

This is the accepted manuscript made available via CHORUS. The article has been published as:

## Effect of $SU(2)$ symmetry on many-body localization and thermalization

Ivan V. Protopopov, Wen Wei Ho, and Dmitry A. Abanin

Phys. Rev. B **96**, 041122 — Published 21 July 2017

DOI: [10.1103/PhysRevB.96.041122](https://doi.org/10.1103/PhysRevB.96.041122)

# The effect of $SU(2)$ symmetry on many-body localization and thermalization

Ivan V. Protopopov,<sup>1,2,3</sup> Wen Wei Ho,<sup>1,2</sup> and Dmitry A. Abanin<sup>1,2</sup>

<sup>1</sup>*Department of Theoretical Physics, University of Geneva,  
24 quai Ernest-Ansermet, 1211 Geneva, Switzerland*

<sup>2</sup>*Kavli Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106, USA*

<sup>3</sup>*L. D. Landau Institute for Theoretical Physics RAS, 119334 Moscow, Russia*

(Dated: June 26, 2017)

The many-body localized (MBL) phase is characterized by a complete set of quasi-local integrals of motion and area-law entanglement of excited eigenstates. We study the effect of non-Abelian continuous symmetries on MBL, considering the case of  $SU(2)$  symmetric disordered spin chains. The  $SU(2)$  symmetry imposes strong constraints on the entanglement structure of the eigenstates, precluding conventional MBL. We construct a fixed-point Hamiltonian, which realizes a non-ergodic (but non-MBL) phase characterized by eigenstates having logarithmic scaling of entanglement with the system size, as well as an incomplete set of quasi-local integrals of motion. We study the response of such a phase to local symmetric perturbations, finding that even weak perturbations induce multi-spin resonances. We conclude that the non-ergodic phase is generally unstable and that  $SU(2)$  symmetry implies thermalization. The approach introduced in this work can be used to study dynamics in disordered systems with non-Abelian symmetries, and provides a starting point for searching non-ergodic phases beyond conventional MBL.

PACS numbers: 72.15.Rn, 75.10.Pq, 05.30.Rt

*Introduction.* Over the past several years, the phenomenon of many-body localization (MBL) has been attracting significant interest, both theoretically [1–17] and experimentally [18–20]. Many-body localization occurs in strongly disordered systems and is driven by a mechanism similar to the (single-particle) Anderson localization in the many-body Hilbert space. Isolated many-body localized systems exhibit zero conductivity and avoid thermalization, and therefore provide the only known, generic example of ergodicity breaking in many-body systems.

MBL eigenstates have low, area-law entanglement entropy [8, 21], in contrast to the excited eigenstates of ergodic systems, which have thermal, volume-law entanglement. The systems in which all states are many-body localized exhibit a new kind of robust integrability: a complete set of quasi-local integrals of motion (LIOMs) emerges [8, 9] (see also [22–24]). Apart from providing a simple physical intuition for the ergodicity breaking in MBL phase, LIOM theory has been used to explain dynamical properties of MBL eigenstates, including logarithmic entanglement growth in a quantum quench setup [7–9], as well as power-law decay [25] and revivals [26] of local observables, which can be tested in cold atoms experiments.

A natural question concerns the role of various symmetries on MBL and thermalization. Previous works focused mostly on MBL in the presence of discrete symmetries, such as  $\mathbb{Z}_2$  symmetry. It was shown [13, 27, 28] that in this case two distinct MBL phases are possible, one of which locally preserves  $\mathbb{Z}_2$  symmetry, while the other phase locally breaks that symmetry. It was also argued that MBL can protect topological [21, 27] and symmetry-protected topological [29] order at finite temperatures. Ref. [30] considered the effect of a particular

non-Abelian discrete symmetry on MBL.

The goal of this paper is to study disordered systems with *continuous non-Abelian symmetries*. We focus on the simplest and experimentally relevant example of such a symmetry –  $SU(2)$  spin rotation symmetry – which is realized in the random Heisenberg spin-1/2 chain:

$$H = \sum_{i=1}^L J_i \mathbf{s}_i \cdot \mathbf{s}_{i+1}, \quad (1)$$

where coupling  $J_i$  are randomly drawn from some distribution, and  $\mathbf{s}_i = (s_i^x, s_i^y, s_i^z)$  are the Pauli operators.

