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Time evolution of the Luttinger model with non-uniform temperature profile

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We study the time evolution of a 1D interacting fermion system described by the Luttinger model starting from a non-equilibrium state defined by a smooth temperature profile $T(x)$. As a specific example we consider the case when $T(x)$ is equal to T_L (T_R) far to the left (right). Using a series expansion in $\epsilon = 2(T_R - T_L)/(T_L + T_R)$, we compute the energy density, the heat current density, and the fermion two-point correlation function for all times $t \geq 0$. For local (delta-function) interaction, the first two are computed to all orders giving simple exact expressions involving the Schwarzian derivative of the integral of $T(x)$. For non-local interaction, breaking scale invariance, we compute the non-equilibrium steady state (NESS) to all orders and the evolution to first order in ϵ. The heat current in the NESS is universal even when conformal invariance is broken by the interaction, and its dependence on $T_{L,R}$ agrees with numerical results for the XXZ spin chain. Moreover, our analytical formulas predict peaks at short times in the transition region between different temperatures and show dispersion effects that, even if non-universal, are qualitatively similar to ones observed in numerical simulations for related models, such as spin chains and interacting lattice fermions.

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I. INTRODUCTION

Experiments on ultracold atomic gases have led to renewed interest in non-equilibrium properties of isolated 1D quantum systems.[1–6] This field also has roots in a rich history of theoretical works studying both classical[7–13] and quantum systems[14–24] out of equilibrium. One often studied protocol is to join, at time $t = 0$, disconnected left and right parts of an infinite system, where each part is in thermal equilibrium with temperatures T_L and T_R, respectively. For $t > 0$ the system is evolved with a fully translational invariant Hamiltonian; this produces a heat current and, for long times, the system tends to a *non-equilibrium steady state* (NESS) if $T_L \neq T_R$. This is usually referred to as the *partitioning protocol*.

Using the above protocol, exact results for the NESS were obtained for simple integrable models such as the XX and XY spin chains using C^*-algebraic methods[25–28] and non-equilibrium Green's functions[29] (Keldysh formalism). When written in terms of fermions, these models are all *non-interacting*: they can be mapped to 1D systems of spinless lattice fermions with Hamiltonians that are quadratic in the fermion fields. For general systems of free lattice fermions, results for the NESS were obtained using a generalized Landauer-Büttiker formula.[18,19]

For *interacting* fermions the partitioning protocol was successfully used to obtain exact results for systems described by conformal field theories (CFTs).[30–34] Beyond CFT there are otherwise few exact results for the NESS, and even fewer for the evolution, of interacting fermions; see, e.g., Refs. 35–41. Using the same protocol, the time evolution and properties of the NESS have been studied extensively numerically[42–44] and by approximate analytical methods[45–47] in various models. Recently, effective hydrodynamic equations for the long-time and large-distance dynamics for Bethe ansatz-solvable models were proposed;[48,49] see also Refs. 50 and 51. We also mention recent studies of the heat current and the thermal Drude weight based on Bethe ansatz,[52] density matrix renormalization group,[53] and hydrodynamics.[54–56]

Most results for systems of interacting fermions, such as those mentioned above, rely on approximate methods or on assumptions, and it is thus interesting to obtain exact results for specific models that can serve as a benchmark. In this paper we present some exact results for the *full time evolution* (not just the NESS) of a continuum system of interacting fermions described by the Luttinger model[57–60] on the real line starting, at $t = 0$, from a *non-equilibrium state* defined by a smooth temperature profile $T(x)$. This is related to but different from the partitioning protocol. Specifically, if $\mathcal{H}(x)$ is the energy density operator defining the Hamiltonian, $H = \int dx\, \mathcal{H}(x)$, then the initial state is given by $\hat{\rho} = e^{-\mathcal{G}}/\operatorname{Tr} e^{-\mathcal{G}}$ with

$$\mathcal{G} = \int dx\, \beta(x)\mathcal{H}(x), \tag{1}$$

where $\beta(x) \equiv T(x)^{-1} = \beta[1 + \epsilon W(x)]$ for some smooth function $W(x)$ with β the average inverse temperature and ϵ the distance from equilibrium. (We use units such that $\hbar = k_B = 1$.) We will mainly be concerned with the case of a step-like profile $T(x)$ equal to T_L (T_R) far to the left (right), e.g., $W(x) = -(1/2)\tanh(x/\delta)$ with $\delta > 0$, where β and ϵ are determined by $\beta(\mp\infty) = T_{L,R}^{-1}$. The evolution of the system is given by H, and we are interested in *non-equilibrium expectation values* ($\epsilon \neq 0$) of local observables $\mathcal{O}$,

$$\langle \mathcal{O}(t) \rangle \equiv \operatorname{Tr} \hat{\rho}\mathcal{O}(t), \tag{2}$$

where $\mathcal{O}(t) = e^{iHt}\mathcal{O}e^{-iHt}$. If $\epsilon = 0$, then $\langle\mathcal{O}(t)\rangle = \langle\mathcal{O}\rangle_\beta$ is an equilibrium expectation value with temperature $T = \beta^{-1}$. For the Luttinger model, such equilibrium properties are well-known since a long time from the celebrated exact solution in Ref. 60 using bosonization; see also, e.g., Refs. 61–68.

