes decoupled clusters with zero entropy density. Since the inter-cluster coupling will renormalize to zero in the infrared limit, this system is expected to be an insulator for $u > 0$. Thus tuning u from negative to positive value will lead to a quantum phase transition between a thermal metal with finite entropy density and energy diffusion, to an insulator with zero entropy density and diffusion constant.

To demonstrate the effect mentioned above, we follow Ref. 37 and perform a Hubbard-Stratonovich transformation for the u-term. The action for the chain becomes:

$$\mathcal{S} = \sum_{s=1}^{L}\int d\tau \frac{1}{2}\chi^s_i\partial_\tau\chi^s_i + \frac{J^s_{ijkl}}{4!}\chi^s_i\chi^s_j\chi^s_k\chi^s_l + \frac{U^{s,s+1}_{ijkl}}{4}\chi^s_i\chi^s_j\chi^{s+1}_k\chi^{s+1}_l + \frac{u}{2}b^sb^s - iuC^s_{jk}b^s\chi^s_j\chi^s_k. \tag{3}$$

With the normalizations in Eq.(2), we can write down the disorder averaged theory. Under the replica diagonal

assumption, the action reads

$$\begin{aligned}\mathcal{S} &= \sum_{s=1}^{L}\int d\tau \left(\frac{1}{2}\chi_i^s\partial_\tau\chi_i^s + \frac{u}{2}(b^s)^2\right) - \sum_{s=1}^{L} u^2\frac{C^2}{N^2}\int d\tau_1 d\tau_2\ b^s(\tau_1)b^s(\tau_2)(\chi_j^s(\tau_1)\chi_j^s(\tau_2))^2 \\ &- \sum_{s=1}^{L}\frac{1}{8N^3}\int d\tau_1 d\tau_2\left[J^2(\chi_i^s(\tau_1)\chi_i^s(\tau_2))^4 + U^2(\chi_i^s(\tau_1)\chi_i^s(\tau_2))^2(\chi_j^{s+1}(\tau_1)\chi_j^{s+1}(\tau_2))^2\right] \end{aligned} \tag{4}$$

Following Ref. 41, we proceed by introducing the fermion Green's function and self-energy on every site s:

$$\begin{aligned}\mathcal{S}_{eff} &= \sum_{s=1}^{L}\int d\tau_1 d\tau_2 \left(\frac{1}{2}\chi_i^s(\tau_1)(\delta(\tau_1-\tau_2)\partial_{\tau_2} - \Sigma^s(\tau_1,\tau_2))\chi_i^s(\tau_2) + \frac{u}{2}b^s(\tau_1)\delta(\tau_1-\tau_2)b^s(\tau_2)\right) \\ &- \sum_{s=1}^{L} u^2C^2\int d\tau_1 d\tau_2\ b^s(\tau_1)b^s(\tau_2)(G^s(\tau_1,\tau_2))^2 + \frac{N}{2}\sum_{s=1}^{L}\int d\tau_1 d\tau_2\ \Sigma^s(\tau_1,\tau_2)G^s(\tau_1,\tau_2) \\ &- \frac{N}{2}\sum_{s=1}^{L}\int d\tau_1 d\tau_2\left[\frac{J^2}{4}(G^s(\tau_1,\tau_2))^4 + \frac{U^2}{4}(G^s(\tau_1,\tau_2))^2(G^{s+1}(\tau_1,\tau_2))^2\right] \end{aligned} \tag{5}$$

Integrating out $\Sigma^s(\tau_1,\tau_2)$ will impose the constraint $G^s(\tau_1,\tau_2) = \sum_i \chi_i(\tau_1)\chi_i(\tau_2)$, which establishes the equivalence between Eq. 4 and Eq. 5. According to previous studies of instabilities of SYK model[37], for $u > 0$, the bosonic field b on each site will develop long-range correlation, thus as a mean field approximation we can replace the composite field $b^s(\tau_1)b^s(\tau_2)$ by its expectation value $\langle b^s(\tau_1)b^s(\tau_2)\rangle = N(w^s)^2$, where w^s is an $O(1)$ number and should be self-consistently determined by the saddle point equations. The large-N saddle-point-mean-field equations for the above action are:

$$\begin{aligned}&\tilde{G}^s(i\omega_n)^{-1} = -i\omega_n - \tilde{\Sigma}^s(i\omega_n) \\ &\Sigma^s(\tau) = 4u^2C^2(w^s)^2G^s(\tau) + G^s(\tau)\left[J^2G^s(\tau)^2 + \frac{U^2}{2}(G^{s-1}(\tau)^2 + G^{s+1}(\tau)^2)\right] \\ &\int_0^\beta d\tau\left(uC^2G^s(\tau)^2 - \frac{1}{2}\delta(\tau)\right)uw^s = 0 \end{aligned} \tag{6}$$

If we only consider translational invariant solutions, we can drop the site indices, and the saddle-point-mean-field equations become:

$$\begin{aligned}&\tilde{G}(i\omega_n)^{-1} = -i\omega_n - \tilde{\Sigma}(i\omega_n) \\ &\Sigma(\tau) = 4u^2C^2w^2G(\tau) + (J^2+U^2)G(\tau)^3 \\ &\int_0^\beta d\tau\left(uC^2G(\tau)^2 - \frac{1}{2}\delta(\tau)\right)uw = 0 \end{aligned} \tag{7}$$