$SU(2)$  symmetry puts severe constraints on the entanglement structure of the eigenstates and on the locality properties of the integrals of motion in a possible non-ergodic phase. Generally, in the presence of  $SU(2)$  symmetry, it is impossible to have eigenstates with area-law entanglement, and therefore conventional MBL cannot occur [29, 31]. As we will argue below, the symmetry does allow eigenstates with entanglement that grows logarithmically with the system size. Simultaneously, at least some integrals of motion must become non-local. The key question then is whether such a non-ergodic (but non-MBL) phase may be stable. Below, we will perform the stability analysis, finding that in general such an entanglement structure is unstable, in the sense that in a sufficiently large system, an arbitrarily weak perturbation of the Hamiltonian inevitably strongly mixes eigenstates, leading to delocalization. We will discuss the delocalization mechanism, finding that thermalization is parametrically slow at strong disorder. Our analysis therefore indicates that  $SU(2)$  symmetry is inconsistent with non-ergodicity, and implies thermalization. We expect that the approach introduced below can be used in fu-

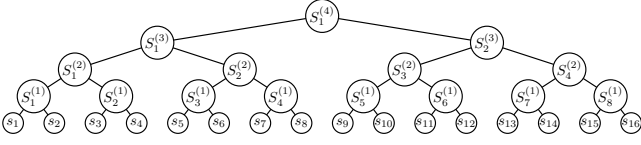


Figure 1. The simplest possible tree-structure and corresponding LIOMs.

ture studies of ergodicity breaking beyond conventional MBL.

We note that two recent works [32, 33] used real-space strong disorder renormalization group for excited states (RSRG-X) to analyze the behaviour of disordered,  $SU(2)$ -symmetric spin chains. Our approach allows us to take into account multi-spin processes, which are not captured by RSRG. Below we show that such processes inevitably lead to delocalization.

*Conventional MBL phase.* Let us start by recalling the description of the conventional MBL phase (no non-Abelian symmetries) in terms of LIOMs. The defining property of MBL is that highly excited eigenstates can be obtained from non-entangled product states by a quasi-local unitary transformation  $U$ ,  $U^\dagger H U = H_{\text{diag}}$ . For the case of spin-1/2 chains (such as random-field XXZ model that has been extensively studied [5, 7, 10, 11, 34]), it is convenient to choose a product state basis in which eigenstates have definite  $s_i^z$  projections. Then, operators  $\tau_i^z = U s_i^z U^\dagger$ , which are dressed spin operators, are quasi-local integrals of motion. In terms of these operators spins, the Hamiltonian takes a simple form [8, 9]:

$$H = \sum_i h_i \tau_i^z + \sum_{i,j} J_{ij} \tau_i^z \tau_j^z + \sum_{i,j,k} J_{ijk} \tau_i^z \tau_j^z \tau_k^z + \dots, \quad (2)$$

where couplings  $J_{ij..k}$  decay exponentially with distance. The Hamiltonian (2) is often viewed as a “fixed-point” Hamiltonian of the MBL phase. Importantly, integrability is robust: if a weak perturbation which is a sum of local, but otherwise arbitrary terms is added to it, a new set of quasi-local integrals of motion can be defined. In what follows, we will show that the presence of  $SU(2)$  symmetry significantly modifies the possible structures of integrals of motion and of the fixed-point Hamiltonian.

*$SU(2)$  symmetry implies non-local integrals of motion.* Let us now discuss the possible structure of the non-ergodic phase in an  $SU(2)$  symmetric spin chain (1). First, we argue that the eigenstates cannot be area-law entangled. Consider two initially disconnected systems,  $\mathcal{L}$  and  $\mathcal{R}$ . The symmetry dictates that each eigenstate  $|\alpha\rangle_{\mathcal{L}}$  of  $\mathcal{L}$  belongs to a multiplet  $m_{\mathcal{L}}$  with some total spin  $S_{\mathcal{L}}$  and degeneracy  $2S_{\mathcal{L}} + 1$ . Similarly, an eigenstate  $|\beta\rangle_{\mathcal{R}}$  of  $\mathcal{R}$  belongs to a  $(2S_{\mathcal{R}} + 1)$ -degenerate multiplet  $m_{\mathcal{R}}$  with spin  $S_{\mathcal{R}}$ . When we couple the two systems, even a very weak coupling will force the eigenstates of the  $\mathcal{L} + \mathcal{R}$  system to transform as an irreducible representation of the  $SU(2)$  symmetry acting on the combined