We use a series expansion in ϵ to compute the time evolution and the NESS for the Luttinger model both in the case of local (delta-function) and non-local interactions starting from a non-equilibrium state. We show that the NESS is factorized in terms of the eigenmodes of the interacting Hamiltonian (plasmons)[60] and not in terms of the fermions; the presence of interaction is manifested by interaction-dependent exponents in the fermion two-point correlation function. In contrast, we find that the final heat current is universal even for an interaction that breaks conformal invariance, and the form of its dependence on $T_{L,R}$ confirms previous numerical results for interacting lattice fermion models, such as the XXZ spin chain studied in Ref. 42. For non-interacting lattice models such results for the temperature dependence were obtained analytically.[18,19,25,29]

For local interaction, our series for the energy and heat current densities can be summed into exact formulas for the time evolution. These results contain the *Schwarzian derivative*[69] of the integral of $T(x)$, which is very suggestive in view of the conformal invariance in the local case. Its presence produces peaks in the energy and heat current densities at zero time in the transition region between different temperatures. These resemble what is found numerically in related models,[42,45] even if the shapes of such peaks clearly are non-universal.

For non-local interaction, breaking conformal invariance, we obtain analytical results for the NESS to all orders and for the time evolution to first order in ϵ. In this case, dispersive effects appear in the evolution, which look qualitatively similar to those seen numerically in lattice models. (Such dispersive effects are absent for local interactions.)

The following two methods are used to compute non-equilibrium expectation values: *Method 1* based on the Dyson series and *Method 2* using one-particle operators; see Sec. V A and Sec. V B, respectively. *Method 1* allows one to compute non-equilibrium results to first order in ϵ from equilibrium ones, and it can be used even for non-exactly solvable models. *Method 2* allows one to compute results for the Luttinger model to all orders in ϵ, and it is in general applicable only to models that are quasi-free.

We consider the Luttinger model given by

$$H = \sum_r \int dx\, {:}\psi_r^+(x)\left(-irv_F\partial_x\right)\psi_r^-(x){:} \qquad (3)$$
$$+\lambda\sum_{r,r'}\int dxdy\, V(x-y){:}\psi_r^+(x)\psi_r^-(x){::}\psi_{r'}^+(y)\psi_{r'}^-(y){:}$$

with fermion fields $\psi_r^-(x)$ and $\psi_r^+(x) = \psi_r^-(x)^\dagger$, where $r = +(-)$ denotes right(left)-moving fermions, $:\cdots:$ indicates Wick (normal) ordering, $v_F > 0$ is the Fermi velocity, $V(x)$ is the interaction potential, and λ is the coupling constant. We use notation similar to Refs. 41 and 60; cf. also Refs. 39 and 67 and references therein. Let $\hat{V}(p) = \int dx\, V(x)e^{-ipx}$ denote the Fourier transform of the potential. The interaction must satisfy $\lambda\hat{V}(p) > -\pi v_F/2$, and $V(x)$ can be local, $V(x) = \pi v_F\delta(x)/2$, which requires renormalizations, or non-local with interaction range $a > 0$, e.g., $V(x) = \pi v_F/[4a\cosh(\pi x/2a)]$. The above examples of potentials are used in Figs. 1 and 2 to illustrate our analytical results, but we emphasize that these results hold true for a large class of interactions.[41,67]

In what follows we study the evolution of the energy density $E(x,t) \equiv \langle\mathcal{H}(x,t)\rangle$, the heat current density $J(x,t) \equiv \langle\mathcal{J}(x,t)\rangle$, and the fermion two-point correlation function $S_r(\xi,\tau,x,t) \equiv \langle\psi_r^+(x+\xi,t+\tau)\psi_r^-(x,t)\rangle$, where $\mathcal{J}(x,t)$ is determined by the continuity equation $\partial_t\mathcal{H}(x,t) + \partial_x\mathcal{J}(x,t) = 0$. We start in Sec. II by presenting results for the NESS. This serves as a useful benchmark for the finite-time results presented in Sec. III for local interaction and in Sec. IV for non-local interaction. Our methods are described in Sec. V, and concluding remarks are given in Sec. VI. Some computational details are deferred to an appendix.

II. NON-EQUILIBRIUM STEADY STATE

It is well-known that the Fourier modes of the fermion densities, $\rho_r(p) \equiv \int dx\, {:}\psi_r^+(x)\psi_r^-(x){:}\, e^{-ipx}$, define boson operators,[60] and that the Luttinger Hamiltonian can be written as $H = H_+ + H_-$ with

$$H_r = \frac{1}{2}\int dq\, v(q){:}\tilde{\rho}_r(-q)\tilde{\rho}_r(q){:} \qquad (4)$$

using Bogoliubov transformed fermion densities $\tilde{\rho}_r(p) = \rho_r(p)\cosh\varphi(p) - \rho_{-r}(p)\sinh\varphi(p)$, where $\tanh 2\varphi(p) = -\lambda\hat{V}(p)/[\pi v_F + \lambda\hat{V}(p)]$, and the renormalized Fermi velocity $v(p) = v_F\sqrt{1 + 2\lambda\hat{V}(p)/\pi v_F}$.[41,60,68] The $\tilde{\rho}_r(p)$ are commonly referred to as *plasmons*, and the Luttinger Hamiltonian is diagonal in terms of these.[60] To find the NESS we write $\hat{\rho}(t) = e^{-iHt}\hat{\rho}e^{iHt} = e^{-\mathcal{G}(-t)}/\operatorname{Tr}(e^{-\mathcal{G}(-t)})$ with $\mathcal{G}(t) = \int dx\,\beta(x)\mathcal{H}(x,t)$ and express $\mathcal{H}(x,t)$ in terms of $\tilde{\rho}_r(p,t) = \tilde{\rho}_r(p)e^{-ir\omega(p)t}$ where $\omega(p) = v(p)p$. Taking $t\to\infty$ in $\hat{\rho}(t)$ by making use of the Riemann-Lebesgue lemma (cf., e.g., Ref. 41), which can be justified for expectation values using *Method 2*, we find

$$\lim_{t\to\infty}\operatorname{Tr}\hat{\rho}(t)\mathcal{O} = \frac{\operatorname{Tr}e^{-\beta_+H_+-\beta_-H_-}\mathcal{O}}{\operatorname{Tr}e^{-\beta_+H_+-\beta_-H_-}} \qquad (5)$$

with $\beta_+ = T_L^{-1}$ and $\beta_- = T_R^{-1}$. This NESS describes a translation invariant state factorized into right- and left-moving plasmons at equilibrium with temperatures $T_\pm = \beta_\pm^{-1}$. A similar NESS was obtained in Refs. 30–32 for CFTs and in Refs. 25–28 for the XX chain; in the latter case the same factorization of the NESS is valid also in terms of right- and left-moving *fermions*, whereas in our

case only the plasmons factorize in such a way but not the fermions.

