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Phys. Rev. B **95**, 214401 — Published 1 June 2017

DOI: 10.1103/PhysRevB.95.214401

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(Dated: May 15, 2017)

We use general hydrodynamic equations to determine the long-wavelength spin excitations in isotropic antiferromagnets in the presence of a homogeneous magnetization. The latter may be induced, such as in antiferromagnets in an external magnetic field, or spontaneous, such as in ferrimagnetic or canted phases that are characterized by the coexistence of antiferromagnetic and ferromagnetic order. Depending on the physical situation, we find propagating spin waves that are gapped in some cases and gapless in others, diffusive modes, or relaxational modes. The excitation spectra turn out to be qualitatively different depending on whether or not the homogeneous magnetization is a conserved quantity. The results lay the foundation for a description of a variety of quantum phase transitions, including the transition from a ferromagnetic metal to an antiferromagnetic one, and the spin-flop transitions that are observed in some antiferromagnets. They also are crucial for incorporating weak-localization and Altshuler-Aronov effects into the descriptions of quantum phases in both clean and disordered magnetic metals.

I. INTRODUCTION

Soft or massless excitations are of paramount importance for the description of condensed-matter systems, since they determine the universal long-wavelength and low-frequency properties of materials that do not depend on microscopic details. A common cause of soft excitations or modes is the presence of a spontaneously broken continuous symmetry in an ordered phase, which leads to static Goldstone modes and related dynamical excitations.[1] They generically couple to various observables and qualitatively change the behavior of both static susceptibilities and equilibrium time-correlation functions; namely, they can lead to power-law instead of exponential decay for large distances or times, a phenomenon known as generic scale invariance.[2]

In magnets, the excitations due to the long-ranged magnetic order are magnons or spin-waves. In simple isotropic ferromagnets and antiferromagnets they are well known to be gapless with a quadratic and linear dispersion relation, respectively, in the long-wavelength limit. This difference is due to the coupling of the antiferromagnetic order parameter, i.e., the staggered magnetization, to the fluctuating homogeneous magnetization, as a consequence of which the two problems do not simply map onto one another.[1]

For antiferromagnets in the presence of a nonzero average homogeneous magnetization, which can be due to the presence of an external magnetic field or coexisting spontaneous ferromagnetic and antiferromagnetic orders, a systematic analysis of the soft modes, or spin excitations in general, in the ordered phase does not exist. This is rather surprising, given the abundance of antiferromagnetic materials, and the importance of the concept for many topics of great current interest, including certain classes of quantum phase transitions[3] and high-T_c as well as iron-based superconductors.[4,5] Early work on the dynamics of antiferromagnets focused on the dynamical critical behavior[6,7] and therefore left out terms that renormalization-group irrelevant near the classical critical point, and later approaches that used spin-wave theory or other solid-state oriented approaches were not systematic and sometimes reached conclusions that are not consistent with basic spin dynamics.

It is the purpose of the present paper to remedy this situation and give a complete classification of the long-wavelength spin excitations in antiferromagnets in the presence of a homogeneous magnetization. We use a hydrodynamic approach that is extremely general and reliable and has been previously applied to helical magnets,[8,9] and to ferromagnets.[10] As we will see, it relies only on the basic equation of motion for a magnetic moment and therefore is more general than approaches based on specific solid-state-oriented models, such as the Heisenberg model. For simplicity, we consider only the case of isotropic magnets, as the problem is fairly complex even in that simple case. If desirable, the symmetry-breaking effects of the spin-orbit interaction can be built in at a later stage. Our results are very general and depend only on the type of order and on conservation laws, rather than on the underlying mechanisms that produce the order. For instance, the spin-wave spectrum is the same irrespective of whether the staggered magnetization and the homogeneous part of the order parameter are produced by electrons in the same band or electrons in different bands, and it the same in what are known as "canted phases" and "fan phases".[11] As we will show, the results are qualitatively different depending on whether the homogeneous magnetization is conserved, or whether

that conservation law is violated, e.g., due to the presence of magnetic impurities.

To conclude these introductory remarks we list some physical problems for which a thorough understanding of the spin dynamics is crucial.

(1) In ferromagnets, and in simple antiferromagnets, the spin waves couple to other observables, e.g., the longitudinal susceptibility and the dynamical structure factor, which is directly observable via neutron scattering. This coupling induces nonanalytic wave-number and frequency dependences that reflect the long-range order in the magnetic phase.[10] Similar effects are expected for the more complicated antiferromagnets discussed here.

(2) More generally, the effects known as weak-localization phenomena and Altshuler-Aronov effects in disordered metals,[2,12,13] as well as their counterparts in clean metals,[14,15] rely on all of the soft modes in the system and their couplings to various observables. A complete list of soft modes is therefore crucial for studying these effects, and in magnetic metals this includes the soft collective spin excitations.

(3) At the phase transition that signals the instability of an ordered phase, the soft modes that characterize the latter disappear, turn into critical modes, or change into modes characteristic of a different type of order, depending on the nature of the phase transition. Knowledge of the soft modes is thus important for describing the transition. In the current context, an interesting example is the quantum phase transition from a ferromagnetic phase to an antiferromagnetic one. There are many experimental examples of such transitions,[16] but no theoretical description exists. Another example of phase transitions for which information about spin waves is important are the metamagnetic transitions known as spin-flop transitions (e.g., from easy-axis to easy-plane) that are commonly observed in antiferromagnets, see Ref. 17 and references therein.

(4) If antiferromagnetic spin fluctuations are behind the pairing mechanism for either high-T_c or iron-based superconductors, as has been suggested,[5,18] then one would expect their spectra to be reflected in tunneling data, just as is the case for the phonon spectra in conventional superconductors. Moreover, the spin fluctuations responsible for the pairing mechanism would be very sensitive to an external magnetic field, which is not the case for phonons. A thorough understanding of spin fluctuations, especially in an external field, is therefore very important in this context. We note that coexistence of antiferromagnetic order and superconductivity has been observed in some materials, see Ref. 19 and references therein, which will make spin excitations in a magnetically ordered state directly relevant. However, even in cases where there is an antiferromagnetic phase nearby in the phase diagram a thorough understanding of the antiferromagnetic parent compound is important.

(5) While magnetic states that have both a antiferromagnetic and a ferromagnetic component have been known for a long time, materials that display such phases have received much attention lately, in part because of their potential technological importance, see, e.g., Refs. 20,21. Their understanding requires information about the spin dynamics of systems in which both order parameters are nonzero.

II. TIME-DEPENDENT GINZBURG-LANDAU THEORY

While the equations of motion for an isotropic Heisenberg antiferromagnet are well known,[1,7,11] many versions in the literature omit terms that are irrelevant for the classical critical behavior, yet contribute to the spin dynamics in the ordered phase. For completeness, we therefore provide a brief derivation.

A. Statics

1. Landau free energy

Consider a general magnetization field $\boldsymbol{M}(\boldsymbol{x})$ of the form

$$\boldsymbol{M}(\boldsymbol{x}) = \boldsymbol{m}(\boldsymbol{x}) + \boldsymbol{\nu}(\boldsymbol{x}) \ , \tag{2.1a}$$

where $\boldsymbol{m}(\boldsymbol{x})$ is a slowly varying function, whereas

$$\boldsymbol{\nu}(\boldsymbol{x}) = \boldsymbol{n}(\boldsymbol{x})\, f(\boldsymbol{x}) \tag{2.1b}$$

with $\boldsymbol{n}(\boldsymbol{x})$ slowly varying and $f(\boldsymbol{x})$ a rapidly oscillating function with zero spatial mean. We defined coarse-grained variables

$$\frac{1}{V_{\mathcal{N}}} \int_{\boldsymbol{y}\in\mathcal{N}(\boldsymbol{x})} d\boldsymbol{y}\, \boldsymbol{M}(\boldsymbol{y}) \approx \boldsymbol{m}(\boldsymbol{x}) \ , \tag{2.2a}$$

$$\frac{1}{V_{\mathcal{N}}} \int_{\boldsymbol{y}\in\mathcal{N}(\boldsymbol{x})} d\boldsymbol{y}\, \boldsymbol{M}(\boldsymbol{y})\, f(\boldsymbol{y}) \approx \boldsymbol{n}(\boldsymbol{x})\, \overline{f^2} \ , \tag{2.2b}$$

where $\mathcal{N}(\boldsymbol{x})$ is a neighborhood of the point $\boldsymbol{x}$ whose volume $V_{\mathcal{N}}$ is large on the microscopic length scale, but small on the macroscopic one, and $\overline{f^2}$ is the spatial average of $f^2(\boldsymbol{x})$. $\boldsymbol{m}$ and $\boldsymbol{n}$ are the magnetization and the staggered magnetization, respectively, and

$$\frac{1}{V_{\mathcal{N}}} \int_{\boldsymbol{y}\in\mathcal{N}(\boldsymbol{x})} d\boldsymbol{y}\, \left(f(\boldsymbol{y})\right)^{n+1} \approx 0 \ , \tag{2.3a}$$

$$\frac{1}{V_{\mathcal{N}}} \int_{\boldsymbol{y}\in\mathcal{N}(\boldsymbol{x})} d\boldsymbol{y}\, \left(f(\boldsymbol{y})\right)^{2n} \equiv \overline{f^{2n}} > 0 \tag{2.3b}$$

for all integer n. By rescaling $\boldsymbol{n}$ we can choose

$$\overline{f^2} = 1 \tag{2.3c}$$

without loss of generality, and we will adopt this choice from now on.

Now consider a Landau free energy that is a functional of $\boldsymbol{M}$. Multiplying out powers of $\boldsymbol{M}$ yields all possible

scalar terms that can be constructed from the two vectors $\boldsymbol{m}$ and $\boldsymbol{n}$. However, all terms that are odd in $\boldsymbol{n}$ are multiplied by odd powers of f and thus vanish upon coarse graining. Up to quartic terms in $\boldsymbol{m}$, $\boldsymbol{n}$, and gradients (for comments on higher-order terms see Sec. IV B 1) we thus obtain a free-energy functional

$$\begin{aligned} F \;=\; \int d\boldsymbol{x}\, \Big[& \frac{r}{2}\,\boldsymbol{n}^2 + \frac{a}{2}\,(\nabla \boldsymbol{n})^2 + \frac{u}{4}\,\big(\boldsymbol{n}^2\big)^2 - \mathfrak{h}\cdot\boldsymbol{n} \\ & + \frac{t}{2}\,\boldsymbol{m}^2 + \frac{c}{2}\,(\nabla \boldsymbol{m})^2 + \frac{v}{4}\,\big(\boldsymbol{m}^2\big)^2 - \boldsymbol{h}\cdot\boldsymbol{m} \\ & + \frac{1}{2}\,w_1\,\boldsymbol{n}^2\boldsymbol{m}^2 + \frac{1}{2}\,w_2\,(\boldsymbol{n}\cdot\boldsymbol{m})^2 \Big] \;. \end{aligned} \tag{2.4}$$

Here $\boldsymbol{h}$ and $\mathfrak{h}$ are a homogeneous and a staggered magnetic field, respectively, and r, a, u, t, etc. are Landau coefficients. The coefficients of terms that result from different powers of $\boldsymbol{M}$ are different even within a bare theory. Moreover, all of the coefficients will behave differently under renormalization, and therefore all Landau coefficients in Eq. (2.4) should be considered independent. In particular, one can have $r < 0$, $t > 0$. This allows for spontaneous antiferromagnetic order, i.e., a nonzero staggered magnetization for $\mathfrak{h} = 0$, with a homogeneous magnetization that vanishes as $\boldsymbol{h} \to 0$. We will also consider the case of coexisting spontaneous order for both the staggered and the homogeneous magnetization.