We arrive practically the same saddle-point-mean-field equations as the single site model studied previously in Ref. 37. For $u < 0$, the only solution for the saddle point equations has $w = 0$. In this limit, the model resembles the feature of the SYK chain model previously studied by Gu *et.al.*[42], and the system is a diffusive metal. For $u > 0$, the solution of the saddle point equations gives w a non-zero value, which for small positive u is approximately given by

$$w \sim \frac{(J^2+U^2)^{1/2}}{uC}\exp\left(-\frac{\sqrt{\pi}(J^2+U^2)^{1/2}}{4uC^2}\right); \tag{8}$$

and the system spontaneously breaks the time reversal symmetry.

In the time-reversal symmetry breaking phase, the physics on each cluster is very similar to Ref.[37]. The first term $4u^2C^2w^2G(\tau)$ in the self energy dominates at low energy. In the deep infrared limit, the fermions on each site behave identically as the SYK_2 model and their scaling dimension $[\chi] = 1/2$. Therefore, the J^s and $U^{s,s+1}$ interactions are irrelevant in the infrared (both terms have scaling dimension -1 and hence irrelevant) and the chain becomes an array

of isolated decoupled SYK_2 clusters. We expect the chain becomes an insulator in this limit, which will be verified with a large-q expansion method in the following section. Thus tuning u from negative to positive drives a continuous phase transition between a diffusive thermal metal and an insulator phase. Notice that w increases continuously from zero starting $u = 0$, but it is exponentially small at small u, which is analogous to the BCS instability of the fermi liquid, and the Kondo effect, and it is a sign of the marginally-relevant nature of u[37].

III. LARGE q ANALYSIS

For $u > 0$, when w develops a non-trivial vacuum expectation value, the system is effectively described by the Hamiltonian H_{chain} but with the second term replaced by on-site SYK_2 terms. This replacement is equivalent to, in the path integral formalism, setting b^s to their vacuum expectation values in Eq. 3 and neglecting their fluctuations which should be unimportant at low energies. Further, to justify that system becomes an insulator in the IR at zero temperature, it is enough to just consider the effective on-site SYK_2 couplings and the U interactions (because the J does not couple different sites). Therefore, we consider the effective model:

$$H_{\text{eff}} = \sum_s \sum_{1\le i_1<i_2\le N} iV^s_{i_1i_2}\chi^s_{i_1}\chi^s_{i_2} + \sum_s \sum_{\substack{1\le i_1<i_2\le N\\ 1\le i_3<i_4\le N}} U^{s,s+1}_{i_1i_2i_3i_4}\chi^s_{i_1}\chi^s_{i_2}\chi^{s+1}_{i_3}\chi^{s+1}_{i_4}, \tag{9}$$

where $V^s_{i_1i_2}$ are independent Gaussian random variables which correspond to $u\langle b_s\rangle C_{i_1i_2}$ in the original model. We also redefined the coefficient of the U term for later convenience. In fact, H_{eff} belongs to a large family of models:

$$H_q = (i)^{q/2}\sum_s \sum_{1\le i_1<i_2\ldots<i_q\le N} V^s_{i_1i_2\ldots i_q}\chi^s_{i_1}\chi^s_{i_2}\cdots\chi^s_{i_q} + (i)^q\sum_s \sum_{\substack{1\le i_1<i_2\ldots<i_q\le N\\ 1\le i_{q+1}<i_{q+2}\cdots<i_{2q}\le N}} U^{s,s+1}_{i_1i_2\ldots i_{2q}}\chi^s_{i_1}\chi^s_{i_2}\cdots\chi^s_{i_q}\chi^{s+1}_{i_{q+1}}\chi^{s+1}_{i_{q+2}}\cdots\chi^{s+1}_{i_{2q}} \tag{10}$$

where we consider SYK_q on-site coupling and SYK_{2q} inter-cluster coupling with an even integer q. The couplings $V^s_{i_1i_2\ldots i_q}$ and $U^{s,s+1}_{i_1i_2\ldots i_{2q}}$ are independent random variables with zero mean and Gaussian distribution:

$$\overline{\left(V^s_{i_1i_2\ldots i_q}\right)^2} = \frac{V^2(q-1)!}{N^{q-1}}\,, \qquad \overline{\left(U^{s,s+1}_{i_1i_2\ldots i_{2q}}\right)^2} = \frac{U^2(q-1)!\;q!}{N^{2q-1}}. \tag{11}$$

The effective Hamiltonian H_{eff} can be recovered by choosing $q = 2$. In fact, we expect H_q for all q to share the same feature that the model effectively becomes isolated SYK_q clusters in the IR, where the inter cluster couplings (the U terms) are irrelevant. Hence, the system should be an insulator at low energy. To justify this expectation, we will solve this family of models using the large-q expansion developed in Ref. 41. It was shown in previous studies[41,44], this large-q expansion method is capable of capturing the thermodynamical properties of the SYK model and its generalizations at low energies.