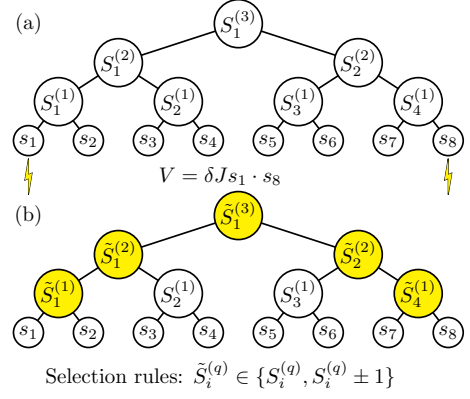


Figure 2. Illustration of selection rules for the matrix elements of the operator  $\mathbf{s}_1 \cdot \mathbf{s}_L$  for a system of 8 spins. The two tree states, (a) and (b), have non-vanishing matrix element only when  $\tilde{S} = S$ ,  $S \pm 1$  while spins in the other nodes coincide.

system. The least entangled (and therefore most non-ergodic) states correspond to the scenario when different multiplets (which are eigenstates of disconnected  $\mathcal{L}$  and  $\mathcal{R}$  systems) do not strongly hybridize once the systems are joined. Assuming that this holds, the eigenstates of the whole system are obtained by adding together a single multiplet  $m_{\mathcal{L}}$  with spin  $S_{\mathcal{L}}$  and a single multiplet  $m_{\mathcal{R}}$  with spin  $S_{\mathcal{R}}$  to form larger multiplets, whose spin can take values  $|S_{\mathcal{L}} - S_{\mathcal{R}}|, |S_{\mathcal{L}} - S_{\mathcal{R}}| + 1, \dots, S_{\mathcal{L}} + S_{\mathcal{R}}$ . Such states have entanglement entropy which is typically of the order  $S_{\text{ent}} \sim \ln(\min(S_{\mathcal{R}}, S_{\mathcal{L}}))$ . Since the total spin of each subsystem grows extensively with its size, we conclude that if a non-ergodic phase exists in the presence of  $SU(2)$  symmetry, the eigenstates cannot be area-law entangled, as in the MBL phase.

Further, as the system size is increased, spins of different subsystems (“blocks”) have to be added up, such that larger and larger spins are formed [29]. Depending on the strength of the coupling between different spins (which are disordered) a spin of a given block should be first added with the spin of the block to the left or to the right of it. Graphically, we can denote adding two spins by connecting corresponding blocks; then, a tree-like structure, an example of which is shown in Fig. 1 emerges. An eigenstate is uniquely specified by the spin values at every leaf of the tree. Such a plausible structure of the eigenstates, described by a tree tensor network, corresponds to logarithmic scaling of entanglement entropy with the system size,  $S_{\text{ent}} \sim \ln L$ . Such entanglement scaling is non-MBL, but also strongly sub-thermal, and therefore describes non-ergodic eigenstates.

The picture described above corresponds to an incomplete set of quasi-local integrals of motion: a total spin at every step is approximately conserved; it becomes a precise IOM if we deform it by a quasi-local unitary operator, which accounts for perturbative mixing between different multiplets. It is instructive to write down a com-

plete set of integrals of motion for the case of a regular tree structure, illustrated in Fig. 1. There, at the first step, spins  $2i - 1$  and  $2i$  are added to form a (possibly larger) spin  $\mathbf{S}_i^{(1)}$ , then at the second step spins  $\mathbf{S}_{2i-1}^{(1)}$ ,  $\mathbf{S}_{2i}^{(1)}$  are added to form spin  $\mathbf{S}_i^{(2)}$ , etc., until we get just one large spin describing a multiplet of the whole system. In this case, the complete set of IOMs is given by:

$$\left[\mathbf{S}_i^{(1)}\right]^2, i = 1, \dots, \frac{L}{2}; \left[\mathbf{S}_i^{(2)}\right]^2, i = 1, \dots, \frac{L}{4}, \dots \quad (3)$$

where  $L$  is the total number of spins  $1/2$  in the chain. The IOMs  $\left[\mathbf{S}_i^{(k)}\right]^2$  become less and less local as  $k$  is increased: they act on  $2^k$  spin- $1/2$ s. This should be contrasted with the conventional MBL phase characterized by a complete

set of quasi-local IOMs.

We note that since the order in which blocks should be merged depends on the value of the total spin of the resulting block, the integrals of motion would have a different structure for different states. However, for strong disorder such an ambiguity in the block merging only arises when the spin of the resulting block is chosen to be small. As the block merging progresses, such a situation becomes less and less probable. We can therefore assume that, at least starting from some high enough level of our hierarchical construction, the order of merging is fixed and does not depend on the state.

