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Wilson operator algebras and ground states of the coupled BF theories

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The multi-flavor BF theories in (3+1) dimensions with cubic or quartic coupling are the simplest topological quantum field theories that can describe fractional braiding statistics between loop-like topological excitations (three-loop or four-loop braiding statistics). In this paper, by canonically quantizing these theories, we study the algebra of Wilson loop and Wilson surface operators, and multiplets of ground states on three torus. In particular, by quantizing these coupled BF theories on the three-torus, we explicitly calculate the \mathcal{S} - and \mathcal{T} -matrices, which encode fractional braiding statistics and topological spin of loop-like excitations, respectively. In the coupled BF theories with cubic and quartic coupling, the Hopf link and Borromean ring of loop excitations, together with point-like excitations, form composite particles.

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I. INTRODUCTION

For more than three decades exotic quantum phases of matter have been extensively studied in condensed matter physics. In particular gapped systems with non-trivial topological order have been of much interest.¹ Topologically ordered phases have properties such as fractional statistics, long-range entanglement, ground-state degeneracy on manifolds with non-trivial topology, and symmetry fractionalization, etc.²⁻⁷ Canonical examples are fractional quantum Hall states in 2+1 dimensions, which have been observed experimentally.

At long wavelengths, topologically ordered phases of matter can be described by topological quantum field theories (TQFTs), for which all correlation functions are topological, i.e., metric independent. For example, many fractional quantum Hall states, as well as simple lattice models such as the Kitaev toric code model,⁸⁻¹⁰ can be described by the Chern-Simons topological quantum field theories. For these examples, fractional braiding statistics between quasiparticles is described in terms of Wilson lines (loops) in the TQFTs, i.e., by the correlation functions of Wilson loops forming a Hopf link in the 2+1-dimensional spacetime.¹¹

The idea of fractional braiding statistics can be generalized to 3 + 1 dimensions. Since particles cannot braid in three spatial dimensions or equivalently, their world-lines cannot link in 3 + 1-dimensions, the simplest

kind of braiding is between point-like and loop-like excitations, which can have non-trivial fractional braiding statistics. This is described by the BF topological field theory and has been studied quite well.^{12–16} Topological phases in $3+1$ -dimensions, however, have richer possibilities in terms of the kind of braiding processes that can exist.^{17–22}

In this work we explore a subset of such processes by using $(3+1)$ -dimensional TQFTs. In particular, we study TQFTs which can be thought of as extensions of the ordinary BF theory. We mainly study two kinds of extensions: The first is the BF theory with a cubic deformation. More precisely, we consider multiple (two or three) copies of the BF theory coupled together via a cubic term. These theories realize non-trivial statistics between three loop excitations whose spacetime world surfaces are linked together, i.e., the so-called three-loop braiding statistics. The second is four (or more) copies of BF theories coupled via quartic terms. These field theories describe four-loop braiding statistics. Similar TQFTs with cubic and quartic coupling terms have been discussed recently in the literature.^{21,23} The coupled BF theories with cubic or quartic coupling can also be obtained by functionally bosonizing (or gauging) bosonic symmetry protected phase (SPT) described in Ref. 24 and 25.

In addition to these TQFTs with a cubic or quartic coupling, we also discuss avatars of these coupled topological field theories which are quadratic but with modified coupling to external currents. We quantize these quadratic theories on the spatial three torus and discuss the algebra of Wilson operators which encodes the topological data, i.e braiding statistics.

A salient feature of topological field theories is bulk boundary correspondence wherein ground states in the bulk Hilbert space are in one-to-one correspondence with twisted partition functions defined for the boundary field theory. In our previous work,²⁶ we studied the two-copies of BF theories coupled by a cubic term, but focused on the gapless surface theory and the boundary-bulk correspondence: We quantized the surface theory and explicitly calculated the partition functions under various twisted boundary conditions. In addition, by performing large diffeomorphism transformations or modular transformations on the twisted partition functions, we extracted the bulk braiding data directly from the gapless surface theory. (As a related work, see Ref. 27 for the bulk-boundary correspondence for gapped topologically ordered surface states.) In this work we study such TQFTs describing three-loop and four-loop braiding in more detail. In particular, we will study various “bulk” properties of these TQFTs, and hence provide a complementary perspective to our previous work.

A. Summary and outline

The summary of our main results, as well as the outline of the paper, is as follows.

Section II is devoted to the coupled BF theories realizing non-trivial three-loop braiding statistics. In Sec. II A and Sec. II B, we introduce these coupled BF theories, and give an overview of their basic properties. In particular, at the classical level, one can read off from the equations of motion that Hopf links play particle-like roles in these two theories. This braiding structure is encoded in the algebra of the dynamical gauge fields in these theories.

In the following Sections II C, II D, and II E, we quantized the quadratic BF theories introduced in Sec. II B, which differ from the ordinary BF theory due to their modified coupling to the quasi-vortex current. The quadratic theory has the same equations of motion as the cubic theories. Moreover, the Wilson operator algebra of the quadratic theories encodes the three-loop braiding statistics. More specifically the commutator, and triple commutator between the respective Wilson operators are relevant to the respective particle-loop, and the three-loop braiding phases (Sec. II C).

Further we quantize the quadratic three-loop braiding field theory on a spatial three torus in Sections II D and II E. We construct the multiplet of ground states of the two (or three) copies of the BF theories at level K put on spatial three torus T^3 , by directly constructing representations of the Wilson operator algebra. The ground state degeneracy is K^2 (or K^3). In Appendix B, an alternative construction of the ground state multiplet by using geometric quantization is given. Furthermore, by calculating various overlaps between ground states, we explicitly compute the modular \mathcal{S} and \mathcal{T} matrices and extract particle-loop and three-loop braiding phases from them. These agree with the braiding phases computed in our previous work from the surface theory,²⁶ as well as with previous bulk calculations in the literature.^{17–20,26}

Much of what is discussed in Sec. II carries over to Sec. III, in which we discuss the coupled BF theories realizing non-trivial four-loop braiding statistics. In these theories, the role played by Hopf links in three-loop braiding theories is played by Borromean rings of loop-like excitations. The role of the triple is replaced by the quadruple commutator of the Wilson operators. This carries information about four loop braiding.

Finally in Sec. IV, we propose condensation mechanisms by which topological field theories describing three-loop and four-loop braiding statistics may arise at long wavelengths. It is known that the simplest continuum topological field theory in $3+1$ dimensions, i.e., the BF theory at level K , describes the deconfined phase of the \mathbb{Z}_K gauge theory. This may arise from a parent (ultraviolet) $U(1)$ gauge theory, if the $U(1)$ gauge symmetry is Higgsed to \mathbb{Z}_K by the abelian Higgs mechanism. Alternatively the BF theory may arise as a result of the magnetic condensation via the Julia-Toulouse mechanism. In

Sec. IV, we discuss how the coupled BF theories realizing three- or four-loop braiding statistics may arise from ultraviolet theories by condensation of some sort. By condensing a composite of electric charge and a Hopf link between $U(1)$ field lines, it can be shown that the long wavelength effective field theory is a topological field theory that describes three-loop braiding. Alternately by condensing a composite of electric charge and a Borromean ring between $U(1)$ field lines, it can be shown that the effective field theory is a topological field theory that describes four-loop braiding.

We conclude in Sec. V with a few words on open issues.

II. THREE-LOOP BRAIDING THEORY

A. The cubic theories

In our previous work,²⁶ we analyzed the coupled BF theory defined by the following action:

$$S = \int_{\mathcal{M}} \left[\frac{K}{2\pi} \delta_{IJ} b^I \wedge da^J + \frac{p_1}{4\pi^2} a^1 \wedge a^2 \wedge da^2 + \frac{p_2}{4\pi^2} a^2 \wedge a^1 \wedge da^1 - \delta_{IJ} b^I \wedge J_{qv}^J - \delta_{IJ} a^I \wedge J_{qp}^J \right], \quad (1)$$

where a^I and b^I are one- and two-form gauge fields, respectively; $I, J = 1, 2$; \mathcal{M} is the (3+1)-dimensional spacetime manifold, and we will mostly assume $\mathcal{M} = \Sigma \times \mathbb{R}$ where Σ/\mathbb{R} is a spatial/temporal part of the manifold. K and $p_{1,2}$ are the parameters of the theory; The “level” K is an integer, whereas $p_{1,2}$ are an integer multiple of K and are given by

$$p_1 = q_1 K, \quad p_2 = q_2 K, \quad q_{1,2} = 0, \dots, K-1. \quad (2)$$

Finally, the three-form J_{qp} and two-form J_{qv} represent quasi-particle and quasi-vortex (loop-like) currents, which are treated as a non-dynamical background. For a quasi-particle whose world line is given by $\mathcal{C} \subset \mathcal{M}$, and for a quasi-vortex whose world surface is given by $\mathcal{S} \subset \mathcal{M}$, J_{qp} and J_{qv} are given as

$$J_{qp} = \delta(\mathcal{C}), \quad J_{qv} = \delta(\mathcal{S}), \quad (3)$$

respectively, where the delta function forms $\delta(\mathcal{C})$ and $\delta(\mathcal{S})$ are defined such that $\int_{\mathcal{M}} \delta(\mathcal{C}) \wedge a = \int_{\mathcal{C}} a$ and $\int_{\mathcal{M}} \delta(\mathcal{S}) \wedge b = \int_{\mathcal{S}} b$ for arbitrary one- and two-form a and b , respectively. (For properties of the delta function forms, see Ref. 26.)

The action (1) describes topological gauge theories of various kinds with gauge group $G = \mathbb{Z}_K \times \mathbb{Z}_K$. Following the seminal work of Dijkgraaf and Witten,²⁸ we know that topological gauge theories in $d+1$ -dimensions with a discrete gauge group G are classified by the group cohomology $H^{d+1}(G, U(1))$. Since $H^4(\mathbb{Z}_K \times \mathbb{Z}_K, U(1)) = \mathbb{Z}_K \times \mathbb{Z}_K$, we expect there are K^2 distinct theories. Within

the coupled BF theory (1), these are parametrized by $p_{1,2}$ (or equivalently $q_{1,2}$).

For later use, we record the equations of motion derived from (1):

$$\begin{aligned} \frac{K}{2\pi} da^I &= J_{qv}^I, \\ \frac{K}{2\pi} db^I &= -\frac{p_I}{4\pi^2} a^{\bar{I}} \wedge da^{\bar{I}} + \frac{p_{\bar{I}}}{2\pi^2} a^{\bar{I}} \wedge da^I \\ &\quad - \frac{p_{\bar{I}}}{4\pi^2} da^{\bar{I}} \wedge a^I + J_{qp}^I, \end{aligned} \quad (4)$$

where we introduced the notation $\bar{1} = 2$ and $\bar{2} = 1$, and the repeated capital Roman indices are not summed over here.

In addition to the two flavors of BF theories (1), we will also discuss three flavors of BF theories and couple them by introducing a cubic term. This leads to the action

$$S = \int_{\mathcal{M}} \left[\frac{K}{2\pi} \delta_{IJ} b^I \wedge da^J + p a^1 \wedge a^2 \wedge da^3 - \delta_{IJ} b^I \wedge J_{qv}^J - \delta_{IJ} a^I \wedge J_{qp}^J \right], \quad (5)$$

where the flavor indices I, J run over 1, 2, 3. As before, K and p are the parameters of the theory. This three-flavor theory shares similar properties as the two-flavor theory (1), and can be discussed in parallel with the two-flavor theory. In particular, both two-flavor and three-flavor theories realize non-trivial three-loop braiding statistics.

1. Gauge invariance

Let us now discuss the gauge symmetries of the theory (1). (We will focus on infinitesimal or small gauge transformations here; we will discuss large gauge transformations in detail later.) We first switch off the coupling to currents J_{qp} and J_{qv} . The action (1) is invariant under

$$\begin{aligned} b^I &\rightarrow b'^I = b^I - \frac{p_{\bar{I}}}{2\pi K} (a^{\bar{I}} \wedge d\varphi^I + d\varphi^{\bar{I}} \wedge a^I), \\ a^I &\rightarrow a'^I = a^I + d\varphi^I, \end{aligned} \quad (6)$$

where φ^I is a scalar. This transformation is a generalization of the usual 1-form gauge symmetry that the ordinary BF theory has. As in the ordinary BF theory, the action (1) is invariant under an additional 2-form gauge symmetry

$$b^I \rightarrow b'^I + d\zeta^I \quad (7)$$

where ζ^I is a one-form. Formally, these transformations can be read off by identifying the operators that generate the Gauss law constraints.

Naively it seems that the coupling to sources in Eq. (1) is not gauge invariant. Upon gauge transformation,

the source terms transform as

$$\begin{aligned}
& \delta_{IJ} a^I \wedge J_{qp}^J + \delta_{IJ} b^I \wedge J_{qv}^J \\
& \longrightarrow \delta_{IJ} a^I \wedge J_{qp}^J + \delta_{IJ} b^I \wedge J_{qv}^J \\
& + d\varphi^1 \wedge \left[J_{qp}^1 + \frac{P_2}{K^2} d^{-1} J_{qv}^2 \wedge J_{qv}^1 - \frac{P_1}{K^2} d^{-1} J_{qv}^2 \wedge J_{qv}^2 \right] \\
& + d\varphi^2 \wedge \left[J_{qp}^2 + \frac{P_1}{K^2} d^{-1} J_{qv}^1 \wedge J_{qv}^2 - \frac{P_2}{K^2} d^{-1} J_{qv}^1 \wedge J_{qv}^1 \right] \\
& + \delta_{IJ} d\zeta^I \wedge J_{qv}^J, \tag{8}
\end{aligned}$$

where we have used the equation of motion (4) to write $a^I = (2\pi/K)(d^{-1}J_{qv}^I)$. Demanding the gauge invariance, we can read off the conservation law of currents,

$$\begin{aligned}
d \left[J_{qp}^I + \frac{P_1}{K^2} d^{-1} J_{qv}^I \wedge J_{qv}^I - \frac{P_2}{K^2} d^{-1} J_{qv}^I \wedge J_{qv}^I \right] &= 0, \\
dJ_{qv}^I &= 0. \tag{9}
\end{aligned}$$

Here, for static configuration of currents J_{qv}^I , $d^{-1}J_{qv}^I \wedge J_{qv}^I$, once integrated over space, is the Hopf linking number,

$$\text{Hopf}(J_{qv}^I, J_{qv}^I) = \int_{\Sigma} (d^{-1}J_{qv}^I) \wedge J_{qv}^I, \tag{10}$$

in the spatial manifold Σ . Thus, the composite of the particle current and Hopf linking number current is conserved. This suggests that the Hopf linking number can be treated effectively as a quasiparticle of some sort (Fig. 1).