We first investigate the 2-point Green's function of this model. The saddle point equations of the Green's function $G^s(\tau) = \frac{1}{N}\sum_i\langle\chi^s_i(\tau)\chi^s_i(0)\rangle$ of this model are given by

$$G^s(i\omega)^{-1} = -i\omega - \Sigma^s(i\omega) \tag{12}$$

$$\Sigma^s(\tau) = G^s(\tau)^{q-1}\Big[V^2 + U^2G^{s-1}(\tau)^q + U^2G^{s+1}(\tau)^q\Big] \tag{13}$$

In the large q limit, we can expand the $G^s(\tau)$ as a function of $1/q$:

$$G^s(\tau) = \tfrac{1}{2}\text{sgn}(\tau)\Big[1 + \tfrac{1}{q}g^s(\tau) + ...\Big], \tag{14}$$

where "..." represents higher order terms in $1/q$ which will be neglected in the following. Notice that when $q \to \infty$, the Green's function becomes that of free fermions without any dispersion. In the following, we will obtain an analytical solution of $g^s(\tau)$, that is the $1/q$ correction to the Green's function. At finite temperature β, $g^s(\tau)$ has to satisfy the boundary conditions $g^s(\tau = 0) = g^s(\tau = \beta) = 0$. Given this expansion of $G^s(\tau)$, we can express the self energy via Eq. 13 as

$$\Sigma^s(\tau) = \frac{\text{sgn}(\tau)}{q}e^{g^s(\tau)}\Big[\mathcal{V}^2 + \frac{\mathcal{U}^2}{2}e^{g^{s-1}(\tau)} + \frac{\mathcal{U}^2}{2}e^{g^{s+1}(\tau)}\Big]. \tag{15}$$

where

$$\mathcal{V}^2 \equiv \frac{q}{2^{q-1}}V^2, \quad \mathcal{U}^2 \equiv \frac{2q}{2^{2q-1}}U^2 \tag{16}$$

are the coupling constants kept fixed in the large q limit. Plugging this expression back to the saddle point equation Eq. 12, we can obtain the translationally invariant solution:

$$e^{g_s(\tau)} = \frac{1}{\frac{2\beta^2\mathcal{V}^2}{\pi^2 v^2} + \sqrt{\frac{4\beta^4\mathcal{V}^4}{\pi^4 v^4} + \frac{2\beta^2\mathcal{U}^2}{\pi^2 v^2}}\cos\left(\pi v\left(\frac{|\tau|}{\beta} - \frac{1}{2}\right)\right)}. \tag{17}$$

Here the parameter $v \in (1, 2)$ given by the relation

$$\cos\left(\frac{\pi v}{2}\right) = -\frac{1 - \frac{v^2\pi^2}{2\beta^2\mathcal{V}^2}}{\sqrt{1 + \frac{\mathcal{U}^2}{\mathcal{V}^2}\frac{v^2\pi^2}{2\beta^2\mathcal{V}^2}}} \tag{18}$$

ensures the boundary conditions of $g^s(\tau)$. By taking the low temperature limit $\beta \to \infty$, we can expand v in terms of $(\beta\mathcal{V})^{-1}$:

$$v = 2 - \sqrt{1 + \frac{\mathcal{U}^2}{2\mathcal{V}^2}}\frac{4}{\beta\mathcal{V}} + O\left(\frac{1}{\beta^2\mathcal{V}^2}\right). \tag{19}$$

More importantly, $e^{g_s(\tau)}$ also has a well-defined low temperature limit:

$$e^{g_s(\tau)} = \frac{1}{1 + 2\sqrt{1 + \frac{\mathcal{U}^2}{2\mathcal{V}^2}}\mathcal{V}|\tau| + \mathcal{V}^2\tau^2}. \tag{20}$$

As we can see from this expression, in the long time limit, $e^{g_s(\tau)}$ behaves as $(\mathcal{V}\tau)^{-2}$, which is independent from the coupling constant $\mathcal{U}$. This agrees with the expectation that the low energy physics is governed by the on-site SYK_q couplings and the inter cluster couplings become irrelevant at IR.

To verify the insulating behavior of H_q in the IR, we need to show that all the lower energy modes are localized in space. As is pointed out by Ref. 41,42, the propagation of low energy modes are all captured by the 4-point function

$$\frac{1}{N}\mathcal{F}^{ss'}(\tau_1, \tau_2, \tau_3, \tau_4) \equiv \frac{1}{N^2}\sum_{i,j=1}^{N}\langle T\big(\chi_i^s(\tau_1)\chi_i^s(\tau_2)\chi_j^{s'}(\tau_3)\chi_j^{s'}(\tau_4)\big)\rangle - \delta_{ss'}G^s(\tau_{12})G^{s'}(\tau_{34}), \tag{21}$$