2. Equations of state

We now consider the mean-field equations of state, which are given by

$$(\delta F/\delta \boldsymbol{n})_{\boldsymbol{n}_0} = (\delta F/\delta \boldsymbol{m})_{\boldsymbol{m}_0} = 0 \;. \tag{2.5a}$$

For $\mathfrak{h} = 0$ we have explicitly

$$0 \;=\; r\,\boldsymbol{n}_0 + u(\boldsymbol{n}_0)^2\boldsymbol{n}_0 + w_1(\boldsymbol{m}_0)^2\boldsymbol{n}_0 + w_2(\boldsymbol{n}_0\cdot\boldsymbol{m}_0)\,\boldsymbol{m}_0 \;, \tag{2.5b}$$

$$\boldsymbol{h} \;=\; t\boldsymbol{m}_0 + v(\boldsymbol{m}_0)^2\boldsymbol{m}_0 + w_1(\boldsymbol{n}_0)^2\boldsymbol{m}_0 + w_2(\boldsymbol{n}_0\cdot\boldsymbol{m}_0)\,\boldsymbol{n}_0 \;. \tag{2.5c}$$

a. AFM order only Let us first consider parameter values such that $\boldsymbol{n}_0 \neq 0$ and $\boldsymbol{m}_0(\boldsymbol{h} \to 0) \to 0$. For $\boldsymbol{h} = 0$ we have the simple AFM solution $\boldsymbol{m}_0 = 0$, $\boldsymbol{n}_0 = (0, 0, n_0)$ with $n_0 = \sqrt{-r/u}$. For $\boldsymbol{h} \neq 0$ we need to distinguish between two cases:

Case 1: $w_2 > 0$

In this case $\boldsymbol{n}_0 \cdot \boldsymbol{m}_0 = 0$, and $\boldsymbol{m}_0 = m_0\hat{\boldsymbol{h}}$, with $\hat{\boldsymbol{h}}$ the unit vector in the direction of $\boldsymbol{h}$. Choosing $\boldsymbol{n}_0 = (0, 0, n_0)$, $\boldsymbol{m}_0 = (m_0, 0, 0)$, $\boldsymbol{h} = (h, 0, 0)$, n_0 and m_0 are the solutions of the equations of state

$$(t + w_1 n_0^2)m_0 + v m_0^3 = h \;, \tag{2.6a}$$

$$n_0^2 = -(r + w_1 m_0^2)/u \;, \tag{2.6b}$$

which requires $r < -w_1 m_0^2$. A Gaussian stability analysis (see Sec. II A 3 below) shows that the condition for this state to minimize the free energy is

$$w_1^2 < u\,v + u\,h/2m_0^3 \;. \tag{2.7}$$

We will refer to this case as the transverse-field case. Note that is the field whose direction is chosen in an experiment, and $\boldsymbol{n}$ adjusts such that $\boldsymbol{n}$ and $\boldsymbol{h}$ are perpendicular.

Case 2: $w_2 < 0$

In this case $\boldsymbol{n}_0$, $\boldsymbol{m}_0$, and $\boldsymbol{h}$ are all collinear and the equations of state read

$$\big(t + (w_1 + w_2)n_0^2\big)\,m_0 + v m_0^3 = h \;, \tag{2.8a}$$

$$n_0^2 = -\big(r + (w_1 + w_2)m_0^2\big)/u \;, \tag{2.8b}$$

which requires $r < -(w_1 + w_2)m_0^2$. The stability requirement in this case is

$$(w_1 + w_2)^2 < u\,v \;. \tag{2.9}$$

We will refer to this case as the longitudinal-field case. As in the previous case, $\boldsymbol{n}$ will adjust, in this case such that it is collinear with $\boldsymbol{h}$.

b. Coexisting AFM and homogeneous orders Now consider the case $\boldsymbol{h} = 0$, and parameter values such that both the staggered magnetization and the homogeneous magnetization have nonzero expectation values, $n_0 \neq 0$ and $m_0 \neq 0$.[22] We need to distinguish again between two cases:

Case 1: $w_2 > 0$,

In this case $\boldsymbol{n}_0 \perp \boldsymbol{m}_0$. The equations of state are given by Eqs. (2.6) with $h = 0$, which leads to

$$n_0^2 = -(rv - w_1 t)/(uv - w_1^2) \;, \tag{2.10a}$$

$$m_0^2 = -(tu - w_1 r)/(uv - w_1^2) \;. \tag{2.10b}$$

Coexisting orders thus require $rv - w_1 t < 0$ and $tu - w_1 r < 0$. The stability criterion is given by Eq. (2.7) with $h = 0$, i.e.,

$$w_1^2 < u\,v \;. \tag{2.11}$$

We will refer to this case as the orhogonal-order-parameters case.

Case 2: $w_2 < 0$,

In this case $\boldsymbol{n}_0$ is parallel to $\boldsymbol{m}_0$. The equations of state are given by Eqs. (2.8) with $h = 0$, which leads to

$$n_0^2 \;=\; -\,(rv - (w_1 + w_2)t)\,/\,\big(uv - (w_1 + w_2)^2\big) \;, \tag{2.12a}$$

$$m_0^2 \;=\; -\,(tu - (w_1 + w_2)r)\,/\,\big(uv - (w_1 + w_2)^2\big) \;, \tag{2.12b}$$

and the stability criterion is given by Eq. (2.9). We will refer to this case as the collinear-order-parameters case.

We note that the equations of state for Cases 1 and 2 can be combined to read[23]

$$n_0^2 = -(rv - wt)/(uv - w^2) \ , \tag{2.13a}$$
$$m_0^2 = -(tu - wr)/(uv - w^2) \ , \tag{2.13b}$$

where

$$w = \begin{cases} w_1 & \text{if } w_2 > 0 \\ w_1 + w_2 & \text{if } w_2 < 0 \end{cases} \tag{2.13c}$$

and the stability criterion is

$$w^2 < uv \ . \tag{2.14}$$

The requirements $(rv < wt) \wedge (tu < wr)$ imply that the phase of coexisting orders is always separated from the paramagnetic phase by a purely antiferromagnetic one, except at the tetracritical point $(r, t) = (0, 0)$ where the four phases meet. Observations of coexisting orders are not common; for an example, see Ref. 24.

3. Static susceptibilities; Goldstone modes

Also of interest are the static susceptibilities, which are obtained by expanding the free energy to quadratic order in the Gaussian fluctuations about the solutions of the equations of state. We are in particular interested in the presence of Goldstone modes, which manifest themselves as susceptibilities that diverge as $k \to 0$. The same calculation yields the stability criteria listed in Sec. II A 2 above; they are given by the requirement that all of the static susceptibilities must be positive.

To proceed, we parameterize the fields $\boldsymbol{n}$ and $\boldsymbol{m}$ as follows:

$$\boldsymbol{n} = (-\theta_2, \theta_1, n_0 + \theta_3) \quad , \quad \boldsymbol{m} = \boldsymbol{m}_0 + (\pi_1, \pi_2, \pi_3) \ , \tag{2.15}$$

and expand the free energy, Eq. (2.4), to Gaussian order in the small fluctuations $\theta_{1,2,3}$ and $\pi_{1,2,3}$. This yields a 6×6 eigenvalue problem. Stability requires that all eigenvalues are positive, and eigenvalues that vanish as $k \to 0$ indicate the existence of Goldstone modes.

a. AFM order only

Case 1: $w_2 > 0$ (transverse-field case)

We choose $\boldsymbol{m}_0 = (m_0, 0, 0)$, and $\boldsymbol{h} = (h, 0, 0)$. The 6×6 problem then decomposes into two 2×2 problems (for the pairs (θ_2, π_3) and (θ_3, π_1), respectively, and two single-variable problems for θ_1 and π_2, respectively. Using the equations of state, Eqs. (2.6), we find for the Gaussian-fluctuation contribution to the free energy

$$\begin{aligned} \delta F^{(2)} \ = \ & \frac{1}{2V} \sum_{\boldsymbol{k}} \theta_1(\boldsymbol{k})\, a k^2\, \theta_1(-\boldsymbol{k}) + \frac{1}{2V} \sum_{\boldsymbol{k}} \pi_2(\boldsymbol{k}) \left(h/m_0 + c k^2 \right) \pi_2(-\boldsymbol{k}) \\ & + \frac{1}{2V} \sum_{\boldsymbol{k}} \left(\theta_2(\boldsymbol{k}), \pi_3(\boldsymbol{k})\right) \begin{pmatrix} w_2 m_0^2 + a k^2 & -w_2 n_0 m_0 \\ -w_2 n_0 m_0 & h/m_0 + w_2 n_0^2 + c k^2 \end{pmatrix} \begin{pmatrix} \theta_2(-\boldsymbol{k}) \\ \pi_3(-\boldsymbol{k}) \end{pmatrix} \\ & + \frac{1}{2V} \sum_{\boldsymbol{k}} \left(\theta_3(\boldsymbol{k}), \pi_1(\boldsymbol{k})\right) \begin{pmatrix} u n_0^2 + a k^2 & w_1 n_0 m_0 \\ w_1 n_0 m_0 & h/m_0 + v m_0^2 + c k^2 \end{pmatrix} \begin{pmatrix} \theta_3(-\boldsymbol{k}) \\ \pi_1(-\boldsymbol{k}) \end{pmatrix} \ . \end{aligned} \tag{2.16}$$

All eigenvalues are positive provided Eq. (2.7) holds, and the one related to θ_1 vanishes as $k \to 0$. We thus have one Goldstone mode,

$$g(\boldsymbol{k}) = \theta_1(\boldsymbol{k}) \ , \tag{2.17a}$$

whose susceptibility is soft, namely

$$\chi_g(k) = \langle g(\boldsymbol{k})\, g(\boldsymbol{k})^* \rangle = 1/a\, k^2 \ . \tag{2.17b}$$

Physically, the field polarizes the homogeneous magnetization $\boldsymbol{m}$, and the w_2 term forces the staggered magnetization $\boldsymbol{n}$ to be perpendicular to $\boldsymbol{m}$, but $\boldsymbol{n}$ is still free to rotate about the field direction, so one of the two transverse $\boldsymbol{n}$ fluctuations do not cost any energy in the long-wavelength limit.

Case 2: $w_2 < 0$ (longtitudinal-field case)

In this case $\boldsymbol{n}_0$, $\boldsymbol{m}_0$, and $\boldsymbol{h}$ are all collinear, so $\boldsymbol{m}_0 = (0, 0, m_0)$ and $\boldsymbol{h} = (0, 0, h)$. The 6×6 problem decomposes into three 2×2 problems for the pairs (θ_1, π_2), (θ_2, π_1), and (θ_3, π_3), respectively. Using Eqs. (2.8) we find for the Gaussian-fluctuation contribution to the free energy

$$\delta F^{(2)} = \frac{1}{2V}\sum_{\boldsymbol{k}} (\theta_1(\boldsymbol{k}),\pi_2(\boldsymbol{k})) \begin{pmatrix} -w_2 m_0^2 + ak^2 & w_2 n_0 m_0 \\ w_2 n_0 m_0 & h/m_0 - w_2 n_0^2 + ck^2 \end{pmatrix} \begin{pmatrix} \theta_1(-\boldsymbol{k}) \\ \pi_2(-\boldsymbol{k}) \end{pmatrix}$$
$$+\frac{1}{2V}\sum_{\boldsymbol{k}} (\theta_2(\boldsymbol{k}),\pi_1(\boldsymbol{k})) \begin{pmatrix} -w_2 m_0^2 + ak^2 & -w_2 n_0 m_0 \\ -w_2 n_0 m_0 & h/m_0 - w_2 n_0^2 + ck^2 \end{pmatrix} \begin{pmatrix} \theta_2(-\boldsymbol{k}) \\ \pi_1(-\boldsymbol{k}) \end{pmatrix}$$
$$+\frac{1}{2V}\sum_{\boldsymbol{k}} (\theta_3(\boldsymbol{k}),\pi_3(\boldsymbol{k})) \begin{pmatrix} un_0^2 + ak^2 & (w_1+w_2) n_0 m_0 \\ (w_1+w_2) n_0 m_0 & h/m_0 - w_2 n_0^2 + ck^2 \end{pmatrix} \begin{pmatrix} \theta_3(-\boldsymbol{k}) \\ \pi_3(-\boldsymbol{k}) \end{pmatrix} . \quad (2.18)$$

All eigenvalues are positive provided Eq. (2.9) holds, and all of them remain positive for $k = 0$. Hence there are no Goldstone modes. This reflects the fact that the field polarizes the homogeneous magnetization via the Zeeman term, which in turn polarizes the staggered magnetization via the w_2 term, so any deviation from the collinear field configuration costs energy.