This point of view also played a crucial role in our previous work, Ref. 26. Integrating over the equation of motion (4) over the spatial manifold Σ , again by using $a^I = (2\pi/K)(d^{-1}J_{qv}^I)$, we obtain

$$\begin{aligned}
\frac{K}{2\pi} \int_{\Sigma} db^I &= -\frac{PI}{K^2} \int_{\Sigma} (d^{-1}J_{qv}^I) \wedge J_{qv}^I \\
&+ \frac{P_1}{K^2} \int_{\Sigma} (d^{-1}J_{qv}^I) \wedge J_{qv}^I + \int_{\Sigma} J_{qp}^I, \tag{11}
\end{aligned}$$

where note that in the static configurations considered here, J_{qv} is a delta function two form supporting a spatial loop, whereas J_{qp} is a delta function three form supporting a spatial point. Correspondingly, $d^{-1}J_{qv}$ is a delta function one form supporting a three dimensional manifold. The contributions to the flux $\int_{\Sigma} db^I$ coming from quasivortex loops, $\int_{\Sigma} (d^{-1}J_{qv}^I) \wedge J_{qv}^I$, are given in terms of their Hopf linking number. By using the Stokes theorem, Eq. (11) can be used to link the twisted partition functions on the boundary and the quantum numbers in the bulk, and hence to establish the bulk-boundary correspondence.



FIG. 1. Hopf links (a) and Borromean rings (b) as an effective quasiparticle. The red dot represents an “ordinary” point-like quasiparticle.

B. The quadratic theory

In Ref. 26, an alternative to the cubic theory (1), the quadratic theory, is proposed:

$$\begin{aligned}
S &= \frac{K}{2\pi} \int \delta_{IJ} b^I \wedge da^I - \int \delta_{IJ} a^I \wedge J_{qp}^I \\
&- \int \left[b^1 + \frac{P_2}{2\pi K} a^1 \wedge a^2 \right] \wedge J_{qv}^1 \\
&- \int \left[b^2 + \frac{P_1}{2\pi K} a^2 \wedge a^1 \right] \wedge J_{qv}^2. \tag{12}
\end{aligned}$$

Comparing the cubic and quadratic theories, in the cubic theory, the canonical commutation relations differ from the ordinary BF theory, while they remain the same in the quadratic theory. On the other hand, the set of Wilson loop and surface operators in the cubic theory is conventional (i.e., identical to the ordinary BF theory) while it is modified in the quadratic theory, as seen from the coupling to J_{qv}^I (see below in Eq. (17)). In spite of these differences, the algebra of Wilson loop and surface operators of the two theories appear to be identical. We will use the cubic and quadratic theories somewhat interchangeably; When discussing the Wilson operator algebra and ground state wave functions (functionals), we will use the quadratic theories, while when discussing the condensation picture, we will use the cubic theory.

1. Gauge invariance

One can derive the infinitesimal gauge transformations from the source-free part of the action (12). Since in this case the theory is identical to the ordinary BF theory, there are two conserved charges $(K/2\pi)db^I$ and $(K/2\pi)da^I$. These are 3-form density-like and 2-form vorticity-like charge operators, respectively. The gauge transformations are generated by these charge operators and are given by

$$a^I \rightarrow a^I + d\varphi^I, \quad b^I \rightarrow b^I + d\zeta^I. \tag{13}$$

Similar to the cubic theory discussed earlier, demanding the invariance under (13), one can read off the conservation law of current, which is identical to (9).

C. Three-loop braiding statistics

To see the three-loop braiding statistics, we need to quantize the coupled BF theory (either the cubic theory or its quadratic avatar). In this section, we consider the coupled BF theory on topologically trivial spacetimes, e.g., $\Sigma = \mathbb{R}^3$, $\mathcal{M} = \mathbb{R}^3 \times \mathbb{R}$, and study the properties of the Wilson loop and Wilson surface operators. In the next section, we put the coupled BF theory on the spatial manifold with non-trivial topology, the three torus, $\Sigma = T^3$.

As one of the simplest and quickest way to see the three-loop braiding statistics, let us start by integrating over a^I and b^I , on both cubic and quadratic theories. One then obtains the effective action of the currents

$$\mathcal{Z}[J_{qp}^I, J_{qv}^J] = e^{iS_{eff}[J_{qp}^I, J_{qv}^J]} = \int \mathcal{D}[a^I, b^I] e^{iS} \quad (14)$$

where

$$\begin{aligned} S_{eff} = & -\frac{2\pi}{K} \int (d^{-1} J_{qv}^I) \wedge J_{qp}^I \\ & + \frac{2\pi p_1}{K^3} \int (d^{-1} J_{qv}^1) \wedge (d^{-1} J_{qv}^2) \wedge J_{qv}^2 \\ & + \frac{2\pi p_2}{K^3} \int (d^{-1} J_{qv}^2) \wedge (d^{-1} J_{qv}^1) \wedge J_{qv}^1. \end{aligned} \quad (15)$$

The first term in the effective action describes, as in the ordinary BF theory, the quasiparticle-quasivortex braiding statistics. It is given in terms of the linking number of

$$\text{Link}(J_{qv}^I, J_{qp}^J) = \int_{\mathcal{M}} (d^{-1} J_{qv}^I) \wedge J_{qp}^J, \quad (16)$$

in the spacetime \mathcal{M} . On the other hand, the second and third terms include topological linking among three quasivortex loops, i.e., three-loop braiding statistics.

The three-loop braiding statistics can also be discussed by quantizing the theory and using the Wilson loop and Wilson surface operators. Let us now take the quadratic theory (12). From the coupling to the currents, we read off the Wilson loop and Wilson surface operators in the theory:

$$A_L^I := \exp \left[i \int_L a^I \right], \quad W_S^I := \exp \left[i \int_S \Lambda^I \right], \quad (17)$$

where L and S are arbitrary closed loop and surfaces in the spatial manifold Σ , respectively, and

$$\Lambda^I := b^I + \frac{q_I}{2\pi} a^I \wedge a^{\bar{I}}. \quad (18)$$

The commutation relations between these Wilson operators can be computed from the canonical commutation relation

$$[a_i^I(x), b^{Jj}(y)] = \frac{2\pi i}{K} \delta^{IJ} \delta_i^j \delta^{(3)}(x-y) \quad (19)$$

where $a^I = a_i^I dx^i$, $b^I = (1/2)b_{ij}^I dx^i \wedge dx^j$, and $b^{Ii} := (1/2)\epsilon^{ijk} b_{jk}^I$. (We have adopted the temporal gauge $a_0^I = b_{0i}^I = 0$.) The exponents of the Wilson operators satisfy

$$\begin{aligned} \left[\int_C a^I, \int_S \Lambda^J \right] &= \frac{2\pi i}{K} \delta^{IJ} I(C, S), \\ \left[\int_S \Lambda^I, \int_{S'} \Lambda^J \right] &= \frac{2i}{K^2} \int_{S\#S'} (p_J \delta^{IJ} a^J - p_J \delta^{IJ} a^{\bar{J}}) \end{aligned} \quad (20)$$

and as before the repeated capital Roman indices are not summed over. Here,

$$I(C, S) = \int_{\Sigma} \delta(C) \wedge \delta(S) \quad (21)$$

is the intersection number between C and S , and $S\#S'$ is the intersection of S and S' . Note that (20) is gauge invariant as $S\#S'$ is a contractible 1-cycle.

The three-loop braiding statistic is encoded in the following product of Wilson operators²⁹

$$\begin{aligned} & (\hat{W}_{S'}^{J\dagger} \hat{W}_S^{I\dagger} \hat{W}_{S'}^J \hat{W}_S^I) \hat{W}_{S''}^{K\dagger} (\hat{W}_S^{I\dagger} \hat{W}_{S'}^{J\dagger} \hat{W}_S^I \hat{W}_{S'}^J) \hat{W}_{S''}^K \\ & = \exp \left([[i \int_S \hat{\Lambda}^I, i \int_{S'} \hat{\Lambda}^J], i \int_{S''} \hat{\Lambda}^K] \right) \end{aligned} \quad (22)$$

where the triple commutator is given by

$$\begin{aligned} & \left[\left[\int_S \Lambda^I, \int_{S'} \Lambda^J \right], \int_{S''} \Lambda^K \right] \\ & = \frac{4\pi p_{\bar{J}}}{K^3} (\delta^{IJ} \delta^{\bar{J}K} - \delta^{I\bar{J}} \delta^{JK}) I(S\#S', S''). \end{aligned} \quad (23)$$

Physically, this product of Wilson operators braids loop I with loop J while both I and J are linked with ‘background’ loop K . Notice that the triple commutator satisfies the Jacobi identity:

$$\begin{aligned} & [[\int_S \hat{\Lambda}^I, \int_{S'} \hat{\Lambda}^J], \int_{S''} \hat{\Lambda}^K] + [[\int_{S'} \hat{\Lambda}^J, \int_{S''} \hat{\Lambda}^K], \int_S \hat{\Lambda}^I] \\ & + [[\int_{S''} \hat{\Lambda}^K, \int_S \hat{\Lambda}^I], \int_{S'} \hat{\Lambda}^J] = 0. \end{aligned} \quad (24)$$

This is equivalent to the cyclic relation for the three-loop braiding phase first derived by Wang and Levin in Ref. 30. An exactly solvable model was provided in Ref. 31

D. Quantization on a closed spatial manifold

In Sec. IIC, the coupled BF theory on topologically trivial spacetime is studied in the presence of background quasiparticle and quasivortex currents. In this section, we quantize the coupled BF theory on a spacetime manifold wherein its spatial part Σ is topologically non-trivial. (Our setting closely parallels with Ref. 15.) In particular, we will focus on Σ which is formal. (See the definition of manifolds being formal below.) The simplest case is $\Sigma = T^3$.

1. Mode decomposition and the zero-mode algebra

We Hodge decompose the gauge fields as a^I and b^I as

$$\begin{aligned} a^I &= d\theta^I + \star dK^{I'} + \alpha_l^I \omega_l, \\ b^I &= dK^I + \star d\theta^{I'} + \beta_l^I \eta_l, \end{aligned} \quad (25)$$

where $d\theta^I, dK^I$ and $\star dK^{I'}, \star d\theta^{I'}$ are the exact and co-exact parts of the decomposition, respectively, and $\{\omega_l\}_l$ and $\{\eta_l\}_l$ are bases of harmonic one- and two-forms, respectively. ($d\omega_l = d\star\omega_l = 0$). The ‘‘zero modes’’, α_l^I and β_l^I , which appear in the Hodge decomposition, play a crucial role later. Let $\{L^m\}$ and $\{S^m\}$ be a set of generators of the first and second homology groups, $H_1(\Sigma; \mathbb{Z})$ and $H_2(\Sigma; \mathbb{Z})$, respectively. We define the linking matrix by

$$M^{mn} = I(S^m, L^n) \quad (26)$$

which counts the signed intersections I_{mn} of S^m and L^n . Furthermore,

$$\int_{L^m} \omega_l = \delta_l^m, \quad \int_{S^m} \eta_l = \delta_l^m, \quad \int_{\Sigma} \omega_l \wedge \eta_k = M_{lk} \quad (27)$$

where M_{lm} is the inverse of the linking matrix of Σ .

For the reason which will become clear momentarily, we will work on a spatial manifold which is *formal*. Here, a Riemannian metric is called (metrically) formal if all wedge products of harmonic forms are harmonic. A closed manifold is called geometrically formal if it admits a formal Riemannian metric.³² In particular, we will focus on one of the simplest formal manifolds; the three-torus, $\Sigma = T^3$.

The Wilson loop/surface operators for L^m and S^m on Σ are written in terms of the zero modes, α_l^I and β_l^I . By noting $\int_{L^i} a^I = M^{mi} \int_{\Sigma} a^I \wedge \eta_m = \alpha_i^I$, the Wilson operators for the gauge field a_l^I are given by

$$A_i^I := \exp i \int_{L^i} a^I = \exp i \alpha_i^I. \quad (28)$$

Similarly, one notes $\int_{S^l} b^I = M^{lm} \int_{\Sigma} b^I \wedge \omega_m = \beta_l^I$. Since Σ is formal,

$$\int_{S^l} a^I \wedge a^J = \alpha_i^I \alpha_j^J \int_{S^l} \omega_i \wedge \omega_j C_{ijl} \quad (29)$$

where the product of the two harmonic one-form $\omega_i \wedge \omega_j$ is given in terms of the harmonic two-form as $\omega_i \wedge \omega_j = C_{ijk} \eta_k$. Thus, we consider the Wilson surface operators

$$\begin{aligned} W_i^I &:= \exp i \int_{S^i} \left(b^I + \frac{q_{\bar{I}}}{2\pi} a^I \wedge a^{\bar{I}} \right) \\ &= \exp i \left(\beta_i^I + \frac{q_{\bar{I}}}{2\pi} C_{lmi} \alpha_l^I \alpha_m^{\bar{I}} \right). \end{aligned} \quad (30)$$

In the following, we canonically quantize the theory, and study the algebra obeyed by the Wilson operators.