where we've used the notation $\tau_{ij} \equiv \tau_i - \tau_j$ and $G^s(\tau)$ takes the form of the solution of the saddle point equation. Since the saddle point solution G^s is translational invariant, site index s on G^s will be ignored hereafter. Physically, $\mathcal{F}^{ss'}(\tau_1, \tau_2, \tau_3, \tau_4)$ captures the propagation, from site s to site s', of all the modes created by a pair of fermion operators (at the same site but at different times). In the large N limit, $\mathcal{F}^{ss'}(\tau_1, \tau_2, \tau_3, \tau_4)$ can be calculated via the ladder diagrams[41,42]. The contribution of the n-rung ladder is denoted as $\mathcal{F}^{(n)}$. First of all, we have

$$\mathcal{F}^{ss'(0)}(\tau_1, \tau_2, \tau_3, \tau_4) = \delta_{ss'}\Big(-G(\tau_{13})G(\tau_{24}) + G(\tau_{14})G(\tau_{23})\Big). \tag{22}$$

The contribution of the n-rung ladder can be obtain from that of the $(n-1)$-rung ladder:

$$\mathcal{F}^{ss'(n)}(\tau_1, \tau_2, \tau_3, \tau_4) = \sum_{s''}\int_0^\beta d\tau d\tau' K^{ss''}(\tau_1, \tau_2, \tau, \tau')\mathcal{F}^{s''s'(n-1)}(\tau, \tau', \tau_3, \tau_4). \tag{23}$$

with the kernel K given by

$$\begin{aligned} &K_{ss'}(\tau_1, \tau_2, \tau_3, \tau_4) \\ &\quad = -G(\tau_{13})G(\tau_{24})\Big[\Big((q-1)V^2G(\tau_{34})^{q-2} + 2(q-1)U^2G(\tau_{34})^{2q-2}\Big)\delta_{ss'} + qU^2G(\tau_{34})^{2q-2}(\delta_{s+1,s'} + \delta_{s-1,s'})\Big] \end{aligned} \tag{24}$$

It is useful to think about $K^{ss'}(\tau_1, \tau_2, \tau_3, \tau_4)$ as a matrix with the left index given by s, τ_1 and τ_2, and the right index given by s', τ_3 and τ_4. From this perspective, Eq. 23 can be viewed as a matrix multiplication. Then, the 4-point function, as a summation of all the ladder diagrams, can be written as

$$\mathcal{F} = \frac{1}{1 - K}\mathcal{F}^{(0)}. \tag{25}$$

The form of the ladder diagram suggested that $\frac{1}{1-K}$ should be viewed as the propagator whose eigenvectors correspond to different orthogonal modes in the system. These orthogonal modes are also eigenvectors of K, among which the ones with eigenvalues close to 1 correspond to the low energy modes of the system. The kernel K is diagonal in momentum space:

$$K_p(\tau_1,\tau_2,\tau_3,\tau_4) = -\frac{\mathrm{sgn}(\tau_{13})}{2}\frac{\mathrm{sgn}(\tau_{24})}{2}\Big[2\mathcal{V}^2 e^{g(\tau_{34})} + 2\mathcal{U}^2 e^{2g(\tau_{34})}(1+\cos(p))\Big], \tag{26}$$

where $K_p \equiv \sum_s K^{ss'} e^{ip(s-s')}$ is the real-space Fourier transform of $K_{ss'}$ and p denotes the momentum in real space. Here, we've plugged in the saddle point solution of $G(\tau)$ and omitted the higher order terms in $1/q$. The eigenvectors $\Psi_p(\tau_1,\tau_2)$ and eigenvalues k_p of K_p are given by the equation (to the leading order in $1/q$):

$$k_p\Psi_p(\tau_1,\tau_2) = -\int d\tau_3 d\tau_4 \frac{\mathrm{sgn}(\tau_{13})}{2}\frac{\mathrm{sgn}(\tau_{24})}{2}\Big[2\mathcal{V}^2 e^{g(\tau_{34})} + 2\mathcal{U}^2 e^{2g(\tau_{34})}(1+\cos(p))\Big]\Psi_p(\tau_3,\tau_4). \tag{27}$$

Following Ref. 41, we can take the derivative $\partial_{\tau_1}\partial_{\tau_2}$ on both sides of the equation and obtain

$$k_p\partial_{\tau_1}\partial_{\tau_2}\Psi_p(\tau_1,\tau_2) = -\Big[2\mathcal{V}^2 e^{g(\tau_{12})} + 2\mathcal{U}^2 e^{2g(\tau_{12})}(1+\cos(p))\Big]\Psi_p(\tau_1,\tau_2). \tag{28}$$

Since this equation is invariant under simultaneous translation of the two time variables τ_1 and τ_2, we can directly work with the ansatz $\Psi_{p,n}(\tau_1,\tau_2) = e^{-i\frac{2\pi n}{\beta}\frac{\tau_1+\tau_2}{2}}\psi_{p,n}(\tau_{12})$ with integer n. Further, by introducing the variables

$$\rho = \cos\Big(\pi v\Big(\tfrac{|\tau_{12}|}{\beta} - \tfrac{1}{2}\Big)\Big) \quad \text{and} \quad \alpha = \sqrt{1 + \tfrac{v^2}{8}\tfrac{\mathcal{U}^2}{\mathcal{V}^2}\tfrac{4\pi^2}{\beta^2\mathcal{V}^2}}, \tag{29}$$

we can simplify Eq. 28 to

$$\left[v^2\left((1-\rho^2)\frac{d^2}{d\rho^2} - \rho\frac{d}{d\rho}\right) + \frac{n^2}{4} - \frac{v^2}{4k_{p,n}}\left(\frac{1}{1+\alpha\rho} + \frac{(\alpha^2-1)(1+\cos p)}{(1+\alpha\rho)^2}\right)\right]\psi_{p,n} = 0. \tag{30}$$

