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Semiclassical theory of spin-orbit torques in disordered multiband electron systems

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We study spin-orbit torques (SOT) in non-degenerate multiband electron systems in the weak disorder limit. In order to have better physical transparency a semiclassical Boltzmann approach equivalent to the Kubo diagrammatic approach in the non-crossing approximation is formulated. This semiclassical framework accounts for the interband-coherence effects induced by both the electric field and static impurity scattering. Using the two-dimensional Rashba ferromagnet as a model system, we show that the antidamping-like SOT arising from disorder-induced interband-coherence effects is very sensitive to the structure of disorder potential in the internal space and may have the same sign as the intrinsic SOT in the presence of spin-dependent disorder. While the cancellation of this SOT and the intrinsic one occurs only in the case of spin-independent short-range disorder.

I. INTRODUCTION

Disorder effects to nonequilibrium properties of Bloch electrons in solids is a basic issue in the condensed matter physics. In many instances a relaxation time approximation is employed to account for the disorder effects [1]. However, this conventional treatment is not enough in some transport phenomena related to the spin-orbit coupling such as the spin Hall and anomalous Hall effects [2, 3]. To explain these phenomena, intriguing disorder-induced interband-coherence effects have been discussed extensively [2–4].

In inversion-asymmetric materials with local magnetization coupled to conduction electrons in spin-orbit coupled bands, an electric field induces a nonequilibrium spin-polarization which exerts a torque on the magnetization. This torque relies on the spin-orbit coupling and is termed spin-orbit torque (SOT) [5]. Disorder effects on the SOT have been treated in most studies by just a constant lifetime approximation [6–11] or a single transport relaxation time [12], leaving the disorder-induced interband-coherence effects largely unexplored [13].

The semiclassical Boltzmann transport theory which works in the weak scattering limit with well-defined multiple-band structure is appealing in its ability to obtain intuitive pictures [1]. Existing semiclassical Boltzmann theories for SOTs account for the intrinsic contribution [7, 9] due to the electric-field-induced interband-coherence effect [8, 14] and a field-like contribution proportional to the relaxation time or electron lifetime [6, 7, 9]. However, the interband-coherence effects induced by static impurity scattering cannot be treated by the conventional Boltzmann equation where the only role of scattering is to equilibrate the acceleration of electrons by the electric field. Successful inclusion of disorder-induced interband-coherences into the semiclassical Boltzmann formalism has recently been realized in the context of the anomalous Hall effect [15–17], but that formalism cannot be directly applied to study other spin-related nonequilibrium phenomena such as the SOT.

In the present paper we focus on SOTs in two-dimensional (2D) Rashba ferromagnets with the magnetization perpendicular to the 2D plane in the case of both Rashba bands partially occupied in the weak disorder limit. This isotropic model enables us to obtain analytical results. We find that the antidamping-like SOT arising from disorder-induced interband-coherence may have the same sign as the intrinsic SOT in the presence of spin-dependent disorder, and the cancellation between them occurs only in the case of spin-independent short-range (pointlike) disorder. Thus a careful analysis of different structures of disorder potentials in the internal space is indispensable for the study of SOT. Moreover, our results imply that, for finite-range or long-range disorder, other fine details of disorder also need to be carefully accounted for beyond simple phenomenological treatment.

In order to have better physical transparency, we formulate the analysis in a semiclassical Boltzmann framework taking into account the interband-coherence effects due to both the electric field and static impurities in non-degenerate multiband electron systems. Besides making use of the modified semiclassical Boltzmann equation [16, 17] developed in the semiclassical theory of anomalous Hall effect, the scattering-induced modification to conduction-electron states plays a vital role in this formalism whose validity is not limited to anomalous Hall effect and SOT. Regarding the disorder-induced interband-coherence contributions, the equivalence between the semiclassical theory and microscopic linear response theory in the weak disorder limit under the non-crossing approximation is established.

The rest of the paper is organized as follows. The semiclassical formulation is present in Sec. II, whereas model calculations are given in Sec. III. We make some discussions and conclude the paper in Sec. IV. Appendices A – C include some supplementary discussions.

II. SEMICLASSICAL PICTURE

In the semiclassical version of linear response analysis, the average value of an observable A (quantum mechanically, Hermitian operator $\hat{A}$, which can represent a vector, scalar, etc) in the presence of a dc weak uniform electric field $\mathbf{E}$ and weak static disorder is given by [1]

$$A = \sum_l f_l A_l. \tag{1}$$

Here f_l is the semiclassical Boltzmann distribution function governed by the linearized semiclassical Boltzmann equation, A_l represents the amount of A carried by the conduction-electron state denoted by index l. In the present paper we consider non-degenerate multiband electron (hole) systems in the weak disorder limit, and do not consider thermal related effects. We will show that, by properly considering f_l and A_l, this semiclassical framework takes into account the interband-coherence effects induced by both the electric field and static impurities.

The presence of weak electric field and impurity scattering modifies the conduction-electron state, making A_l deviate from its pure band-value $A_l^0 \equiv \langle l|\hat{A}|l\rangle$. Here $|l\rangle$ is the eigenstate (Bloch state) of disorder-free Hamiltonian $\hat{H}_0$. In equilibrium, A_l is modified to $A_l = A_l^0 + \delta^{ex} A_l$, where $\delta^{ex} A_l$ is related to the scattering-induced correction to Bloch state $|l\rangle$. Thus the semiclassical expression for the equilibrium value of A is $A_0 = \sum_l f_l^0 \left(A_l^0 + \delta^{ex} A_l\right)$ with f_l^0 the Fermi distribution function. Because $\delta^{ex} A_l$ is at least linear in the impurity concentration, in the weak disorder limit one has the conventional expression $A_0 = \sum_l f_l^0 A_l^0$. However, in the presence of the electric field, the out-of-equilibrium distribution function has a component inversely proportional to the impurity concentration, and thus $\delta^{ex} A_l$ contributes to nonequilibrium phenomena even in the weak disorder limit. Besides, the electric field also induces a correction $\delta^{in} A_l$ to A_l related to the so-called intrinsic contribution [9, 18]. As will be explained in Sec. II. B, in the linear response regime and weak disorder limit, $\delta^{ex} A_l$ and $\delta^{in} A_l$ are independent.

In the rest of this section we present formal expressions for f_l and A_l, and describe how the interband-coherence effects are included into the semiclassical formalism.

A. Semiclassical distribution function f_l

In this subsection we briefly describe the modified semiclassical Boltzmann equation proposed by Sinitsyn et al. [16, 17] to determine the distribution function f_l.

The linearized semiclassical Boltzmann equation for electrons (charge e) in nonequilibrium steady-states in the presence of elastic electron-impurity scattering takes the form [16]:

$$e\mathbf{E}\cdot\mathbf{v}_l^0 \frac{\partial f^0}{\partial \epsilon_l} = -\sum_{l'} \omega_{l,l'} \left(f_l - f_{l'} - \frac{\partial f^0}{\partial \epsilon_l} e\mathbf{E}\cdot\delta\mathbf{r}_{l',l} \right). \tag{2}$$

Here $\mathbf{v}_l^0$ is the band velocity, $\omega_{l,l'}$ is the semiclassical scattering rate ($l' \to l$) calculated by the golden rule, $\delta\mathbf{r}_{l',l}$ denotes the coordinate-shift [15] during the scattering and reads $\delta\mathbf{r}_{l',l} = \langle u_{l'}|i\partial_{\mathbf{k}'}|u_{l'}\rangle - \langle u_l|i\partial_{\mathbf{k}}|u_l\rangle - (\partial_{\mathbf{k}'} + \partial_{\mathbf{k}}) \arg\left(\langle l'|\hat{V}|l\rangle\right)$ in the lowest nonzero Born approximation [15]. $|l\rangle = |\eta\mathbf{k}\rangle$ is the Bloch state with eigenenergy $\epsilon_l \equiv \epsilon_{\mathbf{k}}^{\eta}$, η is the band index and $\mathbf{k}$ the crystal momentum. $\arg(..)$ denotes the phase of a complex number.