We will focus on $\Sigma = T^3$, for which the linking matrix is simply the 3×3 identity matrix,

$$M^{mn} = \delta_{mn}. \quad (31)$$

We also take

$$C_{ijk} = \epsilon_{ijk} \quad (32)$$

Upon canonical quantization, the zero modes, $\hat{\alpha}_i^I$ and $\hat{\beta}_i^I$, now denoted with hat to indicate they are quantum operators, satisfy the commutator

$$[\hat{\alpha}_i^I, \hat{\beta}_j^J] = \frac{2\pi i}{K} \delta_{ij} \delta^{IJ}. \quad (33)$$

Correspondingly, we consider the set of Wilson operators

$$\begin{aligned} \hat{A}_i^I &= \exp(i\hat{\alpha}_i^I), \quad \hat{W}_i^I = \exp(i\hat{\Lambda}_i^I), \\ \text{where } \hat{\Lambda}_i^I &= \hat{\beta}_i^I + \frac{q_{\bar{I}}}{2\pi} \epsilon_{ijk} \hat{\alpha}_j^I \hat{\alpha}_k^{\bar{I}}, \end{aligned} \quad (34)$$

\hat{A}_i^I and \hat{W}_i^I are large gauge invariant and similar to (23), the triple commutators of $\hat{\Lambda}_i^I$ encode topological data, i.e., the three-loop-braiding phases. Here we simply present the non-vanishing triple commutators and defer the details of Wilson operator algebra and large gauge invariance to appendix A. The non-zero triple-linking phase factors are:

$$\begin{aligned} [[\hat{\Lambda}_i^I, \hat{\Lambda}_j^I], \hat{\Lambda}_k^{\bar{I}}] &= -\epsilon_{ijk} \frac{4\pi q_{\bar{I}}}{K^2}, \\ [[\hat{\Lambda}_i^1, \hat{\Lambda}_j^2], \hat{\Lambda}_k^1] &= [[\hat{\Lambda}_i^2, \hat{\Lambda}_j^1], \hat{\Lambda}_k^1] = \epsilon_{ijk} \frac{2\pi q_2}{K^2}, \\ [[\hat{\Lambda}_i^1, \hat{\Lambda}_j^2], \hat{\Lambda}_k^2] &= [[\hat{\Lambda}_i^2, \hat{\Lambda}_j^1], \hat{\Lambda}_k^2] = \epsilon_{ijk} \frac{2\pi q_1}{K^2}. \end{aligned} \quad (35)$$

E. Wave function in terms of Wilson operators

In the previous section, we introduced the Wilson operators for non-contractible loops and surfaces on T^3 . In this section, we construct and label all the ground states on T^3 in terms of these Wilson operators. These ground states are large gauge invariant (even though the commutator algebra of the Wilson operators are only large gauge covariant). (See appendix A for details of large gauge invariance.) Furthermore, we use these ground states to calculate the modular \mathcal{T} and \mathcal{S} matrices, which encode the spin and the braiding statistics of topological excitations.^{17–22}

For this purpose, it is advantageous to construct the three-dimensional version of minimum entropy states (MESs), which are a special choice of the basis for the ground state multiplet.³³ By calculating the overlap between MESs before and after applying the modular \mathcal{S} and \mathcal{T} transformations, we can read off the braiding statistics for particle-loop and three-loop braiding. The MES basis has been constructed before in Refs. 17–20 for microscopic models defined on lattices. We will show that

the \mathcal{S} and \mathcal{T} matrices that we are going to calculate are the same as that for their model, and therefore we verify that our model is the continuum version of the Dijkgraaf-Witten model.²⁸ These \mathcal{S} and \mathcal{T} matrices are also consistent with those calculated from the partition functions of the gapless boundary theory in our previous paper.²⁶

1. The ordinary BF theory

Before we study the \mathcal{S} and \mathcal{T} matrices for the coupled BF theory, as a warm up, we first demonstrate our strategy for the ordinary BF theory on T^3 . The zero modes of the BF theory obey the commutation relation $[\hat{\alpha}_i, \hat{\beta}_j] = \delta_{ij}(2\pi i/K)$, $[\hat{\alpha}_i, \hat{\alpha}_j] = [\hat{\beta}_i, \hat{\beta}_j] = 0$, where $i, j = 1, 2, 3$. The Wilson loop and surface operators for non-contractible loops and surfaces on T^3 are given by $\hat{A}_i = \exp(i\hat{\alpha}_i)$ and $\hat{B}_i = \exp(i\hat{\beta}_i)$, and by taking powers thereof. They satisfy

$$\hat{A}_i \hat{B}_j = \delta_{ij} e^{-2\pi i/K} \hat{B}_j \hat{A}_i. \quad (36)$$

We define and choose a vacuum state (a reference state) $|0\rangle$ such that all \hat{A}_i 's are diagonal. All the other ground states can be generated, starting from $|0\rangle$, by applying \hat{B}_i : $\hat{B}_3^{\lambda} \hat{B}_2^{\lambda_2} \hat{B}_1^{\lambda_1} |0\rangle$. These states are the eigenstate of \hat{A}_i operator. The \mathcal{S} and \mathcal{T} matrices for this basis is the Kronecker delta and do not tell us the information about the spin and braiding statistics at all. To extract the spin and braiding statistics, we construct the three-dimensional version of MESs in z -direction by considering the eigenstates of the Wilson operators \hat{A}_1, \hat{A}_2 and \hat{B}_3 . Namely, we consider the set of states given by

$$|\Psi_{n_1, n_2, n_3}\rangle = \frac{1}{\sqrt{K}} \sum_{\lambda} e^{\frac{2\pi i \lambda n_3}{K}} \hat{B}_3^{\lambda} \hat{B}_1^{n_1} \hat{B}_2^{n_2} |0\rangle \quad (37)$$

where $\lambda, n_1, n_2, n_3 \in \mathbb{Z}_K$. As we check momentarily, the \mathcal{T} matrix acts diagonally on these states – an expected feature for states with definite “topological” or “anyonic” charge.

The \mathcal{T} transformation can be visualized as the shear deformation in the xz plane (as its two-dimensional counterpart on T^2). Hence, under the \mathcal{T} transformation, $\hat{B}_1 \rightarrow \hat{B}_1 \hat{B}_3$. The MESs $|\Psi_{n_i}\rangle$ are transformed under \mathcal{T} as

$$\begin{aligned} \mathcal{T}|\Psi_{n_i}\rangle &= \frac{1}{\sqrt{K}} \sum_{\lambda} e^{\frac{2\pi i \lambda n_3}{K}} \hat{B}_3^{\lambda+n_1} \hat{B}_1^{n_1} \hat{B}_2^{n_2} |0\rangle \\ &= e^{-\frac{2\pi i n_1 n_3}{K}} |\Psi_{n_i}\rangle. \end{aligned} \quad (38)$$

Therefore, \mathcal{T} matrix takes a diagonal form for the MESs, and encodes information related to a (3+1)d analogue of topological spin.

The modular \mathcal{S} transformation is slightly more non-trivial and can be decomposed into \mathcal{S}_{13} and \mathcal{S}_{12} , which are 90° rotation in the xz and xy planes, respectively.

Under the \mathcal{S}_{13} transformation,

$$\mathcal{S}_{13}|\Psi_{n_i}\rangle = \frac{1}{\sqrt{K}} \sum_{\lambda} e^{\frac{2\pi i \lambda n_3}{K}} \hat{B}_1^{-\lambda} \hat{B}_3^{n_1} \hat{B}_2^{n_2} |0\rangle. \quad (39)$$

Therefore, the \mathcal{S}_{13} matrix for the MES basis is calculated as

$$\begin{aligned} \langle \Psi_{n'_i} | \mathcal{S}_{13} | \Psi_{n_i} \rangle &= \frac{1}{K} \sum_{\lambda', \lambda} \langle 0 | e^{\frac{2\pi i}{K}(-\lambda' n'_3 + \lambda n_3)} \\ &\quad \times \hat{B}_2^{-n'_2} \hat{B}_3^{-n'_1} \hat{B}_1^{\lambda'} \hat{B}_3^{\lambda} \hat{B}_1^{n_1} \hat{B}_2^{n_2} |0\rangle \\ &= \frac{1}{K} \delta_{n_2, n'_2} e^{\frac{2\pi i}{K}(n_1 n'_3 + n'_1 n_3)}. \end{aligned} \quad (40)$$

In the above derivation, we use $-\lambda' = n_1$, $\lambda = n'_1$ and $n_2 = n'_2$. Combined with the \mathcal{S}_{12} transformation, we can write down the modular \mathcal{S} matrix

$$\mathcal{S} = \frac{1}{K} \delta_{n_1, n'_2} e^{-\frac{2\pi i}{K}(n'_3 n_2 - n_3 n'_1)}. \quad (41)$$

We can easily generalize the above results to the two decouple copies of BF theories on T^3 . The commutators among zero modes are

$$[\hat{\alpha}_i^I, \hat{\beta}_j^J] = \delta_{ij} \delta_{IJ} \frac{2\pi i}{K}, \quad [\hat{\alpha}_i^I, \hat{\alpha}_j^J] = [\hat{\beta}_i^I, \hat{\beta}_j^J] = 0. \quad (42)$$

The MES basis is given by

$$\begin{aligned} |\Psi_{n_i}^{l_i}\rangle &= \frac{1}{K} \sum_{\lambda_1, \lambda_2} e^{\frac{2\pi i}{K}(\lambda_1 n_3 + \lambda_2 l_3)} (\hat{B}_3^1)^{\lambda_1} (\hat{B}_1^1)^{n_1} (\hat{B}_2^1)^{n_2} \\ &\quad \times (\hat{B}_3^2)^{\lambda_2} (\hat{B}_1^2)^{l_1} (\hat{B}_2^2)^{l_2} |0\rangle. \end{aligned} \quad (43)$$

These states are an eigenstate of $\hat{A}_1^I, \hat{A}_2^I, \hat{B}_3^I$ ($I = 1, 2$). The modular \mathcal{T} and \mathcal{S} matrices are given by

$$\begin{aligned} \mathcal{T} &= \delta_{n_i, n'_i} \delta_{l_i, l'_i} e^{-\frac{2\pi i}{K}(n_1 n_3 + l_1 l_3)}, \\ \mathcal{S} &= \frac{1}{K^2} \delta_{n_1, n'_2} \delta_{l_1, l'_2} e^{-\frac{2\pi i}{K}(n'_3 n_2 - n_3 n'_1) - \frac{2\pi i}{K}(l'_3 l_2 - l_3 l'_1)}. \end{aligned} \quad (44)$$

2. The coupled BF theory: wave functions in terms of $\hat{\alpha}$ and $\hat{\beta}$

For the coupled BF theory realizing three-loop braiding statistics defined in Eq. (12), the commutators between α_i^I and β_i^I are identical to those in the two decoupled copies of BF theories defined in Eq. (42). On the other hand, if we consider Λ_i^I instead of β_i^I , the commutators are

$$[\hat{\alpha}_i^I, \hat{\Lambda}_j^J] = \delta_{ij} \delta_{IJ} \frac{2\pi i}{K}, \quad [\hat{\alpha}_i^I, \hat{\alpha}_j^J] = 0, \quad [\hat{\Lambda}_i^I, \hat{\Lambda}_j^J] \neq 0. \quad (45)$$

In the next two subsections, we will construct two sets of MESs in terms of β_i^I and Λ_i^I .

Let us first construct MESs using $\hat{\beta}_i^I$. Similar to the two decoupled copies of BF theories, $\hat{A}_1^I, \hat{A}_2^I, \hat{B}_3^I$ ($I =$

1, 2) commute with each other. Therefore, we define the eigenstate for these operators as

$$|\Psi_{n_i}^{l_i}\rangle \sim \sum_{\lambda_1, \lambda_2} e^{\frac{2\pi i \lambda_1}{K} (n_3 + \frac{l \times n}{K}) + \frac{2\pi i \lambda_2}{K} (l_3 + \frac{n \times l}{K})} \times (\hat{B}_1^1)^{n_1} (\hat{B}_2^1)^{n_2} (\hat{B}_1^2)^{l_1} (\hat{B}_2^2)^{l_2} (\hat{B}_3^1)^{\lambda_1} (\hat{B}_3^2)^{\lambda_2} |0\rangle, \quad (46)$$

where $l \times n := l_1 n_2 - l_2 n_1$. One verifies that the states constructed in Eq. (46) are invariant under the large gauge transformations (A4), up to a phase factor (which can depend on n_i and l_i). Here for simplicity, we consider $q_1 = q_2 = 1$. Different from the two decoupled copies of BF theories, we require $\lambda_1, \lambda_2, n_i, l_i \in \mathbb{Z}_K$ and n_3 (l_3) is shifted by $(l \times n)/K$ ($(n \times l)/K$). Because of the extra factor of K^{-1} in $l \times n/K$ or $n \times l/K$, it may seem that there are K^{12} different eigenstates, as opposed to K^6 , which is the expected number of ground states for two copies of BF theories. This is however not the case once we properly reorganize these wave functions. Let us introduce

$$\begin{aligned} n_1 &\equiv Kt_1 + \bar{n}_1, & n_2 &\equiv Kt_2 + \bar{n}_2 \\ \bar{n}_3 &\equiv n_3 + s \times \bar{n} + \bar{l} \times t + \bar{l}_2(s_1 - t_1) \pmod{K} \\ l_1 &\equiv Ks_1 + \bar{l}_1, & l_2 &\equiv Ks_2 + \bar{l}_2 \\ \bar{l}_3 &\equiv l_3 + t \times \bar{l} + \bar{n} \times s + \bar{n}_2(t_1 - s_1) \pmod{K} \end{aligned} \quad (47)$$

where $\bar{n}_i, \bar{l}_i, t_1, t_2, s_1, s_2 \in \mathbb{Z}_K$. In terms of these quantum numbers, the wave functions depend only on (and are labeled by) \bar{n}_i and \bar{l}_i , as they can be written as

$$|\Psi_{\bar{n}_i}^{\bar{l}_i}\rangle = \frac{1}{K^3} \sum_{\lambda_1, 2, t_1, 2, s_1, 2} e^{\frac{2\pi i \lambda_1}{K} (\bar{n}_3 + \frac{\bar{l} \times \bar{n}}{K}) + \frac{2\pi i \lambda_2}{K} (\bar{l}_3 + \frac{\bar{n} \times \bar{l}}{K})} \times (\hat{B}_1^1)^{Kt_1 + \bar{n}_1} (\hat{B}_2^1)^{Kt_2 + \bar{n}_2} \times (\hat{B}_1^2)^{Ks_1 + \bar{l}_1} (\hat{B}_2^2)^{Ks_2 + \bar{l}_2} (\hat{B}_3^1)^{\lambda_1} (\hat{B}_3^2)^{\lambda_2} |0\rangle. \quad (48)$$

This construction of the ground states is analogous to the construction of the surface partition functions realizing the three-loop braiding phase in Ref. 26.

