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In a recent letter, J. Cardy, Phys. Rev. Lett. **112**, 220401 (2014), the author made a very interesting observation that complete revivals of quantum states after quantum quench can happen in a period which is a fraction of the system size. This is possible for critical systems that can be described by minimal conformal field theories (CFT) with central charge $c < 1$. In this article, we show that these complete revivals are impossible in microscopic realizations of those minimal models. We will prove the absence of the mentioned complete revivals for the critical transverse field Ising chain analytically, and present numerical results for the critical line of the XY chain. In particular, for the considered initial states, we will show that criticality has no significant effect in partial revivals. We also comment on the applicability of quasi-particle picture to determine the period of the partial revivals qualitatively. In particular, we detect a regime in the phase diagram of the XY chain which one can not determine the period of the partial revivals using the quasi-particle picture.

I. INTRODUCTION

Quantum mechanical version of Poincaré recurrence theorem guarantees that any system with discrete energy eigenstates, after a sufficiently long but finite time, will return to a state which is very close to it's initial state. Although this seems a very natural expectation for many body systems, it usually takes astronomical times to see an (almost) complete revivals. However, in some systems, partial revivals are possible which makes the problem a very interesting subject, see for example[2–14]. The problem of revivals or related phenomena also appears in many other concepts such as dynamical transition and quantum speed limit [15–18]. Quite naturally, one usually is interested in the problem of revivals when the number of particles is limited or in other words when there is a finite size effect. The presence of the finite size effect usually makes the exact calculations difficult, however, it has the benefit of being accessible by the numerical means.

In a recent letter [1], the author has made a very interesting analytical calculation by using conformal field theory for finite system size and observed that the complete revivals are possible for Loschmidt amplitude in critical systems in a period which is a fraction of the system size. This is quite unexpected because any full revival requires a nearly perfect fine-tuning of phase conditions. To the best of our knowledge, this is the first example of the prediction of full revivals in many-body quantum systems in an accessible time. In this brief article, we make a closer look to this phenomena in microscopic systems. In particular, we show that the full revivals of [1] are impossible in microscopic realizations of those conformal field theories that studied in [1]. In the next section, first, we briefly review the arguments in favor of and against the presence of complete revivals in critical systems. Then in section three, we study the Loschmidt echo (fidelity)

in quantum XY chain and find an exact determinant formula for a particular initial state. In section four, we calculate the fidelity at the critical point of the periodic (open) transverse field Ising chain analytically (numerically), and show that the complete revivals discussed in [1] are absent. In section five, we explore the other parts of the phase diagram of the XY chain. In particular, we study the quasi-particle picture for different post-quench Hamiltonians and show that the picture can determine the periods of the revivals just in some part of the phase diagram. Finally, in the last section, we conclude our paper.

II. REVIVALS IN CONFORMAL FIELD THEORIES

Consider a one-dimensional periodic quantum chain of length L and the Hamiltonian H. Assume an initial state which is very close to a conformally invariant state $|B\rangle$. Since the conformal states are non-normalizable, one needs to introduce a parameter β which is called extrapolation length and then one can write the initial state as $|\psi_0\rangle \approx e^{-\frac{\beta}{4}H}|B\rangle$. The extrapolation length is usually of the order of a few lattice sites, in other words, we have $L \gg \beta$. The parameter β can be estimated by calculating the expectation value of the Hamiltonian as $\langle\psi_0|H|\psi_0\rangle = \frac{\pi c L}{6\beta^2}$, see [1]. To show the complete revival, the reference [1] calculates the fidelity defined as

$$F(t) = |\langle\psi_0|e^{-iHt}|\psi_0\rangle| \tag{1}$$

using the well-established recipe, see for example [14]. Based on his argument, for minimal models at large $\frac{L}{\beta}$, there must be always complete revivals at multiples of $t = M\frac{L}{2}$, where $M \sim \frac{24}{1-c}$. Not surprisingly in the regime $L \gg \beta$, there is no effect of the extrapolation length in