The distribution function is decomposed into [16]

$$f_l = f_l^0 + g_l^n + g_l^a \tag{3}$$

with g_l^n equilibrating the acceleration of electrons in the electric field between scattering events and the anomalous distribution function g_l^a describing the effect of electric field working during the coordinate-shift process. The coordinate-shift is a disorder-induced interband-coherence effect [2, 15–17] (i.e., related to interband virtual transitions induced by static impurity scattering) and can be directly related to the momentum-space Berry curvature [19] in some simple cases [15–17]. Thus the anomalous distribution function g_l^a is also related to the disorder-induced interband-coherence.

Under the Gaussian disorder approximation (we restrict to this approximation throughout this paper), g_l^n can be further divided into [17, 20]

$$g_l^n = g_l^{2s} + g_l^{sk-in}, \tag{4}$$

where g_l^{2s} is value of g_l^n in the lowest Born approximation ($\omega_{l,l'} \to \omega_{l,l'}^{2s}$), g_l^{sk-in} is responsible for the so-called intrinsic-skew-scattering arising in higher Born orders due to the asymmetry $\omega_{l,l'} \neq \omega_{l',l}$ under the Gaussian disorder [17, 20]. Here we mention that the intrinsic-skew-scattering is a delicate disorder effects related also to the interband-coherence (More discussions can be found in Appendix B).

In the presence of pointlike scalar impurities, one can easily verify that g_l^{sk-in} and g_l^a do not depend on either the impurity density or the scattering strength [20], and g_l^{2s} is inversely proportional to the impurity density. A systematic analysis of Eq. (2) under the noncrossing approximation in isotropic 2D electron systems with multiple Fermi circles has been presented in Ref. 20. Anisotropy in band structures or impurity potentials complicates the analytical treatment, but is not a severe obstacle in numerical solutions [12].

B. Scattering and electric-field modified A_l

In this subsection we obtain the expression for A_l taking into account the interband-coherences induced by both the electric field and static disorder.

To do this, we firstly deal with the case where there is only the electric field or only disorder. The electric-field-induced correction to A_l^0 reads $\delta^{in} A_l = 2\,\mathrm{Re}\langle l|\hat{A}|\delta^{\mathbf{E}} l\rangle$, arising from the electric-field-induced interband-virtual-transition correction

$$|\delta^{\mathbf{E}} l\rangle = -i\hbar e\mathbf{E}\cdot \sum_{\eta' \neq \eta} |\eta'\mathbf{k}\rangle\langle u_{\mathbf{k}}^{\eta'}|\hat{\mathbf{v}}|u_{\mathbf{k}}^{\eta}\rangle / \left(\epsilon_{\mathbf{k}}^{\eta} - \epsilon_{\mathbf{k}}^{\eta'}\right)^2$$

to the Bloch state $|l\rangle = |\eta \mathbf{k}\rangle$. Here $|\mathbf{k}\rangle$ and $|u^{\eta}_{\mathbf{k}}\rangle$ are the plane-wave and periodic parts of $|\eta\mathbf{k}\rangle$, respectively. $\hat{\mathbf{v}}$ is the velocity operator. $\delta^{in}A_l$ is an interband-coherence effect induced solely by the electric field.

Similarly, the scattering-induced correction $\delta^{ex}A_l$ stems from interband-coherence effects in the scattering process. To obtain this part, one can notice that the Bloch state is also modified by the scattering according to the Lippmann-Schwinger equation $|l^s\rangle = |l\rangle + \left(\epsilon_l - \hat{H}_0 + i\epsilon\right)^{-1}\hat{T}|l\rangle$. Here $\hat{T} = \hat{V} + \hat{V}\left(\epsilon_l - \hat{H}_0 + i\epsilon\right)^{-1}\hat{T}$ is the T-matrix, $\hat{V}$ is the disorder potential. Thus $\delta^{ex}A_l$ is related to $\left\langle 2\,\mathrm{Re}\langle l|\hat{A}|\delta^s l\rangle + \langle \delta^s l|\hat{A}|\delta^s l\rangle\right\rangle_c$, where $|\delta^s l\rangle \equiv |l^s\rangle - |l\rangle$ represents the scattering-induced correction to the Bloch state and $\langle ..\rangle_c$ denotes the average over disorder configurations. Here we only consider [21] disorder potential-free $\hat{A}$. By the Lippmann-Schwinger equation, in the lowest nonzero order of the disorder potential we get

$$\left\langle \langle \delta^s l|\hat{A}|\delta^s l\rangle\right\rangle_c = \sum_{l'l''} \frac{\langle V_{ll'}V_{l''l}\rangle_c \langle l'|\hat{A}|l''\rangle}{(\epsilon_l - \epsilon_{l'} - i\epsilon)(\epsilon_l - \epsilon_{l''} + i\epsilon)}$$

and

$$\left\langle 2\,\mathrm{Re}\langle l|\hat{A}|\delta^s l\rangle\right\rangle_c = 2\,\mathrm{Re}\sum_{l'l''} \frac{\langle V_{l'l''}V_{l''l}\rangle_c \langle l|\hat{A}|l'\rangle}{(\epsilon_l - \epsilon_{l'} + i\epsilon)(\epsilon_l - \epsilon_{l''} + i\epsilon)}.$$

Both of them contain intraband and interband matrix elements of $\hat{A}$ in the band representation. In the weak disorder limit the intraband terms will be ignored because they are just trivial renormalization effects [16]. Only the interband terms are left as nontrivial corrections to A^0_l in the weak disorder limit, because they are interband-coherence effects induced by impurities.

Now we turn to the case where both the electric field and disorder are present. In equilibrium with disorder, $A_l = A^0_l + \delta^{ex}A_l$. The application of the electric field modifies both A^0_l and $\delta^{ex}A_l$. However, in the linear response regime, only the electric-field-induced correction to A^0_l contributes to nonequilibrium phenomena in the weak disorder limit and reads $\delta^{in}A_l = 2\,\mathrm{Re}\langle l|\hat{A}|\delta^{\mathbf{E}}l\rangle$, just the same as that in the absence of disorder. Therefore we conclude that in the linear response regime and the weak disorder limit, the effects of electric field and disorder on A_l are independent and thus can be treated separately. Accordingly, taking into account the electric-field- and scattering-induced interband-coherence effects, A_l can be written as

$$A_l = A^0_l + \delta^{ex}A_l + \delta^{in}A_l. \tag{5}$$

The intrinsic correction due to the electric-field-induced interband-coherence is

$$\delta^{in}A_l = \hbar e \sum_{\eta'\neq\eta} \frac{2\,\mathrm{Im}\langle \eta\mathbf{k}|\hat{A}|\eta'\mathbf{k}\rangle\langle u^{\eta'}_{\mathbf{k}}|\hat{\mathbf{v}}\cdot\mathbf{E}|u^{\eta}_{\mathbf{k}}\rangle}{\left(\epsilon^{\eta}_{\mathbf{k}} - \epsilon^{\eta'}_{\mathbf{k}}\right)^2}, \tag{6}$$

whereas the extrinsic correction due to the scattering-induced interband-coherence reads