With respect to the ground states (48), the \mathcal{T} transformation is diagonal,

$$\mathcal{T}|\Psi_{\bar{n}_i}^{\bar{l}_i}\rangle = e^{-\frac{2\pi i}{K} \bar{n}_1 (\bar{n}_3 + \frac{\bar{l} \times \bar{n}}{K}) - \frac{2\pi i}{K} \bar{n}_1 (\bar{n}_3 + \frac{\bar{l} \times \bar{n}}{K})} |\Psi_{\bar{n}_i}^{\bar{l}_i}\rangle \quad (49)$$

On the other hand, the \mathcal{S}_{13} matrix is

$$\begin{aligned} \langle \Psi_{\bar{n}'_i}^{\bar{l}'_i} | \mathcal{S}_{13} | \Psi_{\bar{n}_i}^{\bar{l}_i} \rangle &= \frac{1}{K^6} \sum_{\lambda_1, 2, \lambda'_1, 2, t'_1, 2, t'_1, 2, s'_1, 2, s'_1, 2} \\ &\times \langle 0 | e^{-\frac{2\pi i \lambda'_1}{K} (\bar{n}'_3 + \frac{\bar{l}' \times \bar{n}'}{K}) - \frac{2\pi i \lambda'_2}{K} (\bar{l}'_3 + \frac{\bar{n}' \times \bar{l}'}{K})} \\ &\times e^{+\frac{2\pi i \lambda_1}{K} (\bar{n}_3 + \frac{\bar{l} \times \bar{n}}{K}) + \frac{2\pi i \lambda_2}{K} (\bar{l}_3 + \frac{\bar{n} \times \bar{l}}{K})} \\ &\times (\hat{B}_2^2)^{-(Ks'_2 + \bar{l}'_2)} (\hat{B}_3^2)^{-(Ks'_1 + \bar{l}'_1)} (\hat{B}_1^2)^{\lambda'_2} \\ &\times (\hat{B}_2^1)^{-(Kt'_2 + \bar{n}'_2)} (\hat{B}_3^1)^{-(Kt'_1 + \bar{n}'_1)} (\hat{B}_1^1)^{\lambda'_1} \\ &\times (\hat{B}_3^1)^{\lambda_1} (\hat{B}_1^1)^{Kt_1 + \bar{n}_1} (\hat{B}_2^1)^{Kt_2 + \bar{n}_2} \\ &\times (\hat{B}_3^2)^{\lambda_2} (\hat{B}_1^2)^{Ks_1 + \bar{l}_1} (\hat{B}_2^2)^{Ks_2 + \bar{l}_2} |0\rangle \\ &= \frac{1}{K^2} \delta_{\bar{n}_2, \bar{n}'_2} \delta_{\bar{l}_2, \bar{l}'_2} e^{i\theta^{\bar{n}'_i, \bar{l}'_i, \bar{n}_i, \bar{l}_i}} \end{aligned} \quad (50)$$

where $\theta^{\bar{n}'_i, \bar{l}'_i, \bar{n}_i, \bar{l}_i}$ is given by

$$\begin{aligned} \theta^{\bar{n}'_i, \bar{l}'_i, \bar{n}_i, \bar{l}_i} &= \frac{2\pi}{K} (\bar{n}_3 \bar{n}'_1 + \bar{n}'_3 \bar{n}_1 + \bar{l}_3 \bar{l}'_1 + \bar{l}'_3 \bar{l}_1) \\ &+ \frac{2\pi}{K^2} [(\bar{l} \times \bar{n})(n'_1 - l'_1) + (\bar{l}' \times \bar{n}')(n_1 - l_1)]. \end{aligned} \quad (51)$$

From the calculation of \mathcal{S}_{13} , we can further calculate the modular \mathcal{S} matrix,

$$\begin{aligned} \mathcal{S} &= \frac{1}{K^2} \delta_{n_1, n'_2} \delta_{l_1, l'_2} e^{-\frac{2\pi i}{K} (n'_2 n_2 - n_3 n'_1) - \frac{2\pi i}{K} (l'_2 l_2 - l_3 l'_1)} \\ &\times e^{-\frac{2\pi i}{K^2} [(n_1 + l_1)(n_2 l'_1 + n'_1 l_2) - 2n_2 n'_1 l_1 - 2n_1 l_2 l'_1]}. \end{aligned} \quad (52)$$

The \mathcal{S} and \mathcal{T} matrices obtained in this way are the same as those obtained for the surface partition functions in our previous work, and other bulk calculations.^{17–20,26}

3. The coupled BF theory: wave function in terms of $\hat{\alpha}$ and $\hat{\Lambda}$

While we have succeeded, by using B_i^I , in constructing the ground state wave functions and in computing the \mathcal{T} and \mathcal{S} matrices, it is also worth trying to use Λ_i^I instead of B_i^I to construct wave functions. One motivation for this is that $\exp i\Lambda_i^I$ are the Wilson surface operators, while $\exp iB_i^I$ are not. The commutators among $\hat{\alpha}_i^I$ and $\hat{\Lambda}_i^I$ are:

$$\begin{aligned} [\hat{\alpha}_i^I, \hat{\alpha}_j^J] &= 0, & [\hat{\alpha}_i^I, \hat{\Lambda}_j^J] &= \frac{2\pi i}{K} \delta_{ij} \delta^{IJ} \\ [\hat{\Lambda}_i^I, \hat{\Lambda}_j^I] &= \frac{2iq_{\bar{I}}}{K} \epsilon_{ijk} \hat{\alpha}_k^{\bar{I}}, \\ [\hat{\Lambda}_i^1, \hat{\Lambda}_j^2] &= \frac{-iq_2}{K} \epsilon_{ijk} \hat{\alpha}_k^1 - \frac{iq_1}{K} \epsilon_{ijk} \hat{\alpha}_k^2. \end{aligned} \quad (53)$$

Although Λ_i^I may not commute with each other, $\hat{A}_1^1, \hat{A}_1^2, \hat{W}_3^1, \hat{A}_1^2, \hat{A}_2^2$ and \hat{W}_3^2 still commute with each other, and we can write down the eigenstates for them,

$$|\Psi_{n_i}^{l_i}\rangle = \frac{1}{K} \sum_{\lambda_1, 2} e^{\frac{2\pi i}{K} (\lambda_1 n_3 + \lambda_2 l_3)} \times (\hat{W}_3^1)^{\lambda_1} (\hat{W}_3^2)^{\lambda_2} (\hat{W}_1^1)^{n_1} (\hat{W}_1^2)^{l_1} (\hat{W}_2^1)^{n_2} (\hat{W}_2^2)^{l_2} |0\rangle \quad (54)$$

where $\lambda_1, \lambda_2, n_i, l_i \in \mathbb{Z}_K$. Since \hat{W}_i^I do not mutually commute, the ordering of \hat{W}_i^I is important when generating a set of wave functions. We choose this particular order so that \mathcal{S} and \mathcal{T} matrices are the same as those calculated in the previous subsection. Notice that since \hat{W}_i^I is invariant under the large gauge transformations, so is this wave function.

The matrix elements of \mathcal{S}_{13} can be calculated as

$$\begin{aligned} \langle \Psi_{n'_i}^{l'_i} | \mathcal{S}_{13} | \Psi_{n_i}^{l_i} \rangle &= \frac{1}{K^2} \sum_{\lambda_{1,2}, \lambda'_{1,2}} e^{\frac{2\pi i}{K}(-\lambda'_1 n'_3 - \lambda'_2 l'_3 + \lambda_1 n_3 + \lambda_2 l_3)} \\ &\times \langle 0 | (\hat{W}_2^2)^{-l'_2} (\hat{W}_1^1)^{-n'_2} (\hat{W}_3^2)^{-l'_1} (\hat{W}_3^1)^{-n'_1} (\hat{W}_1^2)^{\lambda'_2} (\hat{W}_1^1)^{\lambda'_1} \\ &\times (\hat{W}_3^1)^{\lambda_1} (\hat{W}_3^2)^{\lambda_2} (\hat{W}_1^1)^{n_1} (\hat{W}_1^2)^{l_1} (\hat{W}_2^1)^{n_2} (\hat{W}_2^2)^{l_2} | 0 \rangle \\ &= \frac{1}{K^2} \delta_{n_2, n'_2} \delta_{l_2, l'_2} e^{i\theta^{n'_i, n_i, l'_i, l_i}}, \end{aligned} \quad (55)$$

where $\theta^{n'_i, n_i, l'_i, l_i}$ is the same as that in Eq. (51). One can then check that the modular \mathcal{S} matrix also matches with the previous calculation in terms of B_i^I , Eq. (52).

As for the \mathcal{T} transformation, since \hat{W}_1^I and \hat{W}_3^I do not commute with each other, their transformation properties under the \mathcal{T} transformation are more complicated. Using the knowledge that $\Lambda = \beta + \alpha \times \alpha$, we decompose \hat{W}_i^I as

$$\hat{W}_i^I = \hat{B}_i^I \hat{C}_i^I \quad \text{where} \quad \hat{C}_i^I = \exp\left(\frac{iq\bar{I}}{2\pi} \epsilon_{ijk} \hat{\alpha}_j^I \hat{\alpha}_k^{\bar{I}}\right). \quad (56)$$

We propose that under the \mathcal{T} transformation,

$$(\hat{B}_1^I)^{n_1} (\hat{C}_1^I)^{n_1} \rightarrow (\hat{B}_1^I)^{n_1} (\hat{B}_3^I)^{n_1} (\hat{C}_1^I)^{n_1} (\hat{C}_3^I)^{n_1}. \quad (57)$$

The above result can be rewritten in terms of the \hat{W}_i^I operators as

$$(\hat{W}_1^I)^{n_1} \rightarrow (\hat{W}_3^I)^{n_1/2} (\hat{W}_1^I)^{n_1} (\hat{W}_3^I)^{n_1/2}. \quad (58)$$

According to this definition, under the \mathcal{T} transformation, $|\Psi_{n_i}^{l_i}\rangle$ are transformed as

$$\begin{aligned} \mathcal{T} |\Psi_{n_i}^{l_i}\rangle &= \frac{1}{K} \sum_{\lambda_{1,2}} e^{\frac{2\pi i}{K}(\lambda_1 n_3 + \lambda_2 l_3)} \\ &\times (\hat{W}_3^1)^{\lambda_1} (\hat{W}_3^2)^{\lambda_2} (\hat{W}_3^1)^{n_1/2} (\hat{W}_1^1)^{n_1} (\hat{W}_3^1)^{n_1/2} \\ &\times (\hat{W}_3^2)^{l_1/2} (\hat{W}_1^2)^{l_1} (\hat{W}_3^2)^{l_1/2} (\hat{W}_2^1)^{n_2} (\hat{W}_2^2)^{l_2} | 0 \rangle. \end{aligned} \quad (59)$$

Therefore, the \mathcal{T} matrix is given by

$$\mathcal{T} = \delta_{n_i, n'_i} \delta_{l_i, l'_i} e^{-\frac{2\pi i}{K}(n_3 n_1 + l_3 l_1) - \frac{2\pi i}{K^2}(l \times n)(n_1 - l_1)}. \quad (60)$$

This also matches with the previous calculation Eq. (49).

We conclude this section with some comments. Above, we read off the Wilson operators for the three-loop braiding theory from the coupling to sources. Infinitesimal gauge transformations could be derived directly from the charge or Gauss law operators of the theory. As for the large gauge transformations, they were obtained by demanding the invariance of the Wilson operators (Appendix A). We showed that the triple-commutator of these Wilson operators encoded the three-loop braiding phase. Although the commutator of Wilson-operators themselves are large gauge covariant, the ground states could be written down explicitly and were large gauge invariant.

III. FOUR-LOOP BRAIDING THEORY

A. The quartic theory

In this section, we consider the following BF theory with quartic coupling:

$$\begin{aligned} S = \int_{\mathcal{M}} \left[\frac{K}{2\pi} \delta_{IJ} b^I \wedge da^J + \epsilon_{IJKL} \frac{p}{4!} a^I \wedge a^J \wedge a^K \wedge a^L \right. \\ \left. - \delta_{IJ} b^I \wedge J_{qv}^J - \delta_{IJ} a^I \wedge J_{qp}^J \right], \end{aligned} \quad (61)$$

where $I, J \in 1, 2, 3, 4$. This action can be considered as describing a discrete (lattice) gauge theory with the gauge group $\mathbb{Z}_K \times \mathbb{Z}_K \times \mathbb{Z}_K \times \mathbb{Z}_K$. p is a parameter of the theory, and is given by

$$p = \frac{qK^3}{(2\pi)^3}, \quad q = 0, 1, \dots, K-1. \quad (62)$$

In order to gain some intuition about this quartic theory, it is helpful to compare it to a very similar theory in one lower dimensions; There is a topological field theory in $2+1$ dimensions with a very similar structure to the quartic theory. This is the cubic theory (also known as the type-III Dijkgraaf-Witten theory) with the TQFT action given by

$$S_{\text{cubic}} \propto \int_{\mathcal{M}_3} a_I \wedge a_J \wedge a_K. \quad (63)$$

Clearly such a term could arise when there are three or more flavors of gauge fields, i.e., when the discrete gauge group is given by $G = \prod_{i=1}^k \mathbb{Z}_{N_i}$, $i \geq 3$. Although this is a gauge theory built from an Abelian group, it is known that it has an underlying non-abelian structure in disguise. This can be understood by studying the spectrum of the theory within group cohomology models^{17,34} or by analyzing the Wilson operators³⁵. In either case one finds excitations with quantum dimension $d > 1$ and non-trivial fusion channels. In analogy one expects the quartic theory in $3+1$ dimensions to have an underlying non-abelian structure. This has been studied partially in¹⁸ by explicitly constructing representations for this particular group cohomology model.

In the following, however, we will not go into the non-abelian structure. Although the theory is non-abelian one can gain significant insight by studying its Wilson operator algebra. For example one can learn that the topological invariant corresponding to this theory goes beyond a spacetime Hopf link and in fact captures a spacetime Borromean link³⁴. Similarly in the analysis below, we show that the quartic theory furnishes a topological invariant that goes beyond three-loop braiding and can in fact be understood as non-trivial four-loop braiding. Our reasoning presented below parallels our discussion on three-loop braiding statistics realized in the cubic theory.

1. Equations of motion

The first term in the action (61) describes the particle-loop braiding process, as in the ordinary BF theory. On the other hand, as we will discuss, the second term describes four-loop braiding process. To develop understanding of the four-loop braiding process, let us first write down the equations of motion

$$\begin{aligned} \frac{K}{2\pi} da^I &= J_{qv}^I, \\ \frac{K}{2\pi} db^I + \frac{P}{6} \epsilon_{IJKL} a^J \wedge a^K \wedge a^L &= J_{qp}^I. \end{aligned} \quad (64)$$

Let us consider a fixed static quasiparticle and quasivortex configuration and integrate the equation of motion over space. By solving the first equation of motion as $a^I = (2\pi/K)(d^{-1}J_{qv}^I)$, plugging the solution to the other equations of motion, and integrating over space Σ ,

$$\begin{aligned} \frac{K}{2\pi} \int_{\Sigma} db^I &= \int_{\Sigma} J_{qp}^I \\ - \frac{q}{6} \epsilon_{IJKL} \int_{\Sigma} (d^{-1}J_{qv}^J) \wedge (d^{-1}J_{qv}^K) \wedge (d^{-1}J_{qv}^L). \end{aligned} \quad (65)$$

The second term on the right-hand side of the above equation comes from

$$\begin{aligned} \text{Borr}(J_{qv}^I, J_{qv}^J, J_{qv}^K) \\ = \int_{\Sigma} (d^{-1}J_{qv}^I) \wedge (d^{-1}J_{qv}^J) \wedge (d^{-1}J_{qv}^K). \end{aligned} \quad (66)$$

and involves three quasivortex loops. If any two of them are mutually unlinked, i.e., $d(a^I \wedge a^J) = 0$, this term describes the triple linking number of the Borromean ring configuration and is a topological invariant.^{36,37} As in the three-loop braiding theory, the equation of motion (65), suggests the Borromean ring ‘dresses’ the I -th quasiparticle (Fig. 1).