the revival times. Although there is nothing wrong in the CFT calculations in [1], this effect can not be seen in a microscopic quantum chain. There are two good reasons: The first reason, which is already noticed in [14], is related to the presence of the excited states in any global quantum quench which in principle can not be described by CFT. The second reason is that [1] assumes that there is a one to one correspondence between an initial state in a microscopic system and conformal boundary states which in general is not true. There are many discrete initial states, probably exponentially growing with the system size, that flow to the same conformal boundary states either with the same β or different extrapolation lengths. This means that although in a CFT setup the system comes back to itself with probability one, in the discrete model it can be in a state which is completely different but still with the same continuum description. Although in principle, this problem can be resolved by considering all the possible irrelevant perturbations of the CFT, see for example [1, 19] it will eventually affect the complete revivals anyway. The reference [1] discusses, in particular, a quench from the disordered phase in the transverse field Ising chain and shows that there should be complete revivals at times $t = \frac{nL}{2}$ for even n, while for odd n the complete revivals are suppressed. In the next section, we will show that the complete revivals are absent in the critical transverse field Ising chain.

III. LOSCHMIDT ECHO IN QUANTUM XY CHAIN

In this section, we study the revivals in the quantum spin chain when the initial state is the case with all spins σ^z are up or down. The Hamiltonian of XY-chain is as follows:

$$H_{XY} = \tag{2}$$
$$-J\sum_{j=1}^{L}\left[(\frac{1+a}{2})\sigma_j^x\sigma_{j+1}^x + (\frac{1-a}{2})\sigma_j^y\sigma_{j+1}^y\right] - h\sum_{j=1}^{L}\sigma_j^z.$$

Different phases of the model for $J = 1$ are shown in the Figure 1. The line $a = 1$ is the transverse field Ising chain. The $h = 1$ line is critical for all the values of a and we call it XY critical line. On the circle $a^2 + h^2 = 1$, the wave function of the ground state is factorized into a product of single spin states [20].

After using the Jordan-Wigner transformation $c_j^\dagger = \prod_{l<j}\sigma_l^z\sigma_j^+$, which maps the Hilbert space of a quantum chain of a spin 1/2 into the Fock space of spinless fermions, the new Hamiltonian becomes

$$H = J\sum_{j=1}^{L-1}(c_j^\dagger c_{j+1} + ac_j^\dagger c_{j+1}^\dagger + h.c.) - \sum_{j=1}^{L}h(2c_j^\dagger c_j - 1)\tag{3}$$
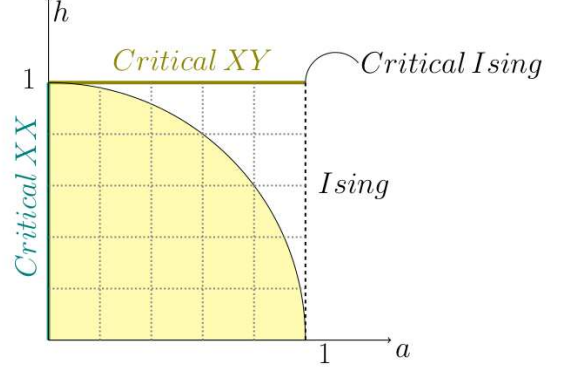$$+\mathcal{N}J(c_L^\dagger c_1 + ac_L^\dagger c_1^\dagger + h.c.),$$

FIG. 1: Color online) Different regions in the phase diagram of the quantum XY chain. The critical XX chain has central charge $c = 1$ and critical XY line has $c = \frac{1}{2}$. The region $a^2 + h^2 < 1$ is depicted with the yellow color.

where $c_{L+1}^\dagger = 0$ and $c_{L+1}^\dagger = \mathcal{N}c_1^\dagger$ for open and periodic boundary conditions respectively with $\mathcal{N} = \prod_{j=1}^{L}\sigma_j^z = \pm 1$. The above Hamiltonian can be written as:

$$\mathbf{H} = \mathbf{c}^\dagger.\mathbf{A}.\mathbf{c} + \frac{1}{2}\mathbf{c}^\dagger.\mathbf{B}.\mathbf{c}^\dagger + \frac{1}{2}\mathbf{c}.\mathbf{B}^T.\mathbf{c} - \frac{1}{2}\text{Tr}\mathbf{A}, \tag{4}$$

with appropriate $\mathbf{A}$ and $\mathbf{B}$ matrices as:

$$\mathbf{A} = \begin{pmatrix} -2h & J & 0 & \dots & \mathcal{N}J \\ J & -2h & J & 0 & 0 \\ 0 & J & -2h & J & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \mathcal{N}J & 0 & \dots & J & -2h \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} 0 & aJ & 0 & \dots & -aJ\mathcal{N} \\ -aJ & 0 & aJ & & 0 \\ 0 & -aJ & 0 & aJ & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ aJ\mathcal{N} & 0 & \dots & -aJ & 0 \end{pmatrix}. \tag{5}$$