In the limit of a vanishing field, $h = m_0 = 0$, two of the positive eigenvalues in Case 2 vanish at $k = 0$, and Case 1 yields one additional zero eigenvalue. This reflects the two zero-field AFM Goldstone modes that are represented by the transverse fluctuations of the staggered magnetization:

$$g_{1,2}(\boldsymbol{k}) = \theta_{1,2}(\boldsymbol{k}) \quad (2.19a)$$

with susceptibilities

$$\chi_{g_1}(k) = \chi_{g_2}(k) = 1/a\,k^2 . \quad (2.19b)$$

b. Coexisting AFM and homogeneous order

Case 1: $w_2 > 0$ (orthogonal-order-parameters case)

This case is obtained from the transverse-field case, Eq. (2.16), by taking the limit $h \to 0$ at fixed m_0. The eigenvalue that corresponds to the π_2 fluctuations now vanishes in the $k \to 0$ limit in addition to the one that corresponds to the θ_1 fluctuations, and the (θ_2, π_3) system contributes a third zero eigenvalue. We thus have three Goldstone modes, namely

$$\begin{aligned} g_1(\boldsymbol{k}) &= \theta_1(\boldsymbol{k}) , \\ g_2(\boldsymbol{k}) &= \pi_2(\boldsymbol{k}) , \\ g_3(\boldsymbol{k}) &= \theta_2(\boldsymbol{k}) + i\pi_3(\boldsymbol{k}) , \end{aligned} \quad (2.20a)$$

with susceptibilities

$$\begin{aligned} \chi_{g_1}(k) &= 1/a\,k^2 , \\ \chi_{g_2}(k) &= 1/c\,k^2 , \\ \chi_{g_3}(k) &= \frac{n_0^2 + m_0^2}{n_0^2 c + m_0^2 a}\,\frac{1}{k^2} . \end{aligned} \quad (2.20b)$$

Physically, the situation is as follows. In the absence of a coupling between $\boldsymbol{n}$ and $\boldsymbol{m}$ the transverse fluctuations of both order parameters would be soft. However, the w_2 coupling enforces the condition that the two order parameters are orthogonal. This leads to one constraint, which reduces the number of soft modes from four to three.

Case 2: $w_2 < 0$ (collinear-order-parameters case)

This case is obtained from the longitudinal-field case, Eq. (2.18), by taking the limit $h \to 0$ at fixed m_0. Of the six eigenvalues, two vanish in this limit at $k = 0$. There are two Goldstone modes,

$$\begin{aligned} g_1(\boldsymbol{k}) &= \theta_1(\boldsymbol{k}) + i\pi_2(\boldsymbol{k}) , \\ g_2(\boldsymbol{k}) &= \theta_2(\boldsymbol{k}) + i\pi_1(\boldsymbol{k}) , \end{aligned} \quad (2.21)$$

with susceptibilities

$$\chi_{g_1}(k) = \chi_{g_2}(k) = \frac{n_0^2 + m_0^2}{n_0^2 c + m_0^2 a}\,\frac{1}{k^2} . \quad (2.22)$$

Physically, the homogeneous magnetization gets slaved to the staggered magnetization by the w_2 coupling, and the soft-mode structure is that of an antiferromagnet.

B. Dynamics

The dynamics of the order-parameter fields $\boldsymbol{n}$ and $\boldsymbol{m}$ are governed by the basic equation of motion that describes the precession of the magnetic moment $\boldsymbol{M}$ in an effective magnetic field,[25,26]

$$\partial_t \boldsymbol{M}(\boldsymbol{x},t) = \boldsymbol{M}(\boldsymbol{x},t) \times \frac{\delta F}{\delta \boldsymbol{M}(\boldsymbol{x})}\Big|_{\boldsymbol{M}(\boldsymbol{x},t)} . \quad (2.23)$$

We put the gyromagnetic ratio equal to unity, which amounts to measuring the magnetization in units of the magnetic moment. By using

$$\begin{aligned} \frac{\delta F}{\delta M_i(\boldsymbol{x})} &= \int d\boldsymbol{y} \left(\frac{\delta F}{\delta m_j(\boldsymbol{y})}\frac{\delta m_j(\boldsymbol{y})}{\delta M_i(\boldsymbol{x})} + \frac{\delta F}{\delta \nu_j(\boldsymbol{y})}\frac{\delta \nu_j(\boldsymbol{y})}{\delta M_i(\boldsymbol{x})} \right) \\ &= \frac{\delta F}{\delta m_i(\boldsymbol{x})} + \frac{\delta F}{\delta \nu_i(\boldsymbol{x})} \end{aligned} \quad (2.24)$$

and coarse-graining Eq. (2.23) we obtain

$$\partial_t \boldsymbol{m}(x) = \boldsymbol{m}(x) \times \frac{\delta F}{\delta \boldsymbol{m}(\boldsymbol{x})}\Big|_{\boldsymbol{m}(x)} + \boldsymbol{n}(\boldsymbol{x},t) \times \frac{\delta F}{\delta \boldsymbol{n}(\boldsymbol{x})}\Big|_{\boldsymbol{n}(x)} , \quad (2.25a)$$

where $x = (\boldsymbol{x}, t)$. Multiplying Eq. (2.23) by $f(\boldsymbol{x})$ and coarse-graining we obtain

$$\partial_t \boldsymbol{n}(x) = \boldsymbol{n}(x) \times \frac{\delta F}{\delta \boldsymbol{m}(\boldsymbol{x})}\bigg|_{\boldsymbol{m}(x)} + \boldsymbol{m}(x) \times \frac{\delta F}{\delta \boldsymbol{n}(\boldsymbol{x})}\bigg|_{\boldsymbol{n}(x)} , \tag{2.25b}$$

where we have used Eq. (2.3c). Adding dissipative terms we finally obtain[6,27]

$$\partial_t \boldsymbol{n} \;=\; -\Gamma_0 \frac{\delta F}{\delta \boldsymbol{n}} + \boldsymbol{n} \times \frac{\delta F}{\delta \boldsymbol{m}} + \boldsymbol{m} \times \frac{\delta F}{\delta \boldsymbol{n}} , \tag{2.26a}$$

$$\partial_t \boldsymbol{m} \;=\; \lambda_0 \boldsymbol{\nabla}^2 \frac{\delta F}{\delta \boldsymbol{m}} + \boldsymbol{n} \times \frac{\delta F}{\delta \boldsymbol{n}} + \boldsymbol{m} \times \frac{\delta F}{\delta \boldsymbol{m}} , \tag{2.26b}$$

where Γ_0 and λ_0 are bare kinetic coefficients. The functional form of the dissipative term in Eq. (2.26a) (constant Γ_0) reflects the fact that the staggered magnetization $\boldsymbol{n}$ is not a conserved quantity. The gradient-squared form of the corresponding term in Eq. (2.26b) is valid provided the total magnetization is conserved. If it is not, e.g., due to the presence of magnetic impurities, then this term must also have a constant coefficient and we have, instead of Eq. (2.26b),

$$\partial_t \boldsymbol{m} = -\mu_0 \frac{\delta F}{\delta \boldsymbol{m}} + \boldsymbol{n} \times \frac{\delta F}{\delta \boldsymbol{n}} + \boldsymbol{m} \times \frac{\delta F}{\delta \boldsymbol{m}} . \tag{2.24b'}$$

The last term on the right-hand-side of Eq. (2.26a) is often omitted since it is irrelevant for the critical dynamics of a classical antiferromagnet.[6] Equations (2.26) without this term, and with $v = w_1 = w_2 = 0$ in Eq. (2.4), is often referred to as Model G in the classification of Ref. 7. We also note that in order to calculate correlation functions one needs to add Langevin forces on the right-hand-sides of Eqs. (2.26), see Appendix B. Alternatively, one can calculate response functions in the presence of the fields $\boldsymbol{h}$ and $\mathfrak{h}$. We will take the latter approach; the fluctuation-dissipation theorem can then be used to determine the correlation functions.

III. LINEARIZED EQUATIONS OF MOTION, AND SPIN EXCITATIONS

We now parameterize the fields $\boldsymbol{n}$ and $\boldsymbol{m}$ as in Eq. (2.15) and linearize the kinetic equations (2.26) in the small fluctuations $\theta_{1,2,3}$ and $\pi_{1,2,3}$. This yields a 6×6 system of linear equations. The solutions for $\mathfrak{h} = 0$ give the eigenoscillations of the antiferromagnet. For counting purposes we will treat this as analogous to a mechanical system; that is, we have six eigenvalues and corresponding eigenvectors that characterize six modes. In the case of propagating modes, pairs of modes that propagate in opposite directions form one spin wave.

A. Conserved homogeneous magnetization; AFM order only

1. Zero field

For completeness, we first recall the well-known results for a vanishing external field, $\boldsymbol{h} = 0$. The six equations decouple into two identical pairs of 2×2 systems for θ_1, π_1 and θ_2, π_2, respectively, and two single equations for θ_3 and π_3, respectively. They read

$$\begin{pmatrix} i\Omega + \Gamma_0 a k^2 & -n_0(t + w_1 n_0^2 + c k^2) \\ n_0 a k^2 & i\Omega + \lambda_0(t + w_1 n_0^2 + c k^2)k^2 \end{pmatrix} \begin{pmatrix} \theta_{1,2} \\ \pi_{1,2} \end{pmatrix} = 0 . \tag{3.1}$$

and

$$\left[i\Omega + \Gamma_0(2u n_0^2 + a k^2)\right] \theta_3 \;=\; 0 , \tag{3.2}$$

$$\left[i\Omega + \lambda_0 \left(t + (w_1 + w_2) n_0^2 + c k^2\right) k^2\right] \pi_3 \;=\; 0 . \tag{3.3}$$

Equation (3.1) yields two identical pairs of gapless propagating modes with eigenfrequencies

$$\Omega_{1,\pm} = \Omega_{2,\pm} = \pm c_1 \, k + \frac{i}{2} \, \Gamma_1 a k^2 + O(k^3) . \tag{3.4a}$$

The speed of the propagating modes is

$$c_1 = n_0 \sqrt{a} \sqrt{t + w_1 n_0^2} = n_0 \sqrt{a h / m_0}\big|_{h\to 0} \tag{3.4b}$$

and the damping coefficient is

$$\Gamma_1 = \Gamma_0 + \lambda_0 (c_1 / n_0 a)^2 . \tag{3.4c}$$

The right eigenvectors are

$$\boldsymbol{\nu}_{\mathrm{R1},\pm} \equiv \begin{pmatrix} \theta_1 \\ \pi_1 \end{pmatrix}_{\pm} = \begin{pmatrix} 1 \\ \pm i (n_0 a / c_1) k + O(k^2) \end{pmatrix} \tag{3.4d}$$

and $\boldsymbol{\nu}_{\mathrm{R2},\pm} = \boldsymbol{\nu}_{\mathrm{R1},\pm}$, so the long-wavelength spin waves are transverse θ-fluctuations with a small admixture of π-fluctuations. Also of interest are the left eigenvectors

$$\boldsymbol{\nu}_{\mathrm{L1},\pm} \equiv (\theta_1, \pi_1)_{\pm} = \left(\pm i (n_0 a / c_1) k + O(k^2), 1\right) \tag{3.4e}$$

and $\boldsymbol{\nu}_{\mathrm{L2},\pm} = \boldsymbol{\nu}_{\mathrm{L1},\pm}$. Note that the left eigenvectors are structurally very different from the right ones. This is important for calculating time correlation functions, and for ensuring that the fluctuation-dissipation theorem holds, see Appendix B.