To see that Borr is a topological invariant, let us introduce $g^K = \epsilon^{IJK} a^I \wedge a^J$. If we require that any two of the flux loops are mutually unlinked, i.e., $d(a^I \wedge a^J) = 0$, this constraint leads to $g^K = du^K$, where u^K is a one-form gauge field and describes the effective magnetic flux loop formed by a^I and a^J . Then, Borr can be written as

$$\begin{aligned} \int_{\Sigma} a^1 \wedge a^2 \wedge a^3 \\ = \int_{\Sigma} a^1 \wedge du^1 = \int_{\Sigma} a^2 \wedge du^2 = \int_{\Sigma} a^3 \wedge du^3. \end{aligned} \quad (67)$$

This is equivalent to a Chern-Simons integral and describes the Hopf linking number between da^k and du^k .

2. Gauge invariance

In the absence of sources, there are two sets of gauge transformations that leave the action invariant: The

usual 1-form gauge transformation

$$b^I \rightarrow b^I + d\zeta^I, \quad a^I \rightarrow a^I \quad (68)$$

and a shifted 0-form gauge transformation

$$a^I \rightarrow a^I + d\varphi^I, \quad b^I \rightarrow b^I + \frac{\pi P}{3K} \epsilon^{IJKL} (a^J \wedge a^K) \varphi^L. \quad (69)$$

Formally, these transformations can be read off by identifying the operators that generate the Gauss law constraints.

Similar to the three-loop braiding theory described earlier, it seems that the coupling to currents is gauge non-invariant. However, by demanding gauge invariance, we can read off the topological currents. The terms with coupling to sources transform under the 0-form gauge transformations as

$$\begin{aligned} a^I \wedge J_{qp}^I + b^I \wedge J_{qv}^I \\ \rightarrow a^I \wedge J_{qp}^I + b^I \wedge J_{qv}^I \\ - \varphi^I d \left[J_{qp}^I - \epsilon^{IJKL} \frac{q}{6} d^{-1} J_{qv}^J \wedge d^{-1} J_{qv}^K \wedge d^{-1} J_{qv}^L \right]. \end{aligned} \quad (70)$$

Hence we can read off the current conservation law

$$d \left[J_{qp}^I - \epsilon^{IJKL} \frac{q}{6} d^{-1} J_{qv}^J \wedge d^{-1} J_{qv}^K \wedge d^{-1} J_{qv}^L \right] = 0. \quad (71)$$

If all the pair of quasivortex loops are mutually unlinked, the first term on the right side describes the triple linking number for the borromean ring configuration. The above equations then indicate, as the equation of motion (65), that the effective particle comes from two parts, the real particle excitation and the Borromean ring configuration. On the other hand, the 1-form gauge symmetry furnishes the second ‘ordinary’ conservation law $dJ_{qv}^I = 0$.

3. Four-loop braiding statistics

That the Borromean ring configuration can be treated as an effective particle, as seen from the equation of motion (65) and the conservation law (71) suggests the theory may realize non-trivial statistics involving four loop-like excitations (four-loop braiding statistics). Following three-loop braiding process, we postulate the four-loop braiding process as shown in Fig. 2. In Fig. 2, we consider the loop L_1 and L_2 form an effective base loop L_{12} , with loop 3 and 4 are linked to L_{12} . Braiding L_3 around L_4 gives rise to a non-trivial phase $\sim n_1 n_2 n_3 n_4 / K$. Furthermore, we can also understand this braiding process by treating loop L_1 as an base loop, with loops L_2, L_3 and L_4 linked to L_1 (Fig. 2 (a)). Loop L_2 braids around L_3 and L_4 . We will verify this argument shortly by computing the algebra of Wilson operators.

The last point of view can be better understood by considering dimensional reduction to one lower dimension as in Fig. 2 (c). The dimensional reduction of the $(3+1)$

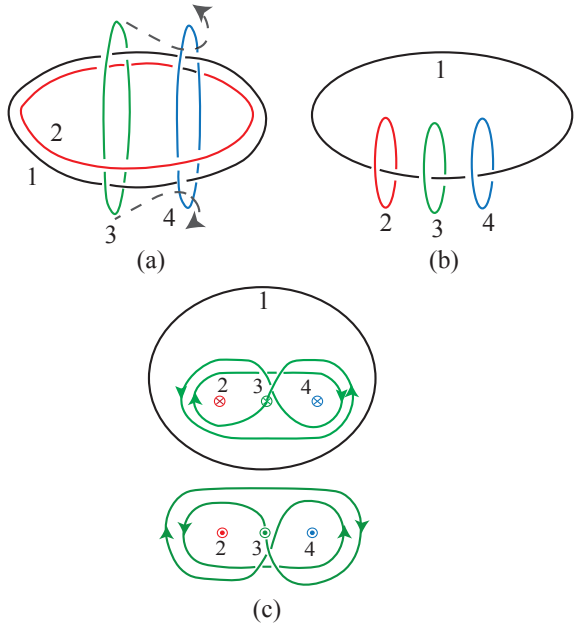


FIG. 2. Four-loop braiding process in 3 + 1 dimensions. (a) Loop 1, 2 and 3 form a Borromean ring configuration. Similarly, loop 1, 2 and 4 form a Borromean ring configuration. Alternatively, L_1 and L_2 form an effective loop L_{12} , with loop 3 and 4 are linked to L_{34} . In this case, it is the same as the three loop braiding process with L_{12} as the base loop. Braiding L_3 around L_4 gives rise to a non-trivial phase $2\pi n_1 n_2 n_3 n_4 / K$. (b) Loop L_1 is the base loop. L_2 , L_3 and L_4 are linked with L_1 . This braiding process can be understood by dimensional reduction to the (2 + 1) dimensions in Fig. 2 (c).

dimensional quartic theory leads to the following (2 + 1) dimensional cubic theory,

$$S = \int_{\mathcal{M}} \left[\frac{K}{2\pi} \delta_{IJ} b^I \wedge da^J + p a^1 \wedge a^2 \wedge a^3 \right] \quad (72)$$

where $I, J = 1, 2, 3$, b^I and a^I are one-form, and p equals to $p = qK^2/(2\pi)^2$ where $q = 0, 1, \dots, K - 1$. The first term is the BF theory and is related to the Hopf linking number for the particle current loops in (2 + 1) dimensions, which describes the particle-particle braiding process. For the second term, if any two of particle current loops are mutually unlinked, it is the Borromean ring and describes the braiding process involving three particles. This braiding process has been discussed in Ref. 17 and can be understood as in Fig. 3.

B. The quadratic theory

As we did for the coupled BF theories realizing the three-loop braiding, we can also consider an alternative quartic theory instead of the quartic theory. Let us con-

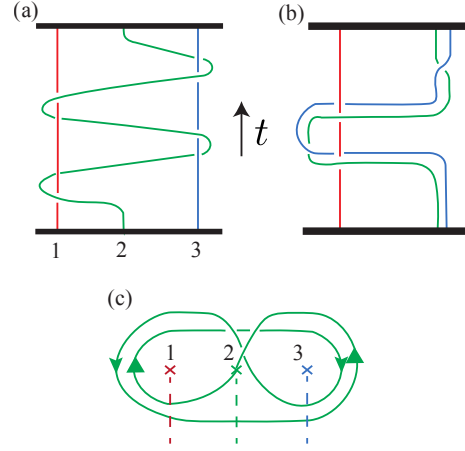


FIG. 3. Three-particle braiding in 2 + 1 dimensions. In (a), particle 1, 2 and 3 are labeled by three different colors red, green and blue. We braid particle 2 around 1 and 3 four times. The Wilson loops for particles 1, 2 and 3 are mutually unlinked. For instance, if there is no Wilson loop for particle 3, the braiding between 1 and 2 is trivial. Nevertheless, the three Wilson loops 1, 2 and 3 together form a Borromean ring in 2 + 1 dimensions. In (b), we treat particle 2 and 3 as an effective particle and braid it around particle 1. This process is topologically equivalent to (a). (c) is the projection of (a) to the two dimensional spatial plane. The braiding of particle 2 around 1 is trivial if there is no particle 3.

sider:

$$S = \frac{K}{2\pi} \int \delta_{IJ} b^I \wedge da^J - \int \delta_{IJ} a^I \wedge J_{qp}^J - \int \delta_{IJ} \Lambda^I \wedge J_{qv}^J \quad (73)$$

where $I, J = 1, \dots, 4$ and

$$\Lambda^I := b^I - \frac{p}{3!} \epsilon^{IJKL} d^{-1} (a^J \wedge a^K \wedge a^L). \quad (74)$$

The equations of motion are the same as Eq. (65). Here, the precise meaning of the term $\int d^{-1} (a^J \wedge a^K \wedge a^L) \wedge J_{qv}^I$ can be understood by taking $J_{qv} = \delta(S)$, which gives rise to for example $\int_S [b^1 - p d^{-1} (a^2 \wedge a^3 \wedge a^4)]$. Looking for a volume V which satisfies $\partial V = S$, this can be written as $\int_S b^1 - p \int_V (a^2 \wedge a^3 \wedge a^4)$.

Using the quadratic theory, let us now discuss the algebra of the Wilson operators. The canonical commutators are the same as the ordinary BF theory and hence $[\int_C a^I, \int_S \Lambda^J] = (2\pi i/K) \delta^{IJ} I(C, S)$. On the other hand,

the multiple commutators among $\int_S \Lambda^I$ are

$$\begin{aligned}
& \left[\int_{S_1} \Lambda^I, \int_{S_2} \Lambda^J \right] \\
&= (-p\epsilon^{IJPQ}) \left[\frac{-2\pi i}{K} \right] \int_{\partial(V_1 \# V_2)} a^P \wedge a^Q, \\
& \left[\left[\int_{S_1} \Lambda^I, \int_{S_2} \Lambda^J \right], \int_{S_3} \Lambda^K \right] \\
&= (-4p\epsilon^{IJKQ}) \left[\frac{-2\pi i}{K} \right]^2 \int_{\partial(V_1 \# V_2) \# S_3} a^Q, \\
& \left[\left[\left[\int_{S_1} \Lambda^I, \int_{S_2} \Lambda^J \right], \int_{S_3} \Lambda^K \right], \int_{S_4} \Lambda^L \right] \\
&= (-4p\epsilon^{IJKL}) \left[\frac{-2\pi i}{K} \right]^3 I(\partial(V_1 \# V_2) \# S_3, S_4), \quad (75)
\end{aligned}$$

where we noted $d(\delta(V_1) \wedge \delta(V_2)) = \delta(S_1) \wedge \delta(V_2) + \delta(V_1) \wedge \delta(S_2)$. The four-loop braiding phase is encoded in the following product of Wilson operators

$$\begin{aligned}
& \left[(W^2 W^1)^\dagger W^1 W^2 \cdot W^3 \cdot (W^1 W^2)^\dagger W^2 W^1 \cdot W^{3\dagger} \right] \\
& \cdot W^4 \cdot [\dots]^\dagger \cdot W^{4\dagger} \\
&= \exp \left(\left[\left[\left[i \int_{S_1} \Lambda^1, i \int_{S_2} \Lambda^2 \right], i \int_{S_3} \Lambda^3 \right], i \int_{S_4} \Lambda^4 \right] \right). \quad (76)
\end{aligned}$$

C. The Wilson operator algebra on T^3

It is also instructive to construct the Wilson operator algebra on a closed spatial manifold with non-trivial topology, e.g., $\Sigma = T^3$. We will work in the setting identical to the previous section, and quantize the theory on $\Sigma = T^3$. As before, we expand a^I and b^I by using the Hodge decomposition as $a^I = \dots + \alpha_i^I \omega_i$, and $b^I = \dots + \beta_p^I \eta_p$, where α_i^I and β_p^I are the zero modes. Also, as before, we consider Wilson operators associated to the generators $\{L^m\}$ and $\{S^m\}$ of the first and second homology groups. For L^i , we consider the Wilson loop operators

$$\hat{A}_i^I := \exp i \int_{L^i} \hat{a}^I = \exp i \hat{\alpha}_i^I \quad (77)$$

As for S^m 's, we consider Wilson surface operators

$$W_i^1 := \exp i \left(\int_{S^i} b^1 - p \int_{\Sigma} a^2 \wedge a^3 \wedge a^4 \right) \quad \text{etc.} \quad (78)$$

The cubic term can be written as, assuming Σ is formal, $\int_{\Sigma} a^I \wedge a^J \wedge a^K = \alpha_i^I \alpha_j^J \alpha_k^K \int_{\Sigma} \omega_i \wedge \omega_j \wedge \omega_k = \epsilon_{ijk} \alpha_i^I \alpha_j^J \alpha_k^K$. Hence, the Wilson surface operators associated to S^i are

$$\begin{aligned}
& \hat{W}_i^I = \exp i \hat{\Lambda}_i^I \\
& \text{where } \Lambda_i^I = \beta_i^I - p\epsilon^{IJKL} \epsilon_{ijk} \alpha_i^J \alpha_j^K \alpha_k^L \quad (79)
\end{aligned}$$

The Wilson operator algebra can be computed as

$$\begin{aligned}
& (\hat{W}_2^2 \hat{W}_1^1)^\dagger \hat{W}_1^1 \hat{W}_2^2 \\
&= \exp([i\Lambda_1^1, i\Lambda_2^2]), \\
& (\hat{W}_2^2 \hat{W}_1^1)^\dagger \hat{W}_1^1 \hat{W}_2^2 \cdot \hat{W}_3^3 \cdot (\hat{W}_1^1 \hat{W}_2^2)^\dagger \hat{W}_2^2 \hat{W}_1^1 \cdot \hat{W}_3^{3\dagger} \\
&= \exp([i\Lambda_1^1, i\Lambda_2^2], i\Lambda_3^3), \\
& \left[(\hat{W}_2^2 \hat{W}_1^1)^\dagger \hat{W}_1^1 \hat{W}_2^2 \cdot \hat{W}_3^3 \cdot (\hat{W}_1^1 \hat{W}_2^2)^\dagger \hat{W}_2^2 \hat{W}_1^1 \cdot \hat{W}_3^{3\dagger} \right] \\
& \times \hat{W}_1^4 \times [\dots]^\dagger \times \hat{W}_1^{4\dagger} \\
&= \exp([i\Lambda_1^1, i\Lambda_2^2], i\Lambda_3^3, i\Lambda_4^4), \quad (80)
\end{aligned}$$

where the repeated commutators are given by

$$\begin{aligned}
& [\Lambda_1^1, \Lambda_2^2] = \frac{2\pi i p}{K} (-\alpha_1^3 \alpha_3^4 + \alpha_2^3 \alpha_3^4 + \alpha_3^3 \alpha_1^4 - \alpha_3^3 \alpha_2^4), \\
& [[\Lambda_1^1, \Lambda_2^2], \Lambda_3^3] = \frac{4\pi^2 p}{K^2} (\alpha_1^4 - \alpha_2^4), \\
& [[[\Lambda_1^1, \Lambda_2^2], \Lambda_3^3], \Lambda_4^4] = \frac{8\pi^3 i p}{K^3}. \quad (81)
\end{aligned}$$

The last equation in Eq. (80) with the quadruple commutator is related to the four-loop braiding statistical process. This four-loop braiding process after dimensional reduction becomes three-particle braiding process and is described by the cubic term defined in Eq. (63). As we discussed before, this three-particle braiding process is due to the non-abelian braiding statistics between the gauge flux excitations and is shown in Ref. 17. Therefore we expect that the four loop braiding process is also related with the non-abelian property of the gauge flux loop. We leave studying the non-abelian structure of this quartic theory as future work.