The long time limit of expectation values for all local observables can be computed using (5) by straightforward generalizations of well-known equilibrium computations. By recalling that $\int dx\,\mathcal{H}(x) = \sum_r H_r$ with H_r in (4) and using the continuity equation to show that $\int dx\,\mathcal{J}(x) = \sum_r (r/2)\int dq\,(d\omega(q)/dq)v(q){:}\tilde\rho_r(-q)\tilde\rho_r(q){:}$, we obtain

$$\begin{aligned}\lim_{t\to\infty} E(x,t) &= w_\lambda + \sum_r \int_{\mathbb{R}^+} \frac{dq}{2\pi}\frac{\omega(q)}{e^{\beta_r\omega(q)}+1},\\ \lim_{t\to\infty} J(x,t) &= \sum_r r\int_{\mathbb{R}^+}\frac{dq}{2\pi}\frac{d\omega(q)}{dq}\frac{\omega(q)}{e^{\beta_r\omega(q)}+1},\end{aligned} \tag{6}$$

where w_λ is the ground state energy density,[41,60] using that the NESS is translation invariant. Similarly, for the fermion two-point correlation function, using the well-known bosonization formula expressing fermions as exponentials of plasmons (see, e.g., Refs. 41, 65–67 and references therein), we find

$$\begin{aligned}\lim_{t\to\infty} S_r(\xi,\tau,x,t) &= \frac{i}{2\pi u_r}\exp\left(\int_{\mathbb{R}^+}\frac{dq}{q}\left\{e^{iqu_r(q)}-e^{iqu_r}\right\}\right)\\ &\times\exp\left(\int_{\mathbb{R}^+}\frac{dq}{q}\sinh^2\varphi(q)\left\{e^{iqu_r(q)}+e^{iqu_{-r}(q)}-2e^{-q0^+}\right\}\right)\\ &\times\exp\left(\int_{\mathbb{R}^+}\frac{dq}{q}\left[\cosh^2\varphi(q)\frac{2\{\cos(qu_r(q))-1\}}{e^{\beta_r\omega(q)}-1}\right.\right.\\ &\left.\left.+\sinh^2\varphi(q)\frac{2\{\cos(qu_{-r}(q))-1\}}{e^{\beta_{-r}\omega(q)}-1}\right]\right),\end{aligned} \tag{7}$$

where $u_r(p)\equiv r[\xi - rv(p)\tau]+i0^+$ and $u_r = u_r(0)$.

The second integral in (6) gives the final energy flow and appears to depend on the interaction. However, by the change of variables $u=\beta_r\omega(q)$ we obtain

$$\lim_{t\to\infty} J(x,t) = \sum_r r\frac{\pi T_r^2}{12} = \frac{\pi}{12}(T_L^2 - T_R^2)\equiv J \tag{8}$$

due to the presence of the group velocity $d\omega(q)/dq$ in the integrand (assuming $d\omega(q)/dq>0$, which is true for a large class of interaction potentials[41]). It follows that the final heat current only depends on $T_{L,R}$ and is independent of microscopic details. Such universal behavior, previously observed in CFTs,[30–32] thus remains true for the Luttinger model even when scale invariance is broken by the interaction. This results supports the conjecture, based on numerical simulations of the XXZ chain,[42] that for interacting fermions, $J = f(T_L) - f(T_R)$, where, in general, f is a non-universal function tending to the universal CFT result[30] in the low-temperature limit.

For non-interacting fermions, the temperature dependence $J = f(T_L) - f(T_R)$ corresponds to the above mentioned factorization of the NESS and was previously obtained analytically by different methods.[18,19,25,29] In fact, using these analytical results, the function f for the XX chain can be computed analytically: $f(T) = (\pi/12)T^2[1-R(b_+)-R(b_-)]$ with non-universal corrections $R(b_\pm) = (6/\pi^2)\int_{b_\pm}^\infty dx\,x/(e^x+1)$ governed by $b_\pm = (v_F/Ta_0)[1\pm\cos(\nu\pi)]/\sin(\nu\pi)$, where a_0 is the lattice spacing and $0<\nu<1$ is the filling factor (specifying the Fermi momentum $k_F = \nu\pi/a_0$). If Ta_0/v_F is small, the corrections are exponentially suppressed, and the universal result becomes exact in the *scaling limit* $Ta_0/v_F\to 0$.

The first integral in (6) expresses the energy density in the NESS as a sum of energy densities at equilibrium with temperatures $T_{L,R}$ and is non-universal. Indeed, it depends on the interaction, and only in the local case, when $v(p)=v$ and $\varphi(p)=\varphi$ are constant, does it simplify to

$$\lim_{t\to\infty} E(x,t) = \sum_r \frac{\pi}{12v}T_r^2 = \frac{\pi}{12v}(T_L^2+T_R^2) \tag{9}$$

after an additive renormalization corresponding to subtracting the (diverging) constant w_λ. Similarly, the two-point correlation function in the local case, after a multiplicative renormalization of the fermion fields (not needed in the non-local case), becomes

$$\begin{aligned}&\lim_{t\to\infty} S_r(\xi,\tau,x,t)\\ &= \frac{1}{2\pi\tilde\ell}\left(\frac{i\pi T_r\tilde\ell/v}{\sinh(\pi T_r u_r/v)}\right)^{1+\eta/2}\left(\frac{i\pi T_{-r}\tilde\ell/v}{\sinh(\pi T_{-r}u_{-r}/v)}\right)^{\eta/2},\end{aligned} \tag{10}$$

where $u_r = r[\xi - rv\tau]+i0^+$, with the equilibrium anomalous exponent[60] $\eta = 2\sinh^2\varphi$ and a length parameter $\tilde\ell$ due to the renormalization; cf. also Refs. 41 and 67. This exponent depends on the interaction and is non-zero if the interaction is non-zero. Clearly, unless $\eta=0$, the NESS does *not* factorize into right(left)-moving fermions with temperatures T_L (T_R) as for the XX chain.