*Fixed-point Hamiltonian.* It is possible to construct a quasi-local “fixed-point” Hamiltonian, for which operators (3) are exact IOMs (note that there is a whole family of such fixed-point Hamiltonians):

$$H_{\text{FP}} = \sum_{i=1}^{L/2} J_i^{(0)} \mathbf{s}_{2i-1} \cdot \mathbf{s}_{2i} + \sum_{i=1}^{L/4} J_i^{(1)} \mathbf{S}_{2i-1}^{(1)} \cdot \mathbf{S}_{2i}^{(1)} + \sum_{i=1}^{L/8} J_i^{(2)} \mathbf{S}_{2i-1}^{(2)} \cdot \mathbf{S}_{2i}^{(2)} + \dots \quad (4)$$

where the couplings  $J_i^{(a)}$  are random, and in order for the Hamiltonian to be quasi-local, they should decay exponentially,  $J_i^{(k)} \propto 2^{-\gamma k}$  with  $\gamma > 2$  [35].

*(In)stability of the non-ergodic phase.* The Hamiltonian (4) has eigenstates which are minimally entangled, given the symmetry constraints. The key question is whether such eigenstates are stable with respect to small, but finite,  $SU(2)$ -symmetric perturbations of the fixed-point Hamiltonian (4). Or, equivalently, whether such a structure of eigenstates can naturally arise starting from a generic local Hamiltonian (1). To answer this question, we study the stability of the eigenstates described by a tree tensor network with respect to local perturbations of the Hamiltonian, extending the approach of Ref. [34]. We choose an  $SU(2)$ -symmetric perturbation coupling the ends of the spin chain:

$$\hat{V} = \delta J \mathbf{s}_1 \cdot \mathbf{s}_L. \quad (5)$$

Such a perturbation describes changing the boundary condition from open to periodic.

The problem of finding the eigenstates of  $H_{\text{FP}} + \hat{V}$  can be formulated as a hopping problem on a lattice, where sites are eigenstates  $|\alpha\rangle$  of  $H_{\text{FP}}$ . Each state  $|\alpha\rangle$  is uniquely specified by choosing values of the total spins for each node of the tree, as well as the total  $z$ -projection of the spin at the last node. Each site  $|\alpha\rangle$  has on-site energy  $E_\alpha = \langle \alpha | H_{\text{FP}} | \alpha \rangle$ , and hopping between sites is set by the matrix elements of the perturbation:  $V_{\alpha\beta} = \langle \alpha | \hat{V} | \beta \rangle$ . To analyze the statistics of matrix elements, we first note that due to global  $SU(2)$  symmetry the perturbation does not change the value of the total spin. Further, the symmetry imposes stringent selection rules

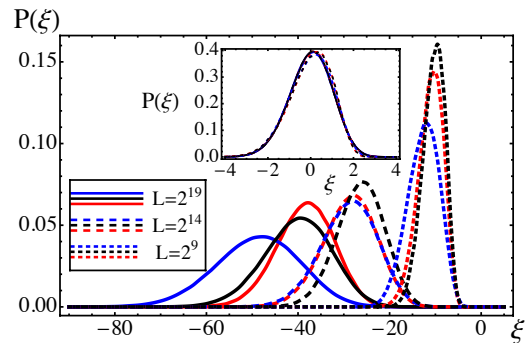


Figure 3. Distribution of  $\zeta = \log_2 |V_{\alpha\beta}/\delta J|$  for different sizes of the system:  $L = 2^9$  (dotted lines),  $L = 2^{14}$  (dashed lines) and  $L = 2^{19}$  (full lines). For each system size three different base states were randomly generated. The corresponding distributions are shown by different colors. The inset shows the result of scaling the distributions to the corresponding widths.

on the matrix elements, see Fig. 2. The spins on the branches of the tree not involving spin  $\mathbf{s}_1$  are not affected by operator  $\mathbf{s}_1$ . Moreover, it is possible to show [36] that  $\mathbf{s}_1$  can change any spin on the branch of the tree which involves spin  $\mathbf{s}_1$  by  $\pm 1$ , or leave it unchanged. Noting that there are  $\log_2 L$  nodes in this branch, as well as the fact that  $\hat{V}$  can also change the spins on the branch involving  $\mathbf{s}_n$ , we obtain that the operator (5) couples a given base state to

$$K(L) \approx 3^{2 \log_2 L} = L^\alpha, \quad \alpha = 2 \log_2 3$$

other states [37].