This equation can be studied perturbatively by treating α and v as independent variables and by viewing $\alpha - 1$ as a small parameter. At the zeroth order, namely $\alpha = 1$, the dependence of this equation on the momentum p simply drops out. We recover the same differential equation studied in Ref. 41 which gives rise to the following eigenvalues

$$\begin{aligned} k^{(0)}_{n,p} &= 1 - \tfrac{3|n|}{2}\Big(1 - \tfrac{v}{2}\Big) + O\Big(\Big(1-\tfrac{v}{2}\Big)^2\Big) \\ &= 1 - \tfrac{3|n|}{\beta\mathcal{V}}\sqrt{1+\tfrac{\mathcal{U}^2}{2\mathcal{V}^2}} + O\Big(\tfrac{1}{\beta^2\mathcal{V}^2}\Big). \end{aligned} \tag{31}$$

where the superscript "(0)" represents the zeroth order in $\alpha - 1$. Here, we've only listed the eigenvalues that are close to 1 since we are only interested in the low energy modes in the theory. The momentum dependence of the eigenvalue $k_{n,p}$ comes into the first order correction in $(\alpha - 1)$. However, since $(\alpha - 1) \sim \frac{1}{\beta^2\mathcal{V}^2}$ in the low temperature limit $\beta\mathcal{V} \to \infty$, even including the first order correction, we still have

$$k^{(1)}_{n,p} = 1 - \frac{3|n|}{\beta\mathcal{V}}\sqrt{1+\frac{\mathcal{U}^2}{2\mathcal{V}^2}} + O\Big(\frac{1}{\beta^2\mathcal{V}^2}\Big), \tag{32}$$

where the $O\Big(\frac{1}{\beta^2\mathcal{V}^2}\Big)$ part also depends on the momentum p. When we view $(1-k_{n,p})^{-1}$ as the propagator of low energy modes, we should rewrite n using the Matsubara frequency $\omega_n = \frac{2\pi n}{\beta}$. It is immediately clear that the leading term in the propagator $(1-k_{n,p})^{-1}$ behaves as $\left(\frac{3\omega_n}{2\pi\mathcal{V}}\sqrt{1+\frac{\mathcal{U}^2}{2\mathcal{V}^2}}\right)^{-1}$ which is independent of the momentum p. Although we cannot calculate the term of order $\frac{1}{\beta^2\mathcal{V}}$ in Eq. 32 exactly, we can generally expect its functional form to be $\frac{\omega_n^2}{\mathcal{V}^2}(c_1 + c_2 p^2)$ in the small momentum limit ($p \ll 1$). Here, c_1 and c_2 are both constants depending on $\mathcal{U}/\mathcal{V}$. In particular, c_2 is proportional to $\mathcal{U}^2/\mathcal{V}^2$ because the momentum dependence of $k^{(1)}_{n,p}$ originates from the first order perturbation that is proportional to $\alpha - 1$ (that is proportional to $\mathcal{U}^2/\mathcal{V}^2$). In contrast, c_1 should stays finite even when $\mathcal{U} = 0$[41], that is when different SYK clusters are completely decoupled. Having established these expectations of the terms of order $\frac{1}{\beta^2\mathcal{V}^2}$ in $k^{(1)}_{n,p}$, and keeping only the lowest order p and ω_n dependent terms, we can write

$$(1-k_{n,p})^{-1} = \left(\frac{3\omega_n}{2\pi\mathcal{V}}\sqrt{1+\frac{\mathcal{U}^2}{2\mathcal{V}^2}} + c_2\frac{\omega_n^2}{\mathcal{V}^2}p^2 + \cdots\right)^{-1}. \tag{33}$$

Therefore, the diffusion constant, which is proportional to the coefficient of the p^2 term, vanishes in the infrared limit.