To calculate the Loschmidt echo, first we decompose $e^{-i\mathbf{H}t}$ using the Balian-Brezin formula [21] as:

$$e^{-i\mathbf{H}t} = e^{\frac{1}{2}\mathbf{c}^\dagger\mathbf{X}\mathbf{c}^\dagger}e^{\mathbf{c}^\dagger\mathbf{Y}\mathbf{c}}e^{-\frac{1}{2}\text{Tr}\mathbf{Y}}e^{\frac{1}{2}\mathbf{c}\mathbf{Z}\mathbf{c}}, \tag{6}$$

where $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ can be calculated from the blocks of matrix $\mathbf{T}$ defined as

$$\mathbf{T} = e^{-it\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{-B} & \mathbf{-A} \end{pmatrix}} = \begin{pmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{pmatrix}, \tag{7}$$

Then we have

$$\mathbf{X} = \mathbf{T}_{12}(\mathbf{T}_{22}^{-1}), \quad \mathbf{Z} = (\mathbf{T}_{22}^{-1})\mathbf{T}_{21}, \quad e^{\mathbf{-Y}} = \mathbf{T}_{22}^T. \tag{8}$$

Note that we always have $\mathbf{Z} = -\mathbf{X}$. Finally, the fidelity for the desired initial state (all the spins up) will be:

$$F(t) = |\langle 0|e^{-i\mathbf{H}t}|0\rangle| = |\det(\mathbf{T}_{22})|^{\frac{1}{2}}. \qquad (9)$$

The same formula is also valid for the case when the initial state is all the spins down, in other words when all the sites are filled with fermions $|1\rangle$. In the next subsections, we will use the above formula to study the revivals in different phases of the quantum spin chain. Note that although we will keep the J coupling explicitly in some of the formulas for numerical calculations we always take $J = 1$.

IV. REVIVALS IN CRITICAL TRANSVERSE FIELD ISING CHAIN

In this section, first we calculate the fidelity for the periodic critical Ising point exactly, then we will study the open chain numerically.

A. Periodic critical transverse field Ising chain

Consider a periodic critical Ising model Hamiltonian in Ramond sector after Jordan-Wigner transformation, equation (3) with $\mathcal{N} = J = a = h = 1$. Since in this case the matrices $\mathbf{A}$ and $\mathbf{B}$ commute, one can calculate the fidelity exactly. Note that in this case the matrix $\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{-B} & \mathbf{-A} \end{pmatrix}$ is a circulant matrix which gaurranties the exact calculation of the eigenvalues of the matrix $\mathbf{T}_{22}$ with classical methods. After expanding (7) we have

$$\mathbf{T}_{22} = \mathbf{T}_{11}^* = \cosh(2t\sqrt{\mathbf{A}}) + \frac{i\sqrt{A}}{2}\sinh(2t\sqrt{\mathbf{A}}), (10)$$

$$\mathbf{T}_{12} = -\mathbf{T}_{21} = -it\mathbf{B}[\frac{\sinh(2t\sqrt{A})}{2t\sqrt{A}}]. \qquad (11)$$

Although it is not needed for our future discussion, we also report the exact form of the matrices $\mathbf{X}$:

$$\mathbf{X} = \frac{-i\mathbf{B}}{2\sqrt{\mathbf{A}}\coth[2t\sqrt{\mathbf{A}}] + i\mathbf{A}}, \qquad (12)$$

Since the eigenvalues of the matrix $\mathbf{A}$ are $\lambda_j = -2 + 2\cos\frac{2\pi j}{L}$, where $j = 1, 2, ..., L$; logarithmic fidelity for the critical periodic Ising chain can be written explicitly as

$$\ln[F(t)] = \frac{1}{4}\sum_{j=0}^{L-1}\ln[1 - \cos^2[\frac{\pi j}{L}]\sin^2[4t\sin[\frac{\pi j}{L}]]]. \qquad (13)$$