Another important ingredient is the level spacing for

the hopping problem. We note that in a system with a quasi-local Hamiltonian, a local perturbation can only significantly couple eigenstates with energy difference of order  $J$  (where  $J$  is the typical interaction scale of the Hamiltonian). Thus, the level spacing in the manifold of states to which a given state is coupled, can be estimated as [38]

$$\Delta(L) \approx \frac{J}{K(L)}. \quad (6)$$

*Matrix elements.* Matrix elements connecting a given base state to the ones allowed by the selection rules can be decomposed in terms of Clebsch-Gordan coefficients [39], and are readily accessible for numerical simulations. In Fig. 3 we show the distribution of the quantity  $\zeta = \log_2 |V_{\alpha\beta}/\delta J|$  for several randomly chosen base states in systems of size  $L = 2^9, 2^{14}$  and  $2^{19}$ . We observe that for each of the bases states the distribution of  $\zeta$  is well behaved and can be characterized by the mean  $\bar{\zeta}$  and standard deviation  $\sigma$  (depending on both the system size and particular base state chosen). Moreover, the inset in Fig. 3 shows that, after shifting by respective  $\bar{\zeta}$  and scaling by the corresponding  $\sigma$ , all those distributions collapse into a single, which is approximately Gaussian:

$$P(\zeta) \propto e^{-(\zeta - \bar{\zeta})/2\sigma^2}. \quad (7)$$

The mean  $\bar{\zeta}$  and standard deviation  $\sigma$  in Eq. (7) are random quantities which depend on the base state. We were able to analyze their statistics in the limit of large  $L$ . We computed the average of  $\bar{\zeta}$  [39], finding that it grows linearly with  $\log L$ :

$$\langle \bar{\zeta} \rangle = -\beta \log_2 L + O(1), \quad \beta = \frac{17 + 4 \ln 2}{9 \ln 2}, \quad (8)$$

while the standard deviation of  $\bar{\zeta}$  and the average of  $\sigma$  obey

$$\sqrt{\langle (\bar{\zeta} - \langle \bar{\zeta} \rangle)^2 \rangle} \propto \sqrt{\log_2 L}, \quad \langle \sigma \rangle = C \sqrt{\log_2 L}. \quad (9)$$

The constant  $C$  is of order one [39].

*Instability with respect to local perturbation.* Next, we use the scalings of the level spacing and typical matrix element, Eqs. (6) and (7),(8),(9) to estimate the probability of finding a (long-range) resonance induced by a local perturbation. To that end, we consider a ratio  $g = \frac{V}{\Delta(L)}$ , which can be viewed as the Thouless parameter. Neglecting the fluctuations in the level spacing, the probability of finding a resonance is given by:

$$P(g > 1) = P(\zeta > A), \quad A = \log_2 \left( \frac{J}{\delta J K(L)} \right). \quad (10)$$

Furthermore, for fixed  $\bar{\zeta}$  and  $\sigma$ , this probability can be transformed as follows:

$$P(\zeta > A | \bar{\zeta}, \sigma) = \frac{1}{2} \operatorname{erfc} \left( \frac{A - \bar{\zeta}}{\sqrt{2}\sigma} \right), \quad (11)$$

where  $\operatorname{erfc}(x)$  is the complementary error function. Using the above results (6),(8), we obtain that the asymptotic behavior of the expression in the r.h.s. of the above equation is given by:

$$\frac{A - \bar{\zeta}}{\sqrt{2}\sigma} \propto \frac{\log_2 \frac{J}{\delta J} - (\alpha - \beta) \log_2 L}{C \sqrt{\log_2 L}}, \quad (12)$$

where  $C$  is the proportionality constant of order 1 in Eq.(9). Interestingly, we find that the parameter  $\gamma = \alpha - \beta \approx 10^{-4}$  is positive, but extremely small. The above expression approaches zero beyond length scale  $L_c$  given by equation

$$\log_2 \frac{J}{\delta J} \approx C \sqrt{\log_2 L_c},$$

which yields

$$L_c \approx \left( \frac{J}{\delta J} \right)^{\frac{\log_2(J/\delta J)}{C^2}}. \quad (13)$$

When  $L > L_c$ , the Thouless parameter becomes  $g \approx 1/2$ , which signals instability of tree-like eigenstates and delocalization, for arbitrarily small  $\delta J$ . This implies that the new eigenstates will have volume-law entanglement entropy, indicating that the non-ergodic phase described above is intrinsically unstable. However, since not all states are resonant with their nearest neighbours, we expect that the system will show non-trivial, and possibly glassy dynamics. [40]

*Relation to RSRG-X.* Previous studies by Vasseur et al. [32] and Agarwal et al. [33] employed RSRG-X to analyze the eigenstates of  $SU(2)$ -symmetric spin chains. In this procedure, strongly coupled pairs of spins are identified, and their spins are added. Such a procedure naturally gives rise to a tree-like structure of the eigenstates described above. We note that Refs. [32, 33] used different RG rules, and seemingly arrived at opposite conclusions: the former study found that the RSRG-X procedure breaks down for  $SU(2)$  chains, concluded that thermalization was inevitable, while the latter one claimed that the procedure remains well-defined.