IV. EFFECTIVE ACTION AND SCALING

The calculation above is actually consistent with a qualitative scaling argument. As was shown in Ref. 41,42, the main fluctuation above the solution of the saddle point equations is the "quasi Goldstone" mode $\epsilon(\tau)$ defined as an infinitesimal time-reparametrization $\tau \to \tau + \epsilon(\tau)$. This reparametrization mode is also responsible for the energy transport of the coupled SYK_4 clusters[42]. Its dynamics is captured by an effective action[48]:

$$\mathcal{S}_\epsilon \sim \sum_{n\neq 0,\pm 1}\sum_p A\omega_n^2\left(\omega_n^2 - \left(\frac{2\pi}{\beta}\right)^2\right)|\epsilon_{n,p}|^2 + Bp^2|\omega_n|^m\left(\omega_n^2 - \left(\frac{2\pi}{\beta}\right)^2\right)|\epsilon_{n,p}|^2. \tag{34}$$

If the entire coupled system had an exact global reparametrization symmetry, then the effective Lagrangian should be zero when momentum $p = 0$, because making a global uniform reparametrization transformation on the system should not change the effective Lagrangian, *i.e.* the reparametrization Goldstone mode is a real Goldstone mode. In this case the only contribution to the action can come from finite momentum, *i.e.* when the system makes spatial dependent reparametrization transformation. The first momentum independent term of the action Eq. 34 mainly comes from the $-i\omega$ term in $G(i\omega)^{-1}$, which explicitly breaks the reparametrization symmetry, and hence makes a nonzero contribution to the action of $\epsilon(\tau)$.

The second term in Eq. 34 depends on momentum p, which can only come from the inter-site coupling. Because the inter-site coupling $U^{s,s+1}$ is completely random, the leading contribution to the p^2 term must be proportional to U^2. Based on its definition, $\epsilon(\tau)$ has the scaling dimension of time. In the models studied in Ref. 42,43, the inter-site coupling is marginal if evaluated at the on-site SYK_4 non-fermi liquid ground state, thus $m = 1$ in Eq. 34, and B has scaling dimension $[B] = 0$. The diffusion constant $D \sim B/A$ is finite. While in our case, since the inter-site coupling $U^{s,s+1}$ has scaling dimension -1 at the conformal saddle point solution, $B \sim U^2$ must have scaling dimension -2 (here we use the convention that frequency ω has scaling dimension $+1$). This implies that in our case $m = 3$ in action Eq. 34, which is consistent with the calculation of $(1 - k_{n,p})^{-1}$ the last section.

Had we switched the role of the two terms in H_q (Eq. 10), *i.e.* had we considered a toy model with an onsite $2q$-body interaction and inter-cluster $q-$body interaction, then the fermion Green's function would be completely dominated by the inter-cluster interaction, and the onsite interaction becomes irrelevant. In this case the physics should return to the case studied in Ref. 42,43, hence the model becomes a diffusive metal in the infrared, and also $m = 1$ in Eq. 34. A similar large$-q$ calculation as we have done in the last section confirms this expectation (see the appendix).

V. DISCUSSION

In this work we presented a model that goes through a continuous MIT by tuning one interaction term in the Hamiltonian. The transition actually already occurs on a single cluster of our model[37], as the infinite degrees of freedom on each cluster support a real quantum phase transition on this zero dimensional system. The metallic phase of this model is essentially SYK_4 clusters connected by a marginal inter-cluster four fermion coupling which obeys an independent Gaussian distribution[42]; the insulator phase is effectively equivalent to SYK_2 clusters coupled by irrelevant random four-fermion interactions, while the on-site SYK_2 term is generated spontaneously through another four fermion interaction.

In addition to time-reversal, our model Eq. 1 also has a local Z_2 fermion parity symmetry: $\chi_j^s \to \eta^s\chi_j^s$, where $\eta^s = \pm 1$. This symmetry forbids a inter-cluster two-body interaction. If this local fermion parity symmetry is broken (either explicitly or spontaneously), an inter-site (random) two-body interaction can potentially be generated. With finite number of degrees of freedom on each site, a two-body Hamiltonian with random hopping should lead to Anderson localization in one dimension. However, if we take the large$-N$ limit before taking the thermodynamics limit, the conductance may still remain finite, but this would be a large$-N$ artifact.

While completing this work, the authors became aware of another work on MIT based on connected SYK model clusters[45], where like the model in Ref. 38, the tuning parameter of the phase diagram is the ratio between the degrees of freedoms of two inequivalent clusters of interacting fermions. It is a very different from the mechanism of phase transition presented in our current work.

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Appendix A: An "opposite" model

In this appendix, we discuss an "opposite" model where the inter-cluster coupling dominates the low energy physics and the on-site SYK interaction becomes irrelevant in the IR. The model is given by a 1D chain of SYK clusters with SYK_{4q} intra-cluster coupling and "SYK_{2q}" inter-cluster couplings:

$$H'_q = (i)^{2q}\sum_s \sum_{1\le i_1<i_2\ldots<i_{4q}\le N} V^s_{i_1 i_2\ldots i_q}\chi^s_{i_1}\chi^s_{i_2}\cdots\chi^s_{i_{4q}} + (i)^q \sum_s \sum_{\substack{1\le i_1<i_2\ldots<i_q\le N\\ 1\le i_{q+1}<i_{q+2}\cdots<i_{2q}\le N}} U^{s,s+1}_{i_1 i_2\ldots i_{2q}}\chi^s_{i_1}\chi^s_{i_2}\cdots\chi^s_{i_q}\chi^{s+1}_{i_{q+1}}\chi^{s+1}_{i_{q+2}}\cdots\chi^{s+1}_{i_{2q}} \tag{A1}$$

with the couplings drawn from Gaussian distributions with zero mean:

$$\overline{\left(V^s_{i_1 i_2\ldots i_{4q}}\right)^2} = \tfrac{V^2(4q-1)!}{N^{4q-1}} \tag{A2}$$

$$\overline{\left(U^{s,s+1}_{i_1 i_2\ldots i_{2q}}\right)^2} = \tfrac{U^2(q-1)!q!}{N^{2q-1}} \tag{A3}$$