IV. CONDENSATION PICTURE

We have so far discussed the coupled BF theories realizing three-loop or four-loop braiding statistics in isolation from physical contexts. In this section, we try to develop physical pictures of the topological field theories discussed above.

A. The BF theory

Let us start with the condensation picture of the single copy of the ordinary BF theory:

$$S = \frac{iK}{2\pi} \int b \wedge da. \quad (82)$$

(In this section, we will work with the Euclidean action.) The BF theory can be thought of as describing the zero correlation length limit of a gapped (topologically ordered) system, which may arise as a result of some sort of condensation.^{14,38-40} There are two complimentary pictures that describe the condensation, which are dual to

each other. In the following, we will develop these pictures by using the duality transformations. (We will use the equations of motion and integration over fields for convenience, but will treat the compactification conditions on the fields somewhat loosely. If necessary, the compactification conditions can be treated rigorously by using the generalized Poisson identity. See Ref. 40 and references therein.)

To discuss the first picture, let us take the equation of motion $\delta S/\delta b = 0$ of the BF theory, which sets $da = 0$. This suggests the Meissner effect and hence the Higgs phase. A convenient action, in which this picture is manifest, can be derived by integrating over b . It is convenient to perturb the BF theory to go away from the strict topological limit by adding

$$\frac{1}{2\lambda} db \wedge \star db + \frac{1}{g^2} da \wedge \star da + i \frac{\Theta}{8\pi^2} da \wedge da. \quad (83)$$

Here, the second and third terms are the Maxwell and axion terms for a , respectively, and the first term is a two-form analogue of the Maxwell term for b . The integration over b can be done by making use of the equation of motion derived by taking the functional derivative $\delta/\delta b$ of the perturbed BF theory, and plug the solution back into the action. The equation of motion can be solved as

$$db = -\frac{i\lambda}{2\pi} \star (d\theta + Ka), \quad (84)$$

where the scalar field θ arises as an ambiguity when integrating the equation of motion to express b in terms of a . Formally, the above manipulation is equivalent to dualizing the two form b to the zero-form θ . The resulting effective Lagrangian is

$$\mathcal{L} = \frac{\lambda}{8\pi^2} (d\theta + Ka)^2 + \frac{1}{g^2} da \wedge \star da + i \frac{\Theta}{8\pi^2} da \wedge da. \quad (85)$$

This is nothing but the Abelian Higgs model.^{14,38,41}

Alternatively, taking the equation of motion $\delta S/\delta a = 0$ of the BF theory sets $db = 0$. This suggests a two-form analogue of the Meissner effect, which can be interpreted as arising from the condensation of monopoles in the dual gauge field v of a . As before, we can integrate over a in the presence of the kinetic term (83). Solving the equation of motion $\delta S/\delta a = 0$, da can be expressed in terms of b as

$$da = \tilde{\tau}_1 (Kb + dv) + i\tilde{\tau}_2 \star (Kb + dv). \quad (86)$$

Here, $\tilde{\tau}_1$ and $\tilde{\tau}_2$ are the dual coupling constants and related to the original coupling constants as

$$\begin{aligned} \tilde{\tau}_1 &= -\frac{\tau_1}{\tau_1^2 + \tau_2^2}, & \tilde{\tau}_2 &= \frac{\tau_2}{\tau_1^2 + \tau_2^2}, \\ \tau_1 &= \frac{\Theta}{2\pi}, & \tau_2 &= \frac{4\pi}{g^2}. \end{aligned} \quad (87)$$

The one form v in (86) arises formally as an ambiguity in solving da in terms b . Plugging the solution back to

the action, we obtain the effective Lagrangian for b and v as

$$\begin{aligned} \mathcal{L} &= \frac{\tilde{\tau}_2}{4\pi} (Kb + dv) \wedge \star (Kb + dv) \\ &+ \frac{i\tilde{\tau}_1}{4\pi} (Kb + dv) \wedge (Kb + dv) + \frac{1}{2\lambda} db \wedge \star db. \end{aligned} \quad (88)$$

This is the Julia-Toulouse-Quevedo-Trugenberg effective action that describes the condensation of monopoles of the dual gauge field v .^{42,43}

It is also instructive to have a comparison with a slightly more microscopic model, which can realize the situation described above. For example, let us consider the Cardy-Rabinovic model⁴⁴

$$Z = \text{Tr}_{a,n,s} \prod_r \delta[\partial_\mu n_\mu(r)] \exp(-S), \quad (89)$$

where a_μ ($\mu = 1, \dots, 4$) is a compact $U(1)$ gauge field (an angular variable) defined on the links of the hypercubic lattice, and n_μ and $s_{\mu\nu}$ are integer-valued fields defined on links and plaquettes, respectively. The integer-valued two-form gauge field $s_{\mu\nu}$ amounts to allowing multivalued configurations of the gauge field. The sum on $s_{\mu\nu}$ corresponds to a sum over topologically non-trivial configurations with magnetic monopoles.⁴⁵ In fact, the monopole current is given explicitly by $m_\mu = (1/2)\epsilon_{\mu\nu\lambda\sigma}\partial_\nu s_{\lambda\sigma}$, where ∂_μ is the lattice difference operator in the μ -direction. On the other hand, we interpret n_μ as the electric current of a charge field. The discrete delta function $\delta[\partial_\mu n_\mu(r)]$ enforces current conservation. The Boltzmann weight is given by

$$\begin{aligned} S &= -iK \sum_L n_\mu a_\mu + \frac{1}{2g^2} \sum_P \Gamma_{\mu\nu} \Gamma_{\mu\nu} \\ &- \frac{iK\theta}{32\pi^2} \sum_{r,r'} f(r-r') \epsilon_{\mu\nu\lambda\sigma} \Gamma_{\mu\nu}(r) \Gamma_{\lambda\sigma}(r'), \end{aligned} \quad (90)$$

where $\Gamma_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu - 2\pi s_{\mu\nu}$ is the field strength. The second and third terms are the Maxwell and axion terms, respectively. (The precise nature of the smearing function $f(r-r')$ is not important here.) The sum over n_μ has the effect of constraining a_μ to take its values restricted to the abelian cyclic group \mathbb{Z}_K , $a_\mu = (2\pi/K)k_\mu$. Because the sum over n_μ is constrained, we can always add any total divergence to a_μ . Thus, the restriction to $a_\mu = (2\pi/K)k_\mu$ represents a partial fixing of the gauge.

For the Cardy-Rabinovic model, in the deconfined phase (charge condensation), there are $2\pi/K$ flux and the braiding with the charge leads to the fractional statistics. The effective theory is described by the BF theory.

B. The three-loop braiding theories

For the three-loop braiding theories, we can repeat the duality transformation, which we carried out for the ordinary BF theory (82) to obtain the Abelian-Higgs model

(85). Dualizing the two-form gauge fields b^I to scalars ϕ^I , we obtain an analogue of the Abelian-Higgs model

$$\mathcal{L} = \sum_I \frac{\lambda_I}{8\pi^2} (a^I + d\theta^I)^2 + C_{IJK} a^I \wedge a^J \wedge da^K + \sum_I \left[\frac{1}{g_I^2} da^I \wedge \star da^I + i \frac{\Theta_I}{8\pi^2} da^I \wedge da^I \right] + \dots \quad (91)$$

where we have introduced the coupling constants λ_I, g_I, Θ_I for each flavor. C_{IJK} describes the cubic coupling and takes different forms for the two- and three-flavor theories.

One can also consider an analogue of the Cardy-Rabinovic theory for the three-loop braiding theories. For example, for the cubic three-flavor theory (5), it may be considered as arising from the following extension of the Cardy-Rabinovic theory:

$$S = \frac{1}{2g^2} \sum_{I=1,2,3} \sum_P \Gamma_{\mu\nu}^I \Gamma_{\mu\nu}^I + iK \sum_{I=1,2} \sum_{L,\mu} a_\mu^I n_\mu^I + iK \sum_{L,\mu} a_\mu^3 \left(n_\mu^3 - \frac{P}{K} \epsilon_{\mu\nu\lambda\rho} a_\nu^1 \partial_\lambda a_\rho^2 \right). \quad (92)$$

The charge condensation phase of this extended Cardy-Rabinovic theory (92) is described by the coupled BF theory (5).

Alternatively, one may try to dualize the gauge fields a^I ; as we have seen, in the ordinary BF theory, dualizing the gauge field a leads to the Julia-Toulouse-Quevedo-Trugenberger effective action (88), and allows us to describe the charge condensation phases as the monopole condensation phase for the dual gauge field v . Due to the cubic coupling, dualizing a^I appears to be rather complicated. The electromagnetic duality exchanges the field strength da and its dual dv , but this does not necessarily mean it works at the level of the connection and exchanges a and v . In the coupled BF theories, the action is not written entirely in terms of the field strength da^I , but the connections a^I appear directly.

While it seems not possible to dualize all a^I , we can nevertheless dualize some of a^I . For example, let us consider the three-flavor theory with cubic coupling defined in (5). The action is written in terms of the field strength da^3 , and hence one can dualize a^3 . As for the first and second flavors, one can dualize $b^{I=1,2}$. The resulting action is

$$\begin{aligned} \mathcal{L} = & \frac{\tilde{\tau}_2}{4\pi} (K\Lambda^3 + dv^3) \wedge \star (K\Lambda^3 + dv^3) \\ & + \frac{i\tilde{\tau}_1}{4\pi} (K\Lambda^3 + dv^3) \wedge (K\Lambda^3 + dv^3) + \frac{1}{2\lambda_3} db^3 \wedge \star db^3 \\ & + \sum_{I=1,2} \left[\frac{\lambda_I}{8\pi^2} (a^I + d\theta^I)^2 \right. \\ & \left. + \frac{1}{g_I^2} da^I \wedge \star da^I + i \frac{\Theta_I}{8\pi^2} da^I \wedge da^I \right] \quad (93) \end{aligned}$$

where

$$\Lambda^3 = b^3 + \frac{2\pi P}{K} a^1 \wedge a^2. \quad (94)$$

Thus, after the dualization, the cubic coupling $a^1 \wedge a^2 \wedge da^3$ disappears, but the magnetic condensation for the dual gauge field v^3 is "dressed" by a^1 and a^2 .

The duality transformations can be also applied to the four-loop braiding theory, where the magnetic monopoles for the dual gauge field (v^4 , say) are dressed by the Borromean ring formed by a^1, a^2 and a^3 . Similar physical picture has been applied in constructing the wave functions for symmetry protected topological (SPT) phases, which can be realized by proliferating domain walls decorated with an SPT phase in one lower dimension.⁴⁶ In this respect, our models here are actually the gauged version of SPT phases.

V. CONCLUSION AND REMARKS

In conclusion, we canonically quantize the multi-flavor BF theories with cubic and quartic coupling. We study the algebra of Wilson operators to understand the three-loop and four-loop braiding processes. Using these Wilson operators, we also construct the multiplet of ground states of the three-loop braiding field theory on T^3 , and calculate the \mathcal{S} and \mathcal{T} matrices, which encode the fractional braiding and spin statistics. We also discuss the topological field theory as the condensation of composite particles from some parent $U(1)$ gauge theory.

We close with a few comments on open issues.

– In (3+1)d, apart from the particle-loop braiding described by the ordinary BF theory, there can be more exotic braiding, including three-loop braiding and four-loop braiding process. In this paper, we study the topological field theory describing the three-loop braiding and four-loop braiding process. By checking the equation of motion in Eqs. (4) and (65), these multiple-loop braiding process can all be understood as an effective particle braiding around the loop excitation. This effective particle can be a Hopf linking configuration, Borromean ring configuration or even more complicated knot configuration.

$$\frac{K}{2\pi} \int_{\Sigma} db^I = \int_{\Sigma} J_{qp}^I + \text{"Knot configuration"} \quad (95)$$

It would be interesting to study more complicated knot-loop braiding process in the future.

– In this paper, we mostly limit ourselves to T^3 as our spatial manifold, which is formal. It would be interesting to study more general cases in which the coupled BF theories are considered on the spacetime or spatial manifolds which are not formal. The coupled BF theories may be able to detect topological aspects (topological invariants) of these manifolds, which cannot be captured by the ordinary BF theory.

– We have carried out constructions of the multiplet of ground states on T^3 and calculated the modular \mathcal{S} and \mathcal{T} matrices, by using the basis of minimal entropy states for the ground state multiplet. Alternatively, the \mathcal{S} and \mathcal{T} matrices may be calculated by first constructing ground states for generic (holomorphic) polarization in geometric quantization. The action of the mapping class group of T^3 , $SL(3, \mathbb{Z})$, on the ground state multiplet can then be calculated by adiabatically changing polarization. We have so far constructed ground states only for the Hodge polarization. (See Appendix B for the definition and more details.) Construction of the ground states for more generic polarization is left as a future problem.

– Finally, we focussed throughout this work on the cases when the level K of BF theory was an integer greater than one. A few comments about the $K = 1$ case are in order. When $K = 1$, there is no topological order, hence there is a unique groundstate on any closed manifold. Furthermore there are no fractionalized excitations therefore no three-loop or four loop braiding for a theory with cubic or quartic interactions.

Given these facts, it seems that $K = 1$ BF theories and their cubic and quartic interacting versions are quite uninteresting and featureless, however this is not true. $K = 1$ BF theories are interesting in their own right and are indeed related to the physics of short-range-entangled phases of matter. For example it is possible to model symmetry protected topological phases of matter using level $K = 1$ BF theories. These theories may have global G symmetries in which case they can be coupled to a flat background G -gauge field A . The partition function in the presence of A is not trivial and can in fact distinguish different SPT phases. For onsite symmetry G with action labelled by $[\omega] \in H^4(BG, U(1))$, the partition function turns out to be

$$\mathcal{Z}_\omega[M, A] = \exp \{i \langle M, A^* \omega \rangle\} \quad (96)$$

where $A^* \omega \in H^4(M, U(1))$, the pull-back of ω to the spacetime manifold and \langle , \rangle is the pairing with the fundamental homology cycle.

Note added: Upon completion of this manuscript, we became aware of a recent work by J. Wang, et al.²¹, which also discusses, among others, the four-loop braiding process and the connection of the quartic theory.

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Appendix A: Wilson operator algebra and large gauge invariance in three-loop braiding field theory

In this appendix, we detail some important properties of the zero-mode Wilson operators introduced in section IID 1.