We emphasize that our approach allows us to analyze multi-spin processes. RSRG-X, in contrast, only captures short-range resonances, which involve a finite number of nearby spins. Our results show that, even if the RSRG-X procedure for  $SU(2)$ -symmetric systems remains well-defined as one increases the system size, delocalization occurs due to multi-spin processes not accounted for by RSRG-X. We therefore expect that a proper analysis of dynamics in disordered  $SU(2)$  spin chains should involve a combination of RSRG-X and the approach introduced in this paper. The role of the perturbation introduced by hand above will be played by smaller terms which are neglected in the RSRG-X approach.

*Summary.* We have studied the effect of  $SU(2)$  symmetry on MBL. We argued that, in contrast to conventional



systems where MBL occurs, such symmetry dictates that the eigenstates have larger than area-law entanglement, and some integrals of motion must become non-local. We have introduced a fixed-point Hamiltonian which gives rise to non-ergodic eigenstates with  $S_{\text{ent}} \sim \ln L$  entanglement. Further, we showed that a weak, local perturbation inevitably introduces resonances between the eigenstates in a sufficiently large system, leading to delocalization. Our results indicate that  $SU(2)$  symmetry necessarily implies delocalization; while proving thermalization is a difficult (if not impossible) task, we expect that  $SU(2)$ -symmetric systems thermalize.

In the future, we will use the approach introduced here to analyze interplay of MBL and other non-Abelian symmetries, including  $SU(n > 2)$ , as well as the limitations of the RSRG-X procedure. More generally, our results are relevant in the context of searching for non-ergodic phases that have a richer entanglement structure than the conventional MBL. In particular, we will consider the stability of eigenstates described by the multi-scale entanglement renormalization ansatz [? ].

**Acknowledgements.** We thank Eugene Demler, Maksym Serbyn, and Romain Vasseur for insightful discussions and helpful comments on the manuscript. This research was supported by the Swiss NSF, and in part by the NSF under Grant No. PHY11-25915. We thank the Kavli Institute for Theoretical Physics at Santa Barbara, where a part of this work was done, for hospitality during the program *Synthetic Quantum Matter*. IP acknowledges support by Russian Science Foundation under the grant No. 14-42-00044.