$$\delta^{ex}A_l = \delta^{inter}_1 A_l + \delta^{inter}_2 A_l, \tag{7}$$

with

$$\delta^{inter}_1 A_l = \sum_{\eta'\mathbf{k}'}\sum_{\eta''\neq\eta'} \times \frac{\left\langle \langle \eta\mathbf{k}|\hat{V}|\eta'\mathbf{k}'\rangle\langle\eta''\mathbf{k}'|\hat{V}|\eta\mathbf{k}\rangle\right\rangle_c \langle \eta'\mathbf{k}'|\hat{A}|\eta''\mathbf{k}'\rangle}{\left(\epsilon^{\eta}_{\mathbf{k}} - \epsilon^{\eta'}_{\mathbf{k}'} - i\epsilon\right)\left(\epsilon^{\eta}_{\mathbf{k}} - \epsilon^{\eta''}_{\mathbf{k}'} + i\epsilon\right)} \tag{8}$$

and

$$\delta^{inter}_2 A_l = 2\,\mathrm{Re}\sum_{\eta'\neq\eta}\sum_{\eta''\mathbf{k}''} \times \frac{\left\langle \langle \eta'\mathbf{k}|\hat{V}|\eta''\mathbf{k}''\rangle\langle\eta''\mathbf{k}''|\hat{V}|\eta\mathbf{k}\rangle\right\rangle_c \langle \eta\mathbf{k}|\hat{A}|\eta'\mathbf{k}\rangle}{\left(\epsilon^{\eta}_{\mathbf{k}} - \epsilon^{\eta'}_{\mathbf{k}} + i\epsilon\right)\left(\epsilon^{\eta}_{\mathbf{k}} - \epsilon^{\eta''}_{\mathbf{k}''} + i\epsilon\right)}. \tag{9}$$

Under a local phase transformation, $\delta^{ex}A_l$ remains unchanged for disorder potential [21] $\hat{V}(\mathbf{r})$. In fact all the three terms of A_l in Eq. (5) are gauge invariant and can be regarded as the basic ingredients of a semiclassical theory.

In the case of $\hat{A} = \hat{\mathbf{v}}$, Eq. (5) is just the velocity of semiclassical electrons appeared in the semiclassical theory of anomalous Hall effect [2, 16]: $\mathbf{v}^0_l$ is the band velocity, $\delta^{in}\mathbf{v}_l$ is the Berry-curvature anomalous velocity [19], and one can show that $\delta^{ex}\mathbf{v}_l$ is consistent with the semiclassical side-jump velocity $\mathbf{v}^{sj}_l$ proposed by Sinitsyn et al. [15–17] as well as Luttinger's quantum transport theory on the anomalous Hall effect [22] (Detailed discussions are present in Appendix A). This consistency indicates that the so-called side-jump velocity can also be understood as arising from scattering-induced modifications to the Bloch state.

More importantly, this consistency implies that $\delta^{ex}A_l$ provides a generalization of the semiclassical side-jump velocity into physical quantities besides the electric current (velocity). In the spin Hall effect where the spin is not conserved in spin-orbit-coupled bands, a semiclassical Boltzmann analysis of disorder-induced interband-coherences is still absent. This is partly due to, in our opinion, the lack of a spin-current-counterpart of the side-jump velocity [23]. Similarly, the lack of a SOT-counterpart of the side-jump velocity has impeded the development of semiclassical Boltzmann theories to SOT. Now the identification of $\delta^{ex}A_l$ provides the counterpart of the side-jump velocity for the case of physical quantities besides the electric current. Also the identification of $\delta^{ex}A_l$ helps establish the equivalence between the semiclassical theory on the disorder-induced interband-coherence transport and diagrammatic perturbation theories in the weak disorder limit under the non-crossing approximation, as demonstrated in Appendix B.

C. Semiclassical expression of linear response in the weak disorder limit

In the linear response regime we get the following semiclassical Boltzmann expression for $\delta A \equiv A - A_0$:

$$\delta A = \sum_l \left(A_l^0 + \delta^{ex} A_l\right)\left(f_l - f_l^0\right) + \sum_l \left(\delta^{in} A_l\right) f_l^0. \quad (10)$$

The first and second terms on the right hand side (rhs) are extrinsic and intrinsic [18] contributions, respectively.

In the weak disorder limit up to the zeroth order of total disorder concentration and scattering strength, one has

$$\begin{aligned}\delta A = &\sum_l A_l^0 g_l^{2s} + \sum_l A_l^0 \left(g_l^a + g_l^{sk-in}\right) \\ &+ \sum_l \left(\delta^{ex} A_l\right) g_l^{2s} + \sum_l \left(\delta^{in} A_l\right) f_l^0. \end{aligned} \quad (11)$$

The first term at the rhs is the conventional Boltzmann result in the lowest Born order, the second term includes contributions from the anomalous distribution function and intrinsic-skew-scattering. The last two terms arise from interband-coherence corrections to the semiclassical value of A_l in Eq. (1).

Below we label the terms on the rhs of Eq. (11) as $\delta^{2s} A = \sum_l A_l^0 g_l^{2s}$, $\delta^{sj} A = \sum_l \left(\delta^{ex} A_l\right) g_l^{2s}$, $\delta^{adis} A = \sum_l A_l^0 g_l^a$, $\delta^{sk-in} A = \sum_l A_l^0 g_l^{sk-in}$ and $\delta^{in} A = \sum_l \left(\delta^{in} A_l\right) f_l^0$. As stated in the past two subsections, disorder-induced interband-coherence effects are included in $\delta^{ex} A_l$, g_l^a and g_l^{sk-in}, the disorder-induced interband-coherence contribution (labeled by $\delta^{SJ} A$) to δA is thus

$$\delta^{SJ} A = \delta^{adis} A + \delta^{sk-in} A + \delta^{sj} A. \quad (12)$$

In the presence of pointlike scalar impurities, all the three terms are independent of both the impurity density and scattering strength [20].

In the semiclassical theory of the anomalous Hall effect formulated recently by Sinitsyn et al. [17], the disorder-induced interband-coherence contribution (called side-jump effect in that context [2, 4]) comprises three ingredients: a side-jump velocity $\mathbf{v}_l^{sj}$, the anomalous distribution function g_l^a and intrinsic-skew-scattering g_l^{sk-in}. As our $\delta^{ex}\mathbf{v}_l$ coincides with $\mathbf{v}_l^{sj}$, when applied to the anomalous Hall effect the present semiclassical formalism is consistent with that by Sinitsyn et al.

Furthermore, we establish (see Appendix B) an one-to-one correspondence between the three semiclassical terms at the rhs of Eq. (12) and special sets of Feynman diagrams representing the disorder-induced interband-coherence transport contributions in the band-eigenstate basis under the non-crossing approximation in the weak disorder limit. This also confirms the validity of our semiclassical framework.

Equation (11) can then be casted into

$$\delta A = \delta^{2s} A + \delta^{SJ} A + \delta^{in} A. \quad (13)$$

Here we can mention that, in the weak scattering limit the widely-used classification of SOT into interband and intraband parts does not take into account $\delta^{SJ} A$, i.e., the disorder-induced interband-coherence effects far beyond the relaxation time approximation [24].

III. MODEL CALCULATION

We consider the case where the SOT is related to the nonequilibrium conduction-electron spin-polarization $\delta\mathbf{S}$ which is coupled to the local magnetization via the s-d exchange coupling. In the simplified treatment adopted here, one only calculates $\delta\mathbf{S}$ in the presence of the driven electric field and disorder [6–10]. In this section we focus on SOTs in 2D Rashba ferromagnets with the magnetization perpendicular to the 2D plane. In Appendix C we also analyze the case of in-plane magnetization and scalar pointlike impurities.