III. FINITE-TIME RESULTS: LOCAL INTERACTION

The Luttinger model with local interaction is conformally invariant, implying that $\mathcal{H}(x,t)$ and $\mathcal{J}(x,t)$ satisfy the wave equation, and thus

$$\begin{aligned}E(x,t) &= \frac{1}{2}\left[G(x-vt)+G(x+vt)\right],\\ J(x,t) &= \frac{v}{2}\left[G(x-vt)-G(x+vt)\right]\end{aligned} \tag{11}$$

for some function $G(x)$. Using *Method 2*, $G(x)$ can be computed as a series expansion in ϵ to all orders (see Appendix A), and, after summation, we obtain the following remarkably simple result:

$$\begin{aligned}G(x) &= \frac{\pi}{6v}\frac{1}{\beta(x)^2}+\frac{v}{12\pi}\left(\frac{\beta''(x)}{\beta(x)}-\frac{1}{2}\left(\frac{\beta'(x)}{\beta(x)}\right)^2\right)\\ &= \frac{\pi}{6v}T(x)^2-\frac{v}{12\pi}\left(\frac{T''(x)}{T(x)}-\frac{3}{2}\left(\frac{T'(x)}{T(x)}\right)^2\right).\end{aligned} \tag{12}$$

The term $(\pi/6v)T(x)^2$ is expected from the equilibrium result for a *uniform* temperature profile, but the presence of the derivative terms has apparently been overlooked in the previous literature. Thus, in the case of a *non-uniform* temperature profile, (11) and (12) show that $E(x,t)$ and $J(x,t)$ also depend on the first and second derivatives of $T(x)$, but not on higher-order ones. This is true even at $t = 0$.

The evolution of the energy flow can be easily understood using (11) and (12). For a step-like $\beta(x) = \beta[1 + \epsilon W(x)]$ with $W(x) = -(1/2)\tanh(x/\delta)$, as in the introduction, the energy profile at $t = 0$ away from $x = 0$ is essentially proportional to the local temperature, i.e., $E(x,0)$ equals $(\pi/12v)T_{L,R}^2$ far to the left and right. However, in the transition region, for small $\delta > 0$ and $\epsilon \neq 0$, the derivative terms in (12) produce peaks; see Fig. 1(a). As t increases a region develops around the origin with a uniform energy density bounded by two *rigid* fronts (their shape does not change with time) that move ballistically to the right (left) with constant velocity v ($-v$); see Fig. 1(b). In the same region the current has a non-vanishing constant value. For large times we recover the results for the NESS in (8) and (9).

As we discuss in Sec. IV, peaks qualitatively similar to those described above are seen in other related models, including interacting lattice models, such as the XXZ chain, and non-interacting models, such as the XX chain. It is important to stress that the shape of the peaks is non-universal and depends on (short-distance) details: this is clear already from the interaction dependence of the derivative terms that appear in (11) due to (12).

It is interesting to note that $G(x)$ can be written as

$$G(x) = \frac{\pi T^2}{6v} g'(x)^2 - \frac{v}{12\pi}(Sg)(x) \tag{13}$$

using the function $g(x) = \int_0^x dx'\, T(x')/T$ and the so-called Schwarzian derivative[69]

$$(Sg)(x) = \frac{g'''(x)}{g'(x)} - \frac{3}{2}\left(\frac{g''(x)}{g'(x)}\right)^2. \tag{14}$$

By recalling that the Luttinger model with local interaction is a CFT with central charge $c = 1$, this result has a simple interpretation as follows. In a CFT, the energy and heat current densities are given by expectation values of the *renormalized energy-momentum tensor* $\mathcal{T}(z)$ and $\bar{\mathcal{T}}(\bar{z})$ as follows:

$$\begin{aligned} E(z,\bar{z}) &= -\frac{1}{2\pi}\left[\langle \mathcal{T}(z)\rangle + \langle \bar{\mathcal{T}}(\bar{z})\rangle\right], \\ J(z,\bar{z}) &= -\frac{iv}{2\pi}\left[\langle \mathcal{T}(z)\rangle - \langle \bar{\mathcal{T}}(\bar{z})\rangle\right] \end{aligned} \tag{15}$$

using $z = x + iv\tau$ and $\bar{z} = x - iv\tau$ with τ denoting imaginary time.[33,69] Moreover, under a conformal transformation $z \to w(z)$, the renormalized energy-momentum tensor in a CFT transforms as $\mathcal{T}(z) \to \mathcal{T}(w)$ with

$$\mathcal{T}(z) = \left(\frac{dw}{dz}\right)^2 \mathcal{T}(w) + \frac{cv}{12}(Sw)(z) \tag{16}$$

using the Schwarzian derivative S.[69] From the above one obtains (11) by a Wick rotation $\tau \to it$ and the identification $G(x) = -\pi^{-1}\langle \mathcal{T}(z)\rangle|_{z=x}$, using that $E(x,t) = E(z,\bar{z})$ and $J(x,t) = -iJ(z,\bar{z})$. Our results in (11) and (13) are therefore equivalent to what one would obtain by a conformal transformation determined by the function $g(x) = \int_0^x dx'\, T(x')/T$ from the equilibrium result $\langle \mathcal{T}(w)\rangle_\beta = \langle \bar{\mathcal{T}}(\bar{w})\rangle_\beta = -c\pi^2 T^2/6v$ (for the latter see, e.g., Ref. 32). As we discuss in Sec. VI, it would be interesting to check if this is true also for other observables.