In addition, there is a pure π_3 mode described by Eq. (3.3) which is diffusive with an eigenfrequency

$$\Omega_3 = i D k^2 + O(k^4) , \tag{3.5a}$$

and a diffusion constant

$$D = \lambda_0 \left(t + (w_1 + w_2) n_0^2\right) . \tag{3.5b}$$

Finally, Eq. (3.2) describes a pure θ_3 mode which is relaxational with eigenfrequency

$$\Omega_4 = i \, 2\Gamma_0 u \, n_0^2 + O(k^2) . \tag{3.6}$$

These are the well-known results for an isotropic Heisenberg antiferromagnet:[1,11,28] There are two spin waves with a linear dispersion relation and quadratic damping, and the dynamics of the longitudinal homogeneous magnetization are diffusive.

Equation (3.6) implies that the longitudinal response function in the limit of long wavelengths and low frequencies is a constant. This result requires a qualification, as it changes qualitatively if one goes beyond the linearized theory: The coupling between the longitudinal and transverse degrees of freedom leads to a longitudinal response function that diverges, in classical antiferromagnets, as $1/k^{d-4}$ in dimensions $d < 4$,[29,30] and as $1/k^{d-3}$ at $T = 0$ in dimensions $d < 3$.[10] This is most easily seen by employing a nonlinear sigma model; alternatively, one can apply the renormalization-group techniques reviewed in Ref. 7 to the present formalism.

2. Transverse field

In Sec. II A we saw that if $w_2 > 0$ and $\boldsymbol{h} \neq 0$, the lowest free-energy state is a configuration where $\boldsymbol{m}_0$ and $\boldsymbol{h}$ are collinear and perpendicular to $\boldsymbol{n}_0$. We choose $\boldsymbol{n}_0 = (0, 0, n_0)$, $\boldsymbol{m}_0 = (m_0, 0, 0)$, and $\boldsymbol{h} = (h, 0, 0)$. The linearized kinetic equations then take the form of two 3×3 systems for $(\theta_1, \theta_3, \pi_1)$ and (θ_2, π_2, π_3), respectively. They read

$$\begin{pmatrix} i\Omega + \Gamma_0 a k^2 & 2n_0^2 m_0 (u - w_1) + m_0 a k^2 & -n_0 \left(h/m_0 + 2m_0^2 (v - w_1) + c\, k^2 \right) \\ -m_0 a k^2 & i\Omega + \Gamma_0 (2u n_0^2 + a k^2) & 2\Gamma_0 w_1 n_0 m_0 \\ n_0 a k^2 & 2\lambda_0 w_1 n_0 m_0 k^2 & i\Omega + \lambda_0 \left(h/m_0 + 2v m_0^2 + c k^2 \right) k^2 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_3 \\ \pi_1 \end{pmatrix} = 0 \ , \tag{3.7}$$

and

$$\begin{pmatrix} i\Omega + \Gamma_0 (w_2 m_0^2 + a k^2) & -n_0 (h/m_0 + c\, k^2) & -\Gamma_0 w_2 n_0 m_0 \\ n_0 a k^2 & i\Omega + \lambda_0 (h/m_0 + c\, k^2) k^2 & h + m_0 c\, k^2 \\ -\lambda_0 w_2 n_0 m_0 k^2 & -(h + m_0 c\, k^2) & i\Omega + \lambda_0 (h/m_0 + w_2 n_0^2 + c\, k^2) k^2 \end{pmatrix} \begin{pmatrix} \theta_2 \\ \pi_2 \\ \pi_3 \end{pmatrix} = 0 \ . \tag{3.8}$$

The former yields one pair of gapless propagating modes with eigenfrequencies

$$\Omega_{1,\pm} = \pm c_1\, k + \frac{i}{2}\, \Gamma_1 a k^2 + O(k^3) \ , \tag{3.9a}$$

and right and left eigenvectors

$$\boldsymbol{\nu}_{\mathrm{R}1,\pm} \equiv \begin{pmatrix} \theta_1 \\ \theta_3 \\ \pi_1 \end{pmatrix}_{\mathrm{R}1,\pm} = \begin{pmatrix} 1 \\ \mp i\, \frac{a w_1 m_0}{u\, c_1}\, k + O(k^2) \\ \pm i\, \frac{n_0\, a}{c_1}\, k + O(k^2) \end{pmatrix} \ , \tag{3.9b}$$

$$\begin{aligned} \boldsymbol{\nu}_{\mathrm{L}1,\pm} \ &\equiv \ (\theta_1, \theta_3, \pi_1)_{\mathrm{L}1,\pm} \\ &= \ \left(\pm i\, \frac{n_0 a}{c_1}\, k, \mp i\, \frac{n_0 m_0 a}{\Gamma_0 c_1} \left(1 - \frac{w_1}{u} \right) k, 1 \right) \\ &\qquad + O(k^2) \end{aligned} \tag{3.9c}$$

This is a generalization of one of the two transverse modes from Eqs. (3.4). The magnetic field leads to a θ_3-component of the eigenoscillation, and it modifies the speed and the damping coefficient of the mode:

$$c_1 \ = \ n_0 \sqrt{a} \sqrt{h/m_0 + 2(v - w_1^2/u) m_0^2} \ , \tag{3.9d}$$

$$\Gamma_1 \ = \ \Gamma_0 + \lambda_0\, (c_1/n_0\, a)^2 + \frac{m_0^2}{\Gamma_0}\, (1 - w_1/u)^2 \ . \tag{3.9e}$$

For $h = 0$ these expressions correctly reduce to Eqs. (3.4). They hold in the limit of asymptotically small k for fixed damping coefficients Γ_0, λ_0, i.e., for $k \ll c_1/a\Gamma_1$. The limits $k \to 0$ and $\Gamma_0, \lambda_0 \to 0$ do not commute, as is obvious from the last term in Eq. (3.9e). In the limit of vanishing damping coefficients at fixed small k the modes remain propagating and gapless, but the speed of the propagation changes. One finds for the eigenfrequencies

$$\tilde{\Omega}_{1,\pm} = \pm \tilde{c}_1\, k + O(k^2, \Gamma_0, \lambda_0) \ , \tag{3.10a}$$

and for the right and left eigenvectors

$$\tilde{\boldsymbol{\nu}}_{\mathrm{R}1,\pm} = \begin{pmatrix} 1 \\ \mp i\, (a\, m_0/\tilde{c}_1) k + O(k^2) \\ \pm i\, (a\, n_0/\tilde{c}_1) k + O(k^2) \end{pmatrix} \tag{3.10b}$$

$$\begin{aligned} \tilde{\boldsymbol{\nu}}_{\mathrm{L}1,\pm} \ &= \ \Big(\pm i c_1 k, -2n_0^2 m_0 (u - w_1), \\ &\qquad n_0 (\frac{h}{m_0} + 2m_0^2 (v - w_1)) \Big) + O(k^2, \Gamma_0, \lambda_0) \end{aligned} \tag{3.10c}$$

with

$$\tilde{c}_1 = n_0 \sqrt{a} \sqrt{h/m_0 + 2(u + v - 2w_1) m_0^2} \ , \tag{3.10d}$$

The second transverse modes are gapped propagating modes with eigenfrequencies

$$\Omega_{2,\pm} = \pm\sqrt{h^2 + c_2^2 k^2} + \frac{i}{2}\,\Gamma_2(h,k) a k^2 + O(\epsilon^3) \tag{3.11a}$$

and right and left eigenvectors

$$\boldsymbol{\nu}_{\mathrm{R}2,\pm} \equiv \begin{pmatrix}\theta_2\\ \pi_2\\ \pi_3\end{pmatrix}_{\mathrm{R}2,\pm} = \begin{pmatrix} 1 \\ \pm i\,\frac{m_0/h}{n_0}\,\sqrt{h^2 + c_2^2 k^2} + O(\epsilon^2) \\ m_0/n_0 + O(\epsilon^2)\end{pmatrix}, \tag{3.11b}$$

$$\begin{aligned}\boldsymbol{\nu}_{\mathrm{L}2,\pm} &\equiv (\theta_2, \pi_2, \pi_3)_{\mathrm{L}2,\pm} \\ &= \left(\frac{n_0 a k^2}{h} + O(\epsilon^2), \frac{\mp i}{h}\,\sqrt{h^2 + c_2^2 k^2} + O(\epsilon^2), 1\right).\end{aligned} \tag{3.11c}$$

where $\epsilon = O(k,h)$. Here

$$c_2 = n_0\sqrt{a\,h/m_0}\ , \tag{3.11d}$$

and

$$\Gamma_2(h,k) = \frac{c_2^2/a}{h^2 + c_2^2\,k^2}\,\left[\Gamma_0 a k^2 + \lambda_0(2h^2 + c_2^2 k^2)/n_0^2\,a\right]\ . \tag{3.11e}$$

We see that in this mode the magnetic field opens a gap of magnitude h. For $h = 0$ we again recover the expressions in Eqs. (3.4). Note that π_3 is part of the linear combination that comprises the gapped propagating modes. For $h = 0$ it decouples and is diffusive, see Eqs. (3.5).

In addition to these propagating modes there are two relaxational modes. One has an eigenfrequency

$$\Omega_3 = i2\Gamma_0 u n_0^2 + O(k^2) \tag{3.12a}$$

and right and left eigenvectors

$$\boldsymbol{\nu}_{\mathrm{R}3} \equiv \begin{pmatrix}\theta_1\\ \theta_3\\ \pi_1\end{pmatrix}_{\mathrm{R}3} = \begin{pmatrix}(1 - w_1/u)m_0/\Gamma_0 + O(k^2)\\ 1\\ O(k^2)\end{pmatrix}\ , \tag{3.12b}$$

$$\boldsymbol{\nu}_{\mathrm{L}3} \equiv (\theta_1, \theta_3, \pi_1)_{\mathrm{L}3} = \left(O(k^2), 1, \frac{w_1 m_0}{u n_0}\right)\ . \tag{3.12c}$$

This is valid in the limit of asymptotically small wavenumber. In the limit of asymptotically small damping one finds instead

$$\tilde{\Omega}_3 = i2\Gamma_0 u n_0^2 c_1^2/\tilde{c}_1^2 \tag{3.13a}$$

and

$$\tilde{\boldsymbol{\nu}}_{\mathrm{R}3} = \begin{pmatrix} 0 \\ 1 \\ \frac{2n_0^2 m_0(u - w_1) + m_0 a k^2}{n_0 h/m_0 + 2n_0 m_0^2(v - w_1) + n_0 c k^2}\end{pmatrix} + O(\Gamma_0, \lambda_0)\ . \tag{3.13b}$$

The other relaxational mode has an eigenfrequency

$$\Omega_4 = i\Gamma_0 w_2 m_0^2 + O(k^2) \tag{3.14a}$$

and right and left eigenvectors

$$\boldsymbol{\nu}_{\mathrm{R}4} \equiv \begin{pmatrix}\theta_2\\ \pi_2\\ \pi_3\end{pmatrix}_{\mathrm{R}4} = \begin{pmatrix}1\\ O(k^2)\\ O(k^2)\end{pmatrix}\ , \tag{3.14b}$$

$$\boldsymbol{\nu}_{\mathrm{L}4} \equiv (\theta_2, \pi_2, \pi_3)_{\mathrm{L}4} = \big(1, O(k^2), -n_0/m_0 + O(k^2)\big)\ . \tag{3.14c}$$