The 2D model Hamiltonian is $\hat{H} = \hat{H}_0 + \hat{V}(\mathbf{r})$, where

$$\hat{H}_0 = \frac{\hat{\mathbf{p}}^2}{2m} + \frac{\alpha_R}{\hbar}\hat{\sigma}\cdot\left(\hat{\mathbf{p}}\times\hat{\mathbf{z}}\right) - J_{ex}\hat{\sigma}\cdot\hat{\mathbf{M}}. \quad (14)$$

Here m is the in-plane effective mass of conduction electron, $\hat{\mathbf{p}} = \hbar\hat{\mathbf{k}}$ the 2D momentum, $\hat{\sigma} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ are the Pauli matrices, α_R is the Rashba coefficient, J_{ex} the exchange coupling. $\hat{\mathbf{M}}$ is the direction of the local magnetization and chosen to be $\hat{\mathbf{M}} = \hat{\mathbf{z}}$ for the in-plane isotropic model. $|u_{\mathbf{k}}^{\eta}\rangle = \frac{1}{\sqrt{2}}\left[\sqrt{1-\eta\cos\theta}, -i\eta\exp(i\phi)\sqrt{1+\eta\cos\theta}\right]^T$ is the inner eigenstate, where $\eta = \pm$, $\tan\phi = \frac{k_y}{k_x}$, $\cos\theta = J_{ex}/\Delta_k$, $\Delta_k = \sqrt{\alpha^2 k^2 + J_{ex}^2}$. We only consider the case $\epsilon_F > J_{ex}$, i.e., both Rashba bands partially occupied. For any energy $\epsilon > J_{ex}$ there are two iso-energy rings corresponding to the two bands: $k_\eta^2(\epsilon) = \frac{2m}{\hbar^2}\left(\epsilon - \eta\Delta_\eta(\epsilon)\right)$ where $\Delta_\eta(\epsilon) \equiv \Delta_{k_\eta(\epsilon)} = \sqrt{\epsilon_R^2 + J_{ex}^2 + 2\epsilon_R\epsilon} - \eta\epsilon_R$ and $\epsilon_R = m\left(\frac{\alpha_R}{\hbar}\right)^2$. The density of states in η band is $D_\eta(\epsilon) = D_0\frac{\Delta_\eta(\epsilon)}{\Delta_\eta(\epsilon)+\eta\epsilon_R}$ with $D_0 = \frac{m}{2\pi\hbar^2}$.

Hereafter the electric field is applied in the y direction. The intrinsic nonequilibrium spin-polarization reads

$$\delta^{in}\mathbf{S} = \sum_l \left(\delta^{in}\mathbf{S}_l\right) f_l^0 = -eE_y\frac{\hbar}{2}\frac{J_{ex}\alpha_R D_0}{J_{ex}^2 + 2\epsilon_R\epsilon_F}\hat{\mathbf{y}}, \quad (15)$$

which is parallel to the electric field and contributes an intrinsic antidamping-like SOT.

It was found in the context of the anomalous Hall effect that the structure of short-range disorder potential (do not consider spin-orbit scattering) in the internal space (internal degrees of freedom such as spin and valley) strongly affects the disorder-induced interband-coherence response [25]. For the in-plane isotropic Rashba model where $\hat{\mathbf{S}} = \frac{\hbar}{2}\hat{\sigma}$, according to the structure of the disorder potential in the 2×2 internal space, the pointlike disorder [21, 26] can be classified following the recipe of Yang et al. [25] as: class A $\hat{V} = V_A\hat{\sigma}_0$, class B $\hat{V} = V_B\hat{\sigma}_z$

and class C $\hat{V} = V_c \hat{\sigma}_{\pm}/\sqrt{2}$. Here $\hat{\sigma}_{\pm} = \hat{\sigma}_x \pm i\hat{\sigma}_y$, $\hat{\sigma}_0$ is the 2×2 identity matrix. Details about the theoretical consideration on this classification in in-plane isotropic systems with 2×2 internal space and the realizations of these scattering classes in practice have been given in Sec. II and Sec. IV of Ref. 25, respectively. It was shown that the disorder-induced interband-coherence contribution to the anomalous Hall effect in in-plane isotropic systems with 2×2 internal space due to class A disorder is quite different from that due to classes B and C disorder, even with opposite signs [25, 27]. While contributions due to class B and C disorder are qualitatively similar [25, 28]. Thus we only take into account class A and class B disorder to calculate the SOT.

A. Class A disorder

According to Eqs. (7,8,9), we obtain $\delta_2^{inter}\mathbf{S}_l = 0$ and

$$\delta^{ex}\mathbf{S}_l = \delta_1^{inter}\mathbf{S}_l = -\frac{\hbar}{2}\frac{\hbar}{\tau}\frac{\eta J_{ex}}{J_{ex}^2 + 2\epsilon_R\epsilon}\frac{\alpha_R \mathbf{k}_\eta(\epsilon)}{2\Delta_\eta(\epsilon)}. \quad (16)$$

$\delta^{ex}\mathbf{S}_l$ contributes a nonequilibrium spin-polarization

$$\delta^{sj}\mathbf{S} = \sum_l g_l^{2s}\delta^{ex}\mathbf{S}_l = eE_y\frac{\hbar}{2}\frac{\alpha_R J_{ex} D_0}{J_{ex}^2 + 2\epsilon_R\epsilon_F}\hat{\mathbf{y}}, \quad (17)$$

which completely cancels the intrinsic contribution. For class A disorder $g_l^{2s} = \left(-\partial_\epsilon f^0\right) e\mathbf{E}\cdot\frac{\hbar\mathbf{k}_\eta(\epsilon)}{m}\tau$ has been obtained before [20], with $\tau = \left(2\pi n_{im}^A V_A^2 D_0/\hbar\right)^{-1}$ the lifetime of Rashba electron with n_{im}^A the density of class A disorder. Moreover, Ref. 20 has shown that the anomalous distribution and the distribution function for the intrinsic skew scattering cancels each other: $g_l^a + g_l^{sk-in} = 0$. Thus $\delta^{adis}\mathbf{S} + \delta^{sk-in}\mathbf{S} = \mathbf{0}$ and the disorder-induced interband-coherence contribution is just $\delta^{SJ}\mathbf{S} = \delta^{sj}\mathbf{S}$, then the total (electric-field-induced plus disorder-induced) interband-coherence contribution to the nonequilibrium spin-polarization vanishes

$$\delta^{SJ}\mathbf{S} + \delta^{in}\mathbf{S} = \mathbf{0}. \quad (18)$$

Thereby $\delta\mathbf{S} = \delta^{2s}\mathbf{S} = \sum_l g_l^{2s}\mathbf{S}_l^0 = -e\alpha_R D_0\tau E_y\hat{\mathbf{x}}$, which is magnetization-independent and coincides with the well-known Edelstein result in the nonmagnetic 2D Rashba model with class A disorder [29]. This $\delta\mathbf{S}$ perpendicular to both the magnetization and electric field contributes a field-like SOT.

B. Class B disorder

In this case, the electron lifetime is still independent of energy and band, and is given by $\tau = \left(\sum_{l'}\omega_{l',l}^{2s}\right)^{-1} = \left(\frac{2\pi}{\hbar}n_{im}^B V_B^2 D_0\right)^{-1}$ with n_{im}^B the density of class B disorder.

According to Eqs. (7,8,9), we get $\delta_2^{inter}\mathbf{S}_l = 0$ and

$$\delta^{ex}\mathbf{S}_l = \delta_1^{inter}\mathbf{S}_l = \frac{\hbar}{2}\frac{\hbar}{\tau}\frac{\eta J_{ex}}{J_{ex}^2 + 2\epsilon_R\epsilon}\frac{\alpha_R \mathbf{k}_\eta(\epsilon)}{2\Delta_\eta(\epsilon)}. \quad (19)$$

We note that, for the same $\mathbf{k}_\eta(\epsilon)$ the sign of $\delta^{ex}\mathbf{S}_l$ is opposite to that in the case of class A disorder.