- 
- [1] D.M. Basko, I.L. Aleiner, and B.L. Altshuler, “Metal-insulator transition in a weakly interacting many-electron system with localized single-particle states,” *Annals of Physics* **321**, 1126 – 1205 (2006).
  - [2] I. V. Gornyi, A. D. Mirlin, and D. G. Polyakov, “Interacting electrons in disordered wires: Anderson localization and low- $t$  transport,” *Phys. Rev. Lett.* **95**, 206603 (2005).
  - [3] Vadim Oganesyan and David A. Huse, “Localization of interacting fermions at high temperature,” *Phys. Rev. B* **75**, 155111 (2007).
  - [4] M. Znidaric, T. Prosen, and P. Prelovsek, “Many-body localization in the heisenberg  $xxz$  magnet in a random field,” *Phys. Rev. B* **77**, 064426 (2008).
  - [5] Arijeet Pal and David A. Huse, “Many-body localization phase transition,” *Phys. Rev. B* **82**, 174411 (2010).
  - [6] Ronen Vosk and Ehud Altman, “Many-body localization in one dimension as a dynamical renormalization group fixed point,” *Phys. Rev. Lett.* **110**, 067204 (2013).
  - [7] Maksym Serbyn, Z. Papić, and Dmitry A. Abanin, “Universal slow growth of entanglement in interacting strongly disordered systems,” *Phys. Rev. Lett.* **110**, 260601 (2013).
  - [8] Maksym Serbyn, Z. Papić, and Dmitry A. Abanin, “Local conservation laws and the structure of the many-body localized states,” *Phys. Rev. Lett.* **111**, 127201 (2013).
  - [9] David A. Huse, Rahul Nandkishore, and Vadim Oganesyan, “Phenomenology of fully many-body-localized systems,” *Phys. Rev. B* **90**, 174202 (2014).
  - [10] Jens H. Bardarson, Frank Pollmann, and Joel E. Moore, “Unbounded growth of entanglement in models of many-body localization,” *Phys. Rev. Lett.* **109**, 017202 (2012).
  - [11] David J. Luitz, Nicolas Laflorencie, and Fabien Alet, “Many-body localization edge in the random-field heisenberg chain,” *Phys. Rev. B* **91**, 081103 (2015).
  - [12] Kartiek Agarwal, Sarang Gopalakrishnan, Michael Knap, Markus Müller, and Eugene Demler, “Anomalous diffusion and griffiths effects near the many-body localization transition,” *Phys. Rev. Lett.* **114**, 160401 (2015).
  - [13] David Pekker, Gil Refael, Ehud Altman, Eugene Demler, and Vadim Oganesyan, “Hilbert-glass transition: New universality of temperature-tuned many-body dynamical quantum criticality,” *Phys. Rev. X* **4**, 011052 (2014).
  - [14] Pedro Ponte, Z. Papić, François Huveneers, and Dmitry A. Abanin, “Many-body localization in periodically driven systems,” *Phys. Rev. Lett.* **114**, 140401 (2015).
  - [15] Achilleas Lazarides, Arnab Das, and Roderich Moessner, “Fate of many-body localization under periodic driving,” *Phys. Rev. Lett.* **115**, 030402 (2015).
  - [16] Dmitry A. Abanin, Wojciech De Roeck, and François Huveneers, “Theory of many-body localization in periodically driven systems,” *Annals of Physics* **372**, 1 – 11 (2016).
  - [17] Vedika Khemani, Achilleas Lazarides, Roderich Moessner, and S. L. Sondhi, “Phase structure of driven quantum systems,” *Phys. Rev. Lett.* **116**, 250401 (2016).
  - [18] Michael Schreiber, Sean S. Hodgman, Pranjali Bordia, Henrik P. Lüschen, Mark H. Fischer, Ronen Vosk, Ehud Altman, Ulrich Schneider, and Immanuel Bloch, “Observation of many-body localization of interacting fermions in a quasirandom optical lattice,” *Science* **349**, 842–845 (2015), <http://science.sciencemag.org/content/349/6250/842.full.pdf>.
  - [19] Jae-yoon Choi, Sebastian Hild, Johannes Zeiher, Peter Schauß, Antonio Rubio-Abadal, Tarik Yefsah, Vedika Khemani, David A. Huse, Immanuel Bloch, and Christian Gross, “Exploring the many-body localization transition in two dimensions,” *Science* **352**, 1547–1552 (2016), <http://science.sciencemag.org/content/352/6293/1547.full.pdf>.
  - [20] J. Smith, A. Lee, P. Richerme, B. Neyenhuis, P. W. Hess, P. Hauke, M. Heyl, D. A. Huse, and C. Monroe, “Many-body localization in a quantum simulator with programmable disorder,” *Nat. Phys.* **12**, 907 (2016).
  - [21] Bela Bauer and Chetan Nayak, “Area laws in a many-body localized state and its implications for topological order,” *Journal of Statistical Mechanics: Theory and Experiment* **2013**, P09005 (2013).
  - [22] John Z. Imbrie, “On many-body localization for quantum spin chains,” *Journal of Statistical Physics* **163**, 998–1048 (2016).
  - [23] V. Ros, M. Müller, and A. Scardicchio, “Integrals of motion in the many-body localized phase,” *Nuclear Physics B* **891**, 420 – 465 (2015).
  - [24] Anushya Chandran, Isaac H. Kim, Guifre Vidal, and Dmitry A. Abanin, “Constructing local integrals of motion in the many-body localized phase,” *Phys. Rev. B*