3. *Longitudinal field*

If $w_2 < 0$, then in a nonzero external field the lowest free-energy configuration is one where $\boldsymbol{m}_0$, $\boldsymbol{h}$, and $\boldsymbol{n}_0$ are all collinear; we choose them to be parallel to $(0,0,1)$. The linearized kinetic equations then decouple into one 2×2 system for (θ_3, π_3),

$$\begin{pmatrix} i\Omega + \Gamma_0(2un_0^2 + ak^2) & 2(w_1 + w_2)\Gamma_0 n_0 m_0 \\ 2(w_1 + w_2)\lambda_0 n_0 m_0 k^2 & i\Omega + \lambda_0(h/m_0 + 2vm_0^2 + ck^2)k^2\end{pmatrix}\begin{pmatrix}\theta_3\\ \pi_3\end{pmatrix} = 0\ , \tag{3.15}$$

and one 4×4 system for $(\theta_1, \theta_2, \pi_1, \pi_2)$,

$$\begin{pmatrix} i\Omega + \Gamma_2(k) & \tilde{m}_0(k) & -\tilde{n}_0(k) & w_2\Gamma_0 n_0 m_0 \\ -\tilde{m}_0(k) & i\Omega + \Gamma_2(k) & -w_2\Gamma_0 n_0 m_0 & -\tilde{n}_0(k) \\ n_0 a k^2 & -w_2\lambda_0 n_0 m_0 k^2 & i\Omega + \tilde{\lambda}_0(k)k^2 & h + m_0 c k^2 \\ w_2\lambda_0 n_0 m_0 k^2 & n_0 a k^2 & -(h + m_0 c k^2) & i\Omega + \tilde{\lambda}_0(k)k^2\end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\\ \pi_1\\ \pi_2\end{pmatrix} = 0\ . \tag{3.16a}$$

Here we have defined

$$\Gamma_2(k) = \Gamma_0(-w_2 m_0^2 + a k^2) \ , \tag{3.16b}$$
$$\tilde{m}_0(k) = m_0 \left(w_2(n_0^2 - m_0^2) + a k^2\right) \tag{3.16c}$$
$$\tilde{n}_0(k) = n_0 \left(h/m_0 - w_2(n_0^2 - m_0^2) + c k^2\right) \tag{3.16d}$$
$$\tilde{\lambda}_0(k) = \lambda_0(h/m_0 - w_2 n_0^2 + c k^2) \tag{3.16e}$$

Equations (3.16) lead to two pairs of gapped propagating modes. For small h and k we find for the eigenfrequencies[31]

$$\Omega_{1,\pm} = \frac{\pm 1}{\sqrt{2}} \left[(1 + w_2^2 n_0^4 m_0^2/h^2) h^2 + 2\tilde{c}^2 k^2 - \sqrt{(1 - w_2^2 n_0^4 m_0^2/h^2)^2 h^4 + 4h^2 (1 + w_2 n_0^2 m_0/h)^2 \tilde{c}^2 k^2} + O(\epsilon^4) \right]^{1/2} - i w_2 \Gamma_0 m_0^2 \left[1 + O(k^2/m_0^2)\right] \ , \tag{3.17a}$$

$$\Omega_{2,\pm} = \frac{\pm 1}{\sqrt{2}} \left[(1 + w_2^2 n_0^4 m_0^2/h^2) h^2 + 2\tilde{c}^2 k^2 + \sqrt{(1 - w_2^2 n_0^4 m_0^2/h^2)^2 h^4 + 4h^2 (1 + w_2 n_0^2 m_0/h)^2 \tilde{c}^2 k^2} + O(\epsilon^4) \right]^{1/2} + i \lambda_0 (h/m_0)\, k^2 \left[1 + O(k^2/m_0^2)\right] \ , \tag{3.17b}$$

where the innermost square root is defined as $\sqrt{x^2} = x$ irrespective of the sign of x,

$$\tilde{c} = n_0 \sqrt{a} \sqrt{h/m_0 - w_2 n_0^2} \tag{3.17c}$$

and ϵ can stand for either h or k. The damping coefficients are easily obtained to $O(\epsilon^2)$, but the results are complicated and we show only the leading terms for $k \to 0$ at fixed h. The corresponding right and left eigenvectors are, at $k = 0$,

$$\boldsymbol{\nu}_{\mathrm{R1},\pm} \equiv \begin{pmatrix} \theta_1 \\ \theta_2 \\ \pi_1 \\ \pi_2 \end{pmatrix}_{\mathrm{R1}\pm} = \begin{pmatrix} 1 \\ \mp i \\ 0 \\ 0 \end{pmatrix} \ , \tag{3.17d}$$

$$\boldsymbol{\nu}_{\mathrm{R2},\pm} \equiv \begin{pmatrix} \theta_1 \\ \theta_2 \\ \pi_1 \\ \pi_2 \end{pmatrix}_{\mathrm{R2}\pm} = \begin{pmatrix} 1 \\ \mp i \\ \pm i m_0/n_0 \\ m_0/n_0 \end{pmatrix} \ , \tag{3.17e}$$

$$\boldsymbol{\nu}_{\mathrm{L1},\pm} \equiv (\theta_1, \theta_2, \pi_1, \pi_2)_{\mathrm{L1},\pm} = (\mp i m_0/n_0, m_0/n_0, 1, \pm i) \ , \tag{3.17f}$$

$$\boldsymbol{\nu}_{\mathrm{L2},\pm} \equiv (\theta_1, \theta_2, \pi_1, \pi_2)_{\mathrm{L2},\pm} = (0, 0, 1, \pm i) \ . \tag{3.17g}$$

Note that the first two modes come with a damping coefficient that does not vanish as $k \to 0$, whereas the other two have a damping coefficient that vanishes as k^2, as in the transverse-field case. Also note that if the coupling constant w_2 were neglected, the first pair of modes would be gapless with a quadratic dispersion relation. Keeping all coupling constants consistent with the symmetry of the problem is thus important for obtaining the correct soft-mode structure.

From the 2×2 system we obtain a diffusive mode with eigenfrequency

$$\Omega_3 = i D_3 k^2 \ . \tag{3.18a}$$

The diffusion constant is given by

$$D_3 = \lambda_0 \left[\frac{h}{m_0} + 2 v m_0^2 - 2(w_1 + w_2)^2 m_0^2/u \right] \ , \tag{3.18b}$$

and the right and left eigenvectors are

$$\boldsymbol{\nu}_{\mathrm{R3}} \equiv \begin{pmatrix} \theta_3 \\ \pi_3 \end{pmatrix}_{\mathrm{R3}} = \begin{pmatrix} \frac{-(w_1+w_2)}{u} \frac{m_0}{n_0} + O(k^2) \\ 1 \end{pmatrix} \ , \tag{3.18c}$$

$$\boldsymbol{\nu}_{\mathrm{L3}} \equiv (\theta_3, \pi_3)_{\mathrm{L3}} = \left(\frac{-(w_1 + w_2)\lambda_0 m_0}{u \Gamma_0 n_0}\, k^2 + O(k^4), 1 \right) \ . \tag{3.18d}$$

This is a generalization of the diffusive π_3 mode in zero field.

Finally, there is a relaxational mode with eigenfrequency

$$\Omega_4 = i 2 \Gamma_0 u n_0^2 + O(k^2) \tag{3.19a}$$

and right and left eigenvectors

$$\boldsymbol{\nu}_{\rm R4} \equiv \begin{pmatrix}\theta_3\\ \pi_3\end{pmatrix}_{\rm R4} = \begin{pmatrix}1\\ \frac{(w_1+w_2)\lambda_0 m_0}{u\Gamma_0 n_0}\,k^2 + O(k^4)\end{pmatrix}\,, \quad (3.19b)$$

$$\boldsymbol{\nu}_{\rm L4} \equiv (\theta_3,\pi_3)_{\rm L4} = \left(1, \frac{w_1+w_2}{u}\,\frac{m_0}{n_0} + O(k^2)\right). \quad (3.19c)$$

which generalizes the relaxational θ_3 mode in zero field.

B. Conserved homogeneous magnetization; coexisting AFM and homogeneous orders

We now consider the modes for the case of coexisting AFM and homogeneous orders. As discussed in Sec. II A 2 we need to distinguish again between $w_2 > 0$, which leads to orthogonal order parameters, and $w_2 < 0$, which leads to collinear order parameters.

1. Orthogonal order parameters

The gapless propagating modes are obtained by taking the straightforward limit $h \to 0$ at fixed m_0 in Eqs. (3.9). The results from Sec. III A 2 remain valid except that the speed of the modes now is

$$c_1 = n_0\sqrt{2a(v - w_1^2/u)m_0^2} \quad (3.20a)$$

in the limit of vanishing wave number, and

$$\tilde{c}_1 = n_0\sqrt{2a(u+v-2w_1)m_0^2} \quad (3.20b)$$

in the limit of vanishing damping coefficients.

For the other oscillating modes the result cannot simply be read off from the results of Sec. III A 2, for two reasons: The oscillation frequency vanishes to $O(k)$ in the limit $h \to 0$ at fixed m_0, and in the damping term the limits $h \to 0$ and $\Gamma_0 \to 0$ do not commute in the coexisting-orders case. An analysis of Eq. (3.8) yields

$$\Omega_{2,\pm} = \pm d_2 k^2\left(1 - \frac{a^2 c n_0^2}{2m_0^2 w_2 d_2^2}\,k^2\right) + \frac{i}{2}\lambda_2 c k^4 + O(k^6) \quad (3.21a)$$

in the limit of asymptotically small wave numbers. Here

$$d_2 = \sqrt{m_0^2 c^2 + n_0^2 a c} \quad (3.21b)$$

and

$$\lambda_2 = \lambda_0(2 + n_0^2 a/m_0^2 c)\,. \quad (3.21c)$$

The corresponding right and left eigenvectors are

$$\boldsymbol{\nu}_{R2,\pm} \equiv \begin{pmatrix}\theta_2\\ \pi_2\\ \pi_3\end{pmatrix}_{R2,\pm} = \begin{pmatrix}1\\ \pm i d_2/n_0 c + O(k^2)\\ m_0/n_0 + a k^2/w_2 n_0 m_0 + O(k^4)\end{pmatrix}. \quad (3.21d)$$

$$\boldsymbol{\nu}_{L2,\pm} \equiv \left(\frac{\pm i a c n_0^2/d_2 + \lambda_0 w_2 n_0^2}{\Gamma_0 w_2 n_0 m_0}\,k^2, \mp i\,\frac{m_0 c}{d_2}, 1\right)\,. \quad (3.21e)$$

These modes have the characteristics of a ferromagnetic magnon, with a quadratic dispersion relation and a damping term that vanishes as k^4. In the limit of vanishing damping coefficients at fixed wave number the oscillation frequency remains the same, but the damping changes. The eigenfrequencies in this limit are

$$\tilde{\Omega}_{2,\pm} = \pm d_2 k^2 + \frac{i}{2}\left(\Gamma_0 a k^2 + \lambda_0 c k^4\right)\,. \quad (3.22)$$

Note the damping term proportional to k^2, which is absent in the limit of asymptotically small wave number.

The previous results for the relaxational modes, Eqs. (3.12) through (3.14), remain valid if one puts $h = 0$ at fixed m_0.

It is not obvious how the results given above cross over from the ones in Sec. III A 2 in the limit $h \to 0$ at fixed m_0. In Appendix A we give a solution of the eigenvalue problem represented by Eq. (3.8) that interpolates between the two results.

2. Collinear order parameters

Of the two pairs of gapped modes shown in Eqs. (3.17), one remains gapped in the limit $h \to 0$ at fixed m_0. However, m_0 should no longer be considered small. To zeroth order in the wave number we find, instead of Eq. (3.17a),

$$\Omega_{1,\pm} = \pm w_2 m_0(n_0^2 - m_0^2) - i w_2 \Gamma_0 m_0^2 + O(k^2)\,. \quad (3.23a)$$

For the other eigenfrequency, Eq. (3.17b), both the propagating part and the damping part vanish to $O(k)$ and $O(k^2)$, respectively, and we need to go to quartic order in k. An elementary but tedious calculation yields

$$\Omega_{2,\pm} = \pm\left(\frac{n_0^2 a + m_0^2 c}{m_0}\,k^2 + \frac{n_0^2 a^2}{w_2 m_0^3}\,k^4\right) + i\lambda_0\,\frac{n_0^2 a + m_0^2 c}{m_0^2}\,k^4 + O(k^6)\,. \quad (3.23b)$$

To lowest order in the wave number the corresponding eigenvectors are still given by Eqs. (3.17d) - (3.17g). We see that the gapped modes are fluctuations of the staggered magnetization that are gapped due to the coupling to the collinear homogeneous magnetization. The gapless mode is a linear combination of staggered and homogeneous fluctuations that are locked together and behave like a ferromagnetic magnon.