The Boltzmann equation (2) is solved following the recipe given by Ref. 20. After lengthy calculations we get

$$\begin{aligned} g_\eta^{2s}(\epsilon) &= \left(-\partial_\epsilon f^0\right) e\mathbf{E}\cdot\frac{\hbar\mathbf{k}_\eta(\epsilon)}{m}\tau\frac{\Delta_{-\eta}^2(\epsilon) + \epsilon_R\epsilon}{J_{ex}^2 + 3\epsilon_R\epsilon}, \\ g_\eta^a(\epsilon) &= \left(-\partial_\epsilon f^0\right)(\hat{\mathbf{z}}\times e\mathbf{E})\cdot\mathbf{k}_\eta(\epsilon)\frac{\eta J_{ex}\alpha_R^2}{2\Delta_\eta(\epsilon)\left(J_{ex}^2 + 3\epsilon_R\epsilon\right)}, \\ g_\eta^{sk-in}(\epsilon) &= \frac{J_{ex}^2 + \epsilon_R\epsilon}{J_{ex}^2 + 3\epsilon_R\epsilon}g_\eta^{adis}(\epsilon). \end{aligned} \quad (20)$$

Then the disorder-induced interband-coherence contributions to the nonequilibrium spin-polarization in Eq. (12) are given by

$$\delta^{sj}\mathbf{S} = \sum_l g_l^{2s}\delta^{ex}\mathbf{S}_l = \frac{J_{ex}^2 + \epsilon_R\epsilon_F}{J_{ex}^2 + 3\epsilon_R\epsilon_F}\delta^{in}\mathbf{S}, \quad (21)$$

$$\delta^{adis}\mathbf{S} = \sum_l g_l^a\mathbf{S}_l^0 = -\frac{\epsilon_R\epsilon_F}{J_{ex}^2 + 3\epsilon_R\epsilon_F}\delta^{in}\mathbf{S}, \quad (22)$$

and

$$\delta^{sk-in}\mathbf{S} = \sum_l g_l^{sk-in}\mathbf{S}_l^0 = \frac{J_{ex}^2 + \epsilon_R\epsilon_F}{J_{ex}^2 + 3\epsilon_R\epsilon_F}\delta^{adis}\mathbf{S}. \quad (23)$$

Thus the total disorder-induced interband-coherence contribution reads

$$\delta^{SJ}\mathbf{S} = \left[2\left(\frac{J_{ex}^2 + 2\epsilon_R\epsilon_F}{J_{ex}^2 + 3\epsilon_R\epsilon_F}\right)^2 - 1\right]\delta^{in}\mathbf{S}. \quad (24)$$

In the large exchange-coupling limit $J_{ex} \gg \sqrt{\epsilon_R\epsilon_F}$ one has $\delta^{SJ}\mathbf{S} \simeq \delta^{in}\mathbf{S}$, the disorder-induced interband-coherence contribution approximately doubles the intrinsic nonequilibrium spin-polarization and the corresponding antidamping-like SOT. While in the opposite limit $J_{ex} \ll \sqrt{\epsilon_R\epsilon_F}$, $\delta^{SJ}\mathbf{S} \simeq -\frac{1}{9}\delta^{in}\mathbf{S}$ and the contribution from disorder-induced interband-coherences partly cancels the intrinsic nonequilibrium spin-polarization. In particular, $\delta^{SJ}\mathbf{S} = 0$ when $J_{ex}^2 = \left(\sqrt{2} - 1\right)\epsilon_R\epsilon_F$.

The total interband-coherence contribution to the nonequilibrium spin-polarization reads

$$\delta^{in}\mathbf{S} + \delta^{SJ}\mathbf{S} = -\hbar e\alpha_R D_0\frac{J_{ex}\left(J_{ex}^2 + 2\epsilon_R\epsilon_F\right)}{\left(J_{ex}^2 + 3\epsilon_R\epsilon_F\right)^2}E_y\hat{\mathbf{y}}, \quad (25)$$

which exerts an antidamping-like torque on the magnetization. Besides, $\delta^{2s}\mathbf{S} = -e\alpha_R D_0\tau\frac{J_{ex}^2 + \epsilon_R\epsilon_F}{J_{ex}^2 + 3\epsilon_R\epsilon_F}E_y\hat{\mathbf{x}}$ leads to a field-like torque proportional to τ.

C. Competition between classes A and B

When the dominant scattering class is tuned (by doping or by varying temperature [25]), rich behaviors of SOT are expected even in the weak disorder limit. In the presence of both class A and class B impurities, we assume [25] $\langle V_A V_B \rangle_c = 0$, and only the main results are given in this case.

Due to $\sum_\eta \frac{D_\eta}{\Delta_\eta} = 0$, the electron lifetime is given by $\tau = \tau_A/(1+\zeta) = \left(\tau_A^{-1} + \tau_B^{-1}\right)^{-1}$, where $1/\tau_{A(B)} = 2\pi n_{im}^{A(B)} V_{A(B)}^2 D_0/\hbar$, $\zeta = \tau_A/\tau_B$.

In this subsection we write $\delta S_\alpha = \chi_{\alpha\beta} E_\beta$, with $\alpha, \beta = x, y$. Lengthy calculations lead to

$$\chi_{xy} = -e\alpha_R D_0 \tau \frac{1 - I_1}{1 - \frac{1-\zeta}{1+\zeta} I_1},$$
$$\chi_{yy} = -\hbar e \alpha_R D_0 \frac{J_{ex}}{J_{ex}^2 + 2\epsilon_R \epsilon_F} \frac{\frac{\zeta}{1+\zeta}}{\left(1 - \frac{1-\zeta}{1+\zeta} I_1\right)^2}, \tag{26}$$

where $I_1 = \frac{\epsilon_R \epsilon_F}{J_{ex}^2 + 2\epsilon_R \epsilon_F}$.

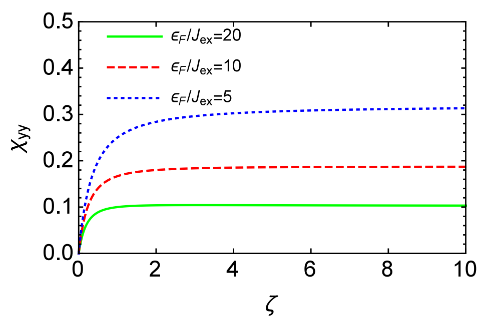

FIG. 1. χ_{yy} versus $\zeta = \tau_A/\tau_B$ for fixed values of ϵ_F/J_{ex}. χ_{yy} is measured in units of $-e\alpha_R D_0 \hbar/J_{ex}$. The plot shows the crossover from the class A dominated regime to the class B dominated regime as ζ increases. Here we set $\epsilon_R/J_{ex} = 0.1$.

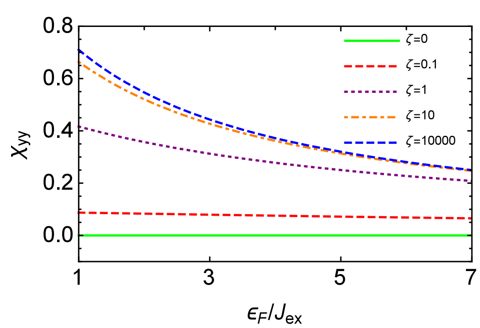

FIG. 2. χ_{yy} versus ϵ_F for fixed values of ζ. χ_{yy} is measured in units of $-e\alpha_R D_0 \hbar/J_{ex}$, and ϵ_F is measured in units of J_{ex}. We have chosen $\epsilon_R/J_{ex} = 0.1$ in plotting the curves.