IV. FINITE-TIME RESULTS: NON-LOCAL INTERACTION

We now consider the Luttinger model with non-local interaction. Such an interaction breaks conformal invariance and gives rise to dispersion effects since the renormalized Fermi velocity $v(p)$ depends on momenta. These effects are qualitatively similar to ones observed in lattice models. (The interaction range introduces a scale similar to the lattice spacing.) We compute quantities only to first order in ϵ using *Method 1*. Comparison with our all-order results for the NESS and for finite times in the local case suggests that such first-order approximation works well for small ϵ: e.g., for $\epsilon = -0.01$, used below in Figs. 1 and 2, first- and all-order results are practically indistinguishable, and thus the deviations seen in these figures between the plots for local and non-local interactions can be fully attributed to dispersive effects.

For the energy and heat current densities we obtain

$$\begin{aligned} E(x,t) &= E_0 + \epsilon E_1(x,t) + O(\epsilon^2), \\ J(x,t) &= \epsilon J_1(x,t) + O(\epsilon^2), \end{aligned} \tag{17}$$

where E_0 is equal to $\lim_{t\to\infty} E(x,t)$ in (6) for $\beta_+ = \beta_- = \beta$,

$$\begin{aligned} E_1(x,t) &= -\sum_{r_1,r_2} \fint_{\mathbb{R}} \frac{dp}{2\pi} \int_{\mathbb{R}} \frac{dq}{4\pi}\, \hat{W}(p) A(p-q,q), \\ J_1(x,t) &= -\sum_{r_1,r_2} \fint_{\mathbb{R}} \frac{dp}{2\pi} \int_{\mathbb{R}} \frac{dq}{4\pi}\, \hat{W}(p) \frac{i}{p}\frac{\partial}{\partial t} A(p-q,q) \end{aligned} \tag{18}$$

with

$$\begin{aligned} A(p_1,p_2) &= e^{i(p_1+p_2)x - i[r_1\omega(p_1)+r_2\omega(p_2)]t} \\ &\times \frac{[r_1 v(p_1) + r_2 v(p_2)]^2}{4v(p_1)v(p_2)} \frac{[r_1 e^{2\varphi(p_1)} + r_2 e^{2\varphi(p_2)}]^2}{4e^{2[\varphi(p_1)+\varphi(p_2)]}} \\ &\times \frac{e^{\beta[r_1\omega(p_1)+r_2\omega(p_2)]} - 1}{r_1\omega(p_1) + r_2\omega(p_2)} \frac{r_1\omega(p_1)}{e^{\beta r_1\omega(p_1)} - 1} \frac{r_2\omega(p_2)}{e^{\beta r_2\omega(p_2)} - 1}. \end{aligned}$$

Similarly, for the two-point correlation function, we obtain

$$S_r(\xi,\tau,x,t) = \langle \psi_r^+(\xi,\tau)\psi_r^-(0,0)\rangle_\beta e^{\epsilon B_{1;r}(\xi,\tau,x,t) + O(\epsilon^2)}, \tag{19}$$

where $\langle \psi_r^+(\xi,\tau)\psi_r^-(0,0)\rangle_\beta$ is equal to $\lim_{t\to\infty} S_r(\xi,\tau,x,t)$ in (7) for $\beta_+ = \beta_- = \beta$,

$$B_{1;r}(\xi,\tau,x,t) = -\sum_{r_1,r_2} \fint_{\mathbb{R}} dp \int_{\mathbb{R}} \frac{dq}{4\pi}\, \hat{W}(p) C(p-q,q) \tag{20}$$

with

$$C(p_1,p_2) = e^{i(p_1+p_2)x - i[r_1\omega(p_1)+r_2\omega(p_2)]t}\theta(-r_1r_2p_1p_2) \times \frac{v(p_1)+v(p_2)}{2}\cosh(\varphi(p_1)-\varphi(p_2)) \times \frac{e^{\beta r[\omega(p_1)+\omega(p_2)]}-1}{r[\omega(p_1)+\omega(p_2)]}F_r^{r_1}(p_1)F_r^{r_2}(p_2)$$

and

$$F_r^{r'}(p') = \frac{e^{-\varphi(p')}+rr'e^{\varphi(p')}}{2}\frac{e^{ir'p'u_{r'}(p')}-1}{e^{\beta r\omega(p')}-1}.$$

The above results agree, to first order in ϵ, with (6) and (7) as $t\to\infty$.