In Figure 3, one can see that although there are partial revivals at multiples of $t = \frac{L}{4}$, which can be understood with the quasi-particle picture, the complete revivals do not happen. Of course if one waits enough

time, there will be always almost complete revivals but they are usually expected to happen in much larger times that are usually inaccessible. Note that for the considered initial state, we have $\langle\psi_0|H|\psi_0\rangle = L$ which means that $\beta = \sqrt{\frac{\pi}{12}}$ or in other words we are in a regime that $\frac{L}{j}$ is very large which is well inside the regime considered
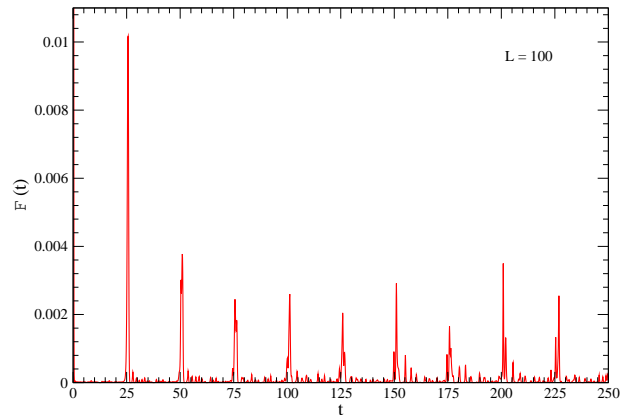

FIG. 2: (Color online) logarithmic fidelity for the periodic critical Ising chain starting from the all aligned spins σ^z initial state.

B. Open critical transverse field Ising chain

In this subsection, we repeat the analyses of the previous section for the open chains to see the effect of the boundary condition on the revivals. Unfortunately, we were not able to provide an exact result in this case so the calculations are based on a numerical evaluation of the determinant in the equation (9). The main reason for our failure at the critical Ising point can be traced back to this fact that in this case the two matrices $\mathbf{A}$ and $\mathbf{B}$ do not commute and so the expansion method gets too complicated after few steps. Also note that in this case the matrix $\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{-B} & \mathbf{-A} \end{pmatrix}$ is not a circulant matrix and so the common methods of diagonalization can not be applied. The numerical results depicted in Figure 3 confirms the absence of the complete revivals introduced in [1] and the usefulness of the quasi-particle picture. We will come back to a more detailed study of the quasi-particle picture in the next section.

V. REVIVALS AND QUASI-PARTICLE PICTURE

In this section, we extend the analyses of the previous section to the other parts of the phase diagram of the XY chain. In addition, we also examine the applicability of the quasi-particle picture in determining the periods of
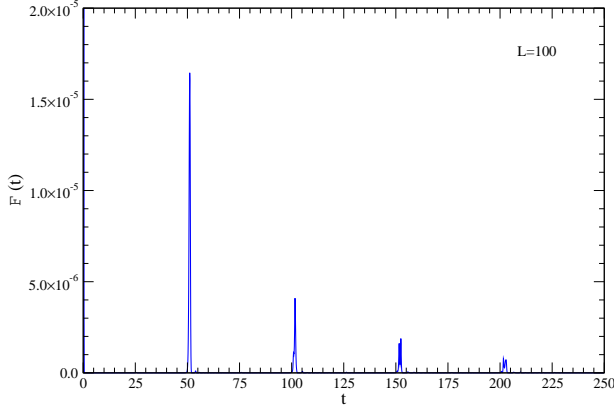
FIG. 3: (Color online) logarithmic fidelity for the open critical Ising chain starting from the all aligned spins σ^z initial state.

the revivals in the Loschmidt echo. First, we make a brief comment on the quasi-particle picture, see [22]. Based on this semi-classical picture the pre-quench state has more energy than the post-quench Hamiltonian ground state and so consequently, the initial state plays the role of a source of quasi-particles. The quasi-particles with the maximum group velocity usually are the ones that can be connected to the saturation of the entanglement entropy [23] or the revivals in the Loschmidt echo [14]. The dispersion relation and the group velocity of the Hamiltonian (3) are

$$\epsilon_k = J\sqrt{(\cos\phi_k - h)^2 + a^2\sin^2\phi_k}, \qquad (14)$$

$$v_g = J\sin\phi_k \frac{2a^2\cos\phi_k - \cos\phi_k + h}{\sqrt{(\cos\phi_k - h)^2 + a^2\sin^2\phi_k}}. \qquad (15)$$

where $\phi_k = \frac{2\pi}{L}(k + \frac{\mathcal{N}-1}{4})$ with $k = 0, ..., L-1$. In Figure 4, we depicted v_g for different values of a and h. Note that for sufficiently large L, there is no significant difference between open and periodic cases. Using the above formula one can derive the maximum group velocity v_g^{max} for different values of a and h, see Figure 5. Having these group velocities for quasi-particles, one can guess the following periods for the appearance of revivals in the Periodic and Open quantum chains:

$$T_p = \frac{L}{2v_g}, \qquad T_o = \frac{L}{v_g}, \qquad (16)$$

where v_g is usually the velocity of the fastest quasi-particles v_g^{max}. However, this is not a rule and sometimes other quasi-particles can carry more information than the fastest quasi-particles as it was discussed in the context of entanglement entropy in [24] and in the context of Loschmidt echo in [14]. In those cases v_g in the equation (16) will be different from v_g^{max}. We are not aware of a criterion which one can use apriory to decide what is the most important group velocity. In the next three subsections, we study the revivals and the quasi-particle picture in different regimes.
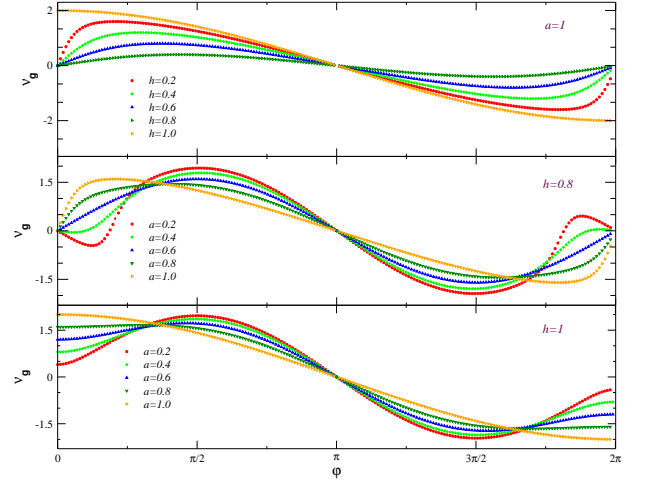

FIG. 4: (Color online) Group velocity v_ϕ with respect to ϕ for different values of a and h.
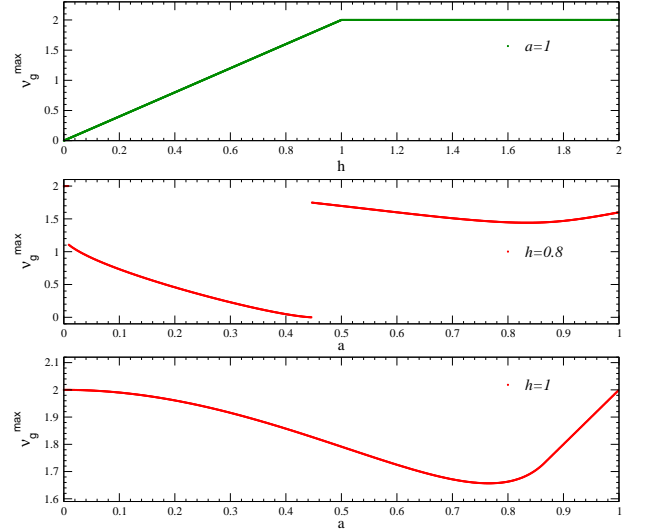

FIG. 5: (Color online) Maximum group velocity for different values of a and h.