Similar to the model in Eq. 10, we will also solve H'_q in the large-q limit[41] where the coupling constants

$$\mathcal{V}^2 \equiv \frac{4qV^2}{2^{4q-1}} \quad \text{and} \quad \mathcal{U}^2 \equiv \frac{2qU^2}{2^{2q-1}} \tag{A4}$$

are kept fixed. In this model, we expect the SYK_{4q} on-site coupling becomes irrelevant in the IR. The low temperature physics should be governed by the SYK_{2q} intra-cluster coupling. To justify this expectation, we first solve the saddle point equations for the 2-point Green's function:

$$G_s(i\omega)^{-1} = -i\omega - \Sigma_s(i\omega) \tag{A5}$$

$$\Sigma_s(\tau) = G_s(\tau)^{q-1}\Big[V^2 G_s(\tau)^{3q} + U^2 G_{s-1}(\tau)^q + U^2 G_{s+1}(\tau)^q\Big]. \tag{A6}$$

Using the ansatz $G_s(\tau) = \frac{1}{2}\text{sgn}(\tau)\Big[1 + \frac{1}{q}g_s(\tau) + ...\Big]$, we obtain the form of the self energy in the large-q limit

$$\Sigma_s(\tau) = \frac{\text{sgn}(\tau)}{q} e^{g_s(\tau)}\Big[\frac{\mathcal{V}^2}{4}e^{3g_s(\tau)} + \frac{\mathcal{U}^2}{2}e^{g_{s-1}(\tau)} + \frac{\mathcal{U}^2}{2}e^{g_{s+1}(\tau)}\Big](1 + ...), \tag{A7}$$

where ... represents the higher order terms in $1/q$ which will be neglected from now on. The solutions of the saddle point equations satisfying the boundary conditions $g^s(0) = g^s(\beta) = 0$ take the form of

$$e^{2g_s(\tau)} = \frac{1}{\frac{4\beta^2\mathcal{U}^2}{\pi^2 v^2} + \sqrt{\frac{16\beta^4\mathcal{U}^4}{\pi^4 v^4} + \frac{\beta^2\mathcal{V}^2}{\pi^2 v^2}}\cos\Big(\pi v\Big(\frac{|\tau|}{\beta} - \frac{1}{2}\Big)\Big)}, \tag{A8}$$

with the condition that

$$\cos\Big(\frac{\pi v}{2}\Big) = -\frac{1 - \frac{v^2\pi^2}{4\beta^2\mathcal{U}^2}}{\sqrt{1 + \frac{\mathcal{V}^2}{\mathcal{U}^2}\frac{v^2\pi^2}{16\beta^2\mathcal{U}^2}}}. \tag{A9}$$

where $v \in (1, 2)$. By taking the low temperature limit $\beta \to \infty$, we can expand v in terms of $(\beta\mathcal{U})^{-1}$:

$$v \sim 2 - \sqrt{1 + \frac{\mathcal{V}^2}{8\mathcal{U}^2}}\frac{2}{\beta\mathcal{U}}. \tag{A10}$$

Also, as $\beta \to \infty$, $e^{2g_s(\tau)}$ reaches it low-temperature limit:

$$e^{2g_s(\tau)} = \frac{1}{1 + 2\sqrt{1 + \frac{\mathcal{V}^2}{8\mathcal{U}^2}}\sqrt{2}\mathcal{U}|\tau| + 2\mathcal{U}^2\tau^2}. \tag{A11}$$

Notice that, in the long time limit, we always have $e^{2g_s(\tau)} \sim \frac{1}{2\mathcal{U}^2|\tau|^2}$, which is independent from $\mathcal{V}$. Physically, it means that the IR physics is dominated by the SYK_{2q} inter-cluster couplings.

To understand the transport properties of this 1D chain given by H'_q, we also need to calculate the kernel $K^{ss'}(\tau_1,\tau_2,\tau_3,\tau_4)$ of the 4-point function[41,42,44]. The kernel for our model is given by

$$\begin{aligned}&K^{ss'}(\tau_1,\tau_2,\tau_3,\tau_4)\\&=-G(\tau_{13})G(\tau_{24})\Big[\Big((4q-1)V^2G(\tau_{34})^{4q-2}+2(q-1)U^2G(\tau_{34})^{2q-2}\Big)\delta_{ss'}+qU^2G(\tau_{34})^{2q-2}(\delta_{s+1,s'}+\delta_{s-1,s'})\Big]\end{aligned}\tag{A12}$$

Switching to the variables $\mathcal{V}$ and $\mathcal{U}$, also switching to the momentum space, we can write the kernel as

$$K_p(\tau_1,\tau_2,\tau_3,\tau_4)=-\frac{\mathrm{sgn}(\tau_{13})}{2}\frac{\mathrm{sgn}(\tau_{24})}{2}\Big[2\mathcal{V}^2e^{4g(\tau_{34})}+2\mathcal{U}^2e^{2g(\tau_{34})}(1+\cos(p))\Big].\tag{A13}$$