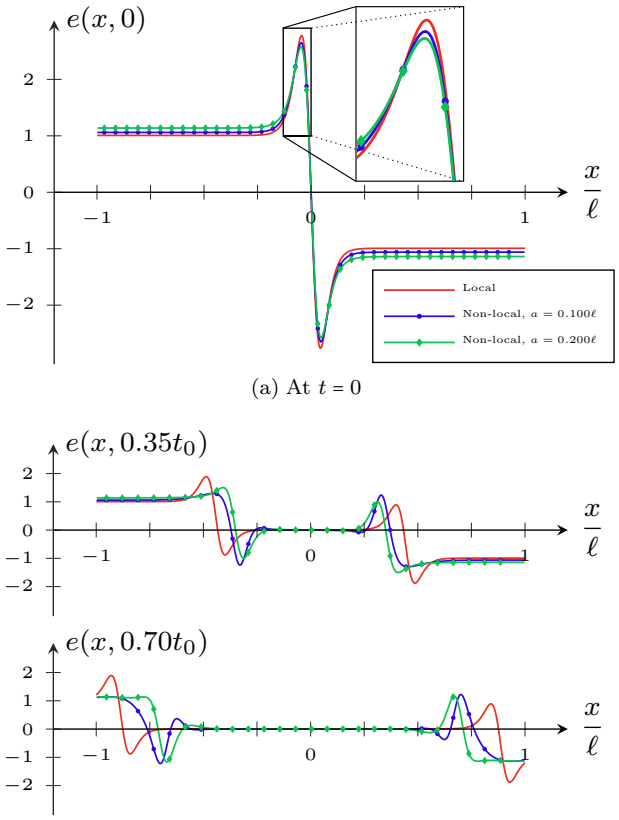
Similar to the discussion for the local case in Sec. III for a step-like $\beta(x)$, our analytical results in (17) and (18) show, for small $\delta>0$ and $\epsilon\neq 0$, that peaks are produced in the transition region between different temperatures; see Fig. 1(a). As t increases a region develops around the origin with a uniform energy density bounded by two ballistically moving *non-rigid* fronts (their shape changes with time); see Fig. 1(b).

In Fig. 2 we plot the current through $x=0$ as a function of time. The plotted results contain an initial peak. As seen from the dotted line in Fig. 2, such a peak is absent in the local case if the second term in (13) is omitted. A qualitatively similar peak is present in numerical results for the XXZ chain; see, e.g., Fig. 1(a) in Ref. 42 (at low temperatures) and Fig. 3 in Ref. 47 (at high temperatures) showing the heat current through the contact point in the partitioning protocol. As emphasized in Sec. III and also in Ref. 42, the shape of such peaks is non-universal: in the Luttinger model the shape depends on the interaction and in the XXZ chain on the anisotropy and the dispersion relation. However, the presence of the peaks seems to be a generic feature.

To further support our claim about the peaks, we also present, as an example for non-interacting lattice fermions, plots of the corresponding results for the XX chain computed to first order in ϵ using *Method 1*; see Figs. 3 and 4. Peaks and dispersion effects that are qualitatively similar to the ones in Figs. 1 and 2 are clearly visible. Moreover, we checked numerically and analytically, to first order in ϵ, that the results for the XX chain approach those of the *non-interacting* Luttinger model in the scaling limit; plots of the latter are given by the red (plain) line in Figs. 3 and 4. This is true even at finite times.

V. METHODS

Our results are based on rigorous bosonization methods well-known from studies of the Luttinger model in equilibrium; see, e.g., Refs. 41, 60, 65, and 67. We work on the circle $-L/2\le x\le L/2$ of length $L>0$ with the fermion fields $\psi_r^\pm(x)$ satisfying anti-periodic boundary



(a) At $t=0$

(b) Evolution for $t>0$

Figure 1. Interacting fermions. Plots of analytical results for the energy density $e(x,t)=v(E(x,t)-E_0)/J$ in an interval $[-\ell,\ell]$ around $x=0$ at times (a) $t=0$ and (b) $t>0$ for the Luttinger model with local and non-local interactions. The results in the local case are given by (11) for $V(x)=\pi v_F\delta(x)/2$ and in the non-local case by (17) for $V(x)=\pi v_F/[4a\cosh(\pi x/2a)]$ with $a=0.100\ell$ and $a=0.200\ell$, respectively. The coupling constant is $\lambda=0.6$, and the other parameters are $\beta=20$, $\epsilon=-0.01$, $\delta=0.06\ell$, $t_0=\ell/v_F$, and $v_F=1$. The value of ϵ is small enough that $O(\epsilon^2)$-corrections are negligible.

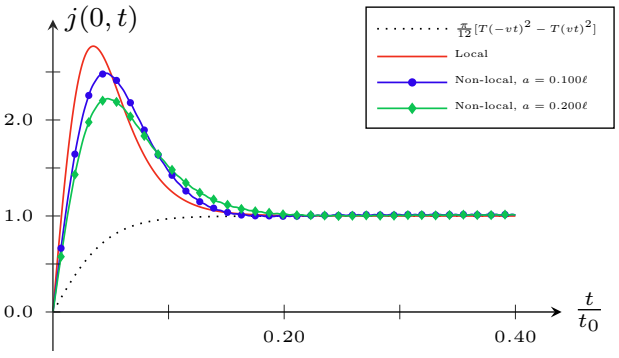
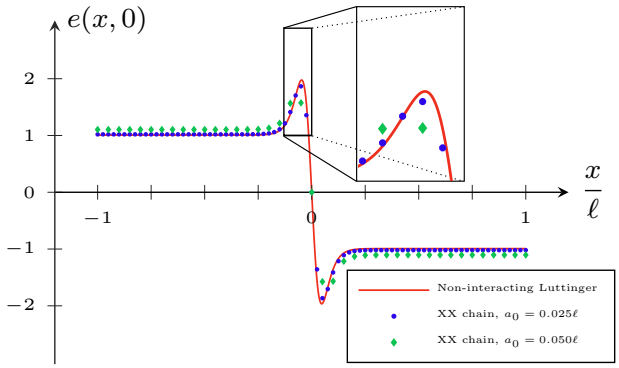
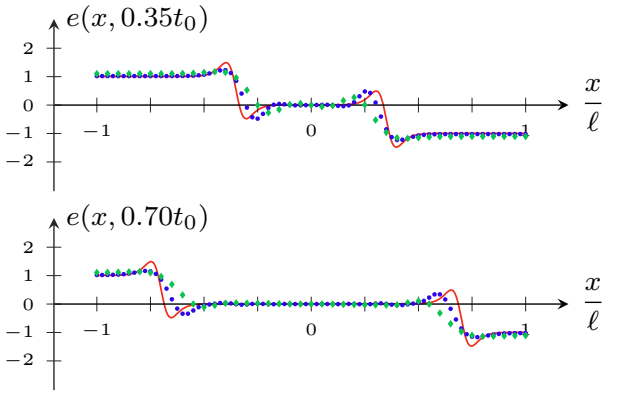


Figure 2. Interacting fermions. Plots of analytical results for the heat current $j(0,t)=J(0,t)/J$ through $x=0$ for the Luttinger model using the same parameters as in Fig. 1. Also included is the local case without the second term in (13) (black dotted line).