The modes shown in Eqs. (3.18) and (3.19) allow for a straightforward limit $h \to 0$ at fixed m_0 to be taken. We thus again obtain a diffusive mode with the diffusion constant given by Eq. (3.18b) with $h = 0$ and the eigenvectors given by Eqs. (3.18c, 3.18d), and a relaxation mode that is still given by Eqs. (3.19).

C. Non-conserved homogeneous magnetization: AFM order only

We now discuss the case of a non-conserved homogeneous magnetization, which means that Eq. (2.26b) gets replaced by Eq. (2.24b'). Although formally this amounts to replacing $\lambda_0 k^2$ by μ_0, the result can in in general not be obtained by performing this substitution in the results of Sec. III A, because the reality properties of the eigenvalue problems may change. As a result the problem gets quite involved, with many different cases depending on parameter values. Since we are mainly interested in soft modes, we will derive and discuss only those in detail. For relaxational modes, and for gapped propagating modes, we will list the eigenfrequencies at zero wave number, but we will not discuss their k-dependence or the corresponding eigenvectors.

1. Zero field

In the absence of an external field, the kinetic equations for the transverse fluctuations now are, instead of Eq. (3.1),

$$\begin{pmatrix} i\Omega + \Gamma_0 a k^2 & -n_0(t + w_1 n_0^2 + c k^2) \\ n_0 a k^2 & i\Omega + \mu_0(t + w_1 n_0^2 + c k^2) \end{pmatrix} \begin{pmatrix} \theta_{1,2} \\ \pi_{1,2} \end{pmatrix} = 0 \ . \tag{3.24}$$

For asymptotically small k this yields two identical diffusive modes with eigenfrequencies

$$\Omega_1 = \Omega_2 = i(\Gamma_0 + n_0^2/\mu_0) a\, k^2 + O(k^4) \ . \tag{3.25a}$$

The corresponding right and left eigenvectors are

$$\boldsymbol{\nu}_{\rm R1} \equiv \begin{pmatrix} \theta_1 \\ \pi_1 \end{pmatrix}_{\rm R1} = \begin{pmatrix} 1 \\ -n_0 a\, k^2/\mu_0(t + w_1 n_0^2) + O(k^4) \end{pmatrix} \ , \tag{3.25b}$$

$$\boldsymbol{\nu}_{\rm R2} \equiv \begin{pmatrix} \theta_2 \\ \pi_2 \end{pmatrix}_{\rm R2} = \boldsymbol{\nu}_{\rm R1} \tag{3.25c}$$

$$\boldsymbol{\nu}_{\rm L1} \equiv (\theta_1, \pi_1)_{\rm L1} = \big(\mu_0/n_0 + O(k^2), 1\big) \ , \tag{3.25d}$$

$$\boldsymbol{\nu}_{\rm L2} \equiv (\theta_1, \pi_1)_{\rm L2} = \boldsymbol{\nu}_{\rm L1} \ . \tag{3.25e}$$

We note in passing that for larger wavenumbers Eq. (3.24) describes two propagating modes, see the discussion in Sec. IV. The other solution of the quadratic equation yields two identical relaxational modes with eigenfrequencies

$$\Omega_3 = \Omega_4 = i\mu_0(t + w_1 n_0^2) + O(k^2) \ , \tag{3.26}$$

In addition, the analogs of Eqs. (3.2) and (3.3) yield two more relaxational modes. One is a pure θ_3 mode with eigenfrequency

$$\Omega_5 = i\Gamma_0(2u n_0^2 + a\, k^2) \ , \tag{3.27}$$

and one is a pure π_3 mode with eigenfrequency

$$\Omega_6 = i\mu_0 \left[t + (w_1 + w_2) n_0^2 + c\, k^2\right] \ . \tag{3.28}$$

Note that the mode spectrum is qualitatively different compared to the case of a conserved homogeneous magnetization: There are no propagating spin waves; instead, the transverse fluctuations form one diffusive mode and one relaxational one.

2. Transverse field

The relevant kinetic equations for this case are obtained by replacing $\lambda_0 k^2$ in Eqs. (3.7, 3.8) by μ_0. The first 3×3 system yields one diffusive mode with eigenfrequency

$$\Omega_1 = i\, D_1 k^2 + O(k^4) \tag{3.29a}$$

where the diffusion constant is given by

$$D_1 = a\left[\Gamma_0 + \frac{m_0^2}{\Gamma_0}(1 - w_1/u) + \frac{n_0^2}{\mu_0}\,\frac{h + 2m_0^3(v - w_1^2/u)}{h + 2m_0^3 v}\right]. \tag{3.29b}$$

The corresponding right and left eigenvectors are

$$\boldsymbol{\nu}_{\rm R1} = \begin{pmatrix}\theta_1\\ \theta_3\\ \pi_1\end{pmatrix}_{\rm R1} = \begin{pmatrix} 1 \\ \left(\frac{1}{2u\Gamma_0 n_0^2} + \frac{w_1/u}{\mu_0(h/m_0+2vm_0^2)}\right) m_0 a k^2 + O(k^4) \\ \frac{-n_0 a k^2}{\mu_0(h/m_0+2vm_0^2)} + O(k^4)\end{pmatrix}, \tag{3.29c}$$

$$\boldsymbol{\nu}_{\rm L1} = (\theta_1,\theta_3,\pi_1)_{\rm L1} = \left(\frac{\mu_0(h/m_0+2m_0^2 v)}{n_0(h/m_0+2m_0^2(v-w_1^2/u))} + O(k^2), \frac{-\mu_0 m_0(h/m_0+2m_0^2 v)(1-w_1/u)}{\Gamma_0 n_0(h/m_0+2m_0^2(v-w_1^2/u))} + O(k^2), 1\right). \tag{3.29d}$$

In addition, we find two relaxational modes with eigenfrequencies

$$\Omega_2 = i2\Gamma_0 u n_0^2 + O(k^2)\ , \tag{3.30}$$

and

$$\Omega_3 = i\mu_0(h/m_0 + 2m_0^2 v) + O(k^2)\ , \tag{3.31}$$

Now consider the 3×3 system that is the analog of Eq. (3.8). Here the energy scale $w_2\mu_0 n_0^2$ competes with h, and for small h there are neither propagating nor diffusive modes. Instead we find three relaxational modes with eigenfrequencies

$$\Omega_4 = i\mu_0 h/m_0 + O(\epsilon^2)\ , \tag{3.32}$$

$$\Omega_5 = i\mu_0(h/m_0 + w_2 n_0^2) + O(\epsilon^2)\ , \tag{3.33}$$

$$\Omega_6 = i w_2 h m_0 \frac{n_0^2+\Gamma_0\mu_0}{\mu_0(h/m_0+w_2 n_0^2)} + O(\epsilon^4)\ , \tag{3.34}$$

with $\epsilon = O(h, m_0)$.

3. Longitudinal field

The 2×2 matrix that is analogous to Eq. (3.15) yields two relaxational modes. To zeroth order in h and m_0 the eigenfrequencies are

$$\Omega_1 = i2\Gamma_0 u n_0^2 + O(\epsilon^2) \tag{3.35}$$

and

$$\Omega_2 = i\mu_0(h/m_0) + O(\epsilon^2)\ . \tag{3.36}$$

From the 4×4 matrix that is the analog of Eq. (3.16a) one finds four additional relaxational modes. To lowest order in $\epsilon = O(h, m_0)$ the corresponding eigenfrequencies are

$$\Omega_3 = \Omega_4 = i\mu_0(h/m_0 - w_2 n_0^2) + O(\epsilon)\ , \tag{3.37}$$

$$\Omega_5 = \Omega_6 = -i w_2 h m_0 \frac{n_0^2+\Gamma_0\mu_0}{\mu_0(h/m_0 - w_2 n_0^2)} + O(\epsilon^3)\ . \tag{3.38}$$

D. Non-conserved homogeneous magnetization: Coexisting AFM and homogeneous orders

1. Orthogonal order parameters

The 3×3 matrix that is analogous to Eq. (3.7) yields one diffusive mode:

$$\Omega_1 = i\left(\Gamma_0 + n_0^2/\mu_0 + m_0^2/\Gamma_0\right) a\, k^2 + O(k^4)\ . \tag{3.39a}$$

The corresponding right and left eigenvectors are

$$\boldsymbol{\nu}_{\rm R1} \equiv \begin{pmatrix}\theta_1\\ \theta_3\\ \pi_1\end{pmatrix}_{\rm R1} = \begin{pmatrix} 1 \\ \frac{w_1\Gamma_0 n_0^2 + v\mu_0 m_0^2}{2\Gamma_0\mu_0 n_0^2 m_0(uv-w_1^2)}\, a k^2 + O(k^4) \\ \frac{-(u\Gamma_0 n_0^2 + w_1\mu_0 m_0^2)}{2\Gamma_0\mu_0 n_0 m_0^2(uv-w_1^2)}\, a k^2 + O(k^4)\end{pmatrix}, \tag{3.39b}$$

$$\boldsymbol{\nu}_{\rm L1} = (\theta_1,\pi_2,\pi_3)_{\rm L1} = \left(\frac{\mu_0}{n_0} + O(k^2), -\frac{\mu_0 m_0}{\Gamma_0 n_0} + O(k^2), 1\right). \tag{3.39c}$$

Note that if we put $\mu_0 = \lambda_0 k^2$ we can *not* relate these expressions to Eqs. (3.9, 3.20), since the nature of the mode has changed.

The other two modes are relaxational with eigenfrequencies

$$\Omega_{2,3} = i\Big(u\Gamma_0 n_0^2 + v\mu_0 m_0^2 \mp\sqrt{(u\Gamma_0 n_0^2 - v\mu_0 m_0^2)^2 + 4w_1^2\Gamma_0\mu_0 n_0^2}\Big)\,. \tag{3.40}$$

From the 3×3 matrix that is analogous to Eq. (3.8) we obtain two soft modes. Depending on parameter values, they can be gapless propagating with both the oscillation frequency and the damping coefficient proportional to k^2:

$$\Omega_{4,\pm} = \frac{\pm k^2}{2(\Gamma_0 m_0^2 + \mu_0 n_0^2)} \sqrt{4d_2^2(\Gamma_0 m_0^2 + \mu_0 n_0^2)^2 - \mu_0^2(\Gamma_0 a - \mu_0 c)^2 n_0^4} + i\mu_0 \frac{\Gamma_0(n_0^2 a + 2m_0^2 c) + \mu_0 n_0^2 c}{2(\Gamma_0 m_0^2 + \mu_0 n_0^2)} k^2 + O(k^4) \ , \tag{3.41a}$$

with d_2 from Eq. (3.21b) and right and left eigenvectors

$$\boldsymbol{\nu}_{\mathrm{R4},\pm} \equiv \begin{pmatrix} \theta_2 \\ \pi_2 \\ \pi_3 \end{pmatrix}_{\mathrm{R4},\pm} = \begin{pmatrix} 1 \\ \pm i\sqrt{m_0^2/n_0^2 + \mu_0/\Gamma_0} + O(k^2) \\ m_0/n_0 + O(k^2) \end{pmatrix} \ , \tag{3.41b}$$

$$\boldsymbol{\nu}_{\mathrm{L4},\pm} \equiv (\theta_2, \pi_2, \pi_3)_{\mathrm{L4},\pm} = \left(\frac{n_0 \mu_0}{m_0 \Gamma_0} + O(k^2), \mp i\sqrt{1 + \mu_0 n_0^2/\Gamma_0 m_0^2} + O(k^2), 1 \right) \ . \tag{3.41c}$$