Here, we've only kept the leading terms in $1/q$. Let k_p and $\Psi_p(\tau_1,\tau_2)$ denote the eigenvalue and eigenfunction of the kernel K_p. By the similar analysis in Sec. III, eigenvalue equation $k_p\Psi_p=K_p\Psi_p$ can be deformed into the following form:

$$k_p\partial_{\tau_1}\partial_{\tau_2}\Psi_p(\tau_1,\tau_2)=-\Big[2\mathcal{V}^2e^{4g(\tau_{12})}+2\mathcal{U}^2e^{2g(\tau_{12})}(1+\cos(p))\Big]\Psi_p(\tau_1,\tau_2).\tag{A14}$$

Since this equation is invariant under simultaneous translation of the two time variables τ_1 and τ_2, we can directly work with the ansatz $\Psi_{p,n}(\tau_1,\tau_2)=e^{-i\frac{2\pi n}{\beta}\frac{\tau_1+\tau_2}{2}}\psi_{p,n}(\tau_{12})$ with integer n. Also, to simplify the equation, we work with the variables:

$$\rho=\cos\Big(\pi v\Big(\tfrac{|\tau|}{\beta}-\tfrac{1}{2}\Big)\Big)=\cos\left(\tfrac{v}{2}(|x|-\pi)\right),\tag{A15}$$

$$\alpha=\sqrt{1+\tfrac{\mathcal{V}^2}{\mathcal{U}^2}\tfrac{v^2\pi^2}{16\beta^2\mathcal{U}^2}}.\tag{A16}$$

The differential equation above can then be written as

$$\left[v^2\left((1-\rho^2)\frac{d^2}{d\rho^2}-\rho\frac{d}{d\rho}\right)+\frac{n^2}{4}-\frac{v^2}{8k_{p,n}}\left(\frac{(1+\cos p)}{1+\alpha\rho}+\frac{4(\alpha^2-1)}{(1+\alpha\rho)^2}\right)\right]\psi_{p,n}=0.\tag{A17}$$

Again, this equation is not exactly solvable for generic values of α. We can only proceed via perturbative approaches. For now, we can temporarily treat α and v as independent variables. We will expand this equation around $\alpha=1$ and treat $\alpha-1$ as a perturbation. When $\alpha=1$, this equation is exactly solvable[41]. The low-lying quantized value of $k^{(0)}_{n,p}$ is given by

$$\begin{aligned}k^{(0)}_{n,p}&=\tfrac{1+\cos p}{2}\left(1-\tfrac{3|n|}{2}\Big(1-\tfrac{v}{2}\Big)+O\Big(\Big(1-\tfrac{v}{2}\Big)^2\Big)\right)\\&=\tfrac{1+\cos p}{2}\left(1-\tfrac{3|n|}{2\beta\mathcal{U}}\sqrt{1+\tfrac{\mathcal{V}^2}{8\mathcal{U}^2}}+O\Big(\tfrac{1}{\beta^2\mathcal{U}^2}\Big)\right).\end{aligned}\tag{A18}$$

where the superscript "(0)" represents the zeroth order in $\alpha-1$. However, we notice that $\alpha-1\sim O\Big(\frac{1}{\beta^2\mathcal{U}^2}\Big)$. So even when we include first order correction of $\alpha-1$ to $k_{n,p}$, we should still have

$$k^{(1)}_{n,p}=\frac{1+\cos p}{2}\left(1-\frac{3|n|}{2\beta\mathcal{U}}\sqrt{1+\frac{\mathcal{V}^2}{8\mathcal{U}^2}}+O\Big(\frac{1}{\beta^2\mathcal{U}^2}\Big)\right).\tag{A19}$$

We know that $(1-k_{n,p})^{-1}$ should be viewed as the propagator of low-lying modes in the system with the integer n identified with the Matsubara frequency ω_n. When we expand this expression at small momentum p, we can write

$$(1-k_{n,p})^{-1}\simeq\left(\frac{p^2}{4}+\frac{3\omega_n}{4\pi\mathcal{U}}\sqrt{1+\frac{\mathcal{V}^2}{8\mathcal{U}^2}}\right)^{-1},\tag{A20}$$

which indicates a pole related to diffusion mode with a finite diffusion constant:

$$D=\frac{\pi\mathcal{U}}{3\sqrt{1+\frac{\mathcal{V}^2}{8\mathcal{U}^2}}},\tag{A21}$$

which indicates a diffusive metallic phase at zero temperature. This result agrees with the expectation that the low-energy physics is governed by the inter-cluster couplings. Interestingly, even though the on-site SYK_{4q} coupling is irrelevant in the IR, the diffusion constant D still depends on its coupling strength $\mathcal{V}$ in a non-trivial way.

We note that a detailed study for the case of $q = 1$ of Eq. A1 with complex fermion will be given in an upcoming work Ref. 46.

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