(a) At $t = 0$



(b) Evolution for $t > 0$

Figure 3. Non-interacting fermions. Plots of analytical results for the energy density $e(x,t) = v(E(x,t)-E_0)/J$ in an interval $[-\ell,\ell]$ around $x = 0$ at times (a) $t = 0$ and (b) $t > 0$ for the non-interacting Luttinger model and for the XX chain. The results for the former are given by (11) and (12) with v_F instead of v. The XX chain is considered close to half filling on a lattice with spacing $a_0 = 0.025\ell$ and $a_0 = 0.050\ell$, respectively. The other parameters are the same as in Fig. 1.

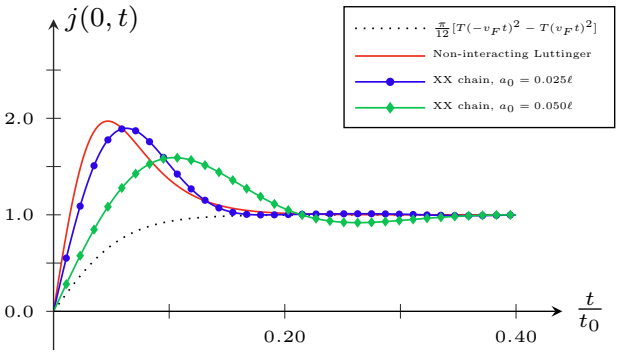


Figure 4. Non-interacting fermions. Plots of analytical results for the heat current $j(0,t) = J(0,t)/J$ through $x = 0$ for the non-interacting Luttinger model and for the XX chain using the same parameters as in Fig. 3.

conditions and take the thermodynamic limit $L \to \infty$ only after computing expectation values for finite $t \geq 0$. The order, first $L \to \infty$ and then $t \to \infty$, is important for computing results in the long time limit.[30,41]

A. Method 1

To compute $\langle \mathcal{O} \rangle$ we write $\mathcal{G}$ in (1) as $\beta(H + \mathcal{W})$ with $\mathcal{W} = \epsilon \int dx\, W(x)\mathcal{H}(x)$ and use the fact that $U(\beta) \equiv e^{\beta H} e^{-\beta(H+\mathcal{W})}$ satisfies

$$\partial_\beta U(\beta) = -e^{\beta H}\mathcal{W}e^{-\beta(H+\mathcal{W})} = -\mathcal{W}(\beta)U(\beta) \qquad (21)$$

with $\mathcal{W}(\beta) = e^{\beta H}\mathcal{W}e^{-\beta H}$. Solving this by iteration we obtain a series expansion in ϵ (the Dyson series):

$$\langle \mathcal{O} \rangle = \langle \mathcal{O} \rangle_\beta - \epsilon\,[\langle \mathcal{C}\mathcal{O} \rangle_\beta - \langle \mathcal{C} \rangle_\beta \langle \mathcal{O} \rangle_\beta] + O(\epsilon^2) \qquad (22)$$

with $\mathcal{C} = \int_0^\beta d\beta' \int dx\, W(x)\mathcal{H}(x, -i\beta')$. It follows that non-equilibrium expectation values are expressed as sums of equilibrium ones. This method can be used for computing non-equilibrium results to first order in ϵ for any model where equilibrium results are computable. Computations of the energy and heat current densities and the two-point correlation functions for the Luttinger model are straightforward but tedious using Wick's theorem; the details will be presented elsewhere.

B. Method 2

Higher-order terms can be computed using general mathematical results for quasi-free models; see, e.g., Ref. 70. For the bosonized Luttinger Hamiltonian we write $H = d\hat{\Gamma}(K)$ to mean boson second quantization of the one-particle operator K, and similarly $\mathcal{W} = d\hat{\Gamma}(W)$ for some W. (We note that the second quantization map $d\hat{\Gamma}$ is in a non-trivial representation of the boson field algebra and that there are certain technical requirements on the one-particle operators[70,71] that are fulfilled in the cases of interest to us.) For $\mathcal{O} = d\hat{\Gamma}(O)$ with some one-particle operator O, one can show (e.g., using results in Ref. 70) that $\langle \mathcal{O} \rangle - \langle \mathcal{O} \rangle_\beta$ can be written as

$$\begin{aligned}
&\frac{\mathrm{Tr}(e^{-\beta d\hat{\Gamma}(K+W)} d\hat{\Gamma}(O))}{\mathrm{Tr}(e^{-\beta d\hat{\Gamma}(K+W)})} - \frac{\mathrm{Tr}(e^{-\beta d\hat{\Gamma}(K)} d\hat{\Gamma}(O))}{\mathrm{Tr}(e^{-\beta d\hat{\Gamma}(K)})} \\
&= \mathrm{tr}\left(\left\{\left[e^{2\beta(K+W)} - 1\right]^{-1} - \left[e^{2\beta K} - 1\right]^{-1}\right\} O\right) \\
&= \frac{1}{\beta}\sum_\nu \mathrm{tr}\left(\left\{[i\nu - 2(K+W)]^{-1} - [i\nu - 2K]^{-1}\right\} O\right) \qquad (23) \\
&= \sum_{n=1}^{\infty} \frac{1}{\beta}\sum_\nu \mathrm{tr}\left([i\nu - 2K]^{-1}(2W[i\nu - 2K]^{-1})^n O\right),
\end{aligned}$$

where tr is the one-particle trace and the ν-sum is over all boson Matsubara frequencies $\nu \in (2\pi/\beta)\mathbb{Z}$. (Note

that the second and third identities in (23) are standard expansions.) The computation of $G(x)$ in (12) using (23) for the Luttinger model with local interaction is explained in Appendix A.