- [91, 085425 \(2015\)](#).
- [25] Maksym Serbyn, Z. Papić, and D. A. Abanin, “Quantum quenches in the many-body localized phase,” *Phys. Rev. B* **90**, 174302 (2014).
  - [26] R. Vasseur, S. A. Parameswaran, and J. E. Moore, “Quantum revivals and many-body localization,” *Phys. Rev. B* **91**, 140202 (2015).
  - [27] David A. Huse, Rahul Nandkishore, Vadim Oganesyan, Arijeet Pal, and S. L. Sondhi, “Localization-protected quantum order,” *Phys. Rev. B* **88**, 014206 (2013).
  - [28] Jonas A. Kjäll, Jens H. Bardarson, and Frank Pollmann, “Many-body localization in a disordered quantum ising chain,” *Phys. Rev. Lett.* **113**, 107204 (2014).
  - [29] Anushya Chandran, Vedika Khemani, C. R. Laumann, and S. L. Sondhi, “Many-body localization and symmetry-protected topological order,” *Phys. Rev. B* **89**, 144201 (2014).
  - [30] Romain Vasseur, Aaron J. Friedman, S. A. Parameswaran, and Andrew C. Potter, “Particle-hole symmetry, many-body localization, and topological edge modes,” *Phys. Rev. B* **93**, 134207 (2016).
  - [31] A. C. Potter and R. Vasseur, “Symmetry constraints on many-body localization,” ArXiv e-prints (2016), [arXiv:1605.03601 \[cond-mat.dis-nn\]](#).
  - [32] R. Vasseur, A. C. Potter, and S. A. Parameswaran, “Quantum criticality of hot random spin chains,” *Phys. Rev. Lett.* **114**, 217201 (2015).
  - [33] Kartiek Agarwal, Eugene Demler, and Ivar Martin, “ $1/f^\alpha$  noise and generalized diffusion in random heisenberg spin systems,” *Phys. Rev. B* **92**, 184203 (2015).
  - [34] Maksym Serbyn, Z. Papić, and Dmitry A. Abanin, “Criterion for many-body localization-delocalization phase transition,” *Phys. Rev. X* **5**, 041047 (2015).
  - [35] Spin  $\mathbf{S}_i^{(k)}$  is obtained by adding  $2^k$  spins  $1/2$ , and is less or equal than  $2^{k-1}$ . Thus, the magnitude of each interaction term at level  $k$  is  $J_i^k \mathbf{S}_{2i-1}^{(k)} \cdot \mathbf{S}_{2i}^{(k)} \sim 2^{-\gamma k} 2^{2(k-1)}$ . We see that in order for this term to decay with  $k$ , condition  $\gamma > 2$  must hold.
  - [36] In the language of group theory operators  $s_1^{x,y,z}$  organize spin 1 representation of  $SU(2)$ . Correspondingly, the matrix elements of  $\mathbf{s}_1$  between two states of the system with total spin  $S$  and  $S'$  can be non-zero only if the product of the  $S, S'$  and spin-1 representations contains trivial representation as a direct summand. This is only the case for  $S - S' = 0, \pm 1$ .
  - [37] Occasionally, in some nodes of the branch connecting spin  $\mathbf{s}_1$  to the top of the tree one can encounter a block spin  $S_1^{(i)} = 0$ . In this case only one (out of three) possibility,  $\tilde{S}_1^{(i)} = 1$ , leads to non-zero matrix element (the transition to  $\tilde{S}_1^{(1)} = 0$  is prohibited by the selection rules). In addition,  $V_{\alpha\beta}$  vanishes also if somewhere along the branch the block spins  $S_1^{(i)}, S_1^{(i+1)}$  and  $S_2^{(i)}$  form a “right triangle” and the transition  $\alpha \rightarrow \beta$  does not change them, see Ref. [39] for more details. However, due to simple entropic reasons, *for states in the middle of the many-body band* the typical value of spin at level  $n$  from the bottom of the tree is  $2^{n/2}$ . Thus, formation of a singlet far from the bottom of the tree as well as other exceptional situations leading to vanishing matrix element are asymptotically highly improbable and do not alter the scaling of the connectivity  $K(L)$ .
  - [38] If the couplings  $J_i^k$  decay as  $2^{-\gamma k}$  with  $\gamma \gg 1$  the in the fixed-point Hamiltonian (4) dictates the scaling of the *minimal* level spacing  $\Delta_{\min} \propto L^{-\gamma+1/2}$ . For  $\gamma < 2 \log_2 3 + 1/2$  we however expect the level spacing to obey Eq. (6).
  - [39] “Supplemental online material,”.
  - [40] We also note that in the limit  $\delta J \rightarrow 0$ , the delocalization occurs at the lengthscale  $L_* \approx (\frac{J}{\delta J})^{1/\gamma}$ , which corresponds to the numerator in the r.h.s. of Eq.(12) becoming negative. However, given the smallness of parameter  $\gamma$ ,  $L_* < L_c$  only for extremely small  $\delta J \lesssim J \cdot 2^{-C^2/\gamma}$ , and therefore practically, the delocalization lengthscale will be given by Eq.(13).