1. The Wilson operator algebra and three-loop braiding statistics

The three-loop braiding phase can be read off from the algebra of Wilson surface operators. To compute the algebra of Wilson operators, we use the Baker-Campbell-Hausdorff formula:

$$e^{\hat{A}} e^{\hat{B}} = \exp \left(\hat{A} + \hat{B} + \frac{1}{2} [\hat{A}, \hat{B}] + \frac{1}{12} [\hat{A} - \hat{B}, [\hat{A}, \hat{B}]] + \dots \right). \quad (A1)$$

Thus, for the products of Wilson operator,

$$\begin{aligned} & \hat{W}_i^{I\dagger} \hat{W}_j^{J\dagger} \hat{W}_i^I \hat{W}_j^J \\ &= \exp \left([i\hat{\Lambda}_i^I, i\hat{\Lambda}_j^J] \right), \\ & (\hat{W}_j^{J\dagger} \hat{W}_i^{I\dagger} \hat{W}_j^J \hat{W}_i^I) \hat{W}_k^{K\dagger} (\hat{W}_i^{I\dagger} \hat{W}_j^{J\dagger} \hat{W}_i^I \hat{W}_j^J) \hat{W}_k^K \\ &= \exp \left([[i\hat{\Lambda}_i^I, i\hat{\Lambda}_j^J], i\hat{\Lambda}_k^K] \right). \end{aligned} \quad (A2)$$

The triple commutator is a phase and the above algebra of Wilson surface operator describes the three-loop braiding phase. This is consistent with previous work on three-loop braiding statistics.²⁹ To have a non-zero three-loop braiding phase, I, J, K cannot be all equal. i, j, k cannot be all equal neither. We list non-zero triple-linking phase factors below:

$$\begin{aligned} [[\hat{\Lambda}_i^I, \hat{\Lambda}_j^I], \hat{\Lambda}_k^{\bar{I}}] &= -\epsilon_{ijk} \frac{4\pi q_I}{K^2}, \\ [[\hat{\Lambda}_i^1, \hat{\Lambda}_j^2], \hat{\Lambda}_k^1] &= [[\hat{\Lambda}_i^2, \hat{\Lambda}_j^1], \hat{\Lambda}_k^1] = \epsilon_{ijk} \frac{2\pi q_2}{K^2}, \\ [[\hat{\Lambda}_i^1, \hat{\Lambda}_j^2], \hat{\Lambda}_k^2] &= [[\hat{\Lambda}_i^2, \hat{\Lambda}_j^1], \hat{\Lambda}_k^2] = \epsilon_{ijk} \frac{2\pi q_1}{K^2}. \end{aligned} \quad (A3)$$

a. Large gauge invariance

Unlike the infinitesimal gauge transformations, the large gauge transformations cannot be derived from the conserved charges or Gauss law constraints of the action. However, the large gauge invariance can be deduced by demanding the invariance of the Wilson operators \hat{A}_i^I and \hat{W}_i^I . (Or vice versa: once the large gauge transformations are properly defined, the Wilson operators are defined as those that are invariant under the large gauge transformations.) Hence the correct large gauge transformations

are

$$\begin{aligned}\hat{\alpha}'_i &\rightarrow \hat{\alpha}''_i = \hat{\alpha}'_i + 2\pi n^I_i, \\ \hat{\beta}'_i &\rightarrow \hat{\beta}''_i = \hat{\beta}'_i - q_I \epsilon_{ijk} \left[n^I_j \hat{\alpha}''_k + \hat{\alpha}'_j n^I_k + 2\pi n^I_j n^I_k \right].\end{aligned}\quad (\text{A4})$$

It is worth noticing that, since $\hat{\beta}''_i$ transforms non-linearly under large gauge transformations, the $[\hat{\beta}''_i, \hat{\beta}''_j]$ commutator is not preserved. In fact,

$$\begin{aligned}[\hat{\beta}''_i, \hat{\beta}''_j] &= -\frac{4\pi i q_I}{K} \epsilon_{ijk} n^I_k, \\ [\hat{\beta}''_i, \hat{\beta}''_j] &= \frac{2\pi i}{K} \epsilon^{ijk} (q_1 n^2_k + q_2 n^1_k),\end{aligned}\quad (\text{A5})$$

However the algebra of observables, i.e the Wilson algebra transforms covariantly under large gauge transformations. E.g.,

$$[\hat{\Lambda}''_i, \hat{\Lambda}''_j] = \frac{-2iq_2}{K} \epsilon_{ijk} (\hat{\alpha}''_k + 2\pi n^2_k) = \frac{-2iq_2}{K} \epsilon_{ijk} \hat{\alpha}''_k. \quad (\text{A6})$$

Therefore, the operator algebra is preserved under the large gauge transformations.

As for the Wilson operators \hat{A}''_i and \hat{W}''_i , they are invariant under the large gauge transformations (A4) by construction. Nevertheless, it should be noted that their product may not be so, as seen in $\hat{W}''_i \hat{W}''_j \hat{W}''_i$ in Eq. (A2) (note the commutators in Eq. (53)), although the algebra of the Wilson operators is gauge covariant; The algebra of the Wilson operators generated by \hat{A}''_i and \hat{W}''_i and that generated by \hat{A}''_i and \hat{W}''_i are isomorphic. While $\hat{W}''_i \hat{W}''_j \hat{W}''_i$ is not gauge invariant, the product $(\hat{W}''_j \hat{W}''_i \hat{W}''_j) \hat{W}''_k \hat{W}''_i \hat{W}''_j$ and the triple commutator $[[i\hat{\Lambda}''_i, i\hat{\Lambda}''_j], i\hat{\Lambda}''_k]$ are large gauge invariant, and so is the three-loop braiding phase.

Appendix B: Ground state wave functionals by geometric quantization

In this section, we will construct (ground state) wave functions (functionals) of the coupled BF theories. The ground state wave functionals of topological quantum field theories such as the (2+1)-dimensional Chern-Simons theories and BF theories can be constructed by using the method of geometric quantization.^{15,47,48}

In geometric quantization, one endows the phase space with a complex line bundle E with curvature Ω (the symplectic two-form) and connection \mathcal{A} (the symplectic connection) such that Ω is expressed as $\Omega = d\mathcal{A}$ (at least locally). Sections of this line bundle form the pre-quantum Hilbert space with element Ψ . To obtain the ‘‘physical’’ Hilbert space which implements unitarity and irreducibility on the Poisson bracket, one further needs to impose a constraint on Ψ . This procedure is called choosing polarization. For more details of geometric quantization, see, Ref. 48, for example.

b. two-flavor v.s. three-flavor theories

In the following, we will construct the ground state wave functions of the coupled BF theories on $\Sigma = T^3$. We will focus on the quadratic avatar of the three flavors of BF theories coupled by a cubic term.

$$\begin{aligned}S = \int_{\mathcal{M}} &\left\{ \frac{K}{2\pi} \delta_{IJ} b^I \wedge da^J - \delta_{IJ} a^I \wedge J^J_{qv} \right. \\ &- \left[b^1 + \frac{q_1}{2\pi} a^2 \wedge a^3 \right] \wedge J^1_{qv} \\ &- \left[b^2 + \frac{q_2}{2\pi} a^3 \wedge a^1 \right] \wedge J^2_{qv} \\ &\left. - \left[b^3 + \frac{q_3}{2\pi} a^1 \wedge a^2 \right] \wedge J^3_{qv} \right\}.\end{aligned}\quad (\text{B1})$$

Furthermore, we will focus on the zero mode sector. (The wave functions of the ‘‘oscillator’’ part of the theory is identical to those in the ordinary BF theory, and can be constructed by following, e.g., Ref. 15.

Working with the three-flavor theory has a technical advantage than the two-flavor theory. To explain the advantage, we split the construction of the ground state wave functions in the following two steps:

(i) One first identifies the symplectic structure of the zero mode phase space. Then, following the generic procedure of the geometric quantization, one chooses the polarization (i.e., the choice of variables to use to write down wave functions). One can then identify the generic structure of the wave functions, inner product, etc. We call the set of wave functions obtained this way the ‘‘large’’ Hilbert space.

(ii) The ‘‘large’’ Hilbert space is not yet of our physical relevance, since they are not invariant under large gauge transformations. To further write down ground state wave functions explicitly, we need to demand the large gauge invariance (the Gauss law constraint). (Since systems of our interest are topological and there is no Hamiltonian. The large gauge invariance is the only guidance to construct physical ground state wave functions.) We demand the set of the wave functions are gauge-singlet (or in fact one can relax this condition a little bit; one may demand the wave functions to form a projective representation of the algebra of the gauge transformations. Such ‘‘generalized’’ gauge invariance is in particular relevant when the level K is a rational number $K = k_1/k_2$. Here, we will focus on the simplest case when $K = \text{integer or } k_2 = 1$).

For the two-flavor theory, the main difficulty is that, the large gauge transformations cannot be represented as a unitary operator within the ‘‘large’’ Hilbert space. This can be seen from the fact that the set of commutators are not preserved by the large gauge transformations. (See Sec. A 1 a.) In other words, the symplectic two-form is not preserved under the large gauge transformation. This should be contrasted to the case of the (2+1)-dimensional Chern-Simons theory and the ordinary BF theories in (3+1) dimensions. That the large gauge transformations cannot be represented as unitary operators within the

large Hilbert space does not mean that it is impossible to construct the “small” or restricted Hilbert space which is gauge invariant. Nevertheless, this difficulty adds some complication in constructing the ground state wave functions.

For the three-flavor theory, there is no such difficulty; the symplectic two-form is manifestly large gauge invariant; the technical reason why we will work with the three-flavor theory in this section.

c. choice of polarization

There is another complication in quantizing and constructing wave functions in coupled BF theories, which is associated to the choice of polarization. In the (2+1)-dimensional Chern-Simons theories and BF theories, it is convenient to choose a generic holomorphic polarization. In the case of the Chern-Simons theory, this is convenient when making a contact with (1+1)-dimensional conformal field theories. In the coupled BF theory, however, we will focus on a specific polarization, the “Hodge” polarization following the terminology in Ref. 49. In this polarization, we construct wave functions in terms of the zero modes α_i^I . One reason for this is that we found it is somewhat technically involved to construct the wave function by using the holomorphic polarization. However, on the other hand, the comparison with wave functions constructed in Sec. III E can be easily made for the wave functions in the Hodge polarization.

1. Geometric quantization of the BF theory

We now move on to the construction of wave functions by geometric quantization. We start by taking the ordinary BF theory on $\mathcal{M} = T^3 \times \mathbb{R}$ as an example. Our setting is described in Sec. III D. As mentioned earlier, we will focus on the zero mode sector. The zero modes of the BF theory satisfy the Poisson bracket

$$\{\alpha_i, \beta_j\} = \frac{2\pi}{K} \delta_{ij}. \quad (\text{B2})$$

a. the holomorphic polarization

Let us first construct wave functions in the holomorphic polarization following Ref. 15. In the holomorphic polarization, we introduce complex coordinates

$$\begin{aligned} \gamma_i &:= \alpha_i + \rho_{ij} \beta_j, \\ \bar{\gamma}_i &:= \alpha_i + \bar{\rho}_{ij} \beta_j, \end{aligned} \quad (\text{B3})$$

where ρ is an arbitrary symmetric 3×3 complex-valued matrix, whose imaginary part is negative-definite. ρ can be thought of as parametrizing a complex structure on

$H^1(\Sigma; \mathbb{R}) \oplus H^2(\Sigma; \mathbb{R})$ forming the multi-dimensional complex space of the γ variables.¹⁵ The inverse transformations are

$$\begin{aligned} \beta_i &= \frac{1}{2i} R_{ij} (\gamma - \bar{\gamma})_j, \\ \alpha_i &= \frac{-1}{2i} (\bar{\rho} R \gamma - \rho R \bar{\gamma})_i, \end{aligned} \quad (\text{B4})$$

where we introduced the notation

$$(\text{Im } \rho)_{ij}^{-1} = R_{ij} \quad (\text{B5})$$

The complex coordinates satisfy the Poisson bracket

$$\{\gamma_i, \bar{\gamma}_j\} = \frac{2\pi}{K} (-2i \text{Im } \rho_j^i). \quad (\text{B6})$$

The symplectic 2-form is

$$\Omega = -\frac{iK}{4\pi} R_{ij} d\bar{\gamma}_i \wedge d\gamma_j \quad (\text{B7})$$

We choose the symplectic potential as

$$\mathcal{A} = +\frac{K}{8\pi} (\bar{\gamma} - \gamma)_i R_{ij} (\bar{\rho} R d\gamma - \rho R d\bar{\gamma})_j, \quad (\text{B8})$$

which satisfies $d\mathcal{A} = \Omega$.

As a first step of constructing ground state wave functions, we choose a particular polarization and impose the condition:

$$\begin{aligned} \left(\frac{\partial}{\partial \bar{\gamma}_i} + i\mathcal{A}_{\bar{\gamma}_i} \right) \Psi &= 0 \\ \Rightarrow \left(\frac{\partial}{\partial \bar{\gamma}_i} - i\frac{K}{8\pi} [(\bar{\gamma} - \gamma) R \rho R]_i \right) \Psi &= 0. \end{aligned} \quad (\text{B9})$$

Solutions to this constraint are given by

$$\Psi(\gamma, \bar{\gamma}) = \exp \left[-i\frac{K}{16\pi} (\bar{\gamma} - \gamma) R \rho R (\bar{\gamma} - \gamma) \right] f(\gamma), \quad (\text{B10})$$

where f is a function of γ only. The set of all wave functions of the above form constitute what we have called the “large” Hilbert space.

We now construct a set of ground state wave functions by imposing the invariance under large gauge transformations

$$\gamma \rightarrow \gamma + 2\pi(n + \rho m). \quad (\text{B11})$$

In the following, we present two slightly different construction of the wave functions.