A. Ising line: $a = 1$

In Figure 6, we depicted the Loschmidt echo for different values of h. Two comments are in order: first of all, at non-critical points similar to the critical point we have partial revivals. Apart from the period of the revivals, there is no significant difference in the form of the Loschmidt echo at and outside of the critical point. Secondly, the period of the revivals can be understood by taking the maximum group velocity $v_g^{max} = 2h$ as the relevant velocity.
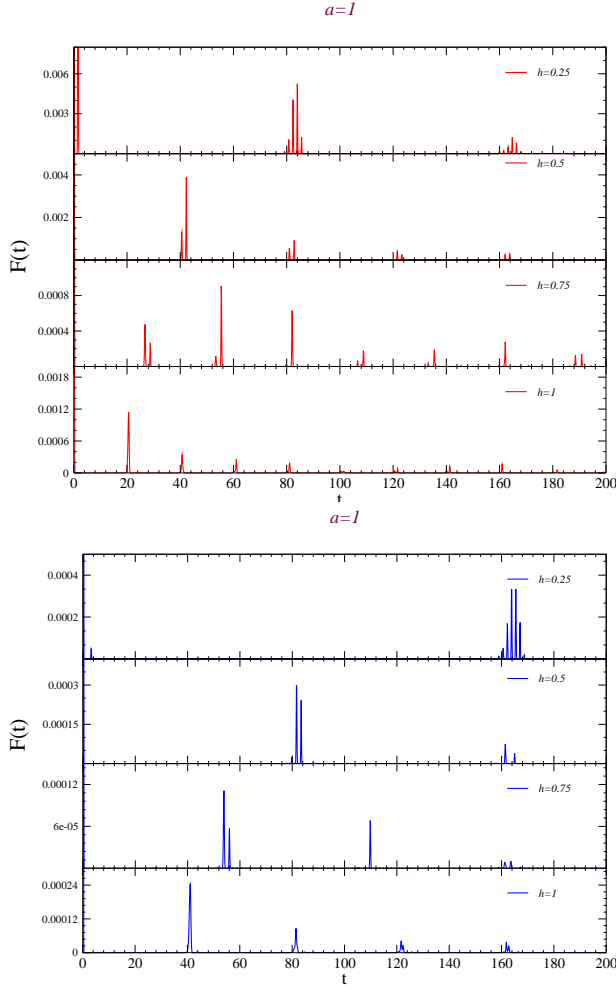
FIG. 6: (Color online) Loschmidt echo on the Ising line for periodic and open chains in top and bottom panel respectively($L = 80$). The revivals are compatible with quasi-particle picture with maximum group velocity $v_g^{max} = 2h$.

FIG. 7: (Color online) Loschmidt echo on the XY critical line for periodic and open chains in top and bottom panel respectively($L = 80$). The revivals are compatible with quasi-particle picture with the Fermi group velocity $v_g^f = 2a$.

B. Critical XY line: $h = 1$

This is a critical line which it is in the same universality class as the critical Ising chain. The results for the Loschmidt echo on different points are shown in Figure 7. The interesting fact is that in this case, the relevant group velocity is clearly $|v_g^f| = 2a$ which is the Fermi velocity. As it was already discussed in the context of the Loschmidt echo after local quenches in [14], it is not the maximum group velocity for $a < \frac{\sqrt{3}}{2}$. This is an interesting example of a case which the relevant group velocity is different from the maximum group velocity.

C. Non-critical regime: $0 < h, a < 1$

Based on the results of the above two subsections one might be tempted to guess that since this region is non-critical, the quasi-particle picture with the maximum
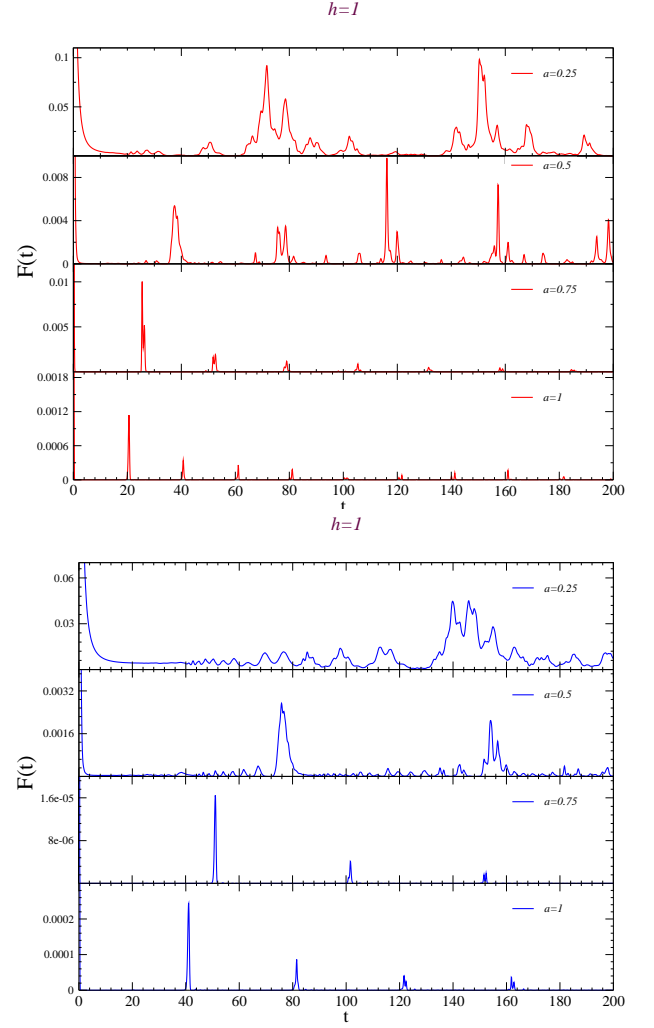
group velocity should be appropriate to guess the form of the partial revivals. However, strikingly as one can see in Figure 8, there are two completely different regimes with very different behaviors. In the region $a^2 + h^2 > 1$ the quasi-particle picture with the maximum group velocity works perfectly, however, in the region $a^2 + h^2 \leq 1$ there seems to be no clean way to attribute the revivals to quasi-particles with fixed velocities. One might understand it as a regime that there are more than one type of important quasi-particles which their velocity difference kill clean periodic revivals. It is absolutely unclear why the line $a^2 + h^2 = 1$ should separate these two regimes. To keep the figure simple, we have only depicted four different points, however we have checked many other different points and confirmed numerically that indeed this line is at the border between the two different regimes.