One can observe that in the limit $\zeta \to 0$ or $\zeta \to \infty$, our previous results in Sec. III. A and Sec. III. B are recovered, and the values of nonequilibrium spin-polarizations (and thus SOTs) vary continuously as ζ changes between these two limits. In Fig. 1 we plot χ_{yy} as a function of ζ for fixed values of ϵ_F. One can see that χ_{yy} increases monotonically as ζ increases from the class A dominated regime to the class B dominated regime. The class B dominated regime is reached at smaller ζ for larger ϵ_F. In Fig. 2 we plot χ_{yy} as a function of ϵ_F for different values of ζ. As ζ increases from zero, the curve of χ_{yy} is shifted upward from the class A dominated regime due to the increasing contribution from class B scattering. For the chosen parameter $\epsilon_R/J_{ex} = 0.1$, we find that $\zeta = 10$ is quite approaching the class B dominated case $\zeta = 10000$. This is consistent with the trend shown in Fig. 1.

IV. DISCUSSION AND CONCLUSION

A. Comparison to other theories

In the context of the anomalous Hall effect, it has been realized [16, 20, 30, 31] that, in the weak disorder limit under the non-crossing approximation the semiclassically obtained disorder-induced interband-coherence contribution (side-jump) is equivalent to the ladder vertex correction to the bare bubble representing the intrinsic contribution [32] in the non-chiral basis (just the spin-σ_z basis for two-band models such as the Rashba model [20, 31] and Dirac model [16, 30], while the chiral basis means the band-eigenstate basis [4, 16]) in Kubo diagrammatic theories. Only few papers addressed vertex corrections to the intrinsic SOT [33–36], and these calculations do not give pictures of interband-coherence effects due to employment of the non-chiral basis.

Regarding the present model, the cancellation between the intrinsic and disorder-induced interband-coherence contributions in the case of scalar short-range disorder is consistent with that obtained by calculating the vertex correction in quantum transport theories [33, 34]. For the case of class B disorder, we have also performed a Kubo diagrammatic calculation [37] under the non-crossing approximation and obtained the same weak-disorder-limit result for the SOT as that of the present semiclassical theory.

B. Relative magnitude of antidamping-like and field-like SOTs

In the Rashba system with both bands partially occupied, under the good-metal condition ($\epsilon_F \tau/\hbar \gg 1$) there are still two different limits often discussed in literatures. One is the weak disorder limit where the disorder broadening is much smaller than the band splitting due to Rashba and exchange couplings [6, 9, 10], the other is the opposite limit – diffusive limit [38]. If the Rashba and exchange couplings are both weak, the system may be near the diffusive limit, where the Boltzmann theory does not work.

In the weak disorder limit, the antidamping-like SOT from the intrinsic and disorder-induced interband-coherence contributions is smaller than the field-like one. However, unlike the longitudinal conductivity whose leading contribution under the good-metal condition is always proportional to $\epsilon_F \tau/\hbar$, the field-like SOT (proportional to χ_{xy}) is not proportional to ϵ_F even in the weak disorder limit. Thus as the system evolves from the weak disorder limit to the diffusive limit, while the longitudinal electric conductivity remains large, the field-like SOT may become much smaller and may not remain dominant over the antidamping-like one. This attracting possibility will be investigated in a separate paper.

C. Neglected contributions

Very recently, the diagrammatic calculation of the anomalous Hall effect under the Gaussian disorder has been improved by going beyond the non-crossing approximation [39]. The resulting additional contribution is also independent of both disorder density and scattering strength in the case of scalar pointlike impurities in the weak disorder limit. There should also be corresponding additional contribution to the SOT. However, it is now not clear whether this contribution can be included into the semiclassical Boltzmann theory. This issue is left for future discussion.

We assumed Gaussian disorder as in Refs. 4, 33–35. Non-Gaussian disorder is not included in this paper. In the context of the anomalous Hall effect, non-Gaussian disorder leads to skew scattering contributions which depend on the scattering time [2]. In the field of the SOT, the effects of non-Gaussian disorder can be calculated by the same method as that applied to the anomalous Hall effect [2].

D. Summary

In summary, we have studied spin-orbit torques in non-degenerate multiband electron systems by formulating a semiclassical Boltzmann framework in the weak disorder limit. This semiclassical formulation accounts for interband-coherence effects induced by both the electric field and static impurity scattering, and is equivalent to the Kubo diagrammatic approach under the non-crossing approximation in the weak disorder limit. Using the 2D Rashba ferromagnets as an example, we showed that the disorder-induced interband-coherence effects contribute an antidamping-like torque, which is very sensitive to the structure of disorder potential in the internal space (spin space for the considered model) and may have the same sign as the intrinsic spin-orbit torque in the presence of spin-dependent disorder.

We expect these findings are helpful also in understanding spin-orbit torques in the 2D anti-ferromagnetic Rashba model [40]. The semiclassical framework proposed in this paper can be employed to treat other nonequilibrium phenomena related to disorder-induced interband-coherence effects.

ACKNOWLEDGMENTS

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Appendix A: Consistency of our formulas and the side-jump velocity

In the well-established semiclassical Boltzmann theory of anomalous Hall effect [15, 16] the side-jump velocity is obtained by linking it to the coordinate-shift $\mathbf{v}_l^{sj} = \sum_{l'} \omega_{l',l}^{2s} \delta \mathbf{r}_{l',l}$. Here we prove that our $\delta^{ex}\mathbf{v}_l = \delta_1^{inter}\mathbf{v}_l + \delta_2^{inter}\mathbf{v}_l$ is consistent with this $\mathbf{v}_l^{sj}$. Due to [21] $\hat{\mathbf{v}} = \frac{1}{i\hbar}\left[\hat{\mathbf{r}}, \hat{H}_0\right]$, we have

$$\delta_1^{inter}\mathbf{v}_l = \sum_{l',l''\neq l'} \frac{1}{i\hbar} \left\langle \frac{V_{ll'}\langle l'|\hat{\mathbf{r}}|l''\rangle V_{l''l}}{\epsilon_l - \epsilon_{l'} - i\delta} \frac{\epsilon_{l''} - \epsilon_{l'}}{\epsilon_l - \epsilon_{l''} + i\delta} \right\rangle_c = 2\,\mathrm{Re} \sum_{l',l''\neq l'} \frac{i}{\hbar} \left\langle V_{ll'} \frac{\langle l'|\hat{\mathbf{r}}|l''\rangle V_{l''l}}{\epsilon_l - \epsilon_{l'} - i\delta} \right\rangle_c \tag{A1}$$

and

$$\begin{aligned}\delta_2^{inter}\mathbf{v}_l &= \mathrm{Re} \sum_{l'\neq l,l''} \frac{2}{i\hbar} \left\langle \frac{\epsilon_{l'} - \epsilon_l}{\epsilon_l - \epsilon_{l'} + i\delta} \frac{\langle l|\hat{\mathbf{r}}|l'\rangle V_{l'l''} V_{l''l}}{\epsilon_l - \epsilon_{l''} + i\delta} \right\rangle_c \\ &= 2\,\mathrm{Re} \sum_{l'\neq l,l''} \frac{i}{\hbar} \left\langle \langle l|\hat{\mathbf{r}}|l'\rangle \frac{V_{l'l''} V_{l''l}}{\epsilon_l - \epsilon_{l''} + i\delta} \right\rangle_c = 2\,\mathrm{Re} \sum_{l',l''\neq l} \frac{-i}{\hbar} \left\langle \langle l''|\hat{\mathbf{r}}|l\rangle \frac{V_{l'l''} V_{ll'}}{\epsilon_l - \epsilon_{l'} - i\delta} \right\rangle_c ,\end{aligned} \tag{A2}$$

thus

$$\delta^{ex}\mathbf{v}_l = 2\,\mathrm{Re}\sum_{l',l''\neq l'}\frac{i}{\hbar}\left\langle V_{ll'}\frac{\langle l'|\hat{\mathbf{r}}|l''\rangle V_{l''l}}{\epsilon_l-\epsilon_{l'}-i\delta}\right\rangle_c + 2\,\mathrm{Re}\sum_{l',l''\neq l}\frac{-i}{\hbar}\left\langle V_{ll'}\frac{\langle l'|\hat{V}|l''\rangle\langle l''|\hat{\mathbf{r}}|l\rangle}{\epsilon_l-\epsilon_{l'}-i\delta}\right\rangle_c. \tag{A3}$$