In the first construction, we note, under the large gauge transformations, the symplectic potential is transformed as

$$\begin{aligned} \mathcal{A} &\rightarrow \mathcal{A} + d\Lambda \\ \text{where } \Lambda &= -\frac{Ki}{2} m \cdot (\bar{\rho} R \gamma - \rho R \bar{\gamma}) + \text{const.} \end{aligned} \quad (\text{B12})$$

where the constant term can depend on m and n . Physical wave functions, which are gauge invariant, then must satisfy

$$\Psi(\gamma + 2\pi(n + \rho m), \bar{\gamma} + 2\pi(n + \bar{\rho} m)) = e^{i\Lambda} \Psi(\gamma, \bar{\gamma}) \quad (\text{B13})$$

This condition is translated into the condition on f :

$$e^{+iK m \cdot \gamma + i\pi K m \cdot \rho \cdot m} f(\gamma + 2\pi(n + m\rho)) = f(\gamma) \quad (\text{B14})$$

up to an unknown phase factor mentioned above. The solution can be constructed by using the Jacobi theta function:

$$\Psi_q(\gamma) = \Theta\left(\frac{c+q}{K}, \frac{K}{d}\right) \left(\frac{K}{2\pi}|\gamma| - K\rho\right), \quad (\text{B15})$$

where c and d are arbitrary parameters (“twisting angles”). Here, the Jacobi theta function is defined by

$$\Theta\left(\frac{c}{d}\right)(z|\Pi) := \sum_{n_\ell \in \mathbb{Z}^p} \exp\left[i\pi(n+c)^\ell \Pi_{\ell k}(n+c)^k + 2\pi i(n+c)^\ell(z+d)_\ell\right] \quad (\text{B16})$$

where $c^\ell, d_\ell \in [0, 1]$ and $z_\ell \in \mathbb{C}$. The theta function satisfies

$$\Theta\left(\frac{c}{d}\right)(z_\ell + s_\ell + \Pi_{\ell k} t^k |\Pi) := \exp\left[2\pi i c^\ell s_\ell - i\pi t^\ell \Pi_{\ell k} t^k - 2\pi i t^\ell(z+d)_\ell\right] \Theta\left(\frac{c}{d}\right)(z|\Pi) \quad (\text{B17})$$

for integers s_l and t_l , and

$$\Theta\left(\frac{c}{d}\right)(z_\ell + C\Pi_{\ell k} t^k |\Pi) = \exp\left[-i\pi C^2 t^\ell \Pi_{\ell k} t^k - 2\pi i C t^\ell(z+d)_\ell\right] \Theta\left(\frac{c+Ct}{d}\right)(z|\Pi) \quad (\text{B18})$$

for any non-integer $C \in \mathbb{R}$. We note, in particular,

$$\begin{aligned} & \Theta\left(\frac{c+q}{K}, \frac{K}{d}\right) \left(\frac{K}{2\pi}[\gamma + 2\pi(n + \rho m)] - K\rho\right) \\ &= \exp\left[2\pi i c \cdot n + i\pi m \cdot K\rho \cdot m + i2\pi m \cdot d\right] \exp(iK m \cdot \gamma) \Theta\left(\frac{c+q}{K}, \frac{K}{d}\right) \left(\frac{K}{2\pi}|\gamma| - K\rho\right) \end{aligned} \quad (\text{B19})$$

In the second construction, we implement the large gauge transformation by using unitary operators, which we call $U_{m,n}$. This operator sends $\alpha \rightarrow \alpha + 2\pi n$ and $\beta \rightarrow \beta + 2\pi m$:

$$U_{m,n} \alpha U_{m,n}^\dagger = \alpha + 2\pi n, \quad U_{m,n} \beta U_{m,n}^\dagger = \beta + 2\pi m. \quad (\text{B20})$$

The unitary operator can be identified, up to a constant phase factor, as

$$U_{m,n} = \exp\left[-iK(m_i \alpha_i - n_i \beta_i)\right]. \quad (\text{B21})$$

Noting $\bar{\gamma}_i = -(4\pi/K)R_{ij}^{-1}(\partial/\partial\gamma_j)$, the operator implementing the large gauge transformations can be written as

$$\begin{aligned} U_{m,n} &= \exp\left[2\pi(n + m \cdot \rho) \cdot \frac{\partial}{\partial\gamma} + \frac{K}{2}(n + m \cdot \bar{\rho}) \cdot R \cdot \gamma\right] \\ &= e^{\frac{\pi K}{2}(n+m\bar{\rho})R(n+m\rho) + \frac{K}{2}(n+m\bar{\rho})R\gamma} \\ &\quad \times \exp\left[2\pi(n + m\rho) \frac{\partial}{\partial\gamma}\right] \end{aligned} \quad (\text{B22})$$

The action of U on wave functions is

$$U_{m,n} \Psi(\gamma) = e^{\frac{\pi K}{2}(n+m\bar{\rho})R(n+m\rho) + \frac{K}{2}(n+m\bar{\rho})R\gamma} \times \Psi(\gamma + 2\pi(n + \rho m)) \quad (\text{B23})$$

The wave functions that solve this constraint are given by

$$\Psi_q(\gamma) = e^{-\frac{K}{8\pi}\gamma_i R_{ij} \gamma_j} \Theta\left(\frac{c+q}{K}, \frac{K}{d}\right) \left(\frac{K}{2\pi}|\gamma| - K\rho\right), \quad (\text{B24})$$

where c and d are arbitrary parameters (“twisting angles”).

b. the Hodge polarization

We have so far constructed wave functions by using the holomorphic polarization (B3). We now try a different polarization, which we call the Hodge polarization, following, Ref. 49. In this polarization, we attempt to write down the wave function in terms of α_i : $\Psi(\alpha)$. Given the canonical commutation relation $[\alpha_i, \beta_j] = (2\pi i/K)\delta_{ij}$, β_i

acts on the wave functions as $\beta_i = -i(2\pi/K)\partial/\partial\alpha_i$. Demanding (B20), the unitary transformations that implement large gauge transformations can be represented as

$$\begin{aligned} U_{m,n} &= \exp[-iKm \cdot \alpha + iKn \cdot \beta] \\ &= e^{-\pi i Km \cdot n - i Km \cdot \alpha} \exp\left[2\pi n \cdot \frac{\partial}{\partial \alpha}\right]. \end{aligned} \quad (\text{B25})$$

Physical wave functions can be constructed by demanding large gauge invariance:

$$U_{m,n}\Psi(\alpha) = e^{i\Theta_{m,n}}\Psi(\alpha) \quad (\text{B26})$$

where $\Theta_{m,n}$ is a constant phase, which can depend on m and n . I.e.,

$$\begin{aligned} U_{m,n}\Psi(\alpha) &= e^{-\pi i Km \cdot n - i Km \cdot \alpha} \Psi(\alpha + 2\pi n) \\ &= e^{i\Theta_{m,n}}\Psi(\alpha) \end{aligned} \quad (\text{B27})$$

This constraint can be solved by an ansatz

$$\Psi(\alpha) = \sum_{k \in \mathbb{Z}^3} C(k) e^{ik \cdot \alpha}. \quad (\text{B28})$$

From the large gauge invariance, C must satisfy the constraint

$$C(p + Km) = e^{i\theta} C(p), \quad (\text{B29})$$

which can be solved by

$$C_q(p) = \begin{cases} e^{i\theta l} & \text{when } p = q + Kl \\ 0 & \text{otherwise} \end{cases} \quad (\text{B30})$$

To summarize, the solutions are

$$\begin{aligned} \Psi_q(\alpha) &= e^{i\theta \cdot \alpha} e^{iq \cdot \alpha} \sum_l e^{iKl \cdot (\alpha + \phi/K)} \\ &= e^{i\theta \cdot \alpha} e^{iq \cdot \alpha} \frac{2\pi}{K} \sum_m \delta\left(\alpha + \frac{\phi}{K} + \frac{2\pi}{K}m\right). \end{aligned} \quad (\text{B31})$$

The free parameter ϕ and θ are the twisting angle.

The states we have constructed are eigen states of $B_i = \exp i\beta_i = \exp[(2\pi/K)\partial/\partial\alpha_i]$. On the other hand, applying A_i changes the label q as $A_i\Psi_q = \Psi_{q+\hat{n}_i}$, where $\hat{n}_i = (0, \dots, 1, \dots, 0)$.

2. Three-loop braiding theory with three flavors

We now move on to the construction of wave functions of the quadratic three loop braiding BF theory with three flavors. The zero modes of the three-flavor theory satisfy the Poisson bracket

$$\{\alpha_i^I, \beta_j^J\} = \frac{2\pi}{K} \delta_{ij} \delta^{IJ}. \quad (\text{B32})$$

The symplectic form and potential ($I, J = 1, 2, 3$) are given by

$$\Omega = \frac{K}{2\pi} d\beta_i^I \wedge d\alpha_i^I, \quad \mathcal{A} = \frac{K}{2\pi} \beta_i^I d\alpha_i^I. \quad (\text{B33})$$

In the quadratic three-flavor BF theory, the fundamental Wilson surface operators are defined by taking the exponential of

$$\Lambda_i^I = \beta_i^I + r \epsilon_{ijk}^{IJK} \alpha_j^J \alpha_k^K, \quad (\text{B34})$$

as $\exp i\Lambda_i^I$, where we have introduced $\epsilon_{ijk}^{IJK} = \epsilon^{IJK} \epsilon_{ijk}$. Generic Wilson surface operators are given by taking products thereof. The parameter r plays a role similar to $q_{1,2}$ in the two-flavor theory,

There is a set of large gauge transformations that preserve Λ_i^I :

$$\begin{aligned} \alpha_i^I &\rightarrow \alpha_i^I + 2\pi n_i^I \\ \beta_i^I &\rightarrow \beta_i^I - 4\pi r \epsilon_{ijk}^{IJK} n_j^J \alpha_k^K - 4\pi^2 r \epsilon_{ijk}^{IJK} n_j^J n_k^K \end{aligned} \quad (\text{B35})$$

The symplectic form is invariant under these large gauge transformations. Under the large gauge transformations, the symplectic form is transformed as

$$\begin{aligned} \mathcal{A} &\rightarrow \mathcal{A} + d\Lambda \\ \Lambda &= +Km_i^I \alpha_i^I - Kr \epsilon_{ijk}^{IJK} n_j^J \alpha_k^K \alpha_i^I \\ &\quad - 2\pi Kr \epsilon_{ijk}^{IJK} n_j^J n_k^K \alpha_i^I + \text{const.} \end{aligned} \quad (\text{B36})$$

In the following, we will write down a set of ground state wave functions for the quadratic three-flavor BF theory. We present two different constructions. In the first construction, we choose to work with α_i^I and Λ_i^I . Following the previous section, we introduce a holomorphic polarization for these variables. A merit of this construction is that the large gauge transformations act on these variables in a simple fashion. In the second construction, we choose to work with α_i^I and β_i^I , and use the Hodge polarization. Unlike Λ_i^I , the large gauge transformations act on β_i^I non-trivially.

a. using Λ as a variable

Following the holomorphic polarization of the ordinary BF theory (B3), we introduce

$$\begin{aligned} \gamma_i^I &= \alpha_i^I + \rho_{ij} \Lambda_j^I, \\ \bar{\gamma}_i^I &= \alpha_i^I + \bar{\rho}_{ij} \Lambda_j^I. \end{aligned} \quad (\text{B37})$$

The wave functions can be constructed by demanding

$$\left(\frac{\partial}{\partial \bar{\gamma}_i^I} + i\mathcal{A}_{\bar{\gamma}_i^I}\right) \Psi(\gamma, \bar{\gamma}) = 0. \quad (\text{B38})$$

The solutions to this constraint are given by

$$\Psi(\gamma, \bar{\gamma}) = \exp\left[\frac{iK}{16\pi} \Lambda_i^I \rho_{ij} \Lambda_j^I - \frac{iKr}{6\pi} \epsilon_{ijk}^{IJK} \alpha_i^I \alpha_j^J \alpha_k^K\right] f(\gamma), \quad (\text{B39})$$

where $f(\gamma)$ is a function of γ only.

We now impose the large gauge invariance. Up to a constant phase factor, f must transform as

$$\begin{aligned} & e^{+iKm^I \cdot \gamma^I + i\pi Km^I \cdot \rho \cdot m^I - \frac{4i\pi^2 K\epsilon}{3} \epsilon_{IJK} n^I \cdot n^J \times n^K} \\ & \times f(\gamma + 2\pi(n + \rho \cdot m)) \\ & = f(\gamma) \end{aligned} \quad (\text{B40})$$

Up to the phase factor, this constraint is the same as the one in the ordinary BF theory. Hence, the solutions to the gauge constraint are given in terms of the theta function.

b. the Hodge polarization

We now attempt to construct the wave functions by using the Hodge polarization, in which the wave functions are constructed as a function of α_i^I . On these wave functions, β_i^I acts as $\beta_i^I = -i(2\pi/K)\partial/\partial\alpha_i^I$. We first look for unitary operators $U_{m,n}$ that implement the large gauge transformations (B35). Up to a phase factor, the unitary operators $U_{m,n}$ are identified as

$$\begin{aligned} U_{m,n} = \exp \left[-iKm^I \cdot \alpha^I + iKn^I \cdot \beta^I \right. \\ \left. + iK\epsilon^{IJK} n^I \cdot \alpha^J \times \alpha^K \right]. \end{aligned} \quad (\text{B41})$$

$U_{m,n}$ can also be written as

$$U_{m,n} = e^{i\phi_{n,m}} e^A e^B \quad (\text{B42})$$

where

$$\begin{aligned} A &= -iKm^I \cdot \alpha^I + 2\pi iK\epsilon^{IJK} n^I \cdot n^J \times \alpha^K \\ & \quad + iK\epsilon^{IJK} n^I \cdot \alpha^J \times \alpha^K, \\ B &= +2\pi n^I \cdot \frac{\partial}{\partial\alpha^I}, \\ \phi_{n,m} &= -i\pi Km^I \cdot n^I + \frac{4\pi^2 i}{3} K\epsilon^{IJK} n^I \cdot (n^J \times n^K) \end{aligned} \quad (\text{B43})$$

The physical wave functions are constrained by the large gauge invariance and must satisfy: $U_{m,n}\Psi(\alpha) = e^{i\Theta_{m,n}}\Psi(\alpha)$. This large gauge constraint can be solved by the ansatz

$$\Psi(\alpha) = e^{-\frac{iK\epsilon}{6\pi} \epsilon^{IJK} \alpha^I \cdot (\alpha^J \times \alpha^K)} \prod_{I=1}^3 \sum_{k_I} C_I(k_I) e^{ik^I \cdot \alpha^I} \quad (\text{B44})$$

Observe that this wave function can be also written as

$$\begin{aligned} \Psi(\alpha) &= \prod_{I=1}^3 \sum_{k_I} C_I(k_I) e^{i(k^I - \frac{K\epsilon}{18\pi} \epsilon^{IJK} \alpha^J \times \alpha^K) \cdot \alpha^I} \\ &= \sum_k C_1(k^1) e^{i(k^1 - \frac{K\epsilon}{\pi} \alpha^2 \times \alpha^3) \cdot \alpha^1} \prod_{I=1}^2 \sum_{k_I} C_I(k_I) e^{ik^I \cdot \alpha^I}. \end{aligned} \quad (\text{B45})$$

The large gauge invariance constrains C_I to satisfy

$$C_I(p + Km) = e^{i\theta} C_I(p), \quad (\text{B46})$$

which can be solved by the same ansatz as in the ordinary BF theory,

$$C_{Iq}(p) = e^{i\theta l} \quad \text{when } p = q + Kl \quad (\text{B47})$$

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