For $\mu_0 \to \lambda_0 k^2$ this is consistent with Eqs. (3.21). However there is no guarantee that the expression under the square root in Eq. (3.41a) is positive. If it is not, then one has instead two diffusive modes,

$$\tilde{\Omega}_{4,\pm} = iD_{4,\pm} k^2 + O(k^4) \tag{3.42a}$$

with diffusion coefficients

$$D_{4,\pm} = \frac{1}{2(\Gamma_0 m_0^2 + \mu_0 n_0^2)} \left[\Gamma_0 \mu_0 (n_0^2 a + 2m_0^2 c) + \mu_0^2 n_0^2 c \pm \sqrt{\mu_0^2(\Gamma_0 a - \mu_0 c)^2 n_0^4 - 4d_2^2(\Gamma_0 m_0^2 + \mu_0 n_0^2)^2} \right] \ . \tag{3.42b}$$

In addition, there is one relaxational mode with eigenfrequency

$$\Omega_5 = i w_2 (\Gamma_0 m_0^2 + \mu_0 n_0^2) \ . \tag{3.43}$$

2. Collinear order parameters

The 4×4 problem that is analogous to Eq. (3.16a) yields a pair of gapless propagating modes with eigenfrequencies

$$\begin{aligned} \Omega_{1,\pm} \ &= \ \pm m_0(m_0^2 c + n_0^2 a) \frac{(m_0^2 - n_0^2)^2 + \Gamma_0^2 m_0^2 + \mu_0^2 n_0^2 + 2\Gamma_0 \mu_0 n_0^2}{m_0^2(m_0^2 - n_0^2)^2 + (\Gamma_0 m_0^2 + \mu_0 n_0^2)^2} k^2 \\ &\quad + i\mu_0(m_0^2 c + n_0^2 a) \frac{(m_0^2 - n_0^2)^2 + \Gamma_0^2 m_0^2 + \Gamma_0 \mu_0 n_0^2}{m_0^2(m_0^2 - n_0^2)^2 + (\Gamma_0 m_0^2 + \mu_0 n_0^2)^2} k^2 + O(k^4) \ . \end{aligned} \tag{3.44a}$$

The right and left eigenvectors are, at $k = 0$,[32]

$$\boldsymbol{\nu}_{\mathrm{R1},\pm} \ \equiv \ \begin{pmatrix} \theta_1 \\ \theta_2 \\ \pi_1 \\ \pi_2 \end{pmatrix}_{\mathrm{R1},\pm} = \begin{pmatrix} 1 \\ \mp i \\ \pm i\, m_0/n_0 \\ m_0/n_0 \end{pmatrix} \ , \tag{3.44b}$$

$$\begin{aligned} \boldsymbol{\nu}_{\mathrm{L1},\pm} \ &\equiv \ (\theta_1, \theta_2, \pi_1, \pi_2)_{\mathrm{L1},\pm} \\ &= \ \left(\frac{\Gamma_0 \mu_0 n_0 m_0 \mp i\mu_0 n_0 (n_0^2 - m_0^2)}{(n_0^2 - m_0^2)^2 + m_0^2 \Gamma_0^2}, \frac{\mu_0 n_0 (n_0^2 - m_0^2) \pm i\Gamma_0 \mu_0 n_0 m_0}{(n_0^2 - m_0^2)^2 + m_0^2 \Gamma_0^2}, \mp i, 1 \right) \ . \end{aligned} \tag{3.44c}$$

In addition, there is a pair of gapped propagating modes with eigenfrequencies

$$\Omega_{2,\pm} = \pm w_2 m_0 (m_0^2 - n_0^2) - i w_2 (\Gamma_0 m_0^2 + \mu_0 n_0^2) + O(k^2) \ . \tag{3.45}$$

Note that the damping coefficient does not vanish at $k = 0$.

The 2×2 problem that is analogous to Eq. (3.15) now yields two relaxational modes with eigenfrequencies

$$\Omega_{3,4} = i \left(u\Gamma_0 n_0^2 + v\mu_0 m_0^2 \pm \sqrt{(u\Gamma_0 n_0^2 - v\mu_0 m_0^2)^2 + 4(w_1 + w_2)^2 \Gamma_0 \mu_0 n_0^2 m_0^2} \right) \ . \tag{3.46}$$

IV. SUMMARY, AND DISCUSSION

In summary, we have used time-dependent Ginzburg-Landau theory, with the basic equation of motion for magnetic moments the only input, to determine the long-

wavelength spin dynamics of antiferromagnets in various physical situations. We have considered purely antiferromagnetic order subject to an external magnetic field, and have distinguished between the cases of a conserved and a non-conserved homogeneous magnetization. We have also considered the case of coexisting antiferromagnetic and ferromagnetic orders, as it occurs, for example, in ferrimagnets and canted magnets. Our results are summarized in Tables I and II.

Table I: Number of modes of various types for antiferromagnetic order with the homogeneous magnetization conserved/non-conserved.

Field	Modes			
	propagating* gapless	propagating* gapped	diffusive	relaxational
zero	4 / 0	0 / 0	1 / 2	1** / 4
transverse	2 / 0	2 / 0	0 / 1	2 / 5
longitudinal	0 / 0	4 / 0	1 / 0	1 / 6

* Two propagating modes form one spin wave.
** See the comments after Eq. (3.6).

Table II: Number of modes of various types for coexisting antiferromagnetic and ferromagnetic orders with the homogeneous magnetization conserved/non-conserved.

Order Parameters	Modes			
	propagating gapless	propagating gapped	diffusive	relaxational
orthogonal	4 / 2 or 0	0 / 0	0 / 1 or 3	2 / 3
collinear	2 / 2	2 / 2	1 / 0	1 / 2

A. Nature of spin excitations

One striking aspect of these results is the qualitative difference between the cases of a conserved and a non-conserved homogeneous magnetization, respectively. In the former case in a zero external field the transverse fluctuations form two pairs of gapless propagating modes, or two spin waves, with a linear dispersion relation. The longitudinal order-parameter fluctuations are relaxational, whereas the longitudinal fluctuations of the homogeneous magnetization are diffusive. In a transverse field one of the spin-wave pairs remains gapless with a linear dispersion, the other one acquires a gap that is proportional to the field, and there is no diffusive mode. In a longitudinal field one finds two pairs of gapped propagating modes, or two spin waves, and one diffusive mode. In all cases there thus are two propagating spin waves. If the homogeneous magnetization is not conserved, in contrast, there are no propagating spin waves and the only soft modes are two diffusive modes in zero field and one diffusive mode in the transverse-field case. (To avoid misunderstandings we reiterate that there is no experimental control over the orientation of the field with respect to the order parameter; the sign of the Landau parameter w_2 determines which case is realized in any given system.) This case is realized, for instance, in an antiferromagnet with magnetic impurities. The results of Ref. 33 obtained by applying linear spin-wave theory to this case, which found propagating modes, are thus not valid in the long-wavelength limit. Propagating spin waves are reconstituted, however, above a threshold wave number, as we discuss next.

An interesting feature of the antiferromagnet in zero field is the crossover between the cases of a conserved and a non-conserved homogeneous magnetization in the limit $\mu_0 \to 0$ or, for fixed small μ_0, with increasing wave number. Consider the eigenproblem posed by Eq. (3.24), and assume that the damping is small in the sense that $\mu_0\Gamma_0 \ll n_0^2$ and $\mu_0^2 \ll n_0^2 a/c$. Then the reality properties of the quadratic equation change at a critical value of the wave number k, and the diffusive modes become propagating for $k > k_c$ where $k_c = \mu_0\sqrt{(t + w_1 n_0^2)/a n_0^2}$. In more physical terms, the threshold wave number is $k_c/k_0 = 1/E_0\tau_{mag}$, with k_0 and E_0 microscopic wavenumber and energy scales, respectively, and $1/\tau_{mag} \propto \mu_0$ the relaxation rate associated with the magnetic impurities. Similarly, the two relaxational modes from Eq. (3.26) become propagating, and collectively these four modes cross over to the four propagating modes (two spin waves) that characterize the antiferromagnet with a conserved homogeneous magnetization, Eq. (3.4a). For metals at low temperatures we expect k_0 and E_0 to be on the order of the Fermi wave number and the Fermi energy, respectively, and the value of k_c can be quite small.

Alternatively, let $\mu_0 \to 0$ in the limit of asymptotically small k. Then the two diffusive modes and two of the relaxational modes listed in Table I for the zero-field non-conserved case turn into the four propagating modes of the conserved case, one of the relaxational modes becomes diffusive, and the remaining mode remains relaxational.

B. Properties of propagating spin waves

1. *Models, and their restrictions*

The long-wavelength and low-frequency properties of the Heisenberg model for an antiferromagnetic nearest-neighbor coupling are captured by an effective field theory that takes the form of a nonlinear sigma model.[34–36] It needs to be noted that this model *always* has the staggered magnetization point in a direction perpendicular to an external magnetic field, i.e., the simple Heisenberg model captures only the case $w_2 > 0$ (the transverse-

field case) in Eq. (2.4) and is less general than the Landau theory we have used. The sigma model correctly describes the two gapless and two gapped modes shown in Table I.[36] In Ref. 37 it was pointed out that the field-dependence of the dispersion relations predicted by the sigma model does not agree with the results of spin-wave theory, even though the qualitative features of the spin waves are correct. A comparison with the coefficients c_1 and c_2 in Eqs. (3.9d) and (3.11d) shows that their field dependence as obtained from the hydrodynamic equations is still substantially more complicated than the results of the spin-wave theory employed in Ref. 37. In the sigma model, both of these coefficients are replaced by their values at $h = 0$.

We have truncated the Landau free energy, Eq. (2.4), at biquadratic order. The property that the staggered magnetization is either parallel or perpendicular to the homogeneous magnetization is a consequence of this truncation. For instance, keeping a term proportional to $(\boldsymbol{n} \cdot \boldsymbol{m})^4$ allows for states where $\boldsymbol{n}$ and $\boldsymbol{m}$ are neither collinear nor orthogonal to each other. This can become important in large magnetic fields, when $\boldsymbol{m}$ is no longer small, see, e.g., Ref. 38. If this is important for a specific purpose one can keep higher order terms in the free-energy functional and repeat the analysis of the hydrodynamic equations, which is completely general.

We also note that our discussion applies to systems in spatial dimensions $d \geq 2$. Antiferromagnetic spin chains show qualitatively different behavior that requires a special treatment; see, e.g., Ref. 39.

2. Damping

In Eqs. (2.26) we have used the standard Landau-Lifshitz form for the damping terms.[7,25,26] Landau and Lifshitz considered a ϕ^4-theory and enforced a time-independent modulus of the magnetic moment by writing Eq. (2.23) with the damping term added as

$$\begin{aligned} \partial_t \boldsymbol{M} &= \boldsymbol{M} \times \frac{\delta F}{\delta \boldsymbol{M}} - \Gamma \left[\frac{\delta F}{\delta \boldsymbol{M}} - \boldsymbol{M} \left(\boldsymbol{M} \cdot \frac{\delta F}{\delta \boldsymbol{M}} \right) / \boldsymbol{M}^2 \right] \\ &= \boldsymbol{M} \times \frac{\delta F}{\delta \boldsymbol{M}} + \Gamma \boldsymbol{M} \times \left(\boldsymbol{M} \times \frac{\delta F}{\delta \boldsymbol{M}} \right) / \boldsymbol{M}^2 \ . \quad (4.1) \end{aligned}$$

Gilbert later proposed to replace the $\boldsymbol{M} \times (\delta F / \delta \boldsymbol{M})$ in the damping term by $\partial_t \boldsymbol{M}$.[40] The resulting Landau-Lifshitz-Gilbert equation,

$$\partial_t \boldsymbol{M} = \boldsymbol{M} \times \frac{\delta F}{\delta \boldsymbol{M}} + \Gamma \boldsymbol{M} \times \partial_t \boldsymbol{M} / \boldsymbol{M}^2 \ , \quad (4.2)$$

is very popular on phenomenological grounds, but its consistency with basic principles of irreversible thermodynamics is questionable, see, e.g., Ref. 41. Equation (4.2) can be mapped onto Eq. (4.1) at the expense of making the prefactor of the Bloch term depending on the damping coefficient.[42] This observation underscores the fact that the Gilbert modification does not have the standard hydrodynamic form, but it also means that, as far as the nature of the spin excitations is concerned, the difference between Eqs. (4.1) and (4.2) is irrelevant.