Only the interband matrix elements (in the band-eigenstate basis) of the position operator are relevant here. Utilizing $\langle\eta\mathbf{k}|\hat{\mathbf{r}}|\eta'\mathbf{k}'\rangle = i\frac{\partial}{\partial\mathbf{k}}\delta_{\mathbf{k}\mathbf{k}'}\delta_{\eta\eta'}+\mathbf{J}^{ll'}$ with $\mathbf{J}^{ll'} = \langle u_{\mathbf{k}}^{\eta}|\frac{\partial}{\partial\mathbf{k}}|u_{\mathbf{k}}^{\eta'}\rangle\delta_{\mathbf{k}\mathbf{k}'}$ and $\mathbf{J}^{l}\equiv\mathbf{J}^{ll}$, we get

$$\delta^{ex}\mathbf{v}_l = \mathrm{Re}\sum_{l'}\frac{2}{\hbar}\left\langle V_{ll'}\frac{[V,\mathbf{J}]_{l'l}}{\epsilon_l-\epsilon_{l'}-i\delta}\right\rangle_c + \sum_{l'}\frac{2\pi}{\hbar}\left\langle|V_{ll'}|^2\right\rangle_c\delta\left(\epsilon_l-\epsilon_{l'}\right)\left[i\mathbf{J}^{l'}-i\mathbf{J}^{l}\right], \tag{A4}$$

where we define $[V,\mathbf{J}]_{l'l}\equiv\sum_{l''}\left[V_{l'l''}\mathbf{J}^{l''l}-\mathbf{J}^{l'l''}V_{l''l}\right]$. This quantity can be greatly simplified by using $\langle\eta\mathbf{k}|\hat{\mathbf{r}}|\eta'\mathbf{k}'\rangle = i\frac{\partial}{\partial\mathbf{k}}\delta_{\mathbf{k}\mathbf{k}'}\delta_{\eta\eta'}+\mathbf{J}^{ll'}$:

$$[V,\mathbf{J}]_{l'l} = -i\langle l'|\left[\hat{V},\hat{\mathbf{r}}\right]|l\rangle + \sum_{l''}\left[\partial_{\mathbf{k}}\left(V_{l'l''}\delta_{\mathbf{k}\mathbf{k}''}\delta_{\eta\eta''}\right)+\partial_{\mathbf{k}'}\left(\delta_{\mathbf{k}'\mathbf{k}''}\delta_{\eta'\eta''}V_{l''l}\right)\right] = \left(\partial_{\mathbf{k}}+\partial_{\mathbf{k}'}\right)V_{l'l}, \tag{A5}$$

thereby

$$\delta^{ex}\mathbf{v}_l = \sum_{l'}\frac{2\pi}{\hbar}\left\langle|V_{ll'}|^2\right\rangle_c\delta\left(\epsilon_l-\epsilon_{l'}\right)\left[i\mathbf{J}^{l'}-i\mathbf{J}^{l}\right] + \mathrm{Re}\sum_{l'}\frac{2}{\hbar}\left\langle\frac{V_{ll'}\hat{\mathbf{D}}V_{l'l}}{\epsilon_l-\epsilon_{l'}-i\delta}\right\rangle_c \tag{A6}$$

with $\hat{\mathbf{D}} = \partial_{\mathbf{k}}+\partial_{\mathbf{k}'}$. This quantity is just the second term of Eq. (2.38) in Luttinger's classical paper on the quantum transport theory of anomalous Hall effect [22]. In fact, the first term of Luttinger's Eq. (2.38) just corresponds to the Berry-curvature anomalous velocity. Luttinger called his Eq. (2.38) the off-diagonal velocity because its calculation involved interband matrix elements of the velocity operator. This is the same thought presented here. On the other hand, the second term of Luttinger's Eq. (2.38) has been cited by Sinitsyn et al. [15] to confirm the validity of their pictorial definition of semiclassical side-jump velocity $\mathbf{v}_l^{sj} = \sum_{l'}\omega_{l',l}^{2s}\delta\mathbf{r}_{l',l}$ which contributes an anomalous Hall current $\mathbf{j}^{sj} = \sum_l\mathbf{v}_l^{sj}g_l^{2s}$. The validity of this definition of the side-jump velocity is finally confirmed by the correspondence to Luttinger's quantum transport theory [22] and by one-to-one correspondence to special sets of Feynman diagrams [2, 4, 16] as well as by successful calculations of anomalous Hall effect in some model systems [16, 20]. The last term of Eq. (A6) can be split into two terms with one related to $\mathrm{Im}\left\langle V_{ll'}\hat{\mathbf{D}}V_{l'l}\right\rangle_c$ and the other related to $\hat{\mathbf{D}}\left\langle|V_{ll'}|^2\right\rangle_c$. The first one is related to the phase of the disorder potential and is thus nontrivial. While the latter one that does not break any symmetry is just a trivial renormalization to $\mathbf{v}_l^0$. It does not contribute to the Hall current in the leading order of perturbation theory and can be ignored [15, 41]. In fact, in Rashba model (14) with short-range disorder and both bands partially occupied, this term vanishes. Then one gets the relation $\delta^{ex}\mathbf{v}_l = \mathbf{v}_l^{sj}$:

$$\delta^{ex}\mathbf{v}_l = \sum_{l'}\frac{2\pi}{\hbar}\left\langle|V_{ll'}|^2\right\rangle_c\delta\left(\epsilon_l-\epsilon_{l'}\right)\left[i\mathbf{J}^{l'}-i\mathbf{J}^{l}-\hat{\mathbf{D}}\arg V_{l'l}\right]\equiv\sum_{l'}\omega_{ll'}^{2s}\delta\mathbf{r}_{l'l}.$$

As an example, considering the anomalous Hall effect in model (14) with both bands partially occupied. By $\langle u_{\mathbf{k}}^{\eta}|\hat{\mathbf{v}}|u_{\mathbf{k}}^{-\eta}\rangle = \frac{\alpha_R}{\hbar}\hat{\mathbf{z}}\times\langle u_{\mathbf{k}}^{\eta}|\hat{\sigma}|u_{\mathbf{k}}^{-\eta}\rangle$ and Eqs. (6-9), we get

$$\delta^{in}\mathbf{v}_l = \frac{\alpha_R/\hbar}{\hbar/2}\hat{\mathbf{z}}\times\delta^{in}\mathbf{S}_l,\ \delta^{ex}\mathbf{v}_l = \frac{\alpha_R/\hbar}{\hbar/2}\hat{\mathbf{z}}\times\delta^{ex}\mathbf{S}_l. \tag{A7}$$

For class A impurities, one thus obtains zero anomalous Hall current under the Gaussian disorder and

$$\delta^{ex}\mathbf{v}_l = \eta\frac{\hbar\mathbf{k}_\eta\left(\epsilon\right)}{m}\times\hat{\mathbf{z}}\frac{J_{ex}\epsilon_R}{\left(J_{ex}^2+2\epsilon_R\epsilon\right)}\frac{\hbar}{2\Delta_\eta\left(\epsilon\right)\tau}. \tag{A8}$$

This result coincides with the side-jump velocity obtained in Ref. 20 from the expression $\mathbf{v}_l^{sj} = \sum_{l'}\omega_{l',l}\delta\mathbf{r}_{l',l}$.