VI. CONCLUSIONS

In this paper, we studied revivals in the XY chain starting from an initial state with all the spins σ^z in the direction of the transverse field. Our conclusions are the followings: first of all, we proved that complete revivals in the times introduced by [1], cannot happen in the microscopic quantum critical chains. Secondly, for the considered initial state we showed that there are three interesting regimes. For $a^2 + h^2 > 1$, one can use the quasi-particle picture to predict the period of the partial revivals. On the critical XY line $h = 1$, one must use the Fermi velocity $v_g^f = 2a$ to calculate the revivals.

However, for the other points, the maximum group velocity v_g^{max} is the important group velocity. For the region $a^2 + h^2 \leq 1$, the revivals do not follow a clean periodic structure which indicates the presence of more than one type of important quasi-particles. It will be interesting to study the structure of revivals in also other models, especially interacting models such as the XXZ chain. Loschmidt echo in the non-critical phase of the XXZ chain has been already studied in [25, 26], however, it seems that the problem of revivals in the critical regimes of interacting models has not been studied in full detail sofar. In particular, it is very important to study the effect of the initial state in these models.
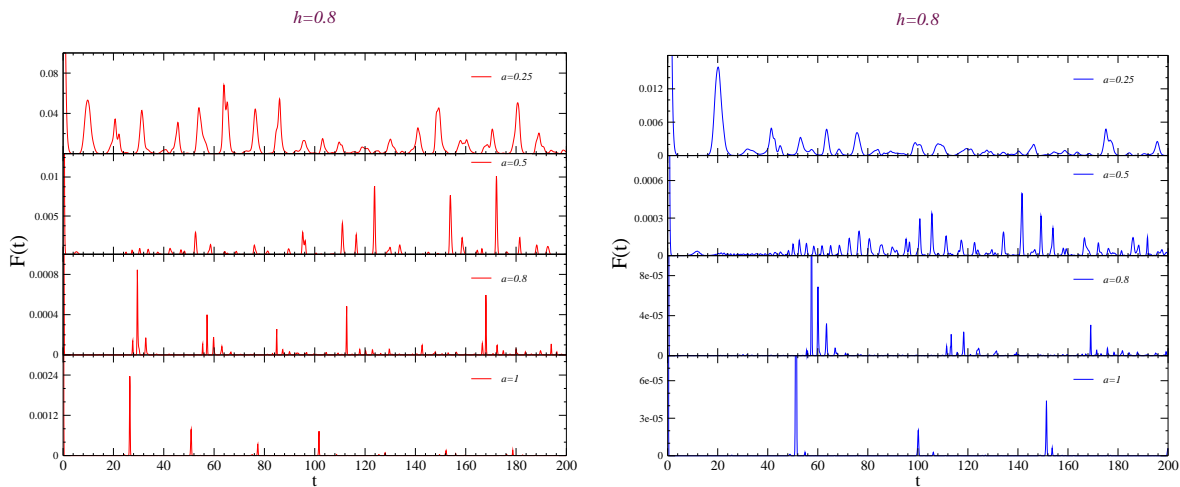

FIG. 8: (Color online) Loschmidt echo on the line $h = 0.8$ for periodic and open chains in left and right panel respectively($L = 80$). The revivals are compatible with quasi-particle picture with maximum group velocity (see Figure 5) as far as $a^2 + h^2 > 1$. For the region $a^2 + h^2 \leq 1$ there are no clean periodic revivals.

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