Regarding the nature of the damping, in a simple antiferromagnet in zero field the damping is quadratic in the wavenumber, and thus always small, in the long-wavelength limit, compared to the oscillation frequency, which is linear in k, see Eq. (3.4a). This is important for ensuring the correct relation between the static Goldstone modes and the time-correlation functions, as we demonstrate in Appendix B. In a simple ferromagnet, the damping is proportional to k^2, and the damping to k^4. For an antiferromagnet with a conserved homogeneous magnetization in a transverse field, the gapless propagating modes still have a linear dispersion relation and both they and the gapped modes have a quadratic damping, see Eqs. (3.9, 3.11), but in a longitudinal field one of the pairs of gapped modes has a damping coefficient that is nozero at $k = 0$, see Eq. (3.17a). For coexisting antiferromagnetic and homogeneous order there are ferromagnon-like spin waves, with a quadratic oscillation frequency and quartic damping, see Eqs. (3.21a) and (3.23b).

In the context of damping it is also interesting to see how the damping term that appears in effective field theories for metals,[43] which is often referred to as Landau damping in analogy to the corresponding effect in a collisionless classical plasma, is related to the hydrodynamic equations. In the case of a ferromagnetic metal with nonmagnetic impurities the Landau-damping term in the paramagnon propagator has the form $|\Omega|/Dk^2$, where D is the diffusion coefficient related to the diffusive dynamics of the conduction electrons in the spin-triplet channel. This corresponds to the damping term in the hydrodynamic equations for the case of a conserved order parameter, see Eq. (2.26b). For an antiferromagnetic metal, or for a ferromagnet with magnetic impurities, the corresponding term is $|\Omega|\tau$, with τ a k-independent relaxation time. This corresponds to the damping term for a non-conserved order parameter, see Eqs. (2.26a) and (2.24b').

In a clean metallic ferromagnet the Landau-damping term has the form $|\Omega|/v_{\rm F}k$, with $v_{\rm F}$ the Fermi velocity.[43] In order to see how this case fits into the hydrodynamic description, we note that Eq. (2.26b) implies a long-wavelength susceptibility of the form

$$\chi(k, i\Omega) \propto -i\Omega/\lambda_0 k^2 + k^2 \ , \quad (4.3)$$

for the conserved and non-conserved cases, respectively. In our case χ is the spin susceptibility, but the following discussion holds more generally for any order-parameter susceptibility. In a clean metallic system at $T = 0$ the kinetic coefficient λ_0 does not exist in the limit of zero frequency and wave number and scales as $\lambda_0 \sim 1/(\Omega + k^{z_\lambda})$, with z_λ the dynamical exponent characteristic of the kinetic coefficient. In the conserved case, $z_\lambda \geq 2$ leads to $\Omega \sim k^2$, and $z_\lambda < 2$ leads to $\Omega \sim k^{4-z_\lambda}$. For the conduction electrons in a metal one has $\Omega \sim k$, and as long as the order parameter couples to the conduction

electrons one therefore expects $z_\lambda = 1$. Effectively, we thus have $\lambda_0 \sim 1/k$,[8] which is consistent with the above form of the Landau-damping term.

C. Outlook: Nonlinear effects, and consequences for quantum phase transitions

The current paper lays the groundwork for several investigations of properties of quantum antiferromagnets. For instance, Ref. 10 considered the coupling of the spin waves to the longitudinal fluctuations, and the resulting behavior of the longitudinal susceptibility and the dynamical structure factor. Within the current formalism, these effects are due to the nonlinearities in the hydrodynamic equations that we have neglected (see also the remark after Eq. (3.6)). Due to limitations inherent in the nonlinear sigma model used in Ref. 10 the only cases considered were those of zero field, and antiferromagnetic order in a transverse field. An analogous investigation of all cases discussed in the present paper would be of interest, especially for the dynamical structure factor, which is directly measurable by neutron scattering.

A renormalized mean-field theory for quantum ferromagnets predicted that the quantum phase transition from a paramagnet to a ferromagnet or ferrimagnet in clean metals is necessarily first order.[44,45] This prediction has been confirmed by experiments on many different materials.[16] The present paper makes possible analogous theories for other quantum phase transitions. For instance, one expects the prediction of a universal tricritical point[45] to hold for canted magnets (the case $w_2 > 0$ in our notation) in addition to ferrimagnets ($w_2 > 0$). Furthermore, it allows for a treatment of the quantum phase transition from a metallic ferromagnet to an antiferromagnet, of which there are various known examples.[16]

Another class of phase transitions that is of interest in this context is the spin-flop transition in uniaxial antiferromagnets as a function of an applied external field. The history of these classical transitions goes back to Neél in the 1930s; they have been studied extensively from the viewpoint of classical phase-transition theory[46] and continue to be of great interest, see Ref. 17 and references therein. They usually are first order, but can be second order in certain materials.[47] An investigation of the corresponding quantum phase transitions would be of interest.

Acknowledgments

This work was supported by the NSF under Grants No. DMR-1401410 and No. DMR-1401449. Part of this work was performed at the Aspen Center for Physics, which is supported by the NSF under Grant No. PHY-1066293.

Appendix A: Interpolating solution for the transverse-field and orthogonal-order-parameters cases

As we saw in Secs. III A 2 and III B 1, the nature of the solution of Eq. (3.8) changes qualitatively if one considers the limit $h \to 0$ for fixed m_0. To see the crossover, one can of course solve the cubic equation exactly, but this is not very illuminating. It is more useful to keep all terms that contribute to leading order in k and h in either the pure AFM case or the coexisting-orders case. The eigenfrequencies then can be found by solving linear equations only. The result is

$$\begin{aligned}\Omega_{2,\pm} &= \pm\sqrt{(h+m_0ck^2)^2+c_2^2k^2+n_0^2ack^4} \\ &\quad +\frac{i}{2}\,k^2\left[\lambda_0(h/m_0+ck^2)+\frac{(h/m_0)(\lambda_0 h^2+\Gamma_0 n_0^2a^2k^2)+\Gamma_0^2\lambda_0 w_2^2 m_0^2(an_0^2+cm_0^2)k^2}{h^2+c_2^2+\Gamma_0^2w_2^2m_0^4}\right] . \end{aligned} \tag{A1a}$$

The corresponding right and left eigenvectors are

$$\boldsymbol{\nu}_{R2,\pm} \equiv \begin{pmatrix}\theta_2\\ \pi_2\\ \pi_3\end{pmatrix}_{R2,\pm} = \begin{pmatrix} 1 \\ \pm i\,\frac{\sqrt{(h+m_0ck^2)^2+c_2^2k^2+n_0^2ack^4}}{n_0(h/m_0+ck^2)} \\ m_0/n_0\end{pmatrix} . \tag{A1b}$$

$$\boldsymbol{\nu}_{L2,\pm} \equiv (\theta_2,\pi_2,\pi_3)_{L2,\pm} = \left(\frac{n_0ak^2}{h+m_0ck^2}, \frac{\mp i}{h+m_0ck^2}\sqrt{(h+m_0ck^2)^2+(h/m_0+ck^2)n_0^2ak^2}, 1\right) . \tag{A1c}$$

These expressions are valid to leading order in k and h for both the AFM case, where the oscillation frequency is of $O(h,k)$ and the damping is of $O(k^2)$, and the coexisting-orders case, where the oscillation frequency is of $O(k^2)$

and the damping is of $O(k^4)$. They correctly interpolate between Eqs. (3.11) and (3.21). To linear order in Γ_0 and λ_0 in the coexisting-orders case one recovers Eq. (3.22).

Appendix B: Time correlation functions, and the fluctuation-dissipation theorem

We mentioned at the end of Sec. II that we have chosen to calculate response functions. It is illustrative to consider the related problem of calculating time correlation functions. To this end we add Langevin forces $\boldsymbol{f}_n$ and $\boldsymbol{f}_m$ on the right-hand sides of Eq. (2.26a) and (2.26b), respectively. These are random forces that are characterized by Gaussian distributions with second moments

$$\langle f_n^i(\boldsymbol{x},t)\, f_n^j(\boldsymbol{x}',t')\rangle \;=\; 2T\,\Gamma_0\,\delta_{ij}\delta(\boldsymbol{x}-\boldsymbol{x}')\,\delta(t-t')\;, \tag{B1a}$$

$$\langle f_m^i(\boldsymbol{x},t)\, f_m^j(\boldsymbol{x}',t')\rangle \;=\; -2T\,\lambda_0\boldsymbol{\nabla}^2\,\delta_{ij}\delta(\boldsymbol{x}-\boldsymbol{x}')\,\delta(t-t'), \tag{B1b}$$

$$\langle f_n^i(\boldsymbol{x},t)\, f_m^j(\boldsymbol{x}',t')\rangle \;=\; 0\;, \tag{B1c}$$

where T is the temperature. For simplicity, we consider only classical systems in this appendix; for a discussion of a quantum Langevin equation, see Ref. 48. These relations guarantee the validity of the fluctuation-dissipation theorem.

We now illustrate the use of this formalism for the simple case of an AFM in zero field, Sec. III A 1; the other cases can be analyzed analogously.

Consider Eq. (3.1) and add the fluctuating forces. Then we have, structurally,

$$M\,\boldsymbol{\psi} = \boldsymbol{f}\;, \tag{B2}$$

where M denotes the 2×2 matrix, $\boldsymbol{\psi}$ comprises the fluctuations θ and π, and $\boldsymbol{f}$ the appropriate components of the fluctuating forces $\boldsymbol{f}_n$ and $\boldsymbol{f}_m$. Multiplying with the left eigenvector $\boldsymbol{\nu}_{\mathrm{L}1,\pm}$, Eq. (3.4e), yields

$$\Lambda_{1\pm}\,\psi_{1\pm} = \pm i(n_0a/c_1)k f_n^1 + f_m^1\;, \tag{B3a}$$

where

$$\psi_{1\pm} = \pm i(n_0a/c_1)k\,\theta_1 + \pi_1 \tag{B3b}$$

and

$$\Lambda_{1\pm} = i\Omega \pm ic_1k + \frac{1}{2}\Gamma_1 ak^2 \tag{B3c}$$

is the eigenvalue of the matrix M that corresponds to the eigenvector $\boldsymbol{\nu}_{\mathrm{L},1\pm}$, see Eqs. (3.4). For the correlation function of $\psi_{1\pm}$ we thus have

$$\langle\psi_{1\pm}\,\psi_{1\pm}^*\rangle_{\Omega,\boldsymbol{k}} = \frac{-2T(n_0a/c_1)^2\Gamma_1k^2}{(\Omega\pm c_1k)^2+\Gamma_1^2a^2k^4/4}\;, \tag{B4}$$

with Γ_1 from Eq. (3.4c). The equal-time correlation function, which is, apart from a factor of T, equal to the $\psi_{1\pm}$-susceptibility χ_ψ by the fluctuation-dissipation theorem, is obtained by integrating over all frequencies:

$$T\chi_\psi = \int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\,\langle\psi_{1\pm}\,\psi_{1\pm}^*\rangle_{\Omega,\boldsymbol{k}} = -2Ta(n_0/c_1)^2\;. \tag{B5}$$

Now consider $k\theta_1 = -i(c_1/2n_0a)(\psi_{1+}-\psi_{1-})$. Equation (B5) yields $k^2\chi_{\theta_1} = 1/a$, or

$$\chi_{\theta_1} = 1/ak^2 \tag{B6}$$

in agreement with Eq. (2.19b).

We see that the structure of the left eigenvector, which is very different from the right one, is crucial for obtaining the correct result for the static susceptibility. The more complicated cases can be analyzed analogously. In particular, we note that the structure of the left eigenvector $\boldsymbol{\nu}_{\mathrm{L}2,\pm}$ in the longitudinal-field case, Eq. (3.17g), makes sure that there in no diverging static susceptibility, in agreement with the absence of any Goldstone modes in the static analysis in Sec. II A 3.

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