Appendix B: Correspondence between semiclassical Boltzmann contributions and Feynman diagrams in the band representation

In the context of the anomalous Hall effect, the one-to-one correspondence between semiclassical contributions and special sets of Feynman diagrams in the band-eigenstate basis under the non-crossing approximation in the weak disorder limit has been established [4, 16]. The diagrams in the band representation for the disorder-induced interband-coherence contributions to the anomalous Hall effect are presented in Fig. 1 of Ref. 4. The correspondence between these diagrams and semiclassical contributions was clearly presented in Refs. 2 and 16. The upper four "interband diagrams" with an interband velocity vertex on the rhs of each diagram correspond to the semiclassical contribution due to the anomalous distribution function g_l^a, the six "intraband diagrams" correspond to the semiclassical contribution due to the intrinsic-skew-scattering g_l^{sk-in}. Whereas the lower four "interband diagrams" with an interband velocity vertex on the left hand side of each diagram are just the semiclassical contribution due to the side-jump velocity $\mathbf{v}_l^{sj}$.

In our case, this kind of correspondence remains unchanged, provided that the left velocity vertex of all these diagrams in the case of the anomalous Hall effect are replaced by the Feynman vertex of A. $\delta^{adis}A$ and $\delta^{sk-in}A$, arising from g_l^a and g_l^{sk-in}, are thus represented by the upper four "interband diagrams" and the six "intraband diagrams", respectively. As for $\delta^{sj}A$ which is related to $\delta^{ex}A_l$, comparing the structure of Eqs. (8,9) with the left interband vertices and the interband-scattering disorder lines in the lower four "interband diagrams", one can verify the correspondence. This correspondence is expected also because $\delta^{ex}A_l$ is a generalization of the side-jump velocity.

The correspondence to the diagrammatic analysis establishes the equivalence between the semiclassical theory and microscopic linear response theories regarding disorder-induced interband-coherence responses under the non-crossing approximation in the weak disorder limit. According to this correspondence, it is clearly seen that $\delta^{adis}A$ and $\delta^{sj}A$ are both related to one interband and one intraband vertex, and are thus interband-coherence disorder effects. While $\delta^{sk-in}A$ is related to two intraband vertices, it also contain interband-coherence disorder effects, i.e., interband off-shell scattering processes, as shown by the middle part of the six "intraband diagrams" in Fig. 1 of Ref. 4. This point can be easily appreciated by considering the case of 2D massive Dirac model [16, 35], where the interband impurity-scattering can only be virtual transition.

Appendix C: SOT in a 2D Rashba ferromagnet with in-plane magnetization and scalar impurities

The model Hamiltonian is [6] $H = \frac{\hbar^2\mathbf{k}^2}{2m} + \alpha_R\hat{\sigma}\cdot(\mathbf{k}\times\hat{\mathbf{z}}) - J_{ex}\hat{\sigma}\cdot\hat{\mathbf{M}} + V_A$, with $\hat{\mathbf{M}} = \cos\theta_M\hat{\mathbf{x}} + \sin\theta_M\hat{\mathbf{y}}$ the direction of the in-plane magnetization. The eigenenergy of the pure system is $\epsilon_{\mathbf{k}}^{\eta} = \frac{\hbar^2k^2}{2m} + \eta\Delta_{\mathbf{k}}$, where $\Delta_{\mathbf{k}} = |\alpha_R(\mathbf{k}\times\hat{\mathbf{z}}) - J_{ex}\hat{\sigma}\cdot\mathbf{M}|$. Note that $k \equiv k(\phi)$ still depends on ϕ due to the anisotropy of the bands arising from the interplay of Rashba effective magnetic field and in-plane magnetization. The spinor eigenstate reads $|u_{\eta\mathbf{k}}\rangle = \frac{1}{\sqrt{2}}[1, -i\eta\exp(i\gamma_{\mathbf{k}})]^T$ with $\cos\gamma_{\mathbf{k}} = \frac{J_{ex}\sin\theta_M + \alpha k_x}{\Delta_{\mathbf{k}}}$ and $\sin\gamma_{\mathbf{k}} = \frac{-J_{ex}\cos\theta_M + \alpha k_y}{\Delta_{\mathbf{k}}}$. In order to make analytical progress, we focus on the limit [6] $\hbar/\tau \ll \alpha k_F \ll J_{ex} \ll \epsilon_F$. The following expressions are obtained by expanding to the first order of $\alpha_R k/J_{ex}$:

$$\Delta_{\mathbf{k}} \simeq J_{ex}\left[1 + \frac{\alpha_R k}{J_{ex}}\sin(\theta_M - \phi)\right],$$

$$\cos\gamma_{\mathbf{k}} \simeq \sin\theta_M + \frac{\alpha_R k}{J_{ex}}\left[\cos\phi - \sin\theta_M\sin(\theta_M - \phi)\right],$$

$$\sin\gamma_{\mathbf{k}} \simeq -\cos\theta_M + \frac{\alpha_R k}{J_{ex}}\left[\sin\phi + \cos\theta_M\sin(\theta_M - \phi)\right],$$

and $\sin(\gamma_{\mathbf{k}'} - \gamma_{\mathbf{k}}) \simeq \frac{\alpha k'}{J_{ex}}\cos(\theta_M - \phi') - \frac{\alpha k}{J_{ex}}\cos(\theta_M - \phi)$, $\cos(\gamma_{\mathbf{k}'} - \gamma_{\mathbf{k}}) \simeq 1$.

Under a weak uniform electric field applied in x direction, the intrinsic nonequilibrium spin-polarization is given by $\delta^{in}\mathbf{S} \simeq \frac{\hbar}{2}eD_0\frac{\alpha_R\cos\theta_M}{J_{ex}}E_x\hat{\mathbf{z}}$. As for $\delta^{ex}\mathbf{S}_l$, in the weak scattering limit the nonzero component in the first order of $\alpha_R k/J_{ex}$ is $\delta^{ex}\mathbf{S}_l \simeq \frac{\hbar}{2}\eta\frac{\hbar}{2J_{ex}\tau}\frac{\alpha_R k}{J_{ex}}\cos(\theta_M - \phi)\hat{\mathbf{z}}$. Thus in the $o(\alpha_R k/J_{ex})$ contribution of $\delta^{sj}\mathbf{S} = \sum_l \delta^{ex}\mathbf{S}_l g_l^{2s}$ the distribution function can be obtained from the Boltzmann equation in the zeroth order of $\alpha_R k/J_{ex}$, just yielding $g_{\eta}^{2s}(\epsilon,\phi) = e\mathbf{E}\cdot\frac{\hbar\mathbf{k}_{\eta}}{m}\tau\left(-\partial_{\epsilon}f^0\right)$. With only in-plane magnetization there is no anomalous distribution function and intrinsic skew scattering. Then the scattering-induced interband-coherence contribution is obtained as $\delta^{SJ}\mathbf{S} = \delta^{sj}\mathbf{S} = -\delta^{in}\mathbf{S}$, which cancels the intrinsic nonequilibrium spin-polarization. This vanishing interband-coherence contribution to the nonequilibrium spin-polarization is consistent with the result in Ref. 33.

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