VI. CONCLUSIONS

We derived analytical results for the NESS and for the full time evolution of the Luttinger model with both local and non-local interactions starting from a non-equilibrium state defined by a smooth non-uniform temperature profile. These results were computed using methods based on a series expansions in the distance ϵ from equilibrium in the initial state. We showed that the NESS is factorised in terms of the eigenmodes of the interacting Hamiltonian and that its fermion two-point correlation function contain interaction-dependent exponents. On the contrary, the final heat current is equal to the universal CFT result[30] even if conformal invariance is broken by the interaction. Moreover, the form of the temperature dependence of the heat current agrees with the one found numerically in Ref. 42 for interacting fermions and analytically in Refs. 18, 19, 25, and 29 for non-interacting fermions.

For local interaction (and thus a priori for the non-interacting case), the series for the energy and heat current densities were computed to all orders in ϵ and summed into simple exact formulas valid at all times. These formulas contain a Schwarzian-derivative term [cf. (11) and (13)], which captures a qualitative feature that appears rather generically, namely the presence of non-universal peaks at short times in the transition region between different temperatures. We also showed that these formulas coincide with the result obtained by a particular conformal transformation from the corresponding equilibrium result. It would be interesting to find an explanation for this and to check if this is true also for other observables and in other CFT models; if true, this would be similar in spirit to results in Ref. 33 but for a different physical situation. Also, it would be interesting to investigate if this can be used to gain some insight into non-equilibrium properties of interacting lattice models, such as the XXZ chain.

For non-local interaction, we computed the time evolution of the energy and heat current densities and of the fermion two-point correlation function to first order in ϵ. This truncated expansion can be seen as a linear-response approach (cf., e.g., Ref. 42) and can, in principle, be used even for models that are not exactly solvable.

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Appendix A: Computational details

For a local interaction, the one-particle operators K and W in (23) are given by $K_{r,r'}(p,p') = (rvp/2)\delta_{r,r'}\delta_{p,p'}$ and $W_{r,r'}(p,p') = \epsilon(rv\,\mathrm{sgn}(p)\sqrt{|pp'|}/2L)\delta_{r,r'}\hat{W}(p-p')$, respectively. Since $G(x) = E(x,0)$ [cf. (11)] it follows that $G(x) = \langle \mathcal{H}(x)\rangle = \langle d\hat{\Gamma}(O)\rangle$ with O given by $O_{r,r'}(p,p') = (rv\,\mathrm{sgn}(p)\sqrt{|pp'|}/2L)\delta_{r,r'}e^{i(p'-p)x}$. Using (23) we obtain $G(x) = \sum_{n=0}^{\infty}\epsilon^n G_n(x)$, where $G_0(x) = \pi/6v\beta^2$ is the equilibrium result and

$$G_n(x) = \int_{\mathbb{R}^{n+1}} \frac{dp_0\dots dp_n}{(2\pi)^{n+1}} \left(\prod_{j=0}^{n-1}\hat{W}(p_j - p_{j+1})\right) \times \frac{1}{2}\sum_r \frac{1}{\beta}\sum_\nu \left(\prod_{j=0}^{n}\frac{rvp_j}{i\nu - rvp_j}\right) e^{i(p_0-p_n)x} \quad \text{(A1)}$$

for $n = 1, 2, \dots$. While this formula can be generalized to non-local interaction, the local case is special in that it is possible to compute the integrals exactly: changing variables to $q_j = p_{j-1} - p_j$ for $j = 1,\dots,n$ and $p = p_n$, and renaming $\nu \to r\nu$, we can write

$$G_n(x) = \frac{v}{4\pi}\int_{\mathbb{R}^n}\frac{dq_1\dots dq_n}{(2\pi)^n} I_n(\mathbf{q})\left(\prod_{j=1}^{n}\hat{W}(q_j)e^{iq_j x}\right) \quad \text{(A2)}$$

with

$$I_n(\mathbf{q}) = \frac{2}{v}\int_{\mathbb{R}} dp\,\frac{1}{\beta}\sum_\nu\left(\prod_{j=0}^{n}\frac{v(p+Q_j)}{i\nu - v(p+Q_j)}\right), \quad \text{(A3)}$$

where $\mathbf{q} = (q_1,\dots,q_n)$ and $Q_j = \sum_{k=j+1}^{n} q_k$. The integral in (A3) can be computed exactly, and, after a lengthy computation, we obtained the following remarkably simple result:

$$I_n(\mathbf{q}) \simeq \frac{(-1)^n}{6}\left\{(n+1)\left(\frac{2\pi}{\beta v}\right)^2 + 2q_1^2 + (n-1)q_1q_2\right\}, \quad \text{(A4)}$$

where $\simeq$ is defined by $q_j q_k \simeq q_1^2$ if $j = k$ and $q_j q_k \simeq q_1 q_2$ if $j \neq k$. Inserting (A4) into (A2) yields

$$G_n(x) = (-1)^n \Bigg(\frac{(n+1)\pi}{6v\beta^2} W(x)^n - \frac{v}{12\pi} \left[W''(x) W(x)^{n-1} + \frac{n-1}{2} W'(x)^2 W(x)^{n-2} \right] \Bigg). \tag{A5}$$

Using this the series $G(x) = \sum_{n=0}^{\infty} \epsilon^n G_n(x)$ can be summed giving the result